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Some Results on Partially Ordered Rings

Jingjing Ma

Department of Mathematical, Applied and Physical Sciences, University of Houston-Clear Lake, Houston, USA
Email: ma@uhcl.edu

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Abstract

The paper presents some results on partially ordered rings. In section 2, it is shown that Archimedean directed maximal partial orders on integral domains must be total orders. Section 3 presents a direct proof that fields not algebraic over $\mathbb{Q}$ admit directed partial orders. Section 4 mainly considers the connection between symmetric partial orders, directed maximal partial orders and full infinite primes for commutative rings that are algebraic over $\mathbb{Z}$. In particular, it is shown that in commutative rings that are algebraic over $\mathbb{Z}$ and do not contain nonzero nilpotent elements, the symmetric partial orders and full infinite primes are in one-to-one correspondence.

Keywords

Algebraic, Archimedean, Infinite Prime, Integral Domain, Maximal Partial Order, Negative Square, Symmetric Partial Order, Total Order

1. Introduction

All rings in the paper are associative, with $1 \neq 0$, and of characteristic 0. A *partially ordered ring* $(R, \geq)$ is a ring R equipped with a partial order $\geq$ that satisfies

$$\forall a,b,c \in R,\ \text{if } a \geq b,\ \text{then } a+c \geq b+c \text{ and if } a \geq 0, b \geq 0,\ \text{then } ab \geq 0.$$

The *positive cone* of the partially ordered ring $(R, \geq)$ is defined as $R^+ = \{a \in R \mid a \geq 0\}$. Clearly R^+ is closed under the addition and multiplication of R, and $R^+ \cap -R^+ = \{0\}$. The elements in R^+ are called *positive*. On the other hand, let R be a ring and P be a subset of R that satisfies $P + P \subseteq P$, $PP \subseteq P$, and $P \cap -P = \{0\}$. Then the partial order defined by for all $x, y \in R$, $y \geq x$ if $y - x \in P$ makes R into a partially ordered ring $(R, \geq)$ with the positive cone P. Because of this connection, we will denote a partially ordered ring either by $(R, \geq)$, where $\geq$ is a partial order, or by (R, P), where P is the positive cone of a partial order, and we will say that $\geq$ or P is a partial order on the ring R. A partial order P on a ring R is called *directed* if for any $a \in R$, there exist $b,c \in P$ such that $a = b - c$. If the partial order P on R is a *lattice order*, then (R,P) is called a *lattice-ordered ring* (ℓ *-ring*). A partial order $\geq$ is called *division closed* if $ab > 0$ (or $ba > 0$) and $a > 0$ implies $b > 0$. In the following, $\mathbb{Z}^+, \mathbb{Q}^+$, and $\mathbb{R}^+$ denote the usual total order on the ring $\mathbb{Z}$ of integers, the field $\mathbb{Q}$ of rational numbers, and the field $\mathbb{R}$ of real numbers, respectively. For more information on partially ordered rings and ℓ-rings, the reader is referred to [1]-[4].

A nonempty subset S of a ring R is called a *preprime* if $S + S \subseteq S$, $SS \subseteq S$, and $-1 \notin S$. A maximal preprime is called a *prime*. By Zorn's Lemma, each preprime is contained in a prime. A prime S is called *infinite* if $1 \in S$, otherwise S is called *finite*. An infinite prime S of R is called *full* if $R = S - S = \{a - b \mid a,b \in S\}$. An infinite prime S (or a partial order P) is called *strong Archimedean* if for any $a \in S$ ($a \in P$), there exists a positive integer n such that $n - a \in S$ ($n - a \in P$). A partially ordered ring $(R,\geq)$ is called *Archimedean* if for any $a,b \in R$, $a \geq nb$ for all integers n implies that $b = 0$. For more information on primes for rings, the reader is referred to [5] [6].

2. Archimedean Maximal Partial Orders on Integral Domains

In ([7], Corollary 4), it is shown that an Archimedean directed maximal partial order on a field must be a total order. In this section, the result is generalized to integral domains. Let D be an integral domain and P be a partial order on D with $1 \in P$. Since $P \setminus \{0\}$ is a multiplicative subset of D, we may form the *quotient ring* of D by $P \setminus \{0\}$, denoted by $P^{-1}D$. Thus

$$P^{-1}D = \{a/b \mid a \in D, b \in P, b \neq 0\},$$

which is an integral domain with the same identity element as D, and D may be considered as a subring of $P^{-1}D$.

Define

$$P_q := \{x \in P^{-1}D \mid x = a/b \text{ for some } a,b \in P, b \neq 0\}.$$

Lemma 1. Let D be an integral domain and $1 \in P$ be a partial order on D. Then P_q is a partial order on $P^{-1}D$ and $P \subseteq P_q$. Moreover, if P is directed on D, then P_q is directed on $P^{-1}D$.

Proof. It is clear that P_q is closed under the addition and multiplication of $P^{-1}D$. Let $x \in P_q \cap -P_q$. Then $x = a/b$, $-x = c/d$ for some $a,b,c,d \in P$, $b \neq 0$ and $d \neq 0$. Then $0 = x + (-x) = a/b + c/d = ad + bc/bd$ implies that $ad + bc = 0$, so $a = c = 0$. Therefore $x = 0$, so $P_q \cap -P_q = \{0\}$ and hence P_q is a partial order on $P^{-1}D$. For any $a \in P$, in $P^{-1}D$, $a = a/1 \in P_q$, so $P \subseteq P_q$.

Let P be a directed partial order on D. For $x \in P^{-1}D$, $x = a/b$ for some $a \in D$ and $b \in P$, $b \neq 0$. Since P is directed on D, $a = a_1 - a_2$ for some $a_1, a_2 \in P$. Then $x = a_1/b - a_2/b$ with $a_1/b, a_2/b \in P_q$. Therefore, P_q is directed on $P^{-1}D$. □

Lemma 2. For an integral domain D and a maximal partial order P on D, P_q is a maximal partial order on $P^{-1}D$ such that $D \cap P_q = P$. Moreover, If P is Archimedean on D, then P_q is Archimedean on $P^{-1}D$.

Proof. By Lemma 1, P_q is a partial order on $P^{-1}D$ and by Zorn's Lemma, $P_q \subseteq P_1$ and P_1 is a maximal partial order on $P^{-1}D$. For $x \in P_1$, $x = a/b$ for some $a \in D$ and $b \in P$, $b \neq 0$. Since $b \in P \subseteq P_q \subseteq P_1$, $a = xb \in P_1$ and hence $a \in D \cap P_1$ that is a partial order on D. On the other hand, since $P \subseteq D \cap P_1$ and P is a maximal partial order on D, we have $P = D \cap P_1$ and hence $a \in P$. Hence $x \in P_q$. It follows that $P_q = P_1$ is a maximal partial order on $P^{-1}D$ and $D \cap P_q = P$.

Let P be Archimedean on D. We show that P_q is Archimedean on $P^{-1}D$. Suppose that $x, y \in P^{-1}D$ such that $nx \leq_{P_q} y$ for all integer n. Let $x = a/b$ and $y = c/d$ for some $a, c \in D$ and $b, d \in P$ with $b \neq 0$, $d \neq 0$. Since $P \subseteq P_q$ and $bd \in P$, $n(ad) = nx(bd) \leq_{P_q} y(bd) = cb$ for all integers n. From $cb - n(ad) \in D \cap P_q = P$ for all integers n, we have $n(ad) \leq_P cb$ for all integers n. Thus, $ad = 0$ since P is Archimedean on D, so $a = 0$ and $x = 0$. Therefore, P_q is Archimedean on $P^{-1}D$. □

Lemma 3. Let D be an integral domain and P be a maximal partial order on D such that any nonzero element in P is a unit of D. Then any subring of D containing P is a field. In particular, D is a field.

Proof. Let R be a subring of D and $P \subseteq R$. Take $a \in R$ and $a \neq 0$. If $a \in P$, then a is a unit by the assumption. Assume that $a \notin P$. Denote by $P[a]$ the set of polynomials in a with coefficients in P. It is clear that $P[a]$ is closed under the polynomial addition and multiplication. If $P[a] \cap -P[a] = \{0\}$, then $P[a]$ is a partial order on D, so $P \subseteq P[a]$ and P is maximal implies $P = P[a]$. Hence $a \in P$, a contradiction.

Since $P[a] \cap -P[a] \neq \{0\}$, there exists $0 \neq w \in P[a] \cap -P[a]$. Then $w = f(a)$ and $-w = g(a)$ for some nonzero polynomials $f(a), g(a)$ with coefficients in P. Therefore, $f(a) + g(a) = w - w = 0$ implies

$$a_n a^n + \cdots + a_1 a + a_0 = 0, \text{ for some } a_n, \cdots, a_1, 0 \neq a_0 \in P, n \geq 1.$$

Since $0 \neq a_0 \in P$, a_0^{-1} exists and in P since P is division closed ([8], Lemma 1), so

$$1 = -a_0^{-1}\left(a_n a^{n-1} + \cdots + a_1\right) a.$$

Thus $a^{-1} = -a_0^{-1}\left(a_n a^{n-1} + \cdots + a_1\right) \in R$ and hence R is a field. □

Theorem 1. Let D be an integral domain and P be an Archimedean maximal partial order on D.

1) If P is directed, then there exists an embedding $\varphi: D \to \mathbb{R}$ such that $P = \varphi^{-1}(\mathbb{R}^+)$. In particular, P is a total order on D.

2) If P is not directed, then there exists an embedding $\phi: D \to \mathbb{C}$ such that $P = \phi^{-1}(\mathbb{R}^+)$ and $\phi(D) \not\subseteq \mathbb{R}$.

Proof. By Lemma 2, P_q is an Archimedean maximal partial order on $P^{-1}D$. Since each nonzero element in P_q is a unit in $P^{-1}D$ and $P^{-1}D$ is an integral domain, by Lemma 3, $P^{-1}D$ is a field.

(1) By Lemma 1, P_q is directed on $P^{-1}D$. From ([7], Corollary 4), there exists an embedding $\sigma: P^{-1}D \to \mathbb{R}$ such that $P_q = \sigma^{-1}(\mathbb{R}^+)$. Define $\varphi = \sigma|_D$, the restriction of σ on D. Then φ is an embedding from D to $\mathbb{R}$. For $a \in D$,

$$a \in P \Leftrightarrow a \in D \cap P_q \Leftrightarrow a \in D, \sigma(a) \in \mathbb{R}^+ \Leftrightarrow \varphi(a) \in \mathbb{R}^+ \Leftrightarrow a \in \varphi^{-1}(\mathbb{R}^+),$$

so $P = \varphi^{-1}(\mathbb{R}^+)$. For $a \in D$, $\varphi(a)$ is either in $\mathbb{R}^+$ or in $-\mathbb{R}^+$, so $a \in P$ or $-a \in P$, that is, P is a total order.

(2) If P is not directed, then P_q is not directed. Otherwise, by ([7], Corollary 4), P_q is a total order on $P^{-1}D$. Since $P = D \cap P_q$, P is a total order on D as well, so P is directed, a contradiction.

By ([7], Corollary 4), there exists an embedding $\delta: P^{-1}D \to \mathbb{C}$ such that $P_q = \delta^{-1}(\mathbb{R}^+)$ and $\delta(P^{-1}D) \not\subseteq \mathbb{R}$. Define $\phi = \delta|_D$, the restriction of δ on D. Then ϕ is an embedding from D to $\mathbb{C}$. For $a \in D$,

$$a \in P \Leftrightarrow a \in D \cap P_q \Leftrightarrow a \in D, \delta(a) \in \mathbb{R}^+ \Leftrightarrow \phi(a) \in \mathbb{R}^+ \Leftrightarrow a \in \phi^{-1}(\mathbb{R}^+),$$

so $P = \phi^{-1}(\mathbb{R}^+)$. If $\phi(D) \subseteq \mathbb{R}$, then $P = \phi^{-1}(\mathbb{R}^+)$ is a total order, a contradiction. Therefore, $\phi(D) \not\subseteq \mathbb{R}$. □

3. Fields that Are Not Algebraic over $\mathbb{Q}$

In 2011, Schwartz and Yang proved that for a field if it has transcendence degree $d \geq 1$, then it has a directed partial order ([9], Corollary 4.2). The result was proved by Dubois in 1970 using the language of infinite primes for fields ([10], 4.12). In this section, based on Dubois' work, we present a direct proof of the result using maximal partial orders. All credit goes to Dubois.

Let F be a field and P be a maximal partial order. Define

$$B_P := \{a \in F \mid \exists n \geq 1 \text{ such that } n - a \in P \text{ and } n + a \in P\}.$$

$$J_P := \{a \in F \mid 1 + ma \in P \text{ for all integers } m\}.$$

$$E_P := P - P.$$

Then B_P is a subring of F and J_P is an ideal of B_P ([10], 6). Clearly $B_P \subseteq E_P$, E_P is a subring of F, and P is directed on F if and only if $E_P = F$.

Lemma 4. Let F be a field and P be a maximal partial order on F.

1) Any subring of F containing P is a subfield of F. In particular, E_P is a subfield of F.

2) P is an infinite prime of F.

3) F is algebraic over E_P.

4) E_P is the quotient field of B_P in F.

Proof. (1) Let $a \in P$, $a \neq 0$. Then $aa^{-1} = 1$ and P is division closed implies $a^{-1} \in P$. By Lemma 3, (1) is true.

(2) Since P is a preprime and $1 \in P$, $P \subseteq S$ for some infinite prime S of F. By (1), the subring $S - S$ is a subfield of F, so $S \cap -S = \{0\}$ since

$S\cap -S$ is an ideal of $S-S$ ([6], Proposition 2.5). Therefore, S is a partial order and hence $P=S$ and P is an infinite prime.

(3) Suppose that $a\in F\setminus E_P$. Denote by $P[a]$ the set of all polynomials in a with coefficients in P. Clearly $P[a]$ is closed under the addition and multiplication of F. Since $P\subseteq P[a]$ and $a\notin P$, $P[a]$ is not a preprime by (2). So, $-1\in P[a]$ and hence a satisfies a nonzero polynomial over E_P. Therefore, F is algebraic over E_P.

(4) Let $a\in B_P$. Then for some positive integer n, $n-a=b\in P$, so $a=n-b\in E_P$. Thus, $B_P\subseteq E_P$ and hence the quotient field of B_P is contained in E_P. For $0\neq b\in P$, $1+b\in P$, so $(1+b)^{-1}\in P$ and hence $1+(1+b)^{-1}\in P$. Also $\left(1-(1+b)^{-1}\right)(1+b)=b\in P$ implies that $1-(1+b)^{-1}\in P$ since P is division closed. Thus, $(1+b)^{-1}\in B_P$. Similarly,

$$1+b(1+b)^{-1}\in P \text{ and } \left(1-b(1+b)^{-1}\right)(1+b)=1\in P$$

imply that $1-b(1+b)^{-1}\in P$. Thus $b(1+b)^{-1}\in B_P$. Therefore, $b=\dfrac{b(1+b)^{-1}}{(1+b)^{-1}}$ is in the quotient field of B_P, so E_P is contained in the quotient field of B_P. Hence E_P is the quotient field of B_P. □

Theorem 2. Let F be a field.

1) For a maximal partial order P on F, if $J_P\neq\{0\}$, then P is directed.

2) (Dubois, Schwartz, Yang) If the transcendence degree of F over $\mathbb{Q}$ is $d\geq 1$. Then F contains directed partial orders.

Proof. (1) Suppose that $E_P\neq F$. We derive a contradiction. Since F is algebraic over E_P and E_P is the quotient field of B_P by Lemma 4, we are able to choose an element $\theta\in F\setminus E_P$ such that θ satisfies an irreducible monic polynomial $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ over B_P with $n\geq 2$. Define

$$P_1:=\left\{p+pz_1\theta+\cdots+pz_{n-1}\theta^{n-1}\mid 0\neq p\in P, z_1,\cdots,z_{n-1}\in J_P\right\}\cup\{0\}.$$

We show that P_1 is a preprime. Let $g(\theta)=p+pz_1\theta+\cdots+pz_{n-1}\theta^{n-1}\in P_1\setminus\{0\}$ and $h(\theta)=p'+p'z_1'\theta+\cdots+p'z_{n-1}'\theta^{n-1}\in P_1\setminus\{0\}$. Then

$$g(\theta)+h(\theta)=(p+p')+(pz_1+p'z_1')\theta+\cdots+(pz_{n-1}+p'z_{n-1}')\theta^{n-1}.$$

Since $p+p'\in P$, $\dfrac{1}{p+p'}\in P$ and since $1-\dfrac{p}{p+p'}=\dfrac{p'}{p+p'}\in P$, $\dfrac{p}{p+p'}\in B_P$. Similarly, $\dfrac{p'}{p+p'}\in B_P$. Thus

$$\frac{pz_i+p'z_i'}{p+p'}=\frac{p}{p+p'}z_i+\frac{p'}{p+p'}z_i'\in J_P,\ i=1,\cdots,n-1,$$

since J_P is an ideal of B_P. Hence

$$g(\theta)+h(\theta)=(p+p')+(p+p')\frac{pz_1+p'z_1'}{p+p'}\theta+\cdots$$
$$+(p+p')\frac{pz_{n-1}+p'z_{n-1}'}{p+p'}\theta^{n-1}$$

is in P_1, so P_1 is closed under the addition of F.

To show that $g(\theta)h(\theta)\in P_1$, we first notice that since $\theta^n=-\left(a_{n-1}\theta^{n-1}+\cdots+a_1\theta+a_0\right)$ with $a_i\in B_P$ and J_P is an ideal of B_P, we have

$$\begin{aligned}g(\theta)h(\theta)&=p\left(1+z_1\theta+\cdots+z_{n-1}\theta^{n-1}\right)p'\left(1+z_1'\theta+\cdots+z_{n-1}'\theta^{n-1}\right)\\&=pp'\left(b_0+b_1\theta+\cdots+b_{n-1}\theta^{n-1}\right),\end{aligned}$$

where $b_1,\cdots,b_{n-1}\in J_P$, and $b_0=1+b$ for some $b\in J_P$. Since $-\frac{1}{n}\le_P b\le_P\frac{1}{n}$ for all positive integers n, $1+b>_P 0$ and since $2-\frac{1}{1+b}=\frac{1+2b}{1+b}\ge_P 0$, $\frac{1}{1+b}\in B_P$. Thus $\frac{1}{1+b}b_1,\cdots,\frac{1}{1+b}b_{n-1}$ are all in J_P, so

$$g(\theta)h(\theta)=pp'\left((1+b)+(1+b)\left(\frac{b_1}{1+b}\right)\theta+\cdots+(1+b)\left(\frac{b_{n-1}}{1+b}\right)\theta^{n-1}\right)$$

is in P_1. Therefore P_1 is closed under the multiplication of F. It is clear that $-1\notin P_1$, so P_1 is a preprime. Since P is an infinite prime of F and $P\subseteq P_1$, we have $P=P_1$.

Let $z\in J_P$ and $z\ne 0$. Then $1+z\theta\in P_1=P$, so $z\theta\in E_P$ and hence $\theta=\frac{z\theta}{z}\in E_P$ by Lemma 4(4), a contradiction. Hence we must have $E_P=F$, that is, P is a directed partial order on F.

(2) Let X be a transcendental basis of F over $\mathbb{Q}$. Then $|X|=d\ge 1$. Take $t\in X$ and consider the polynomial ring $\mathbb{Q}[t]$. Totally order $\mathbb{Q}[t]$ by defining a nonzero polynomial positive if the coefficient of the term of lowest power is a positive rational number. Denote by T the positive cone of the total order on $\mathbb{Q}[t]$. We note that $1+nt\in T$ for all integers n. Since T is a partial order on F, T is contained in a maximal partial order P on F. Then $t\in J_P$, so $J_P\ne\{0\}$ and by (1), P is directed. □

Remark. Dubois actually proved that there exist at least 2^d directed partial orders on the field F ([10], 4.12).

A partially ordered field $(F,\ge)$ is said satisfying the property (AC): if for any $x\in F$, $x+\frac{1}{n}\ge 0$ for all positive integers n, then $x\ge 0$. The property (AC) was introduced by Dubois [5].

Theorem 3. Let F be a field and P be a directed maximal partial order on F. If P satisfies the property (AC), then P is an Archimedean total order on F.

Proof. Since P satisfies the property (AC),

$$J_P:=\{a\in F\mid 1+na\in P,\text{ for all } n\in\mathbb{Z}\}=\{0\}.$$

By ([11], Corollary 1.4), there exists an embedding $\phi:F\to\mathbb{C}$ such that $P=\phi^{-1}\left(\mathbb{R}^+\right)$. Since P is directed, P must be a total order. □

A field is called *directed* O^* if each directed partial order can be extended to a total order [12]. A field is directed O^* if and only if it is algebraic over $\mathbb{Q}$ as shown in the following result.

Corollary 1. For a field F, the following are equivalent.

1) F is algebraic over $\mathbb{Q}$.

2) Each directed maximal partial order on F satisfies the property (AC).

3) F is directed O^*.

Proof. (1)$\Rightarrow$(2) By ([8], Theorem 2), if F is algebraic over $\mathbb{Q}$, then each directed maximal partial order is an Archimedean total order. Suppose that $x+\frac{1}{n}\geq 0$ for all positive integers n and $x<0$. Then $m(-x)\leq 1$ for all integers m, so $-x=0$, a contradiction. Therefore, $x\geq 0$.

(2)$\Rightarrow$(3) By Theorem 3.

(3)$\Rightarrow$(1) Assume that F is not algebraic over $\mathbb{Q}$. Take $x\in F$ that is not algebraic over $\mathbb{Q}$ and form the polynomial ring $A=\mathbb{Z}[x]$. Define the partial order on A as follows.

$$P_A := \{f(x)\in A \mid f(i)\in\mathbb{Z}^+ \text{ and } f(i)\neq 0\}\cup\{0\}, \text{ where } i^2=-1.$$

Clearly P_A is a partial order on A, so a partial order on F. Then P_A is extended to a maximal partial order P_m on F. Let $f(x)=1+x^2\in\mathbb{Z}[x]$. Then for any integer n, $1+nf(x)\in P_A$ since $1+nf(i)=1\in\mathbb{Z}^+$. By Theorem 2, P_m is a directed maximal partial order since $f(x)\in J_{P_m}$. Let $g(x)=-x^2$. Since $g(i)=1\in\mathbb{Z}^+$, $g(x)\in P_A\subseteq P_m$. Therefore, $x^2<_{P_m}0$, so P_m cannot be a total order, a contradiction. □

4. Symmetric Partial Orders, Maximal Partial Orders, and Infinite Primes

For an integral domain that is algebraic over $\mathbb{Z}$, maximal partial orders and infinite primes are identical ([13], Lemma 2). In fact, that R is algebraic over $\mathbb{Z}$ is also a necessary condition for maximal partial orders and infinite primes to be identical.

Lemma 5. Let D be an integral domain. If each infinite prime of D is a partial order on D, then D is algebraic over $\mathbb{Z}$.

Proof. Suppose that D is not algebraic over $\mathbb{Z}$. Take $a\in D$ such that a is not algebraic over $\mathbb{Z}$. Consider the polynomial ring $\mathbb{Z}[a]\subseteq D$. Define

$$S := \{f(a)\in\mathbb{Z}[a] \mid f(0)\in\mathbb{Z}^+\}.$$

Clearly, S is a preprime and $1\in S$, so $S\subseteq S_1$ for an infinite prime S_1 of D. Now $a,-a\in S\subseteq S_1$ implies that $a\in S_1\cap -S_1$, so $S_1\cap -S_1\neq\{0\}$ and S_1 is not a partial order, a contradiction. □

The following result is a direct consequence of Lemma 5 and ([13], Lemma 2).

Corollary 2. Let D be an integral domain. The maximal partial orders on D and infinite primes of D are identical if and only if D is algebraic over $\mathbb{Z}$.

If a commutative ring contains nonzero zero divisors, maximal partial orders and infinite primes are different. In this section, we study the connection between symmetric partial orders, maximal partial orders and full infinite primes for commutative rings that are algebraic over $\mathbb{Z}$.

A partial order P on a ring (may not be commutative) is called *symmetric* if it is directed, division closed, and for any two elements a,b, $0 \neq a+b \in P$ implies either $a \in P$ or $b \in P$. This concept was introduced in [14] and called *symmetric cone* there. For commutative rings that are algebraic over $\mathbb{Z}$, full infinite primes and aymmetric partial orders are associated together as shown in Theorem 5 and a commutative ring with a symmetric partial order has a nice decomposition as shown in Theorem 4.

Let R be a commutative ring and P be a symmetric partial order on R. Define

$$I_P := \{a \in R \mid \text{for any } 0 \neq b \in P, b+na \in P, \forall n \in \mathbb{Z}\}.$$

It is straightforward to check that an element a is in I_P if and only if for any $0 \neq b \in P$, $-b \leq_P na \leq_P b$ for all positive integers n, where $\leq_P$ denotes the partial order with the positive cone P. Recall the *nil-radical* of a commutative ring R is $N := \{a \in R \mid a \text{ is nilpotent}\}$.

Theorem 4. Let R be a commutative ring and P be a symmetric partial order on R.

1) $R = P \cup -P \cup I_P$ and $P \cap I_P = -P \cap I_P = \{0\}$.

2) I_P is a convex ideal of R and R/I_P is a totally ordered ring.

3) If $N = \{0\}$, then I_P is a prime ideal of R.

Proof. (1) Suppose that $a \in R$ and $a \notin P \cup -P$. We show that $a \in I_P$. Take $0 \neq b \in P$. Since $a \notin P$ and P is division closed, for any positive integers k, if $ka \neq 0$, then $ka \notin P$ since $k1 \in P$. Thus $b = (b-ka)+ka \in P$ implies $b-ka \in P$. Thus $b-ka \in P$ for all positive integer k. Similarly, since $-a \notin P$, $b+ka \in P$ for all positive integers k. Hence $b+na \in P$ for all integers n. Since this is true for any $b \in P$, $b \neq 0$. We have $a \in I_P$. It is clear that $P \cap I_P = -P \cap I_P = \{0\}$.

(2) Take $a,b \in I_P$ and assume $a \neq 0$, $b \neq 0$, $a+b \neq 0$ since $a+b \in I_P$ under those cases. If $a+b \in P$, then that P is symmetric implies that $a \in P$ or $b \in P$, a contradiction by (1). If $a+b \in -P$, then $0 \neq -(a+b) = (-a)+(-b) \in P$, so $-a \in P$ or $-b \in P$ and hence $a \in -P$ or $b \in -P$, a contradiction again by (1). Thus $a+b \in I_P$. It is clear that $a \in I_P$ implies that $-a \in I_P$. Hence I_P is a subgroup of $(R,+)$.

Take $a \in I_P$, $r \in R$ and $a \neq 0$, $r \neq 0$, $ra \neq 0$. By (1), we consider $r \in P$, or $r \in -P$, or $r \in I_p$.

(i) $r \in P$. Then P is division closed implies that $ra \notin P$, otherwise $a \in P$, a contradiction by (1). Similarly, $ra \notin -P$. Thus $ra \in I_P$ by (1).

(ii) $r \in -P$. Then $-r \in P$, so by the argument given in (i), $(-r)a \in I_P$. Thus $ra \in I_P$.

(iii) $r \in I_P$. Suppose $ra \in P$. We have $-ra \leq_P 2r \leq_P ra$ and $-1 \leq_P a \leq_P 1$, then

$$(ra+2r)(1-a) \geq_P 0 \Rightarrow ra + 2r - ra^2 - 2ra \geq_P 0$$

$$(ra-2r)(1+a)\geq_P 0\Rightarrow ra-2r+ra^2-2ra\geq_P 0.$$

By adding above two inequalities together, we have $2(-ra)\geq_P 0$. Since $-ra\leq_P 0$, we must have $-ra=0$, so $ra=0$, a contradiction with $ra\neq 0$. Thus $ra\notin P$. Similarly, $ra\notin -P$. Hence $ra\in I_P$. Therefore, for any $r\in R$ and $a\in I_P$, $ra\in I_P$, so I_P is an ideal of R. Let $0\leq_P a\leq_P b\in I_P$. Clearly, if $a\neq 0$, then $a\notin P\cup -P$, so $a\in I_P$. Thus, I_P is a convex ideal of (R,P). Since $R=P\cup -P\cup I_P$, it is clear that R/I_P is a totally ordered ring with positive cone $\overline{P}=\{a+I_P \mid a\in P\}$ [13].

(3) If $I_P=\{0\}$, then P is a total order on R. It is well-known that a totally order ring without nonzero nilpotent elements is a domain, so I_P is a prime ideal. Now we assume that $I_P\neq\{0\}$. Let $a,b\in R$ such that $ab\in I_P$. Suppose that $a\notin I_P, b\notin I_P$. We derive a contradiction. Since $R=P\cup -P\cup I_P$, $a,b\in P$ or $-P$. Without loss of generality, we may assume that $a,b\in P$, so $ab\in P$. Then $ab=0$, since $P\cap I_P=\{0\}$. It follows from $\forall x\in I_P$, $-a\leq_P x\leq_P a$ and $-b\leq_P x\leq_P b$ that $xa=xb=0$, and hence

$$(a+x)(b+x)=x^2\geq_P 0 \text{ and } (a-x)(b+x)=-x^2\geq_P 0.$$

Thus $x^2=0$. Since R contains no nonzero nilpotent elements, $x=0$, so $I_P=\{0\}$, a contradiction. Therefore $ab\in I_P$ implies $a\in I_P$ or $b\in I_P$, that is, I_P is a prime ideal of R. □

We will need following result from [14] that provides a way to construct symmetric partial orders.

Lemma 6. ([14], Theorem 2.3) *Let R be a commutative ring, I be a prime ideal of R, and T be a total order on the quotient ring R/I. Then*

$$P:=\{a\in R \mid a+I\neq 0 \text{ and } a+I\in T \text{ in } R/I\}\cup\{0\}$$

is a symmetric partial order on R.

For an infinite prime S of a commutative ring, define $P_s=[S\setminus(S\cap -S)]\cup\{0\}$.

Theorem 5. Let R be a commutative ring that is algebraic over $\mathbb{Z}$. For each full infinite prime S of R, P_s is a symmetric partial order on R and $S=P_s\cup I_{P_s}$.

Proof. Let $I=S\cap -S$. From ([6], Proposition 2.5), I is a prime ideal of R. Consider the integral domain R/I. Define $\overline{S}:=\{a+I \mid a\in S\}$ in R/I. By ([13], Lemma 5), $\overline{S}$ is a full infinite prime of R/I. Since R/I is an integral domain that is algebraic over $\mathbb{Z}$, by ([13], Lemma 2 and Theorem 1), $\overline{S}$ is a total order on R/I. It is straightforward to verify that

$$P_s=\{a\in R \mid a+I\neq 0 \text{ and } a+I\in\overline{S} \text{ in } R/I\}\cup\{0\}.$$

So, by Lemma 6, P_s is a symmetric partial order on R. Let $0\neq a\in I$. Then $a\notin P_s\cup -P_s$, so $a\in I_{P_s}$ by Theorem 4. Thus $I\subseteq I_{P_s}$. Since $S\subseteq P_s\cup I_{P_s}$ and $P_s\cup I_{P_s}$ is a preprime, $S=P_s\cup I_{P_s}$. □

Corollary 3. Let R be a commutative ring that is algebraic over $\mathbb{Z}$.

1) Each full infinite prime of R is strong Archimedean.

2) Let S be an infinite prime of R. Then there exists a ring homomorphism $\varphi: S-S \to \mathbb{R}$ such that $\mathrm{Ker}(\varphi) = S \cap -S$ and $S = \varphi^{-1}(\mathbb{R}^+)$.

Proof. (1) Let S be a full infinite prime of R. By Theorem 5, $S = P_s \cup I_{P_s}$. Then $I_{P_s} = S \cap -S$, so I_{P_s} is a prime ideal of R. We first show that P_s is strong Archimedean. Since R/I_{P_s} is an integral domain that is algebraic over $\mathbb{Z}$, $\overline{S} = \{a + J_s \mid a \in S\}$ is an Archimedean total order ([13], Theorem 1), so $\overline{S}$ is strong Archimedean. Let $0 \neq a \in P_s$. Then $a + I_{P_s} \in \overline{S}$, so there is a positive integer n such that $0 \neq n(1 + I_{P_s}) - (a + I_{P_s}) \in \overline{S}$. Thus there exists $b \in S$ such that $(n-a) + I_{P_s} = b + I_{P_s}$. Since $b + I_{P_s} \neq 0$, $b \notin I_{P_s}$ and hence $b \in P_s$. Then there exists $i \in I_{P_s}$ such that $n - a = b + i$. If $i = 0$, then $n - a = b \in P_s$. If $i \neq 0$, then $0 \neq -i \in I_{P_s}$, $P_s \cap I_{P_s} = \{0\}$, and $b = (b+i) + (-i) \in P_s$ implies that $n - a = b + i \in P_s$, since P_s is symmetric. Thus, in any case, $n - a \in P_s$, so P_s is strong Archimedean. Now Take $0 \neq a \in S$. If $a \in P_s$, then there is a positive integer n such that $n - a \in P_s \subseteq S$. If $a \in I_{P_s}$, then $1 = (1-a) + a$ and $a \notin P_s$ implies that $1 - a \in P_s \subseteq S$, since again P_s is symmetric. Therefore S is strong Archimedean.

(2) Since $S - S$ is a commutative ring that is algebraic over $\mathbb{Z}$, and S is a full infinite prime of $S - S$, by (1), S is strong Archimedean. Then, by ([6], Proposition 1.7), there exists a ring homomorphism $\varphi: S - S \to \mathbb{R}$ such that $\mathrm{Ker}(\varphi) = S \cap -S$ and $S = \varphi^{-1}(\mathbb{R}^+)$. □

Lemma 7. Let R be a commutative ring that is algebraic over $\mathbb{Z}$ and $N = \{0\}$. Then each symmetric partial order is a maximal partial order.

Proof. Let P be a symmetric partial order on R. Then $P \subseteq P_m$ for some maximal partial order on R. Suppose $P \neq P_m$. We derive a contradiction. Take $a \in P_m \setminus P$. If $a \in -P$, then $-a \in P \subseteq P_m$ as well, a contradiction. Thus by Theorem 4, $a \in I_P$. Let $f(x) = a_n x + \cdots + a_1 x + a_0$ be a nonzero integer polynomial such that $f(a) = 0$. If $a_0 \neq 0$, then $a_0 = -a(a_n a^{n-1} + \cdots + a_1 a) \in I_P$, so $-1 \leq_P n a_0 \leq_P 1$ for all positive integers n, a contradiction. Thus $a_n a^n + \cdots + a_1 a = 0$ for some $n \geq 2$. Suppose that $a_1 = 0$. Then $a_n a^n + \cdots + a_k a^k = 0$ with $n > k \geq 2$ and $a_k \neq 0$. It follows that

$$\left(a_n a^{n-1} + \cdots + a_k a^{k-1}\right)^2 = 0 \text{ since } a\left(a_n a^{n-1} + \cdots + a_k a^{k-1}\right) = 0.$$

Then R does not contain nonzero nilpotent elements implies that $a_n a^{n-1} + \cdots + a_k a^{k-1} = 0$. If $k - 1 \geq 2$, we may repeat the above argument, so without loss of generality, we may assume at the beginning that $a_n a^n + \cdots + a_1 a = 0$ and $a_1 \neq 0$.

Let $e = a_n a^{n-1} + \cdots + a_1$. Then $e^2 = a_1 e$ since $ae = a(a_n a^{n-1} + \cdots + a_1) = 0$. If $e \in P$, then $-e \leq_P a \leq_P e$, so $-e \leq_{P_m} a \leq_{P_m} e$ since $P \subseteq P_m$. If follows from $a \in P_m$ that $-ae = 0 \leq_{P_m} a^2 \leq_{P_m} ae = 0$. Thus $a^2 = 0$ and $a = 0$, a contradiction. Similarly, $e \notin -P$. Hence $e \in I_P$ by Theorem 4. From $a, e \in I_P$, we have $a_1 \in I_P$, so $-1 \leq_P n a_1 \leq 1$ for all positive integers n, a contradiction. Therefore, we must have $P = P_m$ and hence P is a maximal partial order on R. □

For a commutative ring that is algebraic over $\mathbb{Z}$ and $N = \{0\}$, a directed maximal partial order may not be symmetric. The following example demonstrates that not all directed maximal partial orders are symmetric.

Example 1. Let $R = \mathbb{Z} \times \mathbb{Z}$. It is straightforward to check that $P_m = \{(a,b) \mid a,b \in \mathbb{Z}^+\}$ is a directed maximal partial order on R that is not symmetric. We notice that $P = \{(a,b) \mid a > 0\} \cup \{0\}$ is a symmetric partial order and a maximal partial order on R.

By Theorem 4 and Lemma 7, we have the following result.

Theorem 6. Let R be a commutative ring that is algebraic over $\mathbb{Z}$ and $N = \{0\}$. For each full infinite prime S of R, there exists a directed maximal partial order P such that $S = P \cup I_P$.

We notice that in Example 1, P_m is a directed maximal partial order and there exists no full infinite prime S such that $S = P_m \cup I_{P_m}$. However, the situation is different for symmetric partial orders.

Theorem 7. Let R be a commutative ring that is algebraic over $\mathbb{Z}$ and $N = \{0\}$. For each symmetric partial order P on R, there exists a full infinite prime S such that $S = P \cup I_P$.

Proof. We show that $P \cup I_P$ is an infinite prime. It is clear that $P \cup I_P$ is a preprime, so $P \cup I_P \subseteq S$ for some full infinite prime of R. By Theorem 4, I_P is a convex prime ideal and R / I_P is a totally ordered integral domain that is algebraic over $\mathbb{Z}$ with the positive cone $\overline{P} = \{a + I_P \mid a \in P\}$. By ([13], Lemma 5), $\overline{S} = \{a + I_P \mid a \in S\}$ is a full infinite prime of R/I_P and hence $\overline{S}$ is a total order on R/I_P. Thus $\overline{P} = \overline{S}$. Let $0 \neq z \in S$. Then $z + I_P = w + I_P$ for some $w \in P$. It follows that $z = w + i$ for some $i \in I_P$. If $w = 0$ or $i = 0$, then $z \in P \cup I_P$. Suppose that $w \neq 0$ and $i \neq 0$. Then $0 \neq -i \in I_P$ implies that $-i \notin P$, so $w = w + i + (-i) \in P$ and P is symmetric implies that $z = w + i \in P$. Therefore, $S \subseteq P \cup I_P$ and hence $S = P \cup I_P$. □

By Theorems 5 and 7, for a commutative ring that is algebraic over $\mathbb{Z}$ and $N = \{0\}$. The mapping $S \to P_s$ is a one-to-one correspondence between full infinite primes and symmetric partial orders. The correspondence reveals the relation of the infinite primes and symmetric partial orders and provides a foundation for futher investigation of the properties of the infinite primes and symmetric partial orders.

5. Summary

The infinite primes and partial orders for rings are naturally connected. This paper continues exploring the connections between them for general commutative rings with the identity. Using the theory of infinite primes, it is proved that for a field, an Archimedean directed maximal partial order must be a total order, based on it, we were able to show that the result is true for an integral domain. The elementary method of constructing directed maximal partial orders in section 3 was used by Dubois to construct infinite primes. This method can be generalized to produce directed partial orders on other algebras. In section 4, the symmetric partial orders are naturally introduced from the infinite primes. We expect more discovery on the structures of partially ordered rings by using the infinite primes and symmetric partial orders.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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