

On Exponential Diophantine Triples of Order 2 and the Associated C^∞ Differentiable Manifold

Mouftaou Adjibade*, Joachim Moussounda Mouanda 

Mathematics Department, Blessington Christian University, Nkayi, Republic of Congo

Email: *adjibad_m@yahoo.fr

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Abstract

We investigate exponential Diophantine triples of order 2, which are sets of three integers $\{x, y, z\}$, with $x, y, z > 1$, satisfying

$$(x^2 - 1)(y^2 - 1) + 1, (x^2 - 1)(z^2 - 1) + 1, (y^2 - 1)(z^2 - 1) + 1$$

are perfect squares. It is shown that integer points (x, y, z) , with $x, y, z > 1$, of a certain C^∞ differentiable manifold form such triples. This paper establishes a recursive method for generating, via successive mutations (operations analogous to Vieta mutations as in the Markov surface), infinite families of these triples, thereby linking number theory with differential geometry.

Keywords

Diophantine Equation, Exponential Diophantine m -Tuple of Order n , C^∞ Differentiable Manifold, Normalized Triple, Markov Surface, Vieta Mutation

1. Introduction and Main Results

The exponential Diophantine equation of the type

$$(*) (a^n - 1)(b^n - 1) = x^2, n \geq 1,$$

to be solved in positive integers all greater than 1 has been for years studied by many authors. In 2000, Szalay studied first the case $(a, b) = (2, 3)$ and proved that the equation $(*)$ has no positive integer solution [1]. In 2002, Hajdu and Szalay showed that there is no solution for $(a, b) = (2, 6)$ and $(a, b) = (a, ak)$, there is no solution with $k \geq 2$ and $kn > 2$ except for the three cases $(a, n, k) \in \{(2, 3, 2), (3, 1, 5), (7, 1, 4)\}$ [2]. The same year, Cohen proved that there is no solution to $(*)$ when $4 \mid n$, except for $\{a, b\} = \{13, 239\}$ with $n = 0[4]$ [3]. In 2020,

Noubissie, Togbe and Zhang showed that the equation (*) with $b \equiv 3 \pmod{8}$, b prime and a even has no solution in positive integers n, x [4]. One year later, Yasutsugu Fujita and Maohua Le, using some elementary number theory methods, discussed the existence of positive integer solution (x, n) of the polynomial-exponential Diophantine equation (*) $(a^n - 1)(b^n - 1) = x^2$ with $n > 2$. They proved that if $\{a, b\} \neq \{13, 239\}$ and $\text{ord}_2(a^2 - 1) \neq \text{ord}_2(b^2 - 1)$, then (*) has no solution (x, n) with $2 \mid n$ ($\text{ord}_2 t$ denoting the order 2 in the factorization of t) [5]. In 2024, Mouanda's work on matrix solutions of Diophantine equations (Fermat's Diophantine equation, exponential Diophantine equations) shows that these Diophantine equations always admit, in each case, an infinite number of matrix solutions [6]. Later that same year, Mouanda showed that, given n a positive integer, the matrix exponential Diophantine equation

$$(X^n - I_{q \times n})(Y^n - I_{q \times n}) = Z^2, \quad X \neq Y, \quad q \in \mathbb{N},$$

admits at least $4 \times n^2$ different construction structures of matrix solutions and he established the connection between the construction structures of matrix solutions of an exponential Diophantine equation and integer factorization [7]. Still in the same year, Mouanda, Dehainsala, Kangni showed that the structures of the matrix solutions of the matrix elliptic curves $(E): Y^2 = X^3 + \alpha \times I, \alpha \in \mathbb{N}$ allow the construction of the matrix solutions of the equation $(X^{24} - I_{24})(Y^{24} - I_{24}) = Z^2$ [8].

One can generalize equation (*) $(x^n - 1)(y^n - 1) = z^2$ by introducing the equation

$$(x^n - 1)(y^n - 1) + c = z^2,$$

where c is a positive integer constant. For $c = 0$, one recovers equation (*). For $c = 1$, one obtains the exponential Diophantine equation of order n

$$(E_n) \quad (x^n - 1)(y^n - 1) + 1 = z^2, \quad n \in \mathbb{N}^*, \quad x, y, z \in \mathbb{N}^* \setminus \{1\},$$

introduced in 2025 by Mouanda and Dehainsala [9].

Equation (E_n) is motivated by the structure of Diophantine tuples. Setting $X = x^n - 1$ and $Y = y^n - 1$, equation (E_n) reduces to $XY + 1$ is a perfect square, which is the classical defining relation for Diophantine pairs.

In this paper, we study equation (E_n) in the case $n = 2$. The corresponding exponential Diophantine equation is then

$$(E_2) \quad (x^2 - 1)(y^2 - 1) + 1 = z^2, \quad x, y, z \in \mathbb{N}^* \setminus \{1\}.$$

In this case, using Vieta mutations (as in the Markov theory), we generate tree of exponential Diophantine triples of order 2, exactly as in the classical theory of Markov-type equations.

Theorem 1.1. For all integer-valued polynomial of several variables P defined over \mathbb{N} , the set $\{P, P + 1\}$ is an exponential Diophantine pair of order 2.

Theorem 1.2. Let (a, b, c) be a triple of integers, with $1 < a \leq b \leq c$, satisfying the equation

$$(E) \quad a^2 + b^2 + c^2 = 2abc + 2.$$

Then,

(I.) The triple $\{a, b, c\}$ is an exponential Diophantine triple of order 2;

(II.) Defining the transformations

$$\sigma_1 : (a, b, c) \mapsto (a, c, 2ac - b) \text{ and } \sigma_2 : (a, b, c) \mapsto (b, c, 2bc - a),$$

the triples $(a, c, 2ac - b)$ and $(b, c, 2bc - a)$ are normalized triples of non-zero positive integers satisfying equation (E);

(III.) The set $S \subset \mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 2xyz + 2\}$ is a closed, non-compact (unbounded) surface whose integer points (x, y, z) , with $x, y, z > 1$, correspond to exponential Diophantine triples of order 2;

(IV.) $S \subset \mathbb{R}^3$ is a connected, smooth (C^∞) two-dimensional manifold, invariant under the action of the symmetric group S_3 . Moreover, $(0, 1, 1)$ is the unique minimal integer solution of S , and every positive integer point of S descends in finitely many steps to $(0, 1, 1)$.

* The surface S bears a strong resemblance to the classical Markov surface \mathcal{M} , defined by $x^2 + y^2 + z^2 = 3xyz$, and shares with it some fundamental features.

2. Preliminaries

Definition 2.1. A triple (a, b, c) is said to be normalized if

$$a \leq b \leq c.$$

Definition 2.2. A set of m non-zero positive integers $\{a_1, a_2, \dots, a_m\}$ is called a Diophantine m -tuples if

$$a_i a_j + 1 = z^2 \quad \forall 1 \leq i, j \leq m, i \neq j.$$

For example, the set $\{1, 3, 8\}$ is a Diophantine triple since the product of any two elements of this set increased by one unit is a perfect square. In fact, the problem of finding m numbers such that the product of any two elements of them increased by one unit is a perfect square can be traced back to the third century AD. This problem was first solved by the Greek mathematician Diophantus when

he found that $\left\{ \frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16} \right\}$ is a set of four rationals which satisfy the above

property. In the integer case, the first Diophantine quadruple $\{1, 3, 8, 120\}$ has been found by Fermat. Euler was able to extend this set by adding the rational number $\frac{777480}{8288641}$. In his works, Euler discussed certain problems inspired by

Diophantus and explained some elementary constructions for producing Diophantine pairs and highlighted their connection with quadratic equations. He proved the extendability of all Diophantine pair $\{a, b\}$ to a Diophantine quadruple. Indeed, Euler showed that if $\{a, b\}$ is a Diophantine pair, then the set

$$\{a, b, a + b + 2r, 4r(r + a)(r + b) \mid r = \sqrt{ab + 1}\}$$

is a Diophantine quadruple [10]. In the 20th century, several deep contributions appeared. Using transcendental methods, Alan Baker and Harold Davenport

proved in 1969 that Fermat's quadruple cannot be extended to a quintuple, establishing the first non-extendability result for integer quadruples [11]. In 1979, Arkin, Hoggatt and Strauss proved the extendability of all Diophantine triple to a Diophantine quadruple [12]. In January 1999, Philip Gibbs found sets of six positive rationals [13]. It is not known whether larger rational Diophantine m -tuples exist, or if there is an upper bound, but it is known that no infinite set is a Diophantine m -tuples. In 2004, Andrej Dujella proved that there exists at most a finite number of Diophantine quintuples [14]. Recently, in 2016, He, Togbe and Ziegler finally proved that no integer Diophantine quintuple exists, completing the resolution of Fermat's conjecture after nearly 400 years. [15].

Definition 2.3. A set of m positive integers $\{a_1, a_2, \dots, a_m\}$, $a_i > 1$ for all i , is called an exponential Diophantine m -tuples of order n ($n \geq 1$) if

$$(a_i^n - 1)(a_j^n - 1) + 1 = z^2 \quad \forall 1 \leq i, j \leq m, i \neq j.$$

Assume that $n = 1$, we have the exponential Diophantine equation of order 1

$$(E_1) \quad (x-1)(y-1) + 1 = z^2, \quad x, y, z \in \mathbb{N}^* \setminus \{1\}.$$

Setting

$$u = x-1 \quad \text{and} \quad v = y-1.$$

We have

$$uv + 1 = z^2, \quad u, v \geq 1.$$

In this case, the problem of finding exponential Diophantine m -tuples of order 1 comes down to finding Diophantine m -tuples. Indeed, if the set $\{a_1, a_2, \dots, a_m\}$ of m non-zero positive integers is a Diophantine m -tuples, then the set $\{a_1 + 1, \dots, a_m + 1\}$ is an exponential Diophantine m -tuples of order 1. For instance, the set $\{1, 3, 8, 120\}$ is a Diophantine quadruple, then the set $\{2, 4, 9, 121\}$ is an exponential Diophantine quadruple of order 1. As there does not exist Diophantine quintuple, there does not exist exponential Diophantine quintuple of order 1 either.

Assume that $n = 2$, we have the exponential Diophantine equation of order 2

$$(E_2) \quad (x^2 - 1)(y^2 - 1) + 1 = z^2, \quad x, y, z \in \mathbb{N}^* \setminus \{1\}.$$

The sets $\{2, 3, 11\}$, $\{3, 11, 64\}$, $\{2, 41, 153\}$, \dots are exponential Diophantine triples of order 2.

Assume that $n = 3$, we have the exponential Diophantine equation of order 3

$$(E_3) \quad (x^3 - 1)(y^3 - 1) + 1 = z^2, \quad x, y, z \in \mathbb{N}^* \setminus \{1\}.$$

The sets $\{5, 277\}$, $\{11, 13\}$, $\{37, 91\}$, \dots are exponential Diophantine pairs of order 3. It is not yet known whether exponential Diophantine triples of order 3 exist.

For $n \geq 4$, it is not yet known whether the equation (E_4) has any solution.

Definition 2.4. An exponential Diophantine m -tuples $\{a_1, \dots, a_m\}$ of order n is said to be trivial if at least one of the a_i equals 1.

Definition 2.5. Let A be a subset of a topological space X ($A \subset X$).

A is closed if it contains all its limit points; that is, if every convergent sequence (x_n) in X with $x_n \in A$ has its limit also in A .

Definition 2.6. A subset of \mathbb{R}^n is compact if and only if it is closed and bounded (Heine-Borel Theorem).

Remark 2.7. Closed and bounded implies compact in Euclidean spaces (\mathbb{R}^n) but not in general topological spaces.

Definition 2.8. A topological space X is said to be connected if it cannot be written as the union of two nonempty, disjoint, open subsets of X . In other words, if X_1, X_2 are two open subsets of X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$ then $X_1 = \emptyset$ or $X_2 = \emptyset$.

Definition 2.9. The gradient of a function is a vector that points in the direction of the greatest rate of increase of the function at a given point and is calculated by taking the partial derivative of the function with respect to each of its variables. For a differentiable function $f(x_1, \dots, x_n)$, the gradient vector field is given by

$$\nabla f(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Definition 2.10. A regular surface is a subset of a three-dimensional space which can be defined implicitly by an equation $F(x, y, z) = c$ such that the gradient of the function

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

must be non-zero at every point on the surface.

Definition 2.11. A differentiable manifold of dimension n is a Hausdorff, second-countable topological space M that is locally homeomorphic to \mathbb{R}^n , together with an atlas

$$\{(U_i, \varphi_i)\} \quad (U_i \subset M \text{ is open})$$

of coordinate charts such that all transition maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

are C^k (typically C^∞), i.e. they are k -times continuously differentiable.

When the transition maps are all C^∞ , the manifold is called a smooth (or C^∞) differentiable manifold.

Now let us precise what are Charts, atlases, and transition maps.

- Each point on a manifold has a local coordinate system. This means that $\forall x \in M$, there exists an open neighborhood U_x and an homeomorphism

$$\varphi_x : U_x \rightarrow \varphi_x(U_x) \subset \mathbb{R}^n.$$

(U_x, φ_x) is a chart of M or a local coordinate system.

- A specific collection of charts (U_i, φ_i) which covers a manifold is called an atlas of M . An atlas is not unique as all manifolds can be covered in multiple ways using different combinations of charts. Two atlases are said to be equivalent if their union is also an atlas. The atlas containing all possible charts con-

sistent with a given atlas is called the maximal atlas.

- Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. Given two overlapping charts $U_i \cap U_j$, the transition map or the change of coordinates is

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j).$$

Examples: One-dimensional manifolds include lines and circles. Two-dimensional manifolds, also called surfaces, include the plane, the sphere, and the torus.

Remark 2.12. Every regular surface in Euclidean space defines a two-dimensional differentiable manifold.

Definition 2.13. Given an equation $F(x_1, \dots, x_n) = 0$ and a solution (x_1, \dots, x_n) , assume that F is quadratic in one variable, say x_i .

A Vieta mutation in the variable x_i consists in replacing x_i by the second root of the quadratic equation obtained by viewing x_i as the unknown, while keeping the other variables fixed.

The new value of x_i is determined via Viète’s relations (sum and product of the roots), and the resulting n -tuple is again a solution of the equation.

Definition 2.14. An exponential Diophantine triple of order 2 is said to be minimal if it is non-trivial and does not arise from another non-trivial exponential Diophantine triple of order 2 via an increasing Vieta mutation.

3. Proof of the Main Results

In this section, we show that the linear relation $y = x + 1$ implies that any two consecutive integers form an exponential Diophantine pair of order 2. It is a sufficient condition for constructing exponential Diophantine pairs of order 2 of the form $\{P, P + 1\}$ where P is an integer-valued polynomial of several variables defined over \mathbb{N} . We then show, via a Markov-type recurrence, how to construct infinite trees of exponential Diophantine triples of order 2.

Proof of the theorem 1.1.

Considering the exponential Diophantine equation of order 2

$$(E_2) \quad (x^2 - 1)(y^2 - 1) + 1 = z^2, \quad x, y, z \in \mathbb{N}^* \setminus \{1\}.$$

We can write equation (E_2) as

$$(x + 1)(x - 1)(y + 1)(y - 1) = (z + 1)(z - 1).$$

So, one of the sufficient conditions for the equation (E_2) to be true is

$$\begin{cases} (x - 1)(y + 1) = z - 1 & (1) \\ (x + 1)(y - 1) = z + 1 & (2) \end{cases}$$

Expanding the terms on the left side hand in this system of simultaneous equations, we obtain

$$\begin{cases} xy + x - y = z & (1) \\ xy - x + y - 2 = z & (2) \end{cases}$$

Then, this implies that

$$(1)-(2) \Rightarrow \begin{cases} 2x-2y = -2, \\ y = x+1, \end{cases}$$

and

$$(1)+(2) \Rightarrow \begin{cases} 2xy-2 = 2z, \\ z = xy-1. \end{cases}$$

Thus, we have

$$\begin{cases} x \in \mathbb{R}, \\ y = x+1, \\ z = xy-1, \end{cases} \Rightarrow \begin{cases} x \in \mathbb{R}, \\ y = x+1, \\ z = x(x+1)-1. \end{cases}$$

Substituting these values in equation (E_2) , we get

$$(x^2-1)((x+1)^2-1)+1=(x(x+1)-1)^2 \quad \forall x \in \mathbb{R}.$$

Let us now set for $P = P(k_1, k_2, \dots, k_n)$ an integer-valued polynomial of several variables defined over \mathbb{N} ,

$$\begin{cases} x = P(k_1, k_2, \dots, k_n), (k_1, k_2, \dots, k_n) \in \mathbb{N}^n, \\ y = x+1 = P(k_1, k_2, \dots, k_n)+1. \end{cases}$$

The set $S_{(k_1, k_2, \dots, k_n)} = \{P(k_1, k_2, \dots, k_n), P(k_1, k_2, \dots, k_n)+1\}$ is then a polynomial-exponential Diophantine pair of order 2.

Indeed, we have

$$(P^2-1)((P+1)^2-1)+1=(P(P+1)-1)^2 \quad \forall P.$$

Examples of polynomial-exponential Diophantine pairs of order 2:

The following sets are polynomial-exponential Diophantine pairs of order 2.

$$u_k = \{k, k+1\} \quad \forall k \geq 2.$$

$$v_{(k_1, k_2)} = \{3k_1^2 + 2k_2, 3k_1^2 + 2k_2 + 1\} \quad \forall k_1, k_2 \geq 1.$$

$$w_{(k_1, k_2, k_3)} = \{2k_1^3 + 5k_2^2 k_3 - 2k_3, 2k_1^3 + 5k_2^2 k_3 - 2k_3 + 1\} \quad \forall k_1, k_2, k_3 \geq 1.$$

Define the sequences $(u), (v)$ and (w) respectively by

$$(u_k)_{k \geq 2}, (v_{(k,k)})_{k \geq 1}, (w_{(k,k,k)})_{k \geq 1}.$$

Then, the sequences $(u), (v)$ and (w) are sequences of exponential Diophantine pairs of order 2.

Now let us prove our main theorem.

Proof of the theorem 1.2.

Let (a, b, c) be a triple of integers, with

$$1 < a \leq b \leq c,$$

satisfying the equation

$$(E) \quad a^2 + b^2 + c^2 = 2abc + 2.$$

Let us show **(I)**

Let us prove that the triple $\{a, b, c\}$ is an exponential Diophantine triple of order 2.

We have

$$(a^2 - 1)(b^2 - 1) + 1 = (ab)^2 - a^2 - b^2 + 2 = (ab)^2 - (a^2 + b^2) + 2 \quad (\lambda_0)$$

But

$$a^2 + b^2 + c^2 = 2abc + 2 \Rightarrow -(a^2 + b^2) + 2 = c^2 - 2abc.$$

We can write equation (λ_0) as

$$(a^2 - 1)(b^2 - 1) + 1 = (ab)^2 + c^2 - 2abc \quad (\lambda_0)$$

It is straightforward to see that

$$(a^2 - 1)(b^2 - 1) + 1 = (ab - c)^2 \quad (\lambda_0)$$

Proceeding as in the above case, we show that

$$(a^2 - 1)(c^2 - 1) + 1 = (ac - b)^2 \quad (\lambda_1)$$

and that

$$(b^2 - 1)(c^2 - 1) + 1 = (bc - a)^2 \quad (\lambda_2)$$

Hence, we have

$$\begin{cases} (a^2 - 1)(b^2 - 1) + 1 = (ab - c)^2 & (\lambda_0) \\ (a^2 - 1)(c^2 - 1) + 1 = (ac - b)^2 & (\lambda_1) \\ (b^2 - 1)(c^2 - 1) + 1 = (bc - a)^2 & (\lambda_2) \end{cases}$$

Then, $\{a, b, c\}$ is an exponential Diophantine triple of order 2.

We have thus shown that if (a, b, c) is a normalized triple of integers, with $a, b, c > 1$, then

$$(a, b, c) \in S \Rightarrow \{a, b, c\} \in \Gamma,$$

where S denotes the set of solutions of equation (E) and Γ denotes the set of exponential Diophantine triples of order 2.

Let us show **(II.)**

Let us consider the equation

$$(E) \quad a^2 + b^2 + c^2 - 2abc - 2 = 0.$$

Fixing b and c , this equation can be rewritten as a quadratic equation in a :

$$(E_a) \quad a^2 - 2bca + (b^2 + c^2 - 2) = 0.$$

The equation (E_a) is a second-degree equation whose sum of roots is $2bc$ and whose product of roots is $b^2 + c^2 - 2$. If a is a solution of (E_a) , then

$$2bc - a = \frac{b^2 + c^2 - 2}{a}$$

is also a solution. This shows that if (a, b, c) satisfies equation (E) , then

$(2bc - a, b, c)$ is also a solution of (E) .

By considering similarly the quadratic equations in b , denoted (E_b) , and in c , denoted (E_c) , one shows, as above, that if (a, b, c) satisfies equation (E) , then $(a, 2ac - b, c)$ and $(a, b, 2ab - c)$ are also solutions of (E) .

Therefore, if (a, b, c) is a solution of (E) , one can define new solution triples

$$(2bc - a, b, c), (a, 2ac - b, c), (a, b, 2ab - c).$$

This shows that the positive integer solutions can be generated recursively, in a manner analogous to that of Markov triples.

* Triple $(2bc - a, b, c)$

We have

$$0 < \frac{a}{c} \leq 1 \Rightarrow 1 \leq 2b - \frac{a}{c} \Rightarrow 1 \leq \frac{2bc - a}{c} \Rightarrow c \leq 2bc - a \Rightarrow 1 < b \leq c \leq 2bc - a.$$

The normalized form is therefore

$$(b, c, 2bc - a).$$

* Triple $(a, 2ac - b, c)$

Similarly, we have

$$0 < \frac{b}{c} \leq 1 \Rightarrow 1 \leq 2a - \frac{b}{c} \Rightarrow 1 \leq \frac{2ac - b}{c} \Rightarrow c \leq 2ac - b \Rightarrow 1 < a \leq c \leq 2ac - b.$$

The normalized form is therefore

$$(a, c, 2ac - b).$$

* Triple $(a, b, 2ab - c)$

Fixing a and b , equation (E) can be rewritten as a quadratic equation in x :

$$P(x) = x^2 - 2abx + (a^2 + b^2 - 2) = 0.$$

The two roots of the polynomial P are

$$c \text{ and } c' = 2ab - c.$$

Let us compute $P(b)$, the value of the polynomial at $x = b$:

$$\begin{aligned} P(b) &= b^2 - 2abb + (a^2 + b^2 - 2) \\ &= 2b^2 - 2ab^2 + a^2 - 2 \\ &= a^2 + 2b^2(1 - a) - 2. \end{aligned}$$

We have

$$b > a \geq 2.$$

So

$$1 - a \leq -1 \Rightarrow 2b^2(1 - a) \leq -2b^2.$$

Thus,

$$P(b) \leq a^2 - 2b^2 - 2.$$

Since $b > a \Rightarrow 2b^2 > a^2$,

$$a^2 - 2b^2 - 2 < 0.$$

Therefore

$$P(b) < 0.$$

The polynomial P has a positive leading coefficient. Therefore, if it is negative at b , then b lies strictly between its two roots which are $2ab - c$ and c . Hence,

$$2ab - c < b < c.$$

Finally

$$2ab - c < b.$$

Thus, the triple $(a, b, 2ab - c)$ cannot be put into normalized form, since the quantity $2ab - c$ may lie between a and b , or be smaller than a .

Among the three transformations associated with equation (E), only the two transformations

$$\sigma_1 : (a, b, c) \mapsto (a, c, 2ac - b) \text{ and } \sigma_2 : (a, b, c) \mapsto (b, c, 2bc - a)$$

are used in the construction of the solution trees. These transformations strictly increase the maximal component of the triple and therefore generate genuinely new solutions.

The third transformation

$$\sigma_3 : (a, b, c) \mapsto (a, b, 2ab - c)$$

produces a smaller solution, whose maximal component is $b < c$, and thus corresponds to a descent in the tree. Consequently, it is excluded in order to avoid backtracking and to obtain an infinite rooted tree of solutions.

Thus, part (II.) of the theorem is proved; that is, $(a, c, 2ac - b)$ and $(b, c, 2bc - a)$ are normalized triples of non-zero positive integers satisfying equation (E).

Now let us show (III.)

* Define $S \subset \mathbb{R}^3$ by its implicit equation

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$$

where $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a polynomial function defined by

$$F(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 2.$$

(i) The set $\{0\} \subset \mathbb{R}$ is closed since \mathbb{R} is a Hausdorff space.

(ii) $S = F^{-1}(\{0\})$.

(iii) By continuity of F , the inverse image of a closed set is closed. Then, S is closed.

* The set $S \subset \mathbb{R}^3$ is unbounded. Let us prove it.

Considering $x = y = t$ (real parameter t). Substituting these values in the implicit equation of S , we obtain the quadratic equation in z

$$z^2 - 2t^2z + (2t^2 - 2) = 0.$$

The discriminant is

$$\Delta = 4t^4 - 8t^2 + 8 = 4(t^4 - 2t^2 + 2) > 0 \quad \forall t \in \mathbb{R}.$$

Therefore, there always exist two real solutions

$$z(t) = t^2 \pm \sqrt{t^4 - 2t^2 + 2}.$$

For $t \rightarrow +\infty$,

$$\sqrt{t^4 - 2t^2 + 2} = t^2 \sqrt{1 - \frac{2}{t^2} + \frac{2}{t^4}} = t^2 \left(1 + o\left(\frac{1}{t^2}\right) \right).$$

Thus, one of the solutions satisfies

$$z_1(t) = t^2 + \sqrt{t^4 - 2t^2 + 2} \sim 2t^2 \Rightarrow |z_1(t)| \rightarrow +\infty.$$

S contains points of arbitrarily large norm: S is unbounded. Since a compact subset of \mathbb{R}^3 must be closed and bounded, S is not compact.

Finally S is closed, but not bounded (so non compact) since we have found a family of points $(t, t, z_1(t)) \in S$ such that

$$\|(t, t, z_1(t))\| \rightarrow \infty \text{ for } t \rightarrow +\infty.$$

Let (x, y, z) be any integer point of S , with $x, y, z > 1$. By symmetry, we may assume that it is normalized, we have showed in **(I.)** that the triple $\{x, y, z\}$ is an exponential Diophantine triples of order 2.

Let us show **(IV.)**

Surface regularity

We have

$$F(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 2, \quad S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}.$$

Let us search singular points on S : Solving $\nabla F = 0$ on S .

Calculation of the gradient

$$\nabla F = 2(x - yz, y - xz, z - xy).$$

Possible singular points verify the system of simultaneous equation

$$x - yz = 0, y - xz = 0, z - xy = 0.$$

In other words

$$x = yz, y = xz, z = xy. \quad (\star)$$

Analyze (\star)

- First case: if one of the coordinate is 0, say $y = 0$, then $x = yz = 0$ and $z = xy = 0$. We obtain the origin $(0, 0, 0)$, but $F(0, 0, 0) = -2 \neq 0 \Rightarrow (0, 0, 0) \notin S$.
- Second case: $x, y, z \neq 0$. From $x = yz$ and $y = xz$, we obtain $y = (yz)z = z^2$. As $y \neq 0 \Rightarrow z^2 = 1$. Proceeding as in the previous case, we get $x^2 = 1$ and $y^2 = 1$. We deduce that the possible solutions (x_0, y_0, z_0)

of the equation (*) are such that

$$x_0 = \pm 1, y_0 = \pm 1, z_0 = \pm 1.$$

But it is straightforward to see that

$$F(x_0, y_0, z_0) = x_0^2 + y_0^2 + z_0^2 - 2x_0y_0z_0 - 2 = 3 \pm 2 - 2 = 1 \pm 2 \neq 0.$$

So, none of these points (x_0, y_0, z_0) is a point of S .

Finally, there is no point $p \in S$ such that $\nabla F(p) = 0$. In other words, ∇F never takes a zero value on S . Thus, there is no singular point on S . This implies that S is a regular (smooth) surface of class C^∞ (since F is polynomial) in the Euclidean space \mathbb{R}^3 . Therefore, S is a smooth (C^∞) two-dimensional manifold in \mathbb{R}^3 .

Moreover, The equation of the surface S is symmetric in x, y, z , hence if (x, y, z) is a solution, every permutation of these numbers is also a solution. Thus, the surface S is invariant under all permutations of x, y, z . Subsequently, the surface S is invariant under the action of the symmetric group S_3 acting by permutation of the coordinates.

Surface connectedness

Let us prove now that the surface S is connected (cannot be separated in two disjoint non-empty open subsets). We can reason geometrically and analytically.

Let us look at what the surface S looks like.

Write $F(x, y, z) = 0$ as a quadratic equation in z

$$z^2 - (2xy)z + (x^2 + y^2 - 2) = 0.$$

The solutions are

$$z(x, y) = xy \pm \sqrt{(x^2 - 1)(y^2 - 1) + 1}$$

provided that

$$\Delta = (x^2 - 1)(y^2 - 1) + 1 \geq 0.$$

So, for each pair (x, y) such that $\Delta \geq 0$, we have two real values of z (2 sheets)

$$z_+ = xy + \sqrt{\Delta} \quad \text{and} \quad z_- = xy - \sqrt{\Delta}.$$

Define

S_+ the sheet relating to z_+ by

$$S_+ = \{(x, y, z_+(x, y)), x, y, z \in \mathbb{R} \mid \Delta \geq 0\},$$

S_- the sheet relating to z_- by

$$S_- = \{(x, y, z_-(x, y)), x, y, z \in \mathbb{R} \mid \Delta \geq 0\}.$$

For $\Delta = 0$, we have $z_+ = z_- = xy$.

So, for $\Delta = 0$, the two sheets S_+ and S_- intersect and their intersection $(S_+ \cap S_-)$ defines a curve called the junction curve (see **Figure 1**).

Then, the junction curve corresponds to $\Delta = 0$, that is to say

$$x^2y^2 - x^2 - y^2 + 2 = 0.$$

In this case, the two sheets meet and we have $z = xy$.

Explicit equation of the junction curve

$$\begin{cases} x^2y^2 - x^2 - y^2 + 2 = 0, \\ z = xy. \end{cases}$$

It can be considered as a plane curve in the plane (x, y) , and then we move it up in (x, y, z) via $z = xy$.

Junction curve of the surface $S (\Delta=0)$, $z = x \cdot y$

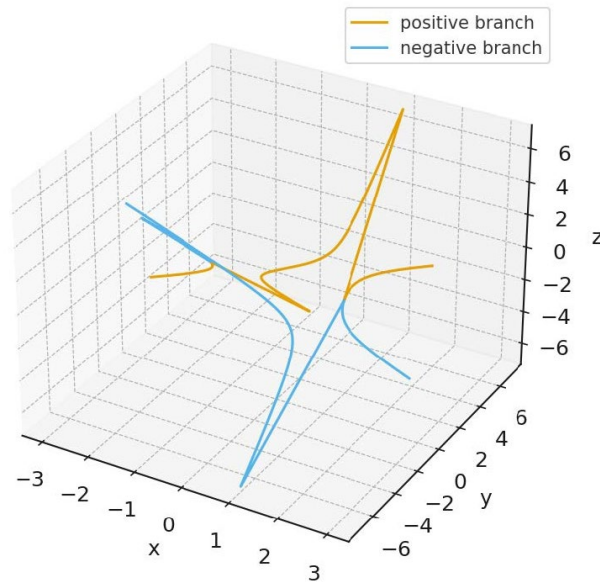


Figure 1. Junction curve.

Continuity between branches

The two signs in

$$z(x, y) = xy \pm \sqrt{x^2y^2 - x^2 - y^2 + 2}$$

do not define two disjoint components, because they meet on the curve where $\Delta = 0$, that is to say

$$x^2y^2 - x^2 - y^2 + 2 = 0.$$

Along this curve, the two real branches coincide ($z = xy$), ensuring the continuity of the surface at the junction. No discontinuity or gap occurs; hence, the two components smoothly connect to form a single continuous surface, as illustrated in **Figure 2**.

Symmetries

The implicit equation

$$x^2 + y^2 + z^2 - 2xyz - 2 = 0$$

is fully symmetric in the variables x, y, z .

Moreover, it is invariant under any even sign change, meaning that the substi-

tution of two among the variables by their opposites leaves the equation unchanged.

Equivalently,

$$(x, y, z) \in S \Rightarrow (-x, -y, z) \in S \Rightarrow (x, -y, -z) \in S, \text{ and so on.}$$

These symmetry operations map one local branch of the surface onto another.

Surface S with its junction curve ($\Delta=0$), $z = x \cdot y$

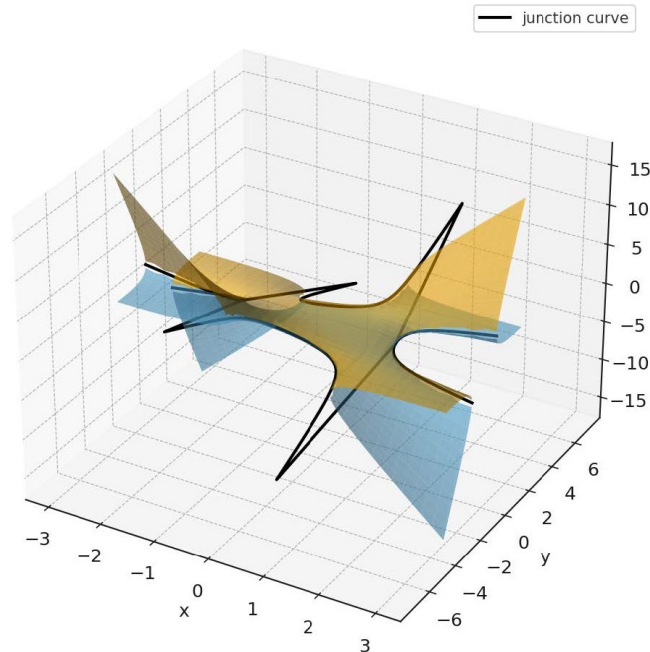


Figure 2. Surface $S : x^2 + y^2 + z^2 = 2xyz + 2$.

Finally, the surface exhibits full symmetry and contains no singularities or discontinuities. The “+” and “-” branches are continuously joined along the boundary defined by $\Delta = 0$. Hence, the surface has no isolated components, and we conclude that S is connected.

Minimal triple on S

Let $(a, b, c) \in \mathbb{N}^3$ be a point of S . By symmetry, we may assume that it is normalized, *i.e.*

$$0 \leq a \leq b \leq c.$$

- case 1: $a > 1$.

We have seen that the Vieta mutation

$$\sigma_3 : (a, b, c) \mapsto (a, b, 2ab - c)$$

strictly decreases the maximal component.

- case 2: $a = 1$.

In this case, it is pretty simple to see that $(a, b, c) = (1, k, k + 1)$, with $k \geq 1$.

The Vieta mutation

$$\sigma_3 : (1, k, k + 1) \mapsto (1, k, k - 1)$$

strictly decreases the maximal component.

- case 3: $a = 0$.

The only solution is $(0, 1, 1)$ and it is easy to see that there is no Vieta mutation that decreases the maximal component of the triple $(0, 1, 1)$.

Therefore, $(0, 1, 1)$ is the only positive integer point of S for which no such decreasing mutation exists.

Let $(a, b, c) \in \mathbb{N}^3$ be a point of S such that

$$0 \leq a \leq b \leq c.$$

If $(a, b, c) \neq (0, 1, 1)$, one can always choose a mutation (σ_3) that decreases the maximum component. Since the components are integers and bounded below, this descent process must terminate after finitely many steps. The only solution for which no such decreasing mutation exists is $(0, 1, 1)$.

This proves that every positive integer point of S descends in finitely many steps to $(0, 1, 1)$. Thus, $(0, 1, 1)$ is the minimal triple that generates all positive integer solutions on the surface, as illustrated in **Figure 3**.

Diagram of the solution tree of S

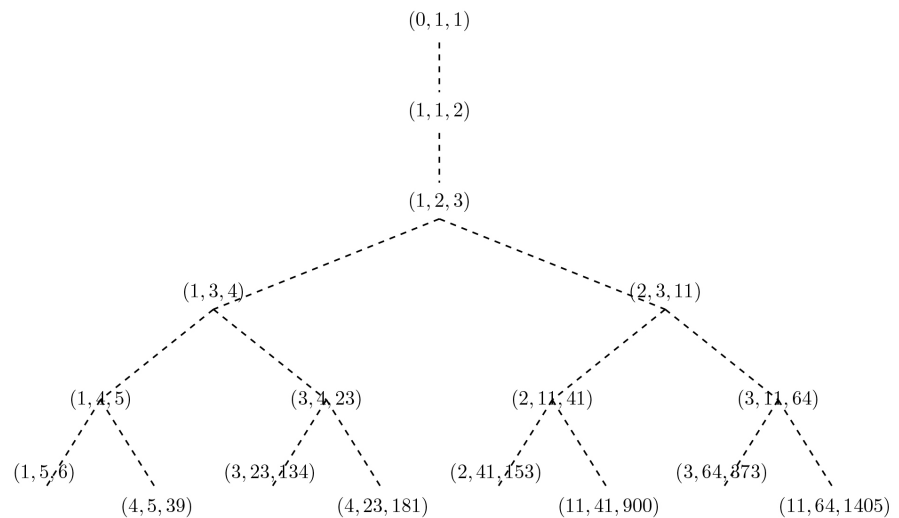


Figure 3. Tree generated from the single minimal triple $(0, 1, 1)$.

4. Construction of the Tree of Exponential Diophantine Triples of Order 2

Let (a, b, c) be a normalized triple of integers, with $a, b, c > 1$, satisfying the equation

$$a^2 + b^2 + c^2 - 2abc - 2 = 0.$$

We define the two transformations:

$$\sigma_1 : (a, b, c) \mapsto (a, c, 2ac - b), \quad \sigma_2 : (a, b, c) \mapsto (b, c, 2bc - a).$$

These transformations have the following properties:

1. They produce new solutions of (E) .

If (a, b, c) satisfies (E) , then so do $\sigma_1(a, b, c)$ and $\sigma_2(a, b, c)$.

2. They are increasing.

For a normalized triple (a, b, c) , the largest component of $\sigma_1(a, b, c)$ and $\sigma_2(a, b, c)$ is strictly greater than c .

Therefore, repeated applications generate triples with strictly increasing maximal elements.

Recursive Generation

Starting from an initial normalized triple (a_0, b_0, c_0) , we can

1. Apply σ_1 and σ_2 to obtain two new normalized triples.
2. Apply σ_1 and σ_2 recursively to each new triple, indefinitely, as shown in

Figure 4.

This process generates:

- * infinite trees of solutions,
- * consisting only of normalized triples,
- * with strictly increasing components, avoiding cycles and repetitions.

Remarks

- * σ_3 is not used in this construction, since it produces smaller triples that have already appeared in the tree and does not contribute to the growing solution tree.
- * This method is analogous to the construction of the Markov triples tree.

Solution tree

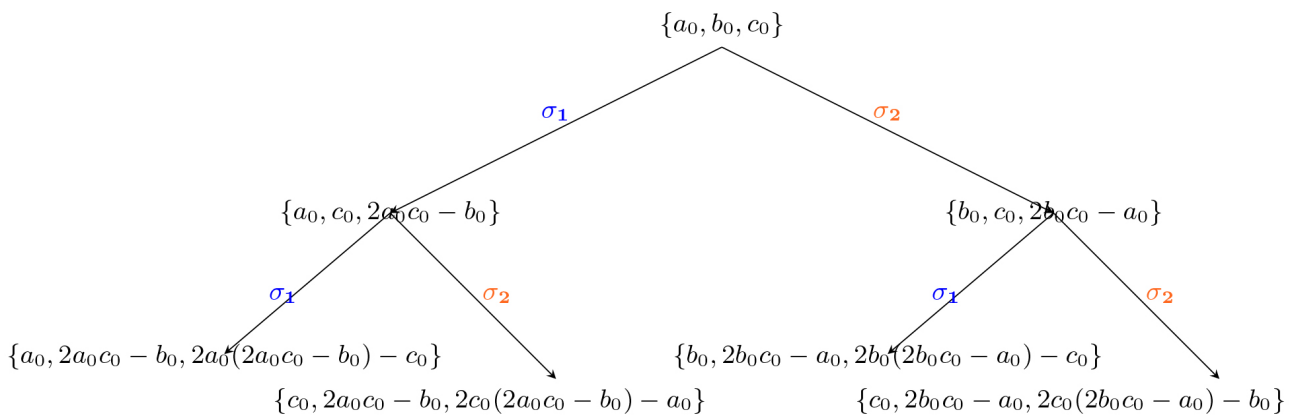


Figure 4. Exponential Diophantine triples of order 2 generated by mutations.

5. Analogy with the Markov Surface \mathcal{M}

Let $S \subset \mathbb{R}^3$ be the surface defined by

$$S: x^2 + y^2 + z^2 = 2xyz + 2.$$

The equation defining S is structurally similar to the classical Markov equation

$$\mathcal{M}: x^2 + y^2 + z^2 = 3xyz,$$

whose integer solutions, the Markov triples, possess deep arithmetic and geometric significance.

The surface S shares with Markov surface \mathcal{M} three fundamental features:

- (i) **A symmetric cubic in three variables**

As in the case of Markov surface \mathcal{M} , the surface S is defined by an equation symmetric in x, y, z , in which the dominant term is the product of the three variables.

(ii) Invariance under the symmetric group S_3

The surface S is invariant under any permutation of the three variables:

$$(x, y, z) \mapsto (y, z, x), (z, x, y), \dots$$

This is exactly the same geometric property as for the Markov surface \mathcal{M} .

(iii) A tree structure generated by mutations (as in Markov theory)

On the Markov surface \mathcal{M} , positive integer solutions are connected by Vieta involutions

$$(x, y, z) \mapsto (x, y, 3xy - z).$$

These mutations generate an infinite tree that contains all positive integer solutions.

On the surface S , a completely analogous phenomenon occurs:

$$(x, y, z) \text{ solution} \Rightarrow (x, y, 2xy - z)$$

is also a solution.

This type of transformation:

- * corresponds to a birational involution,
- * acts as a hidden symmetry of the surface,
- * produces infinitely many solutions by iteration,
- * generates a tree of solutions, exactly as in the classical theory of Markov.

The surface S may be viewed as a modified Markov-type surface \mathcal{M} . The two surfaces share some fundamental features. The Markov surface has the unique minimal integer solution $(1, 1, 1)$, whereas the surface S admits the unique minimal integer solution $(0, 1, 1)$. All integer solutions of S are obtained from this solution by iterating Vieta mutations, and they are organized into a single rooted infinite solution tree, exactly as in the Markov surface \mathcal{M} .

6. Numerical Examples

For all integer-valued polynomial $P > 1$ of several variables over \mathbb{N} , the set $\{P, P+1\}$ is an exponential Diophantine pair of order 2. We can extend this pair to a triple $\{P, P+1, Q\}$ by considering the quadratic equation in Q :

$$(E_Q) \quad P^2 + (P+1)^2 + Q^2 = 2P(P+1)Q + 2.$$

We have two solutions $Q_1 = 1$ (trivial solution) and $Q_2 = 2P^2 + 2P - 1$.

$\{P, P+1, 2P^2 + 2P - 1\}$ is then an exponential Diophantine triple of order 2.

Setting $P(k) = k$, we obtain the infinite parametric family of exponential Diophantine triples of order 2:

- $\{k, k+1, 2k^2 + 2k - 1\}_{k>1}$.
 $\{2, 3, 11\}, \{3, 4, 23\}, \{4, 5, 39\}, \{5, 6, 59\}, \{6, 7, 83\},$
 $\{7, 8, 111\}, \{8, 9, 143\}, \{9, 10, 179\}, \{10, 11, 219\}, \dots$

Applying the transformation σ_1 to $\{k, k+1, 2k^2+2k-1\}$, we obtain the infinite parametric family of exponential Diophantine triples of order 2:

- $\{k, 2k^2+2k-1, 4k^3+4k^2-3k-1\}_{k>1}$.
 $\{2, 11, 41\}, \{3, 23, 134\}, \{4, 39, 307\}, \{5, 59, 584\},$
 $\{6, 83, 989\}, \{7, 111, 1546\}, \{8, 143, 2279\}, \dots$

More generally, from the polynomial-exponential Diophantine triple $\{P, P+1, 2P^2+2P-1\}$ of order 2, we can, exactly as in the construction of the solution tree, using the transformations σ_1 and σ_2 , generate new ones recursively (see **Figure 5**).

Denote by

G_0 : the starting generation and G_n : the nth generation,

$|G_n|$ the number of exponential Diophantine triples at the nth generation.

Then, we have

$$|G_0|=1, |G_1|=2, \dots, |G_n|=2^n \text{ and } \lim_{n \rightarrow +\infty} |G_n| = +\infty.$$

$(|G_n|)_{n \geq 0}$ follows a geometric progression of first term $|G_0|=1$ and ratio 2.

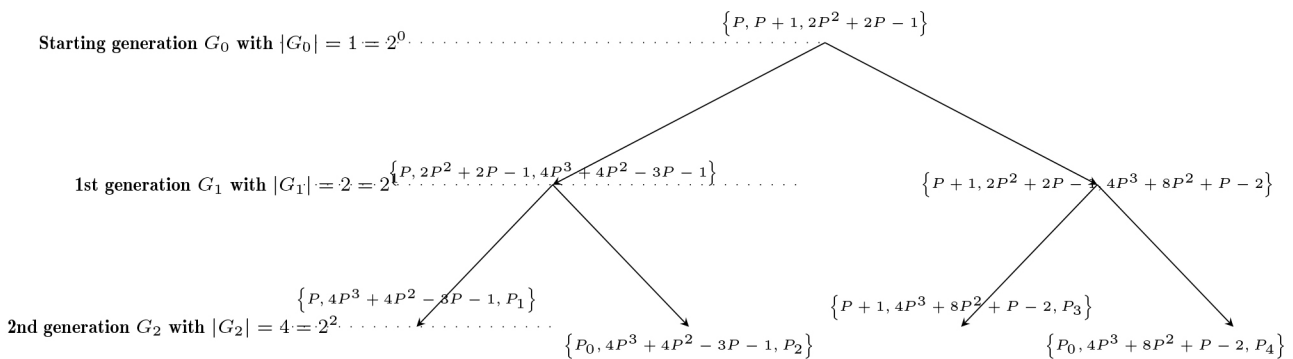


Figure 5. Tree of polynomial-exponential Diophantine triples of order 2.

With:

$$P_0 = 2P^2 + 2P - 1$$

$$P_1 = 2P(4P^3 + 4P^2 - 3P - 1) - (2P^2 + 2P - 1) = 8P^4 + 8P^3 - 8P^2 - 4P + 1$$

$$P_2 = 2(2P^2 + 2P - 1)(4P^3 + 4P^2 - 3P - 1) - P = 16P^5 + 32P^4 - 4P^3 - 24P^2 + P + 2$$

$$P_3 = 2(P+1)(4P^3 + 8P^2 + P - 2) - (2P^2 + 2P - 1) = 8P^4 + 24P^3 + 16P^2 - 4P - 3$$

$$P_4 = 2(2P^2 + 2P - 1)(4P^3 + 8P^2 + P - 2) - (P+1) = 16P^5 + 48P^4 + 28P^3 - 20P^2 - 11P + 3$$

- The family $\{k, k+1, 2k^2+2k-1\}_{k>1}$ is a family of many infinitely minimal exponential Diophantine triples of order 2.

Fixing $k = 2$, we find the minimal triple $\{2, 3, 11\}$ from which infinitely many further triples are generated via mutations, as illustrated in **Figure 6**.

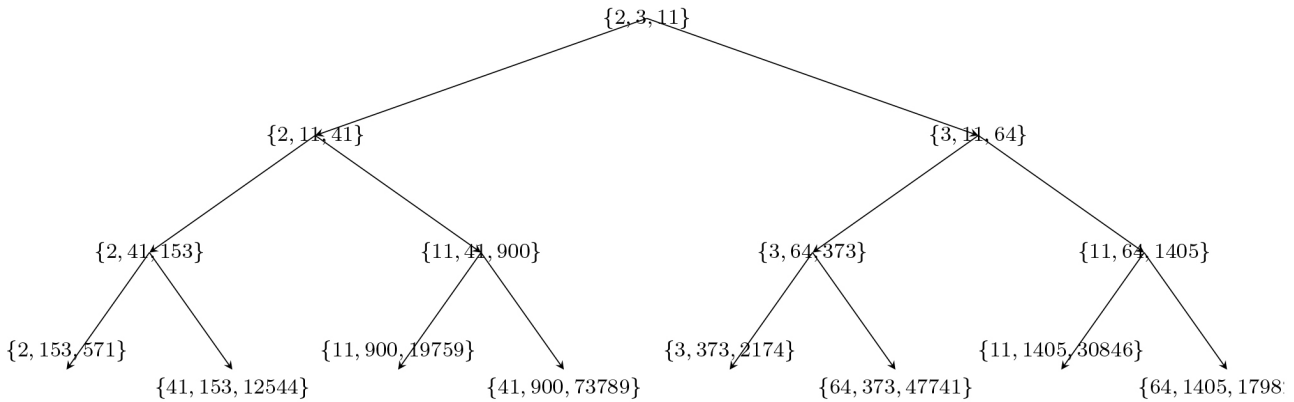


Figure 6. Tree of exponential Diophantine triples of order 2 rooted at $\{2, 3, 11\}$.

The family $(x, y, z) = (k, k + 1, 2k^2 + 2k - 1), k > 1$, defines an algebraic curve $C \subset S$. Eliminating the parameter k , the curve C is given explicitly by

$$C: \begin{cases} y - x = 1, \\ z - 2x^2 - 2x + 1 = 0. \end{cases}$$

Geometrically, the curve C is the intersection of the surface S with the affine plane $y - x = 1$,

$$C = S \cap \{y - x = 1\}.$$

As the intersection of the surface S with the affine plane $y - x = 1$, the curve C is an algebraic subvariety of S . The curve C carries an infinite family of integer points, corresponding to a parametrized family of exponential Diophantine triples of order 2.

In this work, the study of exponential Diophantine triples of order 2 is based on a close interplay between geometry and number theory. A discrete arithmetic problem is reformulated as the study of integer points on a smooth differential surface:

- arithmetic mutations correspond to geometric symmetries of the surface,
- generation and organization of Diophantine solutions are governed by the global geometry of the surface such as its symmetries, connectedness, and smoothness,
- arithmetic parameterizations correspond to well-defined geometric subvarieties.

By interpreting solutions as integer points on a smooth real algebraic surface, a discrete arithmetic problem is embedded into a continuous geometric setting involving tools from geometry such as smoothness, connectedness, unboundedness, S_3 -symmetry, algebraic curves contained in the surface, and Vieta-type birational involutions acting as geometric automorphisms. These structures organize the solutions into arborescence, enable the systematic generation of new solutions by

successive mutations, and yield parametric families of Diophantine solutions. Conversely, Diophantine constraints such as quadratic equations, Markov-type recurrences, and explicit parameterizations uncover algebraic substructures of the surface, notably curves carrying infinitely many integer points, revealing hidden geometric symmetries.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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