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Generator Sets for the Magma Monoid

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Abstract

This paper explores the algebraic structures of the magma monoid $(\mathcal{M}(S),\triangleleft)$, where $\mathcal{M}(S)$ comprises all binary operations on S. We investigate the order and generator set of elements characterizing idempotents, almost constant operations, and units, which facilitates our analysis. A procedure for finding inverses of units is presented, along with key results on almost constant operations. Focusing on $|S|=2$ and $|S|=3$, we classify the elements into various categories, determine their orders, and identify generating sets for the magma monoid. The findings inform a concise methodology for identifying the generator sets of an arbitrary element $*\in\mathcal{M}(S)$.

Keywords

Generator Sets, Magma Monoids, Binary Operations, Semigroup, Monogenic Semigroups, Periodic Semigroups, Idempotents

1. Introduction and Preliminaries

This paper deals with the monoid structure on the set of all binary operations over a fixed set S, as studied in [1] and [2]. Throughout this paper, S represents a nonempty set. A pair $(S,\odot)$, where $\odot$ is a binary operation on S, is said to be a magma. In the literature, magmas are sometimes referred to by various alternative names, such as binars or groupoids. Adopting the notation from [2] and [1], we denote the set of all binary operations on S (all magmas on S) as $\mathcal{M}(S)$ and refer to that collection as *the magma of* S. The use of the word "magma" with these two different but strongly related meanings does not pose a problem and helps the narrative flow smoothly.

Observe that for a finite set S with $|S|=n$, the cardinality of $\mathcal{M}(S)$ is $n^{(n^2)}$. Observing, for example, that

- when $n=2$, $|\mathcal{M}(S)|=2^4=16$,
- when $n=3$, $|\mathcal{M}(S)|=3^9=19683$, and
- when $n=4$, $|\mathcal{M}(S)|=4^{16}=4294967296$,

the reader can see that the size of magma sets grows very rapidly.

The large cardinality of $\mathcal{M}(S)$ represents serious challenges for computational explorations, even for small values of $|S|$. Another symptom of the rapid growth of $|\mathcal{M}(S)|$ is that, when S is infinite, $|\mathcal{M}(S)|$ is always uncountable. When $|S|=n$ is finite, we often write $\mathcal{M}(n)$ instead of $\mathcal{M}(S)$, since clearly sets with the same number of elements induce isomorphic magma monoids.

1.1. Magma Monoids

We consider an operation, $\triangleleft$, which endows $\mathcal{M}(S)$ with a monoid structure. That operation was independently considered in [3] and in [1] and [2].

We follow the notation and terminology of [1] and [2].

Other papers that followed [3] are [4], [5], and [6]. Fayoumi's work in [4] provides an elegant characterization of the center of $\mathcal{M}(S)$ and a generalization of commutative operations through commutative maps was explored in [5]. Nebeský introduced a method of describing graphs using binary operations in [7], [8], and [9], which was further developed in [6].

Following the construction in [10], [11] examines ways to lift the $\triangleleft$ operation from $|\mathcal{M}(S)|$ to sets of graphs whose vertex set is S. The graph-induced operations considered in [12], [13], and [14] include one-value and two-value graph magmas and have applications in the study of amenable bases over infinite-dimensional algebras (e.g., [15]). The graphs inducing such operations are some of the ones explored in [11].

Definition: $\triangleleft$ on $\mathcal{M}(S)$ is defined as follows: for every x,y in S and $*,\circ$ in $\mathcal{M}(S)$,

$$(*\triangleleft\circ)(x,y)=\circ(*(x,y),*(y,x)).$$

The notations $*\triangleleft\circ$ and $\triangleleft(*,\circ)$ are used interchangeably.

Remark: Let π_1 be the binary operation that projects onto the first component, that is, for all $i,j\in S$, $\pi_1(i,j)=i$. Then, $(\mathcal{M}(S),\triangleleft)$ is a non-commutative monoid, having π_1 as its identity element. For the proof, refer to Theorem 2.1 of [2] or Theorem 2 of [3].

We denote the magma monoid resulting from endowing $\mathcal{M}(S)$ with the operation $\triangleleft$ as $(\mathcal{M}(S),\triangleleft)$. This same monoid was denoted $(Bin(S),\square)$ in [3] and related works.

The motivation behind the two independent introductions in the literature of the $\triangleleft$ operation, respectively, in [3] and in [1] and [2], is different from each other. To explain the motivation for [1] and [2], we need the following definition.

Definition: For every operation $\star$ in $\mathcal{M}(S)$, the (left, right, two-sided) outset of $\star$ is defined as the set ($out_l(\star)$, $out_r(\star)$, $out(\star)$), where:

- $out_l(\star)=\{*\in\mathcal{M}(S)\mid\star \text{ is left-distributive over } *\}$;

- $out_r(\star) = \{* \in \mathcal{M}(S) \mid \star \text{ is right-distributive over } *\}$;
- $out(\star) = \{* \in \mathcal{M}(S) \mid \star \text{ is two-sided distributive over } *\}$.

The motivations behind the two independent introductions in the literature [1] and [2] were the following interesting fact:

Remark: For each operation $*$ in the magma monoid $\mathcal{M}(S)$, the outsets $out_l(*), out_r(*)$, and $out(*)$ are submonoids of $(\mathcal{M}(S), \triangleleft)$, inheriting the identity element π_1 from the parent monoid.

In contrast with the above proposition, the motivation in [3] to study the magma monoid was the ability to see it as a natural extension of the monoid of functions $g: S \to S$, under composition. Specifically, we formalize these foundational concepts in Definition 1.1 and Remark 1.1, which are lifted from [3], but with notation adjusted to match what we use in this paper.

Definition: Given a function $g: S \to S$, consider a groupoid $(S, \circ, g)$ where the multiplication is given by the formula

$$\circ(a,b) = g(a).$$

Groupoids of this type, $(S, \circ, g)$, are referred to as leftoids.

Given two leftoids $(S, \circ_1, g)$ and $(S, \circ_2, f)$ the operation "$\triangleleft$" can be defined as follows.

$$(S, \circ_1, g) \triangleleft (S, \circ_2, f) = (S, \triangleleft),$$

where

$$a \triangleleft b = (a \circ_1 b) \circ_2 (b \circ_1 a).$$

Thus, $a \triangleleft b = g(a) \circ g(b) = fg(a)$ and so $(S, \triangleleft)$ is the leftoid $(S, \triangleleft, g \circ f)$. This multiplication corresponds to the composition of functions from S to S. If $g(a) = id_S(a) = a$, then $(S, \circ, id_S)$ has the multiplication $a \circ b = a$. Since the composition of functions is an associative operation, we obtain the proposition below.

Remark: The collection of leftoids with respect to the operation $\triangleleft$ is a semigroup with identity $(S, \circ, id_S)$.

Proof. See Theorem 1 of [3].

It is customary, when introducing a new semigroup, to consider the extent to which Green's relations hold for that semigroup and seek information about its idempotents and regular elements. These questions were partially considered for the magma monoid in [2]. The results on idempotent elements relevant to this paper will be reviewed in Section 2.

Definition:

- An element $*$ of $\mathcal{M}(S)$ is regular if there exists an element $\circ$ satisfying the two conditions that

$$* \triangleleft \circ \triangleleft * = * \text{ and } \circ \triangleleft * \triangleleft \circ = \circ.$$

- In this case, $\circ$ is called a regular conjugate of $*$. It should be clear that, when this happens, then $*$ is a regular conjugate of $\circ$.

Binary operations on a finite set S have a user-friendly representation, first introduced in [1], as numbers corresponding to the base $|S|$ representation of the entries in the operation table. While using these representations, the result of applying the operation to an input pair (i, j) is simply choosing the ij-th entry in the operation's name, with ij interpreted as a number in the base $|S|$.

Example: Operations in $M(S)$ for $S = \{0,1\}$. Each operation is named with the number between 0 and 15 whose binary representation is given by the entries of the table read from the top.

0	1	0
1	0	0
0	0	0

1	1	0
1	0	0
0	0	1

2	1	0
1	0	0
0	1	0

3	1	0
1	0	0
0	1	1

4	1	0
1	0	1
0	0	0

5	1	0
1	0	1
0	0	1

6	1	0
1	0	1
0	1	0

7	1	0
1	0	1
0	1	1

8	1	0
1	1	0
0	0	0

9	1	0
1	1	0
0	0	1

10	1	0
1	1	0
0	1	0

11	1	0
1	1	0
0	1	1

12	1	0
1	1	1
0	0	0

13	1	0
1	1	1
0	0	1

14	1	0
1	1	1
0	1	0

15	1	0
1	1	1
0	1	1

Example: For the set $S = \{0,1,2\}$, the binary operation associated with the number 100 can be represented by its Modulo 3 representation, 000010201. This representation is shown below. The Modulo 3 representations of the other two binary operations, 8229 and 9760, can be similarly determined.

100	2	1	0
2	0	0	0
1	0	1	0
0	2	0	1

8229	2	1	0
2	1	0	2
1	0	2	1
0	2	1	0

9760	2	1	0
2	1	1	1
1	1	0	1
0	1	1	1

Example: When $S = \{0,1\}$, almost all binary operations in $\mathcal{M}(S)$ are regular. The following are the regular conjugates:

1) Self-conjugate operations: $\{0,2,3,4,5,8,10,12,13,14,15\}$;

2) Conjugate pairs: $\{1,7\}$.

The only exceptions are operations **6** and **9** (see Example 1.1), which fail to satisfy the regularity condition, namely $*^3 \neq *$.

1.2. Periodic and Monogenic Semigroups

We introduce here the semigroup terminology from [16] that is pertinent to this paper. For more details, readers may consult a standard reference such as [16],

[17].

For any subset χ of $\mathcal{M}(S)$, the intersection of all subsemigroups of $\mathcal{M}(S)$ that contain χ is a subsemigroup of $\mathcal{M}(S)$ denoted $\langle\chi\rangle$.

The set χ is said to be a generating set of $\mathcal{M}(S)$ if $\mathcal{M}(S)=\langle\chi\rangle$. If $\chi=\{*_1,*_2,\cdots,*_n\}$ for some $n\in\mathbb{N}$, then we write $\langle\chi\rangle=\langle *_1,*_2,\cdots,*_n\rangle$. We consider a special case where the set $\langle\chi\rangle$ is generated by a singleton set $\chi=\{*\}$.

In that case, $\langle *\rangle=\{*,*^2,*^3,\cdots\}$.

Definition: The order of an element $*$ in a semigroup is the number of elements in the subsemigroup $\langle *\rangle$.

Notice that, unlike the case of a group, when $*$ comes from a monoid and is not invertible, then the subsemigroup $\langle *\rangle$ does not contain the identity element of the monoid. An element of a semigroup has order one if and only if it is idempotent.

Definition: A semigroup S is called monogenic if there exists an element $i\in S$ such that $S=\langle i\rangle$.

Given $S=\{i^n \mid n\in\mathbb{N}\}$, if all these powers are distinct, then $\langle i\rangle$ is isomorphic to the additive semigroup of natural numbers. Otherwise, $\langle i\rangle$ is finite, and the number of elements in it is called the order of the semigroup $\langle i\rangle$, as well as the order of the element $i\in S$. If $\langle i\rangle$ is infinite, then i is said to have infinite order.

Definition: A semigroup S is called periodic if to each element a of S there corresponds an idempotent e (*i.e.*, $e=e^2$) and a positive integer m such that $a^m=e$.

A semigroup is periodic if all its elements are of finite order, and every periodic semigroup has at least one idempotent.

In his study of periodic semigroups in [18] (as cited in [19]), Schwarz introduced the relation $\mathcal{K}$ on semigroups, which served as a useful tool in the study of periodic semigroups. We introduce that relation in Definition 1.2. In [19], Miller investigated the properties of periodic semigroups that satisfy the condition $\mathcal{J}\subseteq\mathcal{K}$, where $\mathcal{J}$ is the Green's $\mathcal{J}$ relation, viewed as a subset of $S\times S$. The key results from Miller's study are summarized here for comparison with the critical results of this paper.

Definition: Let S be a periodic semigroup, and denote its set of idempotents E_S. For a,b in S, $a\mathcal{K}b$ if and only if there exist positive integers m,n and an element e in E_S such that $a^m=b^n=e$.

This makes $\mathcal{K}$ an idempotent separating equivalence relation on any periodic semigroup. The $\mathcal{K}$-class containing an element x of S will be denoted K_x. Suppose $e\in E_S$ then K_e is a subsemigroup of S.

Lemma 1 [19]: Let S be a periodic semigroup with $\mathcal{J}\subseteq\mathcal{K}$ and let $e\in E_S$. Then

1) $ab\in K_e$ if and only if $ba\in K_e$ for $a,b\in S$;
2) $ab \ \mathcal{K} \ a^ib$ for $a,b\in S$;
3) K_e is a periodic unipotent subsemigroup of S.

2. Almost-Constant Operations and Units and Idempotents in $(\mathcal{M}(S),\lhd)$

The first step in our exploration of the order of elements in the magma monoid is to study almost constant operations (subsection 2.1), and units and idempotents with respect to $(\mathcal{M}(S),\lhd)$ (respectively, subsection 2.2 and 2.3).

A lower bound for the number of idempotent elements in the magma monoid is stated in Proposition 14, and some examples of idempotent operations are listed in the subsequent remark. In Proposition 2, we prove that if $*$ is an almost-constant operation, then $*^3=*$. We then describe a procedure for finding unit inverses and determine the number of units in $(\mathcal{M}(S),\lhd)$ via Theorem 6.

2.1. Almost Constant Operations

Definition: Given $j\in S$ the binary operation c_j that maps every $(x,y)\in S\times S$ to j, *i.e.*, $c_j(x,y)=j$ for all $x,y\in S$ is called the constant operation determined by j.

Let $\mathcal{K}(S)$ or simply $\mathcal{K}$ represent the set of all constant binary operations on S.

The set of all constant operations, $\mathcal{K}$, is a two-sided ideal of the magma monoid, since for every $*\in\mathcal{M}(S)$ and $\circ\in\mathcal{K}$, we have $*\lhd\circ=\circ$ and $\circ\lhd *=c_{i*i}$.

Definition: Given $i,a\in S$ with $i\neq a$, the operation $c_i(a)$ is defined as:

$$c_i(a)(x,y)=\begin{cases}a, & \text{if } x=y=i,\\ i, & \text{otherwise.}\end{cases}$$

This operation is said to be an *almost-constant* operation.

Example: For a set S with 5 elements, the subsequent operations are identified as almost-constant operations, and we denote them using the notation from Definition 2.1.

$c_4(3)$	4	3	2	1	0
4	$\boxed{3}$	4	4	4	4
3	4	4	4	4	4
2	4	4	4	4	4
1	4	4	4	4	4
0	4	4	4	4	4

$c_3(1)$	4	3	2	1	0
4	3	3	3	3	3
3	3	$\boxed{1}$	3	3	3
2	3	3	3	3	3
1	3	3	3	3	3
0	3	3	3	3	3

Proposition 2: For any $i,a\in S$ with $i\neq a$, if $c_i(a)$ is an almost-constant operation in $\mathcal{M}(S)$, then it satisfies the property: $c_i(a)^3=c_i(a)$.

Proof. Given an almost constant operation $c_i(a)$ in $\mathcal{M}(S)$ let $b_i(a)=c_i(a)\lhd c_i(a)$. Then, straightforward calculations show that $b_i(a)$ exhibits a structure very similar to $c_i(a)$, something that could also be called almost constant, but this time the output a is the norm and i is the exception, with the exception, once again, happening at (i,i).

$$b_i(a)(x,y)=\begin{cases} i, & \text{if } x=y=i, \\ a, & \text{otherwise.}\end{cases}$$

A second round of computation yields the desired result.

□

Definition: A binary operation $*$ is called a unique square (or constant diagonal) operation if it satisfies the property: $*(a,a)=*(b,b)$ for all elements $a,b\in S$.

Proposition 3: A monoid is a unique square operation if and only if it is a group all of whose elements have order two.

2.2. Units in $(\mathcal{M}(S),\triangleleft)$

In our search for units within the magma monoid, we encountered two types of operations that could be classified into two separate categories: distinct constant row operations (DCR) and distinct constant column(DCC) operations.

The name distinct constant row operations reflects the fact that, in a table representation, the rows are constant and distinct from each other. A similar reasoning applies to distinct constant column operations, where columns remain constant and distinct.

Lemma 4: If $*$ is a distinct constant row or a distinct constant column in $\mathcal{M}(S)$, then $*$ is a unit.

Lemma 4 yields the following inner direct sum decomposition: $DCR\oplus\{\pi_1,\pi_2\}=DCC\cup DCR$.

Theorem 5: $DCC\cup DCR$ is a subgroup of $\mathcal{U}$, the group of units in $\mathcal{M}(S)$, and is isomorphic to $\mathcal{S}_n\times\mathbb{Z}_2$.

To further our understanding of the magma monoid, we aim to describe its units in a general setting. We define a key function that is vital for this characterization and present the main results. For further details, see [2].

Definition: For each set S with binary operations $*$ and $\circ$ in the magma monoid, define a map $(*\times\circ\zeta)$ that takes a pair of elements (x,y) in $S\times S$ to a new pair $(x*y,y\circ x)$ $S\times S$.

In light of Definition 0 $*\triangleleft\circ$ can be interpreted as the composition of the operations $\circ$ and $*\times*\zeta$.

Theorem 6: Given $*\in\mathcal{M}(S)$, $*$ is a unit in the magma monoid if, and only if the map $*\times*\zeta$ is bijective.

Proof. See Theorem 3.5 in [2].

Example: For $n=4$, the following operations are units in $(\mathcal{M}(S),\triangleleft)$:

	3	2	1	0
3	2	3	1	3
2	2	1	1	0
1	0	3	0	2
0	0	2	1	3

	3	2	1	0
3	1	3	0	3
2	2	0	3	2
1	1	1	3	1
0	0	0	2	2

Using the insights from Definition 0 and Theorem 6, we can now determine the cardinality of the set of units in the magma monoid, denoted by $\mathcal{U}$, as presented in Proposition 23.

Theorem 7: For a set S of size n, the cardinality of the set of unit elements in $\mathcal{M}(S)$, $\mathcal{U}$, is given by $\binom{n}{2}!\cdot n!\cdot 2^{\binom{n}{2}}$.

Example: For $n=2$, $|\mathcal{U}|=\binom{2}{2}!2!2^{\binom{2}{2}}=4$; For $n=3$, $|\mathcal{U}|=\binom{3}{2}!3!2^{\binom{3}{2}}=144$.

Procedure for Finding Inverse of Units in $(\mathcal{M}(S),\triangleleft)$

The following procedure outlines the steps to determine the inverse of a binary operation $*$ belonging to the set $\mathcal{U}$. Let $x_1,x_2,\cdots,x_n\in S$. Then:

1) To obtain the elements for the diagonals, iterate through each index i in the set $\{1,2,\cdots,n\}$. For each i, compute the product $x_i * x_i$. If the result of this product is x_j (*i.e.*, $x_i * x_i = x_j$), then the inverse operation applied to $x_j * x_j$ yields x_i (*i.e.*, $*^{-1}(x_j,x_j)=x_i$).

2) Given that i is not equal to j and m is not equal to n, consider the expression $((x_i * x_j),(x_j * x_i))$. Suppose this expression evaluates to (x_n,x_m). Then, applying the inverse operation to (x_n,x_m) yields x_i (*i.e.*, $*^{-1}(x_n,x_m)=x_i$), and applying the inverse operation to (x_m,x_n) yields x_j (*i.e.*, $*^{-1}(x_m,x_n)=x_j$).

3) Proceed to Step 2 and systematically vary the values of x_i and x_j until all the cells in the inverse table $*^{-1}$ are completely filled.

Following the procedure outlined above, we successfully determined the inverse of the operation described in Example 2.2.1.

Example: Given the binary operation $*$ defined by the table:

$*$	3	2	1	0
3	2	3	1	3
2	2	1	1	0
1	0	3	0	2
0	0	2	1	3

We can determine the inverse of $*$ by following the three-step procedure outlined earlier.

Step 1: From the operation table for $*$, we observe that $*(3,3)=2$, which implies that $*^{-1}(2,2)=3$. Continuing this process, we find additional inverse relations:

- $*^{-1}(3,3)=0$
- $*^{-1}(1,1)=2$
- $*^{-1}(0,0)=1$

Steps 2 and 3: Reading from the operation table for $*$ we find the following inverse relations:

- $(*(3,2),*(2,3))=(3,2)\Rightarrow *^{-1}(3,2)=3,*^{-1}(2,3)=2$
- $(*(3,1),*(1,3))=(1,0)\Rightarrow *^{-1}(1,0)=3,*^{-1}(0,1)=1$
- $(*(3,0),*(0,3))=(3,0)\Rightarrow *^{-1}(3,0)=3,*^{-1}(0,3)=0$

- $(*(2,1),*(1,2))=(1,3)\Rightarrow *^{-1}(1,3)=2, *^{-1}(3,1)=1$
- $(*(2,0),*(0,2))=(0,2)\Rightarrow *^{-1}(0,2)=2, *^{-1}(2,0)=0$
- $(*(1,0),*(0,1))=(2,1)\Rightarrow *^{-1}(2,1)=1, *^{-1}(1,2)=0$

Following the procedure, we obtain the inverse operation table $*^{-1}$ as:

$$*^{-1}=\begin{array}{c|cccc} & 3 & 2 & 1 & 0 \\ \hline 3 & 0 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 & 1 \end{array}$$

2.3. Idempotents in $(\mathcal{M}(S),\triangleleft)$

Definition: An operation $*\in\mathcal{M}(\mathcal{S})$ is said to be an idempotent operation with respect to $\triangleleft$ if $(*\triangleleft *)(a,b)=*(a,b)$, for all $a,b\in S$.

In brief, $*$ is idempotent if $*^2=*$. The set of all idempotent elements is not a subsemigroup of the $(\mathcal{M}(S),\triangleleft)$ [2].

Our search for idempotent elements in $(\mathcal{M}(S),\triangleleft)$ commences with commutative binary operations within $\mathcal{M}(S)$. We enumerate and classify these operations (Definition 2.3), revealing three distinct categories. Notably, in Proposition 9, we show that idempotent elements are confined to two of these categories. Clearly, if $|S|=n$, then the cardinality of the set of commutative binary operations in the magma monoid, $\mathcal{M}(S)$ is given by $n^{\frac{n(n+1)}{2}}$.

Definition: We define the following sets on the magma monoid:

$$\mathcal{Y}=\{*\in\mathcal{M}(S)\mid * \text{ is commutative and there is a } j\in S \text{ such that } *(i,i)=j \text{ for all } i\in S\}$$

$$\mathcal{W}=\{*\in\mathcal{M}(S)\mid * \text{ is commutative and } *(i,i)=i \text{ for all } i\in S\}$$

$$\mathcal{V}=\{*\in\mathcal{M}(S)\mid * \text{ is commutative and if } *(i,j)=a \text{ then } *(a,a)\neq a \text{ for all } i,j,a\in S\}$$

The set $\mathcal{Y}$ consists of those binary operations, in Definition 0, to which we refer as unique square operations (or constant diagonal operations).

Proposition 8:

1) The set $\mathcal{Y}$ is a two-sided ideal of $\mathcal{M}(S)$.

2) Given $*\in\mathcal{M}(S)$, if $*\in\mathcal{Y}$ then $*^2\in\mathcal{K}$, the set of all constant binary operations.

Proof.

1) Let $*\in\mathcal{Y}$ and choose $k\in S$ such that $*(x,x)=k$ for all $x\in S$. For $\circ\in\mathcal{M}(S)$ with $\circ(k,k)=j$, we have: $(*\triangleleft\circ)(x,x)=j$ for all $x\in S$ making $*\triangleleft\circ$ have a constant diagonal.

For $x\neq y$, letting $*(x,y)=*(y,x)=z$ yields:

$$(*\lhd\circ)(x,y)=\circ(*(x,y),*(y,x))=\circ(z,z) \quad \text{and}$$

$$(*\lhd\circ)(y,x)=\circ(*(x,y),*(y,x))=\circ(z,z).$$

Thus, $*\lhd\circ$ is commutative. Consequently, $*\lhd\circ\in\mathcal{Y}$. A similar argument yields $\circ\lhd*\in\mathcal{Y}$.

2) Given $\circ\in\mathcal{Y}$, if $j\in S$ is such that $\circ(x,x)=j$ for all $x\in S$, we will show that $\circ^2(x,y)=j$, for all $x,y\in S$. We have: $\circ^2(x,y)=\circ(\circ(x,y),\circ(y,x))=\circ(\circ(x,y),\circ(x,y))$, since $\circ$ is commutative. Therefore, $\circ^2=c_j$, a constant operation, as claimed.

Proposition 9: Given $*\in\mathcal{M}(S)$,

1) if $*\in\mathcal{W}$ then $*$ is an idempotent operation.

2) if $*\in\mathcal{Y}$ then $*$ is an idempotent element if and only if it is a constant operation.

Proof.

1) Given $*\in\mathcal{W}$ and $x,y\in S$. We can easily verify that for all $x\in S$, $*^2(x,x)=x=*(x,x)$ and $*^2(x,y)=*(x,y)$. Therefore, $*$ is idempotent.

2) Given $\circ\in\mathcal{Y}$ and $x,y\in S$, we have two cases

(a) if $\circ\in\mathcal{K}$, then $\circ^2=\circ$, so $\circ$ is idempotent;

(b) if $\circ\notin\mathcal{K}$, then by Proposition 8 $\circ^2\in\mathcal{K}$, so $\circ$ itself is not idempotent.

Hence, $\circ$ is idempotent if and only if it is a constant operation.

Proposition 10: Given $*\in\mathcal{M}(S)$, if $*\in\mathcal{V}$ then $*$ is not an idempotent element.

Proof. Given $\circ\in\mathcal{V}$, choose $x,y,j,i\in S$ such that $\circ(x,y)=j$, $\circ(j,j)=i$, and $i\neq j$. Then we have:

$$\circ^2(x,y)=\circ(\circ(x,y),\circ(y,x))=\circ(\circ(x,y),\circ(x,y))=\circ(j,j)=i$$

Thus, $\circ^2(x,y)\neq\circ(x,y)$, since $\circ^2(x,y)=i$ and $\circ(x,y)=j$, and $i\neq j$.

Whereas the idempotency of the commutative binary operations in $(\mathcal{M}(S),\lhd)$ depends on the nature of the diagonal arrangement of the operation, the idempotency of the non-commutative binary operations, in most cases, is independent of the diagonal arrangement.

Proposition 11: Given an idempotent, non-commutative binary operation $*\in\mathcal{M}(S)$ with the properties:

1) $*(x,y)\neq*(y,x)$ for all $x,y\in S$ (non-commutativity);

2) $*(i,i)=i$ for all $i\in S$ (idempotence on the diagonal).

Define a new binary operation $\circ$ on S as follows:

$$\circ(a,b)=\begin{cases} j & \text{if } a=b \\ *(a,b) & \text{otherwise} \end{cases}$$

Then, the operation $\circ$ is also idempotent in $\mathcal{M}(S)$.

Proof. Let $*$ be an idempotent, non-commutative binary operation that satisfies the idempotence on the diagonal condition. Choose $\circ\in\mathcal{M}(S)$ such that $\circ(x,y)=*(x,y)$ for all $x,y\in S$ with $x\neq y$, and $\circ(x,y)=j$ if $x=y$. We show $\circ$ is idempotent, *i.e.*, $\circ^2(x,y)=\circ(x,y)$ for all $x,y\in S$.

We consider two cases:

1) $x\neq y$ for all $x,y\in S$:

$$\circ^2(x,y)=\circ(\circ(x,y),\circ(y,x))=\circ(*(x,y),*(y,x))$$
$$=*(*(x,y),*(y,x))=*^2(x,y)=*(x,y)=\circ(x,y).$$

Hence, $\circ^2(x,y)=\circ(x,y)$.

2) $x=y$ for all $x,y\in S$:

Let $\circ(x,x)=j$ for all $x\in S$. Then $\circ^2(x,x)=\circ(\circ(x,x),\circ(x,x))=\circ(j,j)=j$. Hence, $\circ^2(x,x)=\circ(x,x)$.

Therefore, $\circ$ is idempotent.

Example: For a set S with three elements, the following operations are idempotent in $(\mathcal{M}(S),\lhd)$:

$*$	2	1	0
2	2	2	1
1	1	1	1
0	2	0	0

$\circ_2$	2	1	0
2	2	2	1
1	1	2	1
0	2	0	2

$\circ_1$	2	1	0
2	1	2	1
1	1	1	1
0	2	0	1

$\circ_0$	2	1	0
2	0	2	1
1	1	0	1
0	2	0	0

Proposition 12:

Binary operations in $\mathcal{M}(S)$ that differ from the identity operation π_1 in exactly one entry are idempotent with respect to $\lhd$.

Proof. Suppose there exists a pair (a,b) such that $*(a,b)=t\neq a$ and $*(c,d)=c=\pi_1(c,d)$ for all $(c,d)\neq(a,b)$ then, $*(c,d)=\pi_1(c,d)=c$ and $*\lhd*(c,d)=*(*(c,d),*(d,c))=*(c,d)=c$. And $*\lhd*(a,b)=*(*(a,b),*(b,a))=*(t,b)=t$. □

Theorem 13:

Let T be a subset of S with cardinalities $|S|=n$ and $|T|=m$. Suppose $*$ is an idempotent binary operation in $\mathcal{M}(T)$ with an element $l\in T$ such that $*(l,l)=l$. Then, the binary operation $\odot$ defined on $\mathcal{M}(S)$ as follows is also idempotent:

$$\odot(x,y)=\begin{cases}*(x,y) & \text{if } (x,y)\in T\times T\\ l & \text{if } (x,y)\notin T\times T\end{cases}$$

Proof. See [2] □

We characterize the idempotent and non-idempotent elements in $(\mathcal{M}(S),\lhd)$ in Remark 2.3.

Remark: By applying Theorem 13 and Proposition 12, we can identify the following as some of the idempotent operations in the magma monoid ($|S|\geq 3$):

- All constant operations;
- All the elements in the set $\mathcal{W}$;
- The identity;
- Operations that differ from the identity by only one entry;

- Idempotents derived from the set with $n-1$ elements.

The following operations in $\mathcal{M}(S)$ are non-idempotents:

- Non-identity unitary operations;
- Almost constant operations;
- All elements of the set $\mathcal{V}$;
- All the elements in the set $\mathcal{Y} \setminus \mathcal{K}$;
- Operations that differ from constant operations by exactly one non-diagonal entry.

Despite the empty intersection of all idempotent sets referenced in Remark 0, nonempty intersections exist between some sets, revealing they are not fully pairwise disjoint. As a result, determining a precise lower bound for the number of idempotent elements in the magma monoid poses a significant challenge.

We aim to refine the lower bound established in Proposition 2.9 of [2], as the set of constant operations and the identity have disjoint intersections with all elements in the set $\mathcal{W}$. Consequently, we present an enhanced lower bound for the number of idempotent elements in Proposition 14.

The sets of non-identity unitary operations, almost constant operations, and elements in $\mathcal{Y}$ are pairwise disjoint. By leveraging the cardinalities of these sets, we can establish a tighter upper bound on the number of idempotent operations within the magma monoid.

Proposition 14 gives a bound for the number of idempotents in the magma monoid $\mathcal{M}(S)$, where the underlying set S has n elements.

Proposition 14: For $|S| = n \geq 3$, let $\mathcal{I} \subset (\mathcal{M}(S), \triangleleft)$ denote the set of all idempotent elements in the magma monoid. Then, the following inequality holds:

$$n + \sum_{m=1}^{n} \binom{n}{m} m^{\binom{m}{2}} m \leq |\mathcal{I}| \leq n^{n^2} - \binom{n}{2}! n! 2^{\binom{n}{2}} + 1$$

Example: For $n = 2$, $8 \leq |I| \leq 13$; For $n = 3$, $101 \leq |I| \leq 19540$.

3. Main Results

To gain insight into the structure of $\mathcal{M}(S)$ for arbitrary sets S, we first examine the specific cases of $\mathcal{M}(2)$ and $\mathcal{M}(3)$, where $2 = \{0,1\}$ and $3 = \{0,1,2\}$, as these small, finite cases provide a concrete foundation for understanding the general pattern and properties of $\mathcal{M}(S)$.

3.1. Order of Elements for $\mathcal{M}(2)$ and $\mathcal{M}(3)$

Extending our previous analysis in Section 2, this section presents our results on the order of elements in the magma monoid for $|S| = 2$ and $|S| = 3$. We group the elements into six distinct categories in Proposition 15 and investigate their order within each category, drawing on insights from Lemmas 16 and 17.

Proposition 15: The elements of $\mathcal{M}(S)$ can be categorized into the following six classes for both $|S| = 2$ and $|S| = 3$:

1) Non-identity units

2) Idempotents

3) Almost constant operations

4) Commutative and unique square operations

5) Non-commutative and unique square operations

6) Other operations

Lemma 16: For $|S|=2$, the magma monoid contains seven idempotent elements, which form the set

$$\mathcal{I}=\{0,4,8,12,13,14,15\}$$

These idempotents can be further classified into:

1) Identity: 12

2) Constant operations: $\{0,15\}$

3) Elements from the set $\mathcal{W}$: $\{8,14\}$

4) Operations that are one entry different from the identity: $\{4,13\}$

Lemma 17: For $|S|=2$, the element in the magma monoid of order 2 are categorized as follows:

1) Non-identity elements: $\{3,5,10\}$

2) Almost-constant operations: $\{1,7\}$

3) Commutative and unique square operations: $\{6,9\}$

4) Non-commutative and unique square operations: $\{2,11\}$

Theorem 18 *For* $|S|=2$, *if* $*\in\mathcal{M}(S)$, *then the order of* $*$ *satisfies* $1\leq\langle *\rangle\leq 2$

Proof. By Lemmas 16 and 17, we have exhaustively classified all the possible elements in $\mathcal{M}(S)$. Hence, any element $*\in\mathcal{M}(S)$ must satisfy one of the conditions listed in these lemmas, implying that the order $*$ is either 1 or 2. □

Remark: Note that the above results imply that $\mathcal{M}(S)$ is not a monogenic semigroup when $|S|=2$, since every element has order 1 or 2.

Following from Definition 1.2 and Theorem 18, we can classify the elements of $\mathcal{M}(2)$ into the following idempotent classes, along with their respective elements (see Example 1.1 for the corresponding operations, denoted by numbers):

1) $\mathcal{K}_0=\{0,6\}$

2) $\mathcal{K}_4=\{2,4\}$

3) $\mathcal{K}_8=\{7,8\}$

4) $\mathcal{K}_{12}=\{3,5,10,12\}$

5) $\mathcal{K}_{13}=\{11,13\}$

6) $\mathcal{K}_{14}=\{1,14\}$

7) $\mathcal{K}_{15}=\{9,15\}$

Proposition 19: For any set S with exactly 2 elements and any operation $*\in\mathcal{M}(2)$, the set $\mathcal{K}_*$ forms a commutative subsemigroup of $\mathcal{M}(2)$.

Proposition 20: Given a commutative binary operation $*\in\mathcal{M}(S)$ for $|S|=3$, there exists an idempotent operation $e\in\mathcal{M}(S)$ such that $*^n=e$ for some n with $1\leq n\leq 4$.

Proof. Consider a commutative binary operation $*$ in the semigroup $(\mathcal{M}(S),\lhd)$, and let $e\in\mathcal{M}(S)$ be an idempotent element. The idempotency of

$*$ is influenced by the configuration of its diagonal elements. To explore this, we examine various scenarios related to the diagonal arrangement of the commutative binary operation $*$ within $\mathcal{M}(S)$.

Case 1: If $*(i,i)=i$ for all $i \in S$, then $* \in \mathcal{T}_I \subseteq \mathcal{M}(S)$ and, by Proposition 9 $*$ is idempotent. Hence, $*=e$.

Case 2: If $*(i,i)=j$ for all $i \in S$ with $i \neq j$, then $* \in \mathcal{T}_K$ and, by Proposition 10 $*^2 \in \mathcal{K}$. Hence, $*^2=e$.

Case 3: For $i,j,k \in S$, let $*(i,i)=i, *(j,j)=j$ and $*(k,k) \neq k$. Then, we can easily verify that $*$ is idempotent, *i.e.*, $*^2=*$. Hence, $*=e$.

Case 4: For $i,j,k \in S$, let $*(i,i)=i, *(j,j) \neq j$ and $*(k,k) \neq k$. If we choose $*(j,j)=k$, $*(k,k)=j$, then we have $*^2(j,j)=j$ and $*^2(k,k)=k$, so $*^2(i,i)=i$ for all $i \in S$. Thus $*^2$ is idempotent by Proposition 9. Similarly, if we choose either $*(j,j)=i$ or $*(k,k)=i$, then we have
$*^2(i,i)=*^2(j,j)=*^2(k,k)=i$ and so $*^2 \in \mathcal{T}_K$ and by Proposition 10,
$\left(*^2\right)^2 \in K$. Therefore, $*^4=e$.

Case 5: For $i,j,k \in S$ such that $*(i,i) \neq i$ for all $i \in S$, and $*(i,i), *(j,j)$ and $*(k,k)$ are pairwise distinct. If we choose $*(i,i)=j$, $*(j,j)=k$ and $*(k,k)=i$ then we can verify that $*^3(i,i)=i$ for all $i \in S$. Therefore, $*^3=e$.

Case 6: For $i,j,k \in S$, such that $*(i,i) \neq i$ for all $i \in S$, and two of the diagonal elements $*(i,i), *(j,j)$ and $*(k,k)$ assume the same value. If we choose $*(i,i)=*(k,k)=j$ and $*(j,j)=k$ then $*^2(i,i)=k$, $*^2(j,j)=j$ and $*^2(k,k)=k$. Thus, from Case 2, $*^2$ is idempotent, and therefore $*^2=e$.

The configurations of the diagonal elements examined above cover all possible cases of commutative binary operations.

Theorem 21: For $|S|=3$, if $* \in \mathcal{M}(S)$ is a commutative binary operation, then the order of $*$ satisfies $1 \leq |\langle * \rangle| \leq 7$.

Proof. By Proposition 20, if e is an idempotent element in $\mathcal{M}(S)$ then there exists an integer n such that $*^n=e$ where $1 \leq n \leq 4$. We now examine the following cases to determine the order of $*$:

Case 1: If $*=e$ then $*^n=e$ for all $n \geq 1$, so $\langle * \rangle = \{*\}$ and $|\langle * \rangle|=1$.

Case 2: If $*^2=e$, then $*^{2n}=*^2=e$ for all $n \geq 1$ so we have

$$\langle * \rangle = \begin{cases} \{*, *^2\} & \text{if } *^3=* \\ \{*, *^2, *^3\} & \text{otherwise} \end{cases}$$

so $2 \leq |\langle * \rangle| \leq 3$.

Case 3: If $*^3=e$, then $|\langle * \rangle|$ is either 3 or 5 since

$$\langle * \rangle = \begin{cases} \{*, *^2, *^3\} & \text{if } *^4=* \\ \{*, *^2, *^3, *^4, *^5\} & \text{otherwise} \end{cases}$$

Case 4: If $*^4=e$, then $4 \leq |\langle * \rangle| \leq 7$ since

$$\langle * \rangle = \begin{cases} \{*, *^2, *^3, *^4\} & \text{if } *^5 = * \\ \{*, *^2, *^3, *^4, *^5\} \text{ or } \{*, *^2, *^3, *^4, *^5, *^6\} \text{ or } \{*, *^2, *^3, *^4, *^5, *^6, *^7\} & \text{otherwise} \end{cases}$$

Therefore, we have shown that the order of any commutative binary operation $*$ on a set S with $|S| = 3$ is bounded between 1 and 7, inclusive.

Proposition 22: Given a non-commutative binary operation $* \in \mathcal{M}(S)$ for $|S| = 3$ the following hold:

1) If $*(i,i) = i$ for all $i \in S$ and $*$ is not a unit, then there exists an idempotent operation $e \in \mathcal{M}(S)$ such that $*^n = e$ for some n with $2 \le n \le 4$.

2) If $*(i,i) = j$ for all $i \in S$, then there exists an idempotent operation $e \in \mathcal{M}(S)$ such that $*^n = e$ for some n with $2 \le n \le 4$.

3) If there exists $j \in S$ such $*(i,j) = j = *(j,i)$ for all $i \in S$ then there exists an idempotent operation $e \in \mathcal{M}(S)$, such that $*^2 = e$.

Proof. By employing a similar line of reasoning as in the proof of Proposition 20, it can be shown that all three propositions hold true. □

The order of every unit element in $(\mathcal{M}(S), \triangleleft)$ is less than or equal to 144 (See Example 2.2).

Remark: For any operation $*$ on the set S with $|S| = 3$, the order of the subsemigroup generated by $*$ is at most 144, *i.e.*, $|\langle * \rangle| \le 144$.

3.2. Order of an Element $* \in \mathcal{M}(S)$

Proposition 23: For any element $* \in \mathcal{M}(S)$, the order of the subsemigroup generated by $*$ is strictly less than the cardinality of $\mathcal{M}(S)$, *i.e.*, $|\langle * \rangle| < |\mathcal{M}(S)|$. In other words, $\mathcal{M}(S)$ is not generated by any single element.

Proposition 24: The magma monoid, $(\mathcal{M}(S), \triangleleft)$ is not a monogenic semigroup.

Proposition 25:

1) The order of the subsemigroup generated by an operation $* \in \mathcal{M}(S)$ is 1 if and only if $*$ is idempotent.

2) The order of every almost-constant operation, $c_i(a)$, is 2.

Proposition 26: Let e be an idempotent element in $\mathcal{M}(S)$ if $* \triangleleft * = e$ then, either

1) $*^3 = *$ in which case $*$ generate a subsemigroup of two elements, $\{*, *^2\}$.

2) $*^3 = \circ$ in which case $*$ generates a subsemigroup of three elements, $\{*, *^2, *^3\}$.

Proof. If $* \triangleleft * = e$ then the element $*$ has a cyclic pattern of order 4 where $*^4 = *^2$ and, $*^n = *^{n(\bmod 4)}$ for all $n \ge 5$. This implies that the powers of $*$ are periodic, with $*^n \in \{*, *^2, *^3\}$. Consequently, the subsemigroup generated by $*$ has only two possible forms:

1) If $*^3 = *$ then the subsemigroup is, $\langle * \rangle = \{*, *^2\}$.

2) Otherwise, the subsemigroup is $\langle * \rangle = \{*, *^2, *^3\}$.

3.3. Minimal Spanning Set of $\mathcal{M}(S)$

Let S be a set with $|S| = n$. The monoid $\mathcal{M}(S)$ exhibits the following decomposition properties:

1) $\mathcal{M}(S)$ decomposes into commutative and non-commutative elements.

2) The commutative subset is closed under $\lhd$ and forms a two-sided ideal, implying it cannot generate $\mathcal{M}(S)$.

3) Non-commutative elements split into units and non-units.

4) The set of units is closed and contains an isolated subgroup generated by distinct constant row and column permutations.

5) By Theorem 5, $DCC \cup DCR$ is a finitely generated subgroup of $\mathcal{U}$ generated by precisely two elements.

From these decompositions, we deduce:

- At least two elements generate the units.
- One additional non-unit element generates the non-commutative subset.
- One commutative element is required.

Therefore, the minimal generating set for $\mathcal{M}(S)$ consists of at least four elements.

Proposition 27: For $n = 2$, the set $\mathcal{T} = \{1,2,3,5\}$ spans $\mathcal{M}(S)$, and it is a minimal generating set.

Proof. We can decompose the elements of $\mathcal{M}(2)$ into the following:

1) Units: $\{3,5,10,12\}$ (distinct constant row/column permutations).

2) Non-units, Non-commutative: $\{2,4,11\}$.

3) Commutative: $\{0,1,6,7,8,9,14,15\}$.

The set $\mathcal{T} = \{1,2,3,5\}$ generates $\mathcal{M}(2)$ where

1) Units: $\{3,5\}$ generate $\{3,5,10,12\}$.

2) Non-unit, Non-commutative: $\{2\}$ combined with $\{3,5\}$ generates $\{2,4,11\}$.

3. Commutative: $\{1\}$, combined with $\{1,3,5\}$ generates the commutative elements.

Following the notation in Example 0 and the definition 0, we can do the following calculations to generate all the elements in $\mathcal{M}(2) \setminus \mathcal{T} = \{0,4,6,7,8,9,10,11,12,13,14,15\}$:

$$1 \lhd 2 = 0, 2 \lhd 2 = 4, 2 \lhd (1 \lhd 1) = 6, 3 \lhd 1 = 8$$

$$3 \lhd (1 \lhd 1) = 7, 2 \lhd 1 = 9, 3 \lhd 5 = 10, 2 \lhd 5 = 11$$

$$3 \lhd 3 = 12, 2 \lhd 3 = 13, 1 \lhd 1 = 14, 1 \lhd 2 \lhd 3 = 15.$$

$\mathcal{T} = \{1,2,3,5\}$ forms a minimal generating set for $\mathcal{M}(2)$.

Remark: For $|S| = 2$, the minimal generating set $\mathcal{T}$ is not unique. Specifically, $\mathcal{T} = \{1,3,5,11\}$ also generates $\mathcal{M}(2)$, demonstrating non-uniqueness.

4. Conclusion

This study provides a comprehensive analysis of the magma monoid $(\mathcal{M}(S), \lhd)$, shedding light on the algebraic structures of idempotents, almost constant operations, and units. The classification of elements for $|S|=2$ and $|S|=3$, along with the identification of generator sets, contributes to a deeper understanding of the magma monoid's properties. The presented methodology for finding inverses of units and identifying generator sets offers a valuable tool for further research. This work lays the groundwork for exploring the applications of magma monoids in algebra and computer science(e.g., automata [20]), and highlights potential avenues for future investigation

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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