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Remarks on Different Approaches to the Theory of Higher-Order Types of Asymptotic Variation

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Abstract

As a completion of the previously published theory of higher-order types of asymptotic variation, we point out some remarks about different approaches to construct such an advanced and demanding theory, because there are various (non-equivalent) approaches to the basic concepts of types of asymptotic variation, *i.e.*, the concepts not involving derivatives. After listing the five known approaches to the concept of regular variation and their mutual relationships, we show that, as far as the higher-order theory is concerned, there is essentially "*only one way*" to construct such a theory for differentiable functions.

Keywords

Higher-Order Types of Asymptotic Variation, Hardy's Approach, Karamata's Approach, Bourbaki-Dieudonné's Approach, Zygmund's Approach

1. Introduction

In the previously published theory of higher-order types of asymptotic variation, [1]-[4], we started from the elementary theory of regular and rapid variation for those restricted classes of functions absolutely continuous on some interval $[T,+\infty)$ and such that the "$\lim_{x\to+\infty} xf'(x)/f(x)$" exists as an extended real number α labelled as the "*index of variation*" at $+\infty$. Higher-order regular or rapid variation was then defined by assuming similar asymptotic conditions for some of the derivatives, having preliminarily established precise relationships between the indices of the involved derivatives. A similar procedure was adopted for rapid variation and for three types of exponential variation. This is an approach going back to Hardy [5] for the first order and sketched in Bourbaki ([6], Prop. 2, p. V.40) for higher orders, but in a different context.

The standard Karamata theory of regular (and, for extension, rapid) variation involves the larger classes of functions assumed Lebesgue-measurable on $[T,+\infty)$ and such that "$\lim_{x\to+\infty} f(\lambda x)/f(x)$" exists and equals λ^{α} for all $\lambda>0$, including the limit cases "$\alpha=\pm\infty$" for rapid variation. Other approaches will be mentioned below. In all of these frameworks, all the functions are traditionally assumed positive in a neighborhood of $+\infty$.

In the case of regular variation, Hardy's approach is based on the "*asymptotic differential equation*".

$$xf'(x)=f(x)[\alpha+o(1)],\ \ x\to+\infty,\ \ (\text{for some fixed } \alpha\in\mathbb{R}), \tag{1.1}$$

whereas Karamata's approach starts from the "*asymptotic functional equation*".

$$f(\lambda x)=f(x)[\lambda^{\alpha}+o(1)],\ \ x\to+\infty, (\text{for each } \lambda>0 \text{ and some fixed } \alpha\in\mathbb{R}). \tag{1.2}$$

Solutions of (1.1) form a subclass of the solutions of (1.2), and the class of solutions of (1.2) happens to be the "closure" of the solutions of (1.1) with respect to the equivalence relation of "same growth-order" in the classical sense of

$$f(x)\sim cg(x),\ x\to+\infty, \text{ for some constant } c\neq 0, \tag{1.3}$$

see Propositions 2.1 and 2.3 below.

Something analogous happens for the various concepts of rapid and exponential variation, and the reader of the higher-order theory might ask why Equation (1.2) referred to f' was not used instead of (1.1), with f replaced by f', in constructing the advanced theory. And similar remarks might apply to the other approaches. In this paper, we:

(I) first point out the relationships between the known approaches to the concept of regular variation;

(II) and then show that a higher-order theory of regular variation according to any of these approaches is either of little import or equivalent to the theory developed by the author according to Hardy's approach.

Answering to possible legitimate doubts of some readers, the presented results are no mere curiosity insofar as the time-honoured Karamata theory is so spread in applied mathematics that it would be unfair to discard the endeavor of constructing a Karamata higher-order theory paralleling the already-constructed one with its present and future applications. In fact, it will be shown that nothing is lost in using either Karmata's or Hardy's approaches, so justifying the "*only one way*" mentioned in the abstract. Of course, this conclusion refers to the chosen procedure in constructing the higher-order theory repeatedly using the standard differential operator $\mathrm{d}/\mathrm{d}x$. An approach based on more general operators of type, say,

$$u\to p_n\left[p_{n-1}\left(\cdots\left(p_1\left(p_0u\right)'\right)'\cdots\right)'\right]',$$

with suitable regularity conditions on the strictly-positive functions p_i would in principle be possible as far as the basic results are concerned, but, as the author's first impression, it might cause tremendous trouble in developing the "algebra" of the involved functions, e.g., the properties of composite or inverse functions. But this is another story.

This paper concludes the cycle of articles devoted to the general theory of higher-order types of asymptotic variation.

Essential Notations:

- $\mathbb{R} :=$ real line; $\overline{\mathbb{R}} \equiv$ extended real line $:= \mathbb{R} \cup \{\pm\infty\}$;
- $f \in AC^0(I) \equiv AC(I) \Leftrightarrow f$ is absolutely continuous on each compact subinterval of the interval I;
- $f \in AC^k(I) \Leftrightarrow f^{(k)} \in AC(I)$;
- increasing $\equiv$ nondecreasing; decreasing $\equiv$ nonincreasing;
- In the sequel, all the asymptotic relations between strictly positive functions are meant as $x \to +\infty$:

$$\begin{cases} f(x) = o(g(x)) \Leftrightarrow f(x) \ll g(x) \Leftrightarrow g(x) \gg f(x) \\ \overset{\text{def}}{\Leftrightarrow} \lim_{x\to+\infty} f(x)/g(x) = 0; \\ f(x) \sim cg(x) \ (c = \text{constant} \neq 0) \Leftrightarrow \lim_{x\to+\infty} f(x)/g(x) = c; \\ f(x) = \pm\infty(g(x)) \overset{\text{def}}{\Leftrightarrow} \lim_{x\to+\infty} f(x)/g(x) = \pm\infty. \end{cases} \tag{1.4}$$

$$\begin{cases} f(x) \asymp g(x), x \to +\infty \Leftrightarrow g(x) \asymp f(x), x \to +\infty, \\ \overset{\text{def}}{\Leftrightarrow} 0 < c_1 |f(x)| \leq |g(x)| \leq c_2 |f(x)| \text{ for } x \text{ large enough}, \ (c_1, c_2 > 0). \end{cases} \tag{1.5}$$

- f oscillatory as $x \to +\infty$ means that f vanishes on each interval $[T, +\infty)$;
- A property holds "ultimately" if it holds true on some interval $[T, +\infty)$.

2. Various Approaches to the Concepts of Regular or Rapid Variation

This section contains a quick summary of the various concepts of regular and rapid variation as the independent variable tends to $+\infty$ disregarding a thorough discussion on the historical approaches to the more general concepts of types of asymptotic variation. Each of the involved functions is defined on some interval $[T, +\infty)$.

2.1. The Elementary and Hardy Concepts

Definition 2.1 (I) (The elementary concept). *In the most elementary sense, a function* f *is said to behave as a power at* $+\infty$ *if there exists a number* α *such that:*

$$f(x) \sim cx^\alpha, \ x \to +\infty, \ (c = \text{constant} \neq 0), \tag{2.1}$$

a case wherein f *is said to have "growth-order* α*" (or is of "order* α*") with respect to* x *at* $+\infty$.

(II) (Hardy's concept of "power growth-order"). *If* $f \in AC[T, +\infty)$, f

ultimately $\neq 0$, *and if one of the following asymptotic relations holds true:*

$$\frac{f'(x)}{f(x)}\begin{cases} = o\left(\dfrac{1}{x}\right), & x \to +\infty, \\ \sim \alpha \cdot \left(\dfrac{1}{x}\right), & \alpha \in \mathbb{R} \setminus \{0\}, \; x \to +\infty, \\ = \pm\infty\left(\dfrac{1}{x}\right), & x \to +\infty, \end{cases} \tag{2.2}$$

we say that f *has a specific "growth-order" at* $+\infty$, *namely: zero,* $\alpha \in \mathbb{R} \setminus \{0\}$ *or* $\pm\infty$ *in the respective cases.*

Proposition 2.1 (Asymptotic properties of functions with a given power growth-order). *Let* f *satisfy one of the relations in* (2.2) *and let it ultimately be* $f > 0$ (*by a standard agreement*).

(I) *If* $\alpha \in \mathbb{R}$, *then an integration of the asymptotic relation*

$$f'(x)/f(x) = \alpha x^{-1} + o\left(x^{-1}\right), \quad x \to +\infty, \tag{2.3}$$

yields an integral representation for f, *valid on some neighborhood of* $+\infty$, *of the type*

$$f(x) = c \cdot x^{\alpha} \cdot \exp\left[\int_{T_0}^{x} \epsilon(t) t^{-1} \mathrm{d}t\right], \quad x \geq T_0, \tag{2.4}$$

where c *is a suitable positive constant,* $\epsilon(\cdot)$ *a suitable measurable function such that* $\epsilon(x) = o(1)$, $x \to +\infty$, *and* T_0 *a suitable point,* $T_0 \geq T$. *The following properties are easily inferred from this integral representation.*

(II) *If* $\alpha \neq 0$, *then*

$$\log f(x) \sim \alpha \log x, \quad x \to +\infty; \tag{2.5}$$

$$x^{\alpha-\epsilon} \ll f(x) \ll x^{\alpha+\epsilon}, \quad x \to +\infty, \; \forall\, \epsilon > 0; \tag{2.6}$$

$$f(+\infty) = \begin{cases} +\infty & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha < 0. \end{cases} \tag{2.7}$$

(III) *If* $\alpha = 0$, *then*

$$\log f(x) = o(\log x), \quad x \to +\infty; \tag{2.8}$$

$$x^{-\epsilon} \ll f(x) \ll x^{\epsilon}, \quad x \to +\infty, \; \forall\, \epsilon > 0; \tag{2.9}$$

with no conclusion, generally speaking, about the $\lim_{x \to +\infty} f(x)$ *as shown by the functions* ([1], formula (2.22)),

$$\begin{cases} 1; \; \log x; \; (\log x)^{-1}; \; 2 + \sin\left((\log x)^{\delta}\right), \; 0 < \delta < 1; \\ \exp\left[(\log x)^{\delta} \cdot \sin\left((\log x)^{\delta}\right)\right], \; 0 < \delta < 1/2. \end{cases}$$

(IV) *If instead of* (2.3) f *satisfies the relation*

$$f'(x)/f(x) = \pm\infty\left(x^{-1}\right), \quad x \to +\infty, \tag{2.10}$$

then

$$\log f(x) = \pm\infty(\log x), \quad x \to +\infty; \tag{2.11}$$

whence the estimates:

$$\begin{cases} f(x) \gg x^{\alpha}, \ x \to +\infty, \ \forall\, \alpha \in \mathbb{R} \ \ (\text{if in } (2.10) \text{ there is the sign } +); \\ f(x) \ll x^{-\alpha}, \ x \to +\infty, \ \forall\, \alpha \in \mathbb{R} \ \ (\text{if in } (2.10) \text{ there is the sign } -). \end{cases} \tag{2.12}$$

2.2. Bourbaki-Dieudonné's Concepts of Growth-Order

The following two definitions contain the Bourbaki-Dieudonné equivalent concepts of generalized growth-order for the special case we are treating, *i.e.*, when the comparison function is $g(x) = x$ as $x \to +\infty$. These concepts, like Hardy's, originate from the convenience of grouping together in one class the product of functions with different traditional growth orders, giving the leading role to the factor with the greatest traditional growth order, such as

$$x^{\alpha}, \ x^{\alpha}(\log x)^{\beta}, \ x^{\alpha}(\log x)^{\beta}(\log(\log x))^{\gamma}, \ \cdots.$$

One of the generalizations assumes the estimates in (2.6), whereas the other compares the logarithms as in (2.5).

Definition 2.2 (Generalization of power growth-order based on the leading factor). *Let* f *be a real-valued function defined on some neighborhood of* $+\infty$.

(I) f *has* (*generalized*) *power growth-order* $\alpha \in \mathbb{R}$ *at* $+\infty$ *if the following asymptotic relations hold true*:

$$x^{\alpha-\epsilon} \ll f(x) \ll x^{\alpha+\epsilon}, \ x \to +\infty, \forall\, \epsilon > 0. \tag{2.13}$$

(II) f *has power growth-order* $+\infty$ *as* $x \to +\infty$ *if*

$$f(x) \gg x^{\alpha}, \ x \to x_0, \ \forall\, \alpha > 0 \ (\text{hence, obviously, } \forall\, \alpha \in \mathbb{R}); \tag{2.14}$$

and f *has growth-order* $-\infty$ *as* $x \to +\infty$ *if*

$$f(x) \ll x^{-\alpha}, \ x \to x_0, \ \forall\, \alpha > 0 \ (\text{hence, obviously, } \forall\, \alpha \in \mathbb{R}). \tag{2.15}$$

Definition 2.3 (Generalization of power growth-order based on a double logarithmic scale). *Let f be a real-valued function defined on some neighborhood of* $+\infty$ *and with a constant nonzero sign on a neighborhood of* $+\infty$. *We say that* f *has* (*generalized*) *power growth-order* α *at* $+\infty$ *if*:

$$\lim_{x \to +\infty} \log|f(x)| / \log x = \alpha, \ (-\infty \le \alpha \le +\infty). \tag{2.16}$$

The above concepts in a more general version are stated in a unified manner in ([6], Def. 5, p. V.9), where their equivalence is proved as follows.

Proposition 2.2 *The two concepts of generalized power growth-order are equivalent for a function* f *having a constant nonzero sign on a neighborhood of* $+\infty$.

2.3. Karamata's Approach

Almost contemporaneously with the introduction of Hardy's concept, a completely different approach (at least theoretically) was used by Karamata, starting from the asymptotic functional equation

$$f(\lambda x) \sim \lambda^{\alpha} f(x), \quad x \to +\infty, \text{ for each } \lambda > 0 \text{ and some fixed } \alpha \in \mathbb{R}, \tag{2.17}$$

which is an asymptotic version of the Cauchy multiplicative functional equation

$$f(xy) = f(x) f(y) \quad \forall\, x, y > 0.$$

It is known that this last equation, besides the power functions, has irregular solutions, namely solutions unbounded both above and below on each bounded interval; hence, for useful applications, some kind of regularity must be imposed on the solutions of (2.17). The following proposition contains the characterization of regularly-varying functions according to Karamata ([7], Th. 1.3.1 and its proof, pp. 12-13).

Proposition 2.3 *Let* f *be a real-valued function defined on some interval* $[T, +\infty)$, $T > 0$, *Lebesgue-measurable and* $f(x) > 0$ *for* x *large enough. Then* f *is called regularly varying at* $+\infty$ *in the Karamata sense with index* $\alpha \in \mathbb{R}$ *if the following two equivalent conditions obtain*:

(I) f *admits of a representation of type*

$$f(x) = c(x) \cdot x^{\alpha} \cdot \exp\left[\int_{T_0}^{x} \epsilon(t) t^{-1} \mathrm{d}t\right] \quad \forall\, x \geq T_0 \text{ large enough,} \tag{2.18a}$$

where $c(\cdot)$ *and* $\epsilon(\cdot)$ *are two suitable measurable functions such that*:

$$\lim_{x \to +\infty} c(x) = c \in \left]0, +\infty\right), \quad \lim_{x \to +\infty} \epsilon(x) = 0. \tag{2.18b}$$

(II) f *satisfies the following asymptotic functional equation*:

$$\begin{cases} f(\lambda x) \sim \lambda^{\alpha} f(x), \quad x \to +\infty \text{ for each fixed } \lambda > 0, \\ \text{uniformly with respect to } \lambda \text{ on each compact subinterval of } \left]0, +\infty\right), \end{cases} \tag{2.19a}$$

which, by definition, means that

$$\begin{cases} \lim_{x \to +\infty} \left[\dfrac{f(\lambda x)}{f(x)} - \lambda^{\alpha}\right] = 0 \\ \text{uniformly with respect to } \lambda \text{ on each compact subinterval of } \left]0, +\infty\right). \end{cases} \tag{2.19b}$$

For the index $\alpha = 0$, *the function* f *is also called slowly varying at* $+\infty$ *in the Karamata sense. It is obvious that* f *has an index* α *if and only if*

$$f(x) = x^{\alpha} \cdot \ell(x) \text{ with a suitable slowly-varying function } \ell.$$

An easy consequence of the representation in (2.18) *is the validity of the estimate in* (2.5), *equivalent to those in* (2.6).

Similar approaches may be applied to the Cauchy exponential and logarithmic functional equations.

$$f(x+y) = f(x) f(y), \quad f(xy) = f(x) + f(y),$$

studying suitable asymptotic versions which lead to defining classes of exponentially-varying or logarithmic-varying functions larger than corresponding classes defined by Hardy's approach. In this article, we limit ourselves to considering the case of regular variation because it is the foundational case for developing higher-order theories and suffices to highlight the phenomena we wish to point out. The interested reader may find out to what extent the facts in the next sections have their analogues for the other types of asymptotic variation.

2.4. The Zygmund Property

The last concept in our list was introduced by Zygmund in the framework of trigonometric series.

Definition 2.4 *A real-valued function f, defined and positive on some interval* $[T,+\infty)$, *enjoys the Zygmund property* (*at* $+\infty$) *if:*

$$\begin{cases} x^{\epsilon} f(x) & \text{is ultimately increasing,} \\ x^{-\epsilon} f(x) & \text{is ultimately decreasing,} \end{cases} \quad (\text{for each fixed } \epsilon > 0), \tag{2.20}$$

this pair of conditions implying that f *is measurable for* x *large enough.*

Such a function is also termed "*slowly varying* (*at* $+\infty$)" *in the Zygmund sense.*

A proof of the following fact may be found in ([7], Th. 1.5.5, p. 24].

Proposition 2.4 *A function* f, *positive on some interval* $[T,+\infty)$, *belongs to the Zygmund class if and only if it is absolutely continuous on some neighborhood of* $+\infty$ *and has Hardy's power growth-order zero in the sense of the first asymptotic relation in* (2.2).

3. Comparison between the Various Concepts of Regular Asymptotic Variation

So far, we pointed out five different notions of regular variation at $+\infty$ and are now going to make explicit their mutual relationships. For convenience in the present context, we use notations for the various classes not common in the current literature.

Each of the involved functions is assumed defined on some interval $[T,+\infty)$ and strictly positive thereon.

Hardy's classes. $\mathcal{H}_\alpha(+\infty)$ is the family of functions $f \in AC[T,+\infty)$ and such that

$$f'(x)/f(x) = x^{-1}[\alpha + o(1)], \; x \to +\infty, \; \alpha \in \mathbb{R} \text{ fixed.} \tag{3.1}$$

Karamata's classes. $\mathcal{K}_\alpha(+\infty)$ is the family of functions f measurable on $[T,+\infty)$ and such that

$$f(\lambda x) \sim \lambda^{\alpha} f(x), \; x \to +\infty, \; \text{for each } \lambda > 0, \; \alpha \in \mathbb{R} \text{ fixed.} \tag{3.2}$$

Bourbaki-Dieudonné's classes. $\mathcal{B}_\alpha(+\infty)$ is the family of functions f such that

$$\log f(x) = \alpha \log x + o(\log x), \; x \to +\infty, \; \alpha \in \mathbb{R} \text{ fixed,} \tag{3.3}$$

or, equivalently, the estimates in (2.13) and wherein no regularity property of f is required a priori.

Zygmund's classes. $\mathcal{Z}_\alpha(+\infty)$ is the family of functions f measurable on $[T,+\infty)$ and such that

$$\begin{cases} x^{\alpha+\epsilon} f(x) & \text{is ultimately increasing,} \\ x^{\alpha-\epsilon} f(x) & \text{is ultimately decreasing,} \end{cases} \quad (\text{for each fixed } \epsilon > 0). \tag{3.4}$$

Theorem 3.1 (Inclusion relations between these classes). *The following facts obtain.*

(I)

$$\begin{cases} \mathcal{B}_\alpha(+\infty) \supsetneqq \mathcal{K}_\alpha(+\infty) \underset{\neq}{\supset} \left(\mathcal{K}_\alpha(+\infty) \cap AC[T,+\infty)\right) \supsetneqq \mathcal{H}_\alpha(+\infty) = \mathcal{Z}_\alpha(+\infty); \\ \left(\mathcal{B}_\alpha(+\infty) \cap AC[T,+\infty)\right) \supsetneqq \left(\mathcal{K}_\alpha(+\infty) \cap AC[T,+\infty)\right); \end{cases} \tag{3.5}$$

and $\mathcal{K}_\alpha(+\infty)$ *happens to be the algebraic closure of* $\mathcal{H}_\alpha(+\infty)$ *with respect to the equivalence relation* of "*same growth-order at* $+\infty$" *defined in* (1.3).

(II) *If* $f \in AC[T,+\infty)$ *and if* f' *is monotonic, then*

$$f \in \mathcal{K}_\alpha(+\infty) \Leftrightarrow f \in \mathcal{H}_\alpha(+\infty); \tag{3.6}$$

hence, functions in the set $\{\mathcal{K}_\alpha(+\infty) \setminus \mathcal{H}_\alpha(+\infty)\} \cap AC[T,+\infty)$ *must have an* (*ultimately*) *non-monotonic derivative.*

Proof. (I). First, the asymptotic relation in (2.5), valid for the functions in $\mathcal{K}_\alpha(+\infty)$, as remarked at the end of Proposition 2.3, shows that $\mathcal{K}_\alpha(+\infty) \subset \mathcal{B}_\alpha(+\infty)$.

Second, the strict inclusion is not due to the simple fact that the functions in $\mathcal{B}_\alpha(+\infty)$ have no a-priori regularity restrictions, unlike the measurability of those in $\mathcal{K}_\alpha(+\infty)$, but is a substantial property as stated by the strict inclusion in the second line in (3.5) and proved, e.g., by the function

$$f(x) := x^\alpha \left(2 + \sin\phi(x)\right), \; \phi \in AC[T,+\infty), \; (\alpha \in \mathbb{R}), \tag{3.7}$$

which is in $\mathcal{B}_\alpha(+\infty)$ for any ϕ but is in $\mathcal{K}_\alpha(+\infty)$ only in the very special case that "$2 + \sin\phi(\lambda x) \sim 1, \, x \to +\infty, \forall \lambda > 0$", "$\phi(x) = \text{constant} = (3\pi/2) + 2k\pi$".

The second inclusion in the first line in (3.5) is obvious, whereas the third inclusion, as well as the last claim in part (I), is proved by comparing the representations in (2.4) and (2.18). To visualize this strict inclusion, the reader may check this fact in the following case:

$$\begin{cases} f(x) := x^\alpha (\log x + \sin x) \sim x^\alpha \log x, \quad x \to +\infty, \\ f \in \mathcal{K}_\alpha(+\infty) \setminus \mathcal{H}_\alpha(+\infty), \end{cases} \tag{3.8}$$

wherein the quantity $xf'(x)/f(x) = \alpha + (x\cos x + 1)/(\log x + \sin x)$ has no limit in $\overline{\mathbb{R}}$ as $x \to +\infty$. The last equality in (3.5), $\mathcal{H}_\alpha(+\infty) = \mathcal{Z}_\alpha(+\infty)$ is in Proposition 2.4.

(II). The exact meaning of the monotonicity assumption, explained in ([3], §4), is that f is either concave or convex on $[T,+\infty)$ so that both one-sided derivatives exist as finite numbers on $[T,+\infty)$ and are monotonic thereon. A proof of the equivalence in (3.6) is in ([3], Th. 4.1).

4. One Essential Approach to the Higher-Order Theory of Regular Variation

By Theorem 3.1, we have three non-equivalent notions of regular variation at $+\infty$, with a fixed index $\alpha \in \mathbb{R}$, highlighted by the three classes $\mathcal{H}_\alpha(+\infty)$, $\mathcal{K}_\alpha(+\infty)$,

$\mathcal{B}_\alpha(+\infty)$. In principle, regular variation of higher order is defined by imposing the corresponding condition on some of the derivatives, while suppressing the conventional condition of positivity. The essential features are that the involved functions maintain a constant nonzero sign and exhibit absolute continuity of the appropriate order. For the label concerning the number of the order, we choose the order of the highest involved derivative. The case $n=2$ suffices to point out the phenomenon.

Definition 4.1 *Let* α *be a real number. If in the definitions of the three mentioned classes, each element is assumed to preserve a constant nonzero sign for all* x *large enough, then we have the three enlarged classes of:*

$$\{\mathcal{H}_\alpha(+\infty) \text{ of order } 1\}, \{\mathcal{K}_\alpha(+\infty) \text{ of order } 0\}, \{\mathcal{B}_\alpha(+\infty) \text{ of order } 0\},$$

noticing that in the first of these classes the restriction of "nonzero sign" refers to the given function f *and not to* f' *as in the next concepts.*

Imposing conditions on one further derivative, we define the following classes:

(I)

$$f \in \{\mathcal{H}_\alpha(+\infty) \text{ of order } 2\} \overset{\text{def}}{\Leftrightarrow}$$
$$\begin{cases} f \in AC^1[T,+\infty);\ |f(x)|, |f'(x)| > 0\ \ \forall\, x \text{ large enough;} \\ xf'(x)/f(x) = \alpha + o(1),\ \ x \to +\infty; \\ xf''(x)/f'(x) = \beta + o(1),\ \ x \to +\infty,\ \ \text{with a suitable number } \beta. \end{cases} \tag{4.1}$$

(II)

$$f \in \{\mathcal{K}_\alpha(+\infty) \text{ of order } 1\} \overset{\text{def}}{\Leftrightarrow}$$
$$\begin{cases} f \in AC[T,+\infty);\ |f(x)|, |f'(x)| > 0\ \ \forall\, x \text{ large enough and a.e. for } f'; \\ f(\lambda x) \sim \lambda^\alpha f(x),\ \ x \to +\infty,\ \ \forall\, \lambda > 0; \\ f'(\lambda x) \sim \lambda^\beta f'(x),\ \ x \to +\infty,\ \ \forall\, \lambda > 0,\ \ \text{with a suitable number } \beta. \end{cases} \tag{4.2}$$

<u>*Warning*</u>*:* $f'(\lambda x)$ *stands for* $f'(y)\big|_{y=\lambda x}$ *and not* $\dfrac{\mathrm{d}}{\mathrm{d}x} f(\lambda x)$!

(III)

$$f \in \{\mathcal{B}_\alpha(+\infty) \text{ of order } 1\} \overset{\text{def}}{\Leftrightarrow}$$
$$\begin{cases} f \in AC[T,+\infty);\ |f(x)|, |f'(x)| > 0\ \ \forall\, x \text{ large enough and a.e. for } f'; \\ \lim_{x\to+\infty} \log|f(x)|/\log x = \alpha; \\ \lim_{x\to+\infty} \log|f'(x)|/\log x = \beta,\ \ \text{with a suitable number } \beta. \end{cases} \tag{4.3}$$

To go on it is necessary to know a link between the numbers α, β: a relationship well-known for the class $\{\mathcal{H}_\alpha(+\infty) \text{ of order } 2\}$.

Theorem 4.1 *If* $\alpha \neq 0$ *then* $\beta = \alpha - 1$ *in each of the three classes described in Definition* 4.1-(I)-(II)-(III), *whereas possible exceptions may occur in the case* $\alpha = 0$. *More precisely, for the three mentioned classes, the following inferences obtain:*

$$\begin{cases} \alpha \neq 0 \ \Rightarrow \ \beta = \alpha - 1; \\ \left\{ \alpha = 0 \text{ and } \lim_{x\to+\infty} \left| f(x) \right| = \text{either } 0 \text{ or } +\infty \right\} \ \Rightarrow \ \beta = \alpha - 1; \\ \alpha \in \mathbb{R} \ \Rightarrow \ \beta \le \alpha - 1. \end{cases} \tag{4.4}$$

Proof. (I) For the class $\left\{ \mathcal{H}_\alpha(+\infty) \text{ of order } 2 \right\}$, the result is known: [1], Prop. 2.6.

(II) For the class $\left\{ \mathcal{K}_\alpha(+\infty) \text{ of order } 1 \right\}$, if $\alpha \neq 0$, then the inequalities in (2.6), granted by Proposition 2.3, together with the constant sign of $f(x)$ imply that "$\lim_{x\to+\infty} f(x) =$ either 0 or $\pm\infty$" so that the pertinent L'Hospital rule may be applied in evaluating the limit:

$$\lambda^\alpha = \lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} \overset{H}{=} \lim_{x\to+\infty} \frac{\lambda f'(\lambda x)}{f'(x)} = \lambda^{\beta+1}, \ \forall \lambda > 0, \tag{4.5}$$

and the same argument is obviously valid in the case $\alpha = 0$ under the explicit additional restrictions on $f(+\infty)$. For the last claim in (4.4), assume, if possible, "$\alpha = 0$, $\beta > -1$" which implies "$x\left| f'(x) \right| \in \mathcal{K}_{\beta+1}(+\infty)$", whence, in sequence:

$$\begin{cases} \lim_{x\to+\infty} x\left| f'(x) \right| = +\infty \ \ (\text{being } \beta + 1 > 0); \\ \left| f'(x) \right| \ge C/x \ \text{ for } x \text{ large enough and a positive constant } C; \\ \int^{+\infty} \left| f' \right| = +\infty \text{ and } \left| f(+\infty) \right| = +\infty. \end{cases}$$

But we have just proved that the last property, for $\alpha = 0$, implies $\beta = -1$: a contradiction.

(III) For the class, $\left\{ \mathcal{B}_\alpha(+\infty) \text{ of order } 1 \right\}$ the assumptions imply:

$$\begin{aligned} \log\left(\frac{x\left| f'(x) \right|}{f(x)} \right) &= \log x \cdot \left[1 + \frac{\log\left| f'(x) \right|}{\log x} - \frac{\log f(x)}{\log x} \right] \\ &= \log x \cdot \left[1 + \beta - \alpha + o(1) \right], \ \ x \to +\infty; \end{aligned} \tag{4.6}$$

and suppose, if possible, that "$1 + \beta - \alpha > 0$". This would imply $\lim_{x\to+\infty}\left(\dfrac{x\left| f'(x) \right|}{f(x)} \right) = +\infty$, while an application of the pertinent L'Hospital rule would yield a contradictory result:

$$\alpha = \lim_{x\to+\infty} \frac{\log\left| f(x) \right|}{\log x} \overset{H}{=} \lim_{x\to+\infty} \frac{x\left| f'(x) \right|}{f(x)} = \pm\infty.$$

Hence, $1 + \beta - \alpha \le 0$. □

Counterexamples for the cases $\alpha = 0$. Consider the three functions, reported from [1], formula (2.107):

$$f_1(x) := 1 + \mathrm{e}^{-x}; \ \ f_2(x) := 1 + x^{-\delta} \ \ (\delta > 0); \ \ f_3(x) := 1 + (\log x)^{-\delta} \ \ (\delta > 0). \tag{4.7}$$

The following circumstances occur as $x \to +\infty$:

$$\begin{cases} x f_i'(x) / f_i(x) = o(1); \ \ f_i(\lambda x) / f_i(x) = 1 + o(1); \ \ \log f_i(x) / \log x = o(1); \\ \text{for } i = 1, 2, 3 \text{ hence } \alpha = 0 \text{ in } (4.1), (4.2), (4.3). \end{cases} \tag{4.8}$$

$$\begin{cases} xf_1''(x)/f_1'(x) \to -\infty; \\ f_1'(\lambda x)/f_1'(x) = \exp\left[(1-\lambda)x\right] \to \begin{cases} +\infty \ \text{ for } 0 < \lambda < 1, \\ 0 \ \text{ for } \lambda > 0; \end{cases} \\ \log\left|f_1'(x)\right| / \log x \to -\infty; \\ \text{which means that } f_1' \text{ is rapidly varying of index } -\infty \text{ in each} \\ \text{of the approaches: formally } \beta = -\infty. \end{cases} \tag{4.9}$$

$$\begin{cases} xf_2''(x)/f_2'(x) = (-\delta-1) + o(1); \\ f_2'(\lambda x)/f_2'(x) = \lambda^{-\delta-1} + o(1); \\ \log\left|f_2'(x)\right| / \log x = (-\delta-1) + o(1); \\ \text{hence } \beta = -\delta-1 < -1 \text{ in } (4.1), (4.2), (4.3) \text{ for the function } f_2. \end{cases} \tag{4.10}$$

$$\begin{cases} xf_3''(x)/f_3'(x) = -1 + o(1); \\ f_3'(\lambda x)/f_3'(x) = \lambda^{-1}\left[1 + o(1)\right]; \\ \log\left|f_3'(x)\right| / \log x = -1 + o(1); \\ \text{hence } \beta = -1 \text{ in } (4.1), (4.2), (4.3) \text{ for the function } f_3. \end{cases} \tag{4.11}$$

Next results in the paper—Theorem 4.2 and §5—will show that practically equivalent higher-order theories of asymptotic variation can be built using either Karamata's or Hardy's approaches, whereas applications of the classes $\left\{\mathcal{B}_\alpha(+\infty) \text{ of order } n \geq 1\right\}$ would offer less precise asymptotic results.

Theorem 4.2 (Karamata's and Hardy's higher-order theories).

(I) *The following inclusions obtain for each fixed* $\alpha \neq 0$:

$$\begin{aligned} &\left\{\mathcal{K}_\alpha(+\infty) \text{ of order } 0\right\} \supset \left\{\mathcal{H}_\alpha(+\infty) \text{ of order } 1\right\} \\ &\supset \left\{\mathcal{K}_\alpha(+\infty) \text{ of order } 1\right\} \supset \left\{\mathcal{H}_\alpha(+\infty) \text{ of order } 2\right\}. \end{aligned} \tag{4.12}$$

(II) *Let* $\alpha = 0$ *and f satisfy the conditions in* (4.2) *with some number* $\beta \leq -1$. *If* $\beta < -1$ *then* $f \in \left\{\mathcal{H}_\gamma(+\infty) \text{ of order } 1\right\}$ *where* γ *equals either* $(\beta+1)$ *or zero. If* $\beta = -1$ *then* $f \in \left\{\mathcal{H}_0(+\infty) \text{ of order } 1\right\}$.

(III) *Hence, each function in* $\left\{\mathcal{K}_\alpha(+\infty) \text{ of order } 1\right\}$, $\alpha \in \mathbb{R}$, *belongs to a class* $\left\{\mathcal{H}_\gamma(+\infty) \text{ of order } 1\right\}$ *with a suitable value of* γ, *and this fact implies that nothing is lost in using higher-order classes of regular variation built according to either Karamata's or Hardy's approaches.*

The proof depends on the following

Lemma 4.3 (Regular variation of antiderivatives). *Let* $g \in \mathcal{K}_\gamma(+\infty)$.

(I)

$$\gamma > -1 \ \Rightarrow \begin{cases} \int^{+\infty} |g| = +\infty; \\ \int_T^x g \sim \dfrac{xg(x)}{\gamma+1}, \quad x \to +\infty, \ (T \text{ large enough}); \end{cases} \tag{4.13}$$

$$\gamma < -1 \ \Rightarrow \ \int^{+\infty} |g| < +\infty \text{ and } \int_x^{+\infty} g \sim \frac{xg(x)}{|\gamma+1|}, \quad x \to +\infty. \tag{4.14}$$

(II)

$$\begin{aligned} &\left\{\gamma = -1 \text{ and } \int^{+\infty} |g| = +\infty\right\} \ \Rightarrow \\ &xg(x) = o\left(\int_T^x g\right), \quad x \to +\infty, \ (T \text{ large enough}); \end{aligned} \tag{4.15}$$

$$\left\{\gamma=-1\,\text{and}\int^{+\infty}|g|<+\infty\right\}\;\Rightarrow\; xg(x)=o\left(\int_x^{+\infty}g\right),\;\; x\to+\infty. \tag{4.16}$$

Proof of Lemma 4.3. (I) This is a classical result by Karamata ([7], pp. 26-27).

(II). These claims are cited by Seneta ([8], Exercises 2.1-2.3, pp. 86-87), but the proof of the inference in (4.15) goes back to a paper by Parameswaran ([9], Lemma 1, pp. 220-221). For the reader's convenience, we report the proof of (4.15) and its adaptation to (4.16) because Theorem 4.2 is the main result in this paper. It is much more convenient to use different notations putting $g(x)=L(x)x^{-1}$, with L (strictly positive) slowly varying, and estimating the pertinent antiderivative of g. If k is an integer ≥ 2, then for (4.15) we have

$$M_1(x):=\int_T^x L(t)t^{-1}\mathrm{d}t>\int_{x/k}^x L(t)t^{-1}\mathrm{d}t=L(x)\int_{x/k}^x\frac{L(t)}{L(x)}t^{-1}\mathrm{d}t; \tag{4.17}$$

where the uniformity of the $\lim_{x\to+\infty}L(\lambda x)/L(x)=1$ implies that inside the integral we have the estimate "$L(t)/L(x)>1/2$" choosing x large enough, say $x\geq T_k$, so that

$$M_1(x)>\frac{1}{2}L(x)\int_{x/k}^x t^{-1}\mathrm{d}t=\frac{\log k}{2}L(x)\;\;\forall\, x\geq T_k, \tag{4.18}$$

and the arbitrariness of k implies that $\lim_{x\to+\infty}M_1(x)/L(x)=+\infty$ which, due to the strict positivity of the involved functions, is equivalent to our assertion. To prove (4.16), we have analogously:

$$\begin{aligned}M_2(x)&:=\int_x^{+\infty}L(t)t^{-1}\mathrm{d}t>\int_x^{kx}L(t)t^{-1}\mathrm{d}t=L(x)\int_x^{kx}\frac{L(t)}{L(x)}t^{-1}\mathrm{d}t\\&>\frac{1}{2}L(x)\int_x^{kx}t^{-1}\mathrm{d}t=\frac{\log k}{2}L(x)\;\;\forall\, x\geq T_k,\end{aligned} \tag{4.19}$$

whence $\lim_{x\to+\infty}M_2(x)/L(x)=+\infty$. □

Proof of Theorem 4.2 (I) By Theorem 3.1, the only inclusion to be proved is

$$\left\{\mathcal{K}_\alpha(+\infty)\text{ of order }1\right\}\subset\left\{\mathcal{H}_\alpha(+\infty)\text{ of order }1\right\},$$

which may be proved using the basic result in Lemma 4.3. Let f satisfy the conditions in (4.2) with $\beta=\alpha-1$. For $\alpha>0$, $\beta>-1$, we apply (4.13) to f' so getting "$\int^{+\infty}|f'|=+\infty$" whence "$f(+\infty)=\pm\infty$" and

$$\begin{cases}f(x)\sim\int_T^x f'\left[\equiv f(x)-f(T)\right];\;\int_T^x f'\overset{(4.13)}{\sim}\dfrac{xf'(x)}{\beta+1}\equiv\dfrac{xf'(x)}{\alpha};\\ f'(x)/f(x)\sim f'(x)\Big/\int_T^x f'\sim\dfrac{\alpha f'(x)}{xf'(x)}\equiv\dfrac{\alpha}{x}\;\text{ and }\;xf'(x)/f(x)\sim\alpha.\end{cases} \tag{4.20}$$

For $\alpha<0$, $\beta<-1$, we have

$$\begin{cases}f(+\infty)=0\;\;(\text{due to }\alpha<0),\;\int^{+\infty}|f'|<+\infty\;\text{ and}\\ f(x)=-\int_x^{+\infty}f';\;\int_x^{+\infty}f'\overset{(4.14)}{\sim}\dfrac{xf'(x)}{|\alpha|};\\ f'(x)/f(x)=-f'(x)\Big/\int_x^{+\infty}f'\sim\dfrac{f'(x)}{-xf'(x)}|\alpha|\;\text{ and }\;xf'(x)/f(x)\sim\alpha.\end{cases} \tag{4.21}$$

(II) If $\alpha = 0$ and $\beta < -1$ we have by (4.14):

$$\int^{+\infty} |f'| < +\infty \text{ and } \int_x^{+\infty} f' \sim \frac{xf'(x)}{|\beta+1|}, \tag{4.22}$$

and must separate the two circumstances that the number $f(+\infty)$ be zero or not. In the case $f(+\infty) = 0$, we are in the situation of (4.21) wherein the second relation in the second line is granted by (4.16) with α replaced by $\beta+1$, hence $f \in \{\mathcal{H}_{\beta+1}(+\infty) \text{ of order } 1\}$. In the case "$f(+\infty) = \ell \neq 0$", we have

$$\frac{xf'(x)}{f(x)} \overset{(4.22)}{\sim} \frac{|\beta+1|}{\ell} \int_x^{+\infty} f' = o(1), \tag{4.23}$$

hence, $f \in \{\mathcal{H}_0(+\infty) \text{ of order } 1\}$. It remains the case "$\alpha = 0$, $\beta = -1$" wherein Lemma 4.3-(II), applied to the derivative, will be used once again.

<u>First circumstance:</u> $\int^{+\infty} |f'| = +\infty$. This implies:

$$\begin{cases} |f(+\infty)| = +\infty; \ f(x) = c + \int_T^x f' \sim \int_T^x f'; \\ xf'(x) = o\left(\int_T^x f'\right); \end{cases} \tag{4.24}$$

whence $xf'(x) = o(f(x))$, *i.e.*, $f \in \mathcal{H}_0(+\infty)$.

<u>Second circumstance:</u> $\int^{+\infty} |f'| < +\infty$:

$$\begin{cases} |f(+\infty)| \in \mathbb{R}; \ f(x) = c - \int_x^{+\infty} f'; \\ xf'(x) = o\left(\int_x^{+\infty} f'\right); \end{cases} \tag{4.25}$$

whence the same conclusion as above follows. □

5. Comments on the Bourbaki Classes

As far as a possible higher-order theory is concerned, using the classes $\mathcal{B}_\alpha(+\infty)$, though meaningful in principle, seems to be of minor usefulness in practical asymptotic problems, as such classes would be too large, encompassing oscillatory functions with awkward behaviors. This prevents characterizations of the asymptotic behaviors of derivatives and antiderivatives of <u>all</u> the functions in these classes, at least from the author's trend of thought, which stimulated the higher-order theories and their applications to asymptotic behaviors of Wronskians and asymptotic expansions. In [10] (first lines in §6), we pointed out that mere "$\mathcal{O}$" or "o"-estimates of Wronskians (obtained by simpler means) may be useless for applications to asymptotic expansions because the behaviors of ratios of Wronskians are required. The basic characteristic of any applicable higher-order theory of "types of asymptotic variation" is a precise asymptotic relationship between the higher derivatives and the function itself, which may be lacking in higher-order Bourbaki's classes, as shown by the counterexample below. As an authoritative citation, the reader may notice that, in order to obtain the basic application of the Bourbaki-Dieudonné concept of growth-order, namely the asymptotic behaviors of antiderivatives, the author himself in [6] (Prop. 8, pp. V. 23-24) uses the smaller classes that we denote by $\mathcal{H}_\alpha(+\infty)$. In this context, he uses the locution "comparable of order 1" with an altogether different meaning from ours.

A simple counterexample:

$$\begin{cases} f(x) := x^{\alpha}\left(2+\sin\phi(x)\right) \asymp x^{\alpha}, \ x \to +\infty, \ (\alpha > 0); \\ \log f(x) = \alpha \log x + \log\left(2+\sin\phi(x)\right) \sim \alpha \log x \ \ \forall\, \phi \in C^{1}\left[T, +\infty\right); \\ f'(x) = \alpha x^{\alpha-1}\left[2+\sin\phi(x) + \alpha^{-1}\phi'(x)\, x\cos\phi(x)\right]; \\ \log f'(x) \sim (\alpha-1)\log x \ \text{ hence } f \in \left\{\mathcal{B}_{\alpha}(+\infty) \text{ of order } 1\right\} \ \ \forall\, \phi. \end{cases} \tag{5.1}$$

For any ϕ such that $x\phi'(x) = o(1)$ we have

$$f'(x) \sim \alpha x^{-1} f(x), \quad f \in \left\{\mathcal{H}_{\alpha}(+\infty) \text{ of order } 1\right\};$$

whereas for the special choice, say, $\phi(x) = \alpha \log x$ we have

$$f'(x) \asymp x^{\alpha-1}; \int_{1}^{x} f \asymp \frac{x^{\alpha+1}}{\alpha+1} \asymp \frac{xf(x)}{\alpha+1}; \int_{1}^{x} f' \asymp \frac{x^{\alpha}}{\alpha} \asymp \frac{xf'(x)}{\alpha}; \tag{5.2}$$

see notation in (1.5), with the more precise relation of asymptotic equivalence holding nowhere.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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