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Residue Recurrence and Scaling Properties in the Complex Embedding of Prime Numbers

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Abstract

This paper presents an exploratory analytic framework for examining the distribution of prime numbers through complex exponential embeddings and their associated Residue Recurrence Classes (RRCs). Each prime p is mapped to a unit-magnitude complex oscillator $\omega(p) = \mathrm{e}^{ip}$, allowing the study of angular dynamics modulo 2π within a multiplicative and rotational setting. This representation generalises classical modular analyses and reveals local angular recurrences and clustering within the prime sequence. Residue Recurrence Classes are defined as collections of primes whose angular residues approximate those generated by the integer or fractional roots of smaller primes. Empirical computations suggest that such classes contain numerous primes and exhibit coherent alignments on the unit circle, indicating possible small-scale regularities within the global modular uniformity of primes. We further introduce a quasi-harmonic prime function, formulated as a root-weighted cosine sum over prime angular residues, and analyse the cumulative complex prime centre summation. The latter exhibits an approximate power-law relation between its real and imaginary components, consistent with self-similar or fractal-like scaling. These observations are empirical in nature and do not contradict the asymptotic equidistribution of primes modulo 2π. Rather, they suggest that within finite intervals, the prime sequence may display local harmonic organisation and interference effects that warrant further formal investigation within analytic number theory.

Keywords

Prime Numbers, Complex Exponential Embeddings, Residue Recurrence, Modular Distribution, Quasi-Harmonic Sums, Scaling Relations, Analytic Number Theory, Geometric Representations

1. Introduction

The distribution of prime numbers remains one of the central problems in analytic number theory. While primes are defined multiplicatively as the fundamental building blocks of the integers, their additive and modular properties have long been studied for signs of hidden order within apparent randomness. Classical approaches include Weyl's criterion for equidistribution [1], ergodic and combinatorial methods such as Furstenberg's theorem [2], and probabilistic analogies derived from random matrix theory [3]. Together, these results have progressively deepened our understanding of primes as a sequence that, although seemingly irregular, admits subtle forms of arithmetic and spectral coherence [4].

A related theme in modern mathematical analysis is that nonlinear or geometric representations often reveal regularities in sequences that appear random in their native arithmetic domain. Fractal strings, spectral zeta functions, and geometric embeddings have revealed self-similarity and scaling behaviour in diverse arithmetic and analytic settings [5] [6]. Similar methods have been applied to visualise and study the global geometry of prime distributions [7]-[9].

While earlier visual studies have explored prime distributions in the complex plane, those approaches were primarily descriptive or computational. In contrast, the present work develops a formal analytic structure—defining Residue Recurrence Classes (RRCs) and a quasi-harmonic prime summation function—to characterise angular recurrence, interference, and scaling behaviour within this complex embedding. The aim is not to propose a deterministic law of primes, but to identify reproducible geometric features that may inform further theoretical exploration.

In this work, we propose a complex-analytic framework that treats each prime number as a point on the unit circle,

$$\omega(p) := \mathrm{e}^{ip},$$

where $\omega(p)$ may be interpreted as a unit oscillator with phase p [10]. This mapping enables the study of prime sequences modulo 2π in a rotational and multiplicative context, extending classical results on distributions modulo 1. This approach reveals angular recurrence and clustering phenomena that appear to correspond to the roots and powers of smaller primes.

We define Residue Recurrence Classes (RRCs) as subsets of primes sharing angular residues generated by integer or fractional roots of smaller primes. Empirical analyses indicate that RRCs contain many primes and exhibit coherence along distinct angular branches. Furthermore, we introduce a quasi-harmonic prime function, constructed as a weighted cosine sum over prime residues, and a cumulative complex prime centre summation whose components satisfy a power-law relation suggestive of self-similar scaling.

These findings point to a possible secondary layer of analytic organisation within the prime sequence. Although the results are heuristic and partly empirical, they motivate conjectures regarding the density of RRCs and the asymptotic scaling invariance of prime summations. More broadly, the goal of this study is to provide a coherent analytic framework linking modular recurrence, harmonic structure, and geometric scaling within a unified representation of prime numbers—a perspective that may bridge discrete number theory with ideas from spectral and dynamical analysis.

2. Complex Representation of Prime Numbers

Definition 1 (Prime as Complex Oscillator) Let $p \in \mathbb{P}$. Its complex exponential embedding is

$$\omega(p) := \mathrm{e}^{ip}, \text{ with } |\omega(p)| = 1.$$

This maps the primes to the unit circle in the complex plane.

Definition 2 (Angular Residue) For $p \in \mathbb{P}$, define its modulo residuum as

$$r_p := p \bmod 2\pi.$$

Remark Primes $p < 2\pi$ are mapped directly without angular wrapping as shown in Figure 1. For example,

$$\omega(3) = \mathrm{e}^{i3} \approx -0.989 + 0.141i.$$

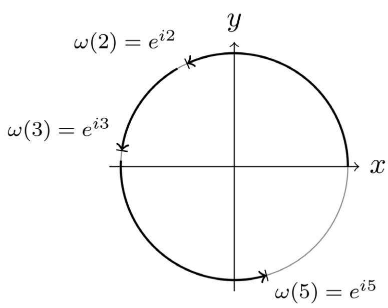

Figure 1. Primes 2, 3, and 5 on the unit circle.

3. Roots and Powers of Prime Oscillators

Theorem 1 (de Moivre's Identity for Primes) Let $p \in \mathbb{P}$ and $n \in \mathbb{Z}$. Then,

$$\left(\cos(p) + i\sin(p)\right)^n = \cos(np) + i\sin(np).$$

Proof. This is followed by induction on n using the trigonometric product-to-sum identities. For negative n, use

$$\left(\cos(p) + i\sin(p)\right)^{-n} = \overline{\left(\cos(p) + i\sin(p)\right)^n} = \cos(-np) + i\sin(-np).$$

Corollary 1 (Complex Roots of Primes) The *n*th root of e^{ip} is given by

$$\sqrt[n]{\mathrm{e}^{ip}} = \mathrm{e}^{i(p+2\pi k)/n}, \ k = 0, 1, \cdots, n-1.$$

This root also lies on the unit circle, evenly dividing the angular position of the original prime representation.

Definition 3 (Residue Recurrence Class (Integer Root Case)) Fix $p \in \mathbb{P}$, $n \in \mathbb{N}$, and let $\varepsilon > 0$ denote an angular tolerance. We define the Residue Recurrence Class with precision ε as

$$\mathcal{RRC}(p,n,\varepsilon) := \{q \in \mathbb{P} \mid |(q - np) \bmod 2\pi| < \varepsilon\}.$$

Remark In our numerical experiments, we used $\varepsilon = 10^{-3}$ radians, which provides a balance between computational sensitivity and numerical stability. This tolerance ensures that only primes with tightly aligned angular residues are included, while maintaining reproducibility of the empirical findings.

For $p = 2$, $n = 2$, we have $\omega(2^{-2}) = e^{-i4} \approx -0.653 + 0.756i$. The following primes are in $\mathcal{RRC}(2,2)$:

$$\{101149, 108959, 517831, 523511, \cdots\}, \text{ with residues} \approx 2.283.$$

Definition 4 (Residue Recurrence Class (Fractional Root Case)) Let $p \in \mathbb{P}$, $\alpha \in \mathbb{Q}_{>0}$, and $\varepsilon > 0$. Then, the fractional-root Residue Recurrence Class is defined as

$$\mathcal{RRC}(p,\alpha,\varepsilon) := \{q \in \mathbb{P} \mid |(q - \alpha p) \bmod 2\pi| < \varepsilon\}.$$

Remark Let $p = 2, \alpha = 0.5 \Rightarrow \theta_{2,0.5} \approx 5.283$ and the conjugate root at $\theta \approx 2.141$. Two recurrence clusters were identified.

- Cluster A: $\{709, 2129, 4259, 4969, 6389, \cdots\}$
- Cluster B: $\{102217, 104437, 105767, 107897, \cdots\}$

These clusters illustrate how fractional prime roots produce multibranch angular recurrences among larger primes. Each angular value corresponds to a distinct branch of the complex root, forming a double-lobed recurrence pattern on the unit circle as shown in Figure 2. This behaviour suggests that $\mathcal{RRC}(p,\alpha)$ is multivalued when $\alpha \notin \mathbb{N}$, reflecting the complex analytical structure of fractional exponents.

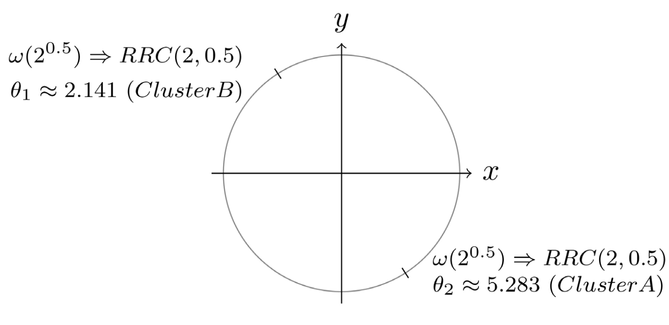

Figure 2. Residue recurrence classes (RRC). Clusters A and B align with the two conjugate roots of $2^{-0.5}$.

4. Conjugate Symmetry and Clustering

Lemma 2 For any $p \in \mathbb{P}$, the conjugate $\bar{\omega}(p) = e^{-ip}$ may coincide with $\omega(q)$ for some distinct prime q, forming a conjugate recurrence pair.

Remark $\omega(17) \approx -0.275 - 0.961i$, and $\bar{\omega}(17) \approx -0.275 + 0.961i \approx \omega(6373)$. Cluster:

$$\{2113, 3533, 4243, 6373, 7793, \cdots\} \Rightarrow r_q \approx 1.849.$$

5. Symmetries and Conjugates

Lemma 3 (Conjugate Pair Cancellation and Prime Representation) Let $\omega(p) = \mathrm{e}^{ip}$ be a complex representation of a prime $p \in \mathbb{P}$. Then, its complex conjugate $\bar{\omega}(p) = \mathrm{e}^{-ip}$ may correspond to a different prime $q \in \mathbb{P}$ such that $\bar{\omega}(p) = \omega(q)$. When both $\omega(p)$ and $\omega(q)$ are included in the quasi-harmonic sum, their effects partially cancel owing to conjugate symmetry:

$$\omega(p) + \bar{\omega}(p) = 2\cos(p).$$

Thus, the full set of primes cannot define the quasi-harmonic prime function; rather, it is supported by a structurally selective subset, leading to quasi-periodicity in the prime function.

Lemma 4 (Clusters of Primes Approximating Complex Conjugates) Let $\omega(p) = \mathrm{e}^{ip}$ be the complex embedding of a prime $p \in \mathbb{P}$ and $\bar{\omega}(p) = \mathrm{e}^{-ip}$ its complex conjugate, then there may exist a finite set of distinct primes $\{q_1, q_2, \cdots, q_k\} \subset \mathbb{P}$ such that

$$\omega(q_j) \approx \bar{\omega}(p), \text{ for each } j = 1, \cdots, k,$$

where $\omega(q_j) = \mathrm{e}^{iq_j}$ and

$$q_j \bmod 2\pi \approx -p \bmod 2\pi.$$

That is, although $p \bmod 2\pi$ differs significantly from the residues of the q_j, the q_j themselves form a tightly clustered set whose angular residues are approximately equal to that of $\bar{\omega}(p)$. These primes form a conjugate residue cluster, suggesting a quasi-linear or recurrent angular alignment among primes under complex embeddings.

6. Quasi-Harmonic Prime Function

Definition 5 (Root-Weighted Residue-Based Prime Oscillator Sum) The root-weighted quasi-harmonic prime function is defined as

$$H_N(x) := \sum_{j=1}^{N} \frac{1}{\sqrt{p_j}} \cdot \cos(x \cdot \theta_j),$$

where $p_j \in \mathbb{P}$ is the *j*th prime, and $\theta_j := p_j \bmod 2\pi$ denotes its angular residue.

Remark The weighting factor $1/\sqrt{p_j}$ serves both theoretical and empirical purposes. From a theoretical standpoint, it parallels normalisations used in Dirichlet-type series and spectral zeta formulations, where amplitudes are inversely related to the square root of frequency or energy to ensure boundedness of oscillatory sums. Empirically, this weighting was observed to stabilise the quasi-periodic interference patterns of $H_N(x)$; unweighted sums, which exhibited erratic oscillations and loss of structure beyond the first few thousand primes.

The function $H_N(x)$ is shown in Figure 3, using the first 10000 prime residues.

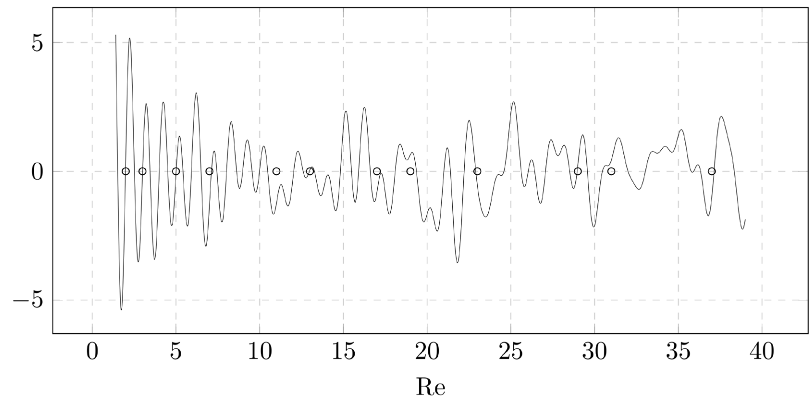

Figure 3. Graphically, the function $H_N(x)$ intersects the real axis near prime values, providing an analytic signal for prime locations.

7. Helical Structure and Scaling Law

Lemma 5 (Non-Zero Center of Angular Sums) Let $S_N := \sum_{j=1}^{N} \omega(p_j)$. Then, $S_N \neq 0$. The cumulative sum of prime oscillators does not cancel, unlike the roots of unity.

Definition 6 (Normalized Prime Center Localization as Quasi-Helical Structure) Let

$$S_{N,n} := \frac{\omega(p_j)}{t \cdot 2\pi}$$

and maps $S_{N,n}$ onto a complex plane with n normalisation.

Remark The trajectory becomes more structured for large t, as shown in Figure 4.

Theorem 6 (Quasi-Helical Structure) The centre point of the prime oscillators is defined as scaled by the real parameter t. As $t \to N$, the centre traces a helical trajectory in 2D space.

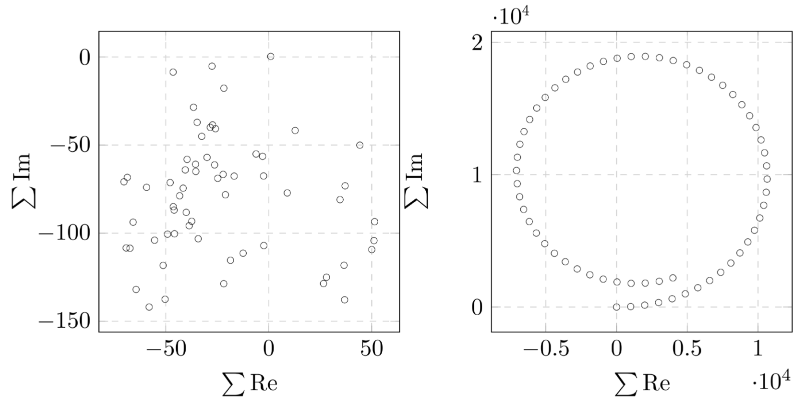

Figure 4. Prime oscillators centre summation as scaled by the real parameter t, shown for $t = 1$ (left panel) and $t = 18000$ (right panel).

Scaling Law of Angular Sums: numerical evaluation of the cumulative complex prime $S_{N,n}$ suggests an approximate scaling relationship between its real $R=\sum Re\left(S_{N,n}\right)$ and imaginary components $I=\sum Im\left(S_{N,n}\right)$,

$$I \approx c \cdot R^{\alpha}, \text{ with fitted parameters } \alpha \approx 2.1138, c \approx 3\times10^{-8}.$$

Remark This observation is based on empirical data fitted to a power law, as shown in **Figure 5**, and is not presented as a proven mathematical theorem. The apparent scaling behaviour may reflect an emergent geometric tendency in the cumulative embedding rather than an exact analytical law. The c exponent suggests a fractal-like power law in the distribution of primes under complex normalisation, consistent with the results observed in [6] [8].

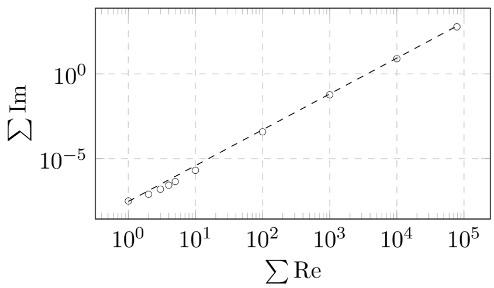

Figure 5. Scaling relationship between real and imaginary parts of prime centre summation, with fitted power law.

8. Conclusions

In this paper, we proposed and numerically explored a geometric-analytic framework for studying the distribution of prime numbers via their embedding on the complex unit circle, $\omega(p)=e^{ip}$. Our analysis revealed two interacting structural layers: 1) angular recurrence linked to integer and fractional roots of smaller primes, and 2) nonlinear summation dynamics exhibiting approximate power-law scaling. The results remain empirical but reproducible across finite prime ranges, suggesting that the apparent randomness of primes may conceal underlying harmonic and geometric tendencies.

The first layer of structure is captured through the introduction of Residue Recurrence Classes (RRCs), which extend classical additive modular studies (e.g., distribution modulo 1) into a multiplicative-rotational setting modulo 2π. This representation reveals coherent angular clustering of primes aligned with the roots of smaller primes, indicating that certain modular substructures may persist within the globally uniform prime sequence.

The second layer involves two complementary analytical constructions: the quasi-harmonic prime function, defined as a root-weighted cosine sum over prime angular residues, which exhibits interference patterns and partial localisation effects; and the complex prime centre summation, whose cumulative trajectory in the complex plane follows an empirical power-law relation between its real and imaginary components. This scaling, while not exact, suggests a self-similar or fractal-like growth tendency in the aggregate embedding.

Remark The coherence described in the examples refers to numerically observed clustering within bounded prime intervals, rather than an asymptotic or proven analytic property. A systematic statistical analysis over extended prime ranges would be necessary to establish quantitative measures of recurrence strength or density.

Remark It is worth noting that, according to Weyl's criterion, the sequence of primes modulo 1 (and thus modulo 2π) is asymptotically equidistributed. The angular clustering effects observed here, therefore, do not contradict equidistribution; they represent local deviations within finite subsets, consistent with small-scale irregularities permitted by the global uniformity predicted by analytic number theory.

Based on the observed patterns, we propose the following conjectures for further mathematical study:

- **Conjecture 1 (Residue Recurrence Density):** For any fixed small prime p and exponent $\alpha \in \mathbb{Q}_{>0}$, the residue recurrence class $\mathcal{RRC}(p,\alpha,\varepsilon)$ has positive lower density within the set of all primes.
- **Conjecture 2 (Scaling Invariance of Prime Summation):** The normalised trajectory of the complex prime centre summation approaches a self-similar attractor curve in the complex plane as the number of terms increases.

Future Work

Further research should seek to formalise the analytic definition and density estimates of RRCs, and to determine whether the observed scaling law admits a closed-form asymptotic formulation. Testing the stability of the scaling exponent over extended computational ranges would also clarify whether this behaviour reflects a genuine number-theoretic regularity or an artefact of finite sampling. Ultimately, such investigations could help determine whether the geometric-analytic representation developed here fits naturally within established analytic frameworks or introduces a new perspective on spectral phenomena in the primes.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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