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Arithmetic Differential Geometry over Mother Number Space

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Abstract

Using elementary Mother space of type e^M, we construct semiring $\mathcal{N}$, ring $\mathcal{Z}$ and field $\mathcal{Q}$ of extended numbers inculuding natural numbers $\mathbb{N}$, rational integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$. We see that $\mathcal{N}$ and $\mathcal{Z}$ have a natural total order structure. They are also countable sets. It can be seen that the ring $\mathcal{Z}$ and the field $\mathcal{Q}$ are partially differentiable rings and fields. As an application, we construct a partially differentiable Riemann manifold over the Mother Pythagoras field K. We also formulate the ABC type conjecture regarding $\mathcal{N}$, $\mathcal{Z}$, $\mathcal{Q}$ and their Mother algebraic extensions L. And we introduce the Diophantine type equations related to the concept of this paper.

Keywords

Monoid, Mother Number, Dark Number, Partially Differentiable Manifold, ABC Type Conjecture

1. Introduction

Just as the existence of unknown elementary particles is predicted in the world of elementary particles, there may also be unknown numbers in mathematics. The goal of this paper is to introduce and discuss unknown numbers. In this paper, this unknown number is called the *dark numbers*. Unknown numbers can be likened to dark matter in physics. The set of dark numbers and ordinary numbers is called the *Mother numbers*. The biggest feature is that a differential concept exists in the world of Mother numbers. That is, a Mother number can be differentiated by a dark number. Note that since ordinary numbers are included in the Mother number, a Mother number can not be differentiated by an ordinary number. Therefore, this differentiation is called a partial differentiation.

We will explain the reason for introducing the concept of Mother numbers.

Kurokawa is studying absolute mathematics in order to challenge the Riemann hypothesis [1]. In particular, Kurokawa has studied the concept of absolute differentiation, which is differentiation by a prime element on a monoid [1]-[3]. Kurokawa's absolute differentiation satisfies the Leibniz rule but does not satisfy linearity. We would like to make Kurokawa's absolute differentiation also satisfy linearity. By extending the concept of number to Mother numbers, we can make it satisfy linearity. This also extends the Kurokawa's category of monoid to the category of commutative ring.

The Mother number considered in this paper is a countable set, and therefore it is a discrete concept. Because of these properties, we want to develop continuous mathematics in discrete mathematics. This corresponds to turning classical physics into quantum physics. Let us explain the Mother number below.

Gotyou has described the concept of elementary Mother space [4]. Gotyou has explained that there are the following types of elementary Mother space:

$$e^M \text{ type}, \quad \frac{1}{1-M} \text{ type}, \quad 1-\log(1-M) \text{ type}.$$

In this paper, we will limit our discussion to the elementary Mother space of type e^M.

Let us explain about elementary Mother space of type e^M.

The generating function of the sequence $\left\{\frac{1}{n!}\right\}_{n\in\mathbb{N}}$ is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots.$$

We substitute a set M for variable x. We also replace sum $+$ with direct sum $\coprod$ of sets, $x^n = x \times \cdots \times x$ with direct product of sets $M^n = M \times \cdots \times M$, divide it in $n!$ with the action of permutation group $\mathfrak{S}_n$ in direct product of set M^n. Hence we have

$$e^M = \{\varnothing\} \coprod M \coprod (M \times M)/\mathfrak{S}_2 \coprod (M \times M \times M)/\mathfrak{S}_3 \coprod \cdots \coprod M^n/\mathfrak{S}_n \coprod \cdots.$$

In this paper, this e^M is called *elementary Mother space*.

We introduce the set $\mathbb{D} = \{1, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \cdots\}$ different from the usual set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \cdots\}$. However, 1 is a common, ordinary 1. We will call $\mathbb{D}$ the set of *dark natural numbers*. $(\mathbb{D}, \times)$ is the monoid. Note that $(\mathbb{D}, \times)$ and $(\mathbb{N}, \times)$ are monoid isomorphisms.

The elementary Mother space of $\mathbb{D}$ is the following.

$$e^{\mathbb{D}} = \{\varnothing\} \coprod \mathbb{D} \coprod (\mathbb{D} \times \mathbb{D})/\mathfrak{S}_2 \coprod (\mathbb{D} \times \mathbb{D} \times \mathbb{D})/\mathfrak{S}_3 \coprod \cdots \coprod \mathbb{D}^n/\mathfrak{S}_n \coprod \cdots.$$

For any unordered pair $(n_1, n_2, \cdots, n_l)^{\mathfrak{S}_l} \in \mathbb{D}^l/\mathfrak{S}_l$, we define the element of $\mathbb{D}^l/\mathfrak{S}_l$ as follows.

$$n_1 \oplus n_2 \oplus \cdots \oplus n_l := (n_1, n_2, \cdots, n_l)^{\mathfrak{S}_l}.$$

By defining the commutative product $\times$ and the commutative sum $+$ of the set e^D, it becomes a commutative semiring $(e^{\mathbb{D}}, \times, +)$ (cf. Section 3).

We introduce the set $\hat{\mathbb{D}}=\{\pm1,\pm\mathbf{2},\pm\mathbf{3},\pm\mathbf{4},\pm\mathbf{5},\cdots\}$ different from the usual set of rational integers without zero $\mathbb{Z}\setminus\{0\}=\{\pm1,\pm2,\pm3,\pm4,\pm5,\cdots\}$. We will call $\hat{\mathbb{D}}$ the set of *dark rational integers*. However, ±1 is a common, ordinary ±1. $(\hat{\mathbb{D}},\times)$ is the monoid. Note that $(\hat{\mathbb{D}},\times)$ and $(\mathbb{Z}\setminus\{0\},\times)$ are monoid isomorphisms.

The elementary Mother space of $\hat{\mathbb{D}}$ is the following.

$$e^{\hat{\mathbb{D}}}=\{\varnothing\}\amalg\hat{\mathbb{D}}\amalg\left(\hat{\mathbb{D}}\times\hat{\mathbb{D}}\right)/\mathfrak{S}_2\amalg\left(\hat{\mathbb{D}}\times\hat{\mathbb{D}}\times\hat{\mathbb{D}}\right)/\mathfrak{S}_3\amalg\cdots\amalg\hat{\mathbb{D}}^n/\mathfrak{S}_n\amalg\cdots.$$

For any unordered pair $(n_1,n_2,\cdots,n_l)^{\mathfrak{S}_l}\in\hat{\mathbb{D}}^l/\mathfrak{S}_l$, we define the element of $\hat{\mathbb{D}}^l/\mathfrak{S}_l$ as follows.

$$n_1\oplus n_2\oplus\cdots\oplus n_l:=(n_1,n_2,\cdots,n_l)^{\mathfrak{S}_l}.$$

By defining the commutative product $\times$, sum $+$ and difference $-$ of the set $e^{\hat{\mathbb{D}}}$, it becomes a commutative ring $\left(e^{\hat{\mathbb{D}}},\times,+,-\right)$ (cf. Section 3).

Each piece $\mathbb{D}^l/\mathfrak{S}_l$ and $\hat{\mathbb{D}}^l/\mathfrak{S}_l$ of $e^{\mathbb{D}}$ and $e^{\hat{\mathbb{D}}}$ is producing a new number $n_1\oplus n_2\oplus\cdots\oplus n_l$.

Each piece of $\mathbb{D}^l/\mathfrak{S}_l$ and $\hat{\mathbb{D}}^l/\mathfrak{S}_l$ is only a set with new elements $n_1\oplus n_2\oplus\cdots\oplus n_l$, but the set $e^{\mathbb{D}}$ of the entire union of each piece is commutative semiring and the set $e^{\hat{\mathbb{D}}}$ is commutative ring.

Furthermore, $e^{\mathbb{D}}$ contains a natural total ordering structure $\le$. We write $\left(e^{\mathbb{D}},\le,\times,+,-\right)$, which is a totally order semiring. And $e^{\hat{\mathbb{D}}}$ contains a natural total ordering structure $\le$. We write $\left(e^{\hat{\mathbb{D}}},\le,\times,+,-\right)$, which is a totally order ring (cf. Section 4 and 5). These are the new totally order semiring and totally order ring corresponding $(\mathbb{N},\le,\times,+)$ and $(\mathbb{Z},\le,\times,+,-)$.

The reason for considering e^M is as follows. The coefficient $\frac{1}{n!}$ that appears in the expansion of e^x corresponds to the fact that the permutation group $\mathfrak{S}_n$ is invariant with respect to e^M. This means that the resulting ring is commutative. In this paper, we will only discuss commutative rings. Furthermore, although the mother number can be defined without using the notation of e^M, using the notation of e^M allows us to express various examples in a unified manner (cf. section 3).

Using $e^{\mathbb{D}}$ as an example, we will explain what kind of number Mother number is,

$$(\text{Mothernumber})=(\text{ordinary number})+(\text{dark number}).$$

According to this decomposition, $e^{\mathbb{D}}$ is

$$e^{\mathbb{D}}=(\mathbb{N}\cup\{0\})+\left(e^{\mathbb{D}}\setminus(\mathbb{N}\cup\{0\})\right).$$

Ordinary numbers in the set $\mathbb{N}\cup\{0\}$ are grouped as follows.

$$(\text{ordinary number})=(\text{ordinary prime})+(\text{ordinary composite number})+(0\text{ and }1).$$

Dark numbers in the set $e^{\mathbb{D}}\setminus(\mathbb{N}\cup\{0\})$ are grouped as follows.

$$(\text{dark number})=(\text{dark prime})+(\text{dark prime element})+(\text{dark composite number}).$$

Both dark prime and dark prime element are numbers that can not be factored into prime factors.

For example, it is

$$(\text{dark number})=(\mathbf{2},\mathbf{3},\mathbf{5},\cdots)+\left(\mathbf{2}\oplus 1,2\oplus \mathbf{3}^2,\mathbf{2}^4\oplus 1,\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{5},\cdots\right)+(\mathbf{4},\mathbf{2}\oplus\mathbf{2},\mathbf{2}\oplus\mathbf{6},\mathbf{6},\cdots),$$

$$\mathbf{4}=\mathbf{2}\times\mathbf{2}=\mathbf{2}^2,\quad \mathbf{2}\oplus\mathbf{2}=\mathbf{2}\times(1\oplus 1)=\mathbf{2}\times 2=2\times\mathbf{2},\quad \mathbf{2}\oplus\mathbf{6}=\mathbf{2}\times(1\oplus\mathbf{3}),\quad \mathbf{6}=\mathbf{2}\times\mathbf{3}.$$

Primes in set $e^{\mathbb{D}}$ of Mother numbers are grouped as follows.

$$(\text{prime})=(\text{ordinary prime})+(\text{dark prime})+(\text{dark prime element}).$$

For example, it is

$$(\text{prime})=(2,3,5,\cdots)+(\mathbf{2},\mathbf{3},\mathbf{5},\cdots)+\left(2\oplus\mathbf{2},\mathbf{2}\oplus\mathbf{3},\mathbf{2}^2\oplus\mathbf{3},\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{3},\mathbf{3}^2\oplus\mathbf{5}^3,\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{5},\cdots\right).$$

One of the advantages of introducing $e^{\mathbb{D}}$ and $e^{\hat{\mathbb{D}}}$ is that they have the following properties.

There are monoid isomorphisms $(\mathbb{N},\times)\simeq(\mathbb{D},\times)$, $(\mathbb{Z},\times)\simeq(\hat{\mathbb{D}},\times)$. Kurokawa's absolute differential exists for these four monoids. Absolute differentiation satisfies the Leibniz rule but does not satisfy the linearity. It is a nonlinear differential (cf. [1]-[3]).

We want to linearly extend the nonlinear absolute differential on $(\mathbb{D},\times)$. For this purpose, linear differentiation can be defined by extending the concept of number to $e^{\mathbb{D}}$. The same goes for $(\hat{\mathbb{D}},\times)$ (cf. Section 7).

We put $(\mathcal{N},\le,\times,+):=\left(e^{\mathbb{D}},\le,\times,+,-\right)$ and $(\mathcal{Z},\le,\times,+,-):=\left(e^{\hat{\mathbb{D}}},\le,\times,+,-\right)$. Furthermore, the quotient field of the commutative ring $\mathcal{Z}$ is written as $\mathcal{Q}:=\left(\mathcal{Z}\setminus\{0\}\right)^{-1}\mathcal{Z}$.

In this paper, we will consider the following expansion of numbers.

$$(\mathbb{N},\le,\times,+)\rightsquigarrow(\mathcal{N},\le,\times,+)$$

$$(\mathbb{Z},\le,\times,+,-)\rightsquigarrow(\mathcal{Z},\le,\times,+,-)$$

$$(\mathbb{Q},\le,\times,\div,+,-)\rightsquigarrow(\mathcal{Q},\le,\times,\div,+,-)$$

We examine various arithmetic, algebraic and geometric properties of $(\mathcal{N},\le,\times,+)$, $(\mathcal{Z},\le,\times,+,-)$ and $(\mathcal{Q},\le,\times,\div,+,-)$. We also discuss the algebraic extension of $\mathcal{Q}$ and the algebraically closed field $\overline{\mathcal{Q}}$.

Each section of this paper is organized as follows.

Section 2 briefly reviews monoids and absolute algebras. Section 3 defines commutative semiring $e^{\mathbb{D}}$ and commutative ring $e^{\hat{\mathbb{D}}}$. And we will introduce various examples of elementary Mother spaces and elementary Mother number spaces. In Sections 4 and 5, we rewrite $e^{\mathbb{D}}$ and $e^{\hat{\mathbb{D}}}$ as isomorphic objects to improve the clarity of the discussion. We also define a total order structure into them. Section 6 describes the fundamental theorem of number theory and the division theorem in $e^{\hat{\mathbb{D}}}$. In Section 7, we define the differential concept on $e^{\mathbb{D}}$, $e^{\hat{\mathbb{D}}}$ and the Mother Pythagoras filed K. There are two important concepts of differentiation. One is point differentiation and the other is functional differentiation. Sections 8, 9 and 10 are the applications in this paper. In Section 8, we define partially differentiable Riemann manifolds with metric topology on K. In Section 9 and 10, we construct the ABC type conjectures about $e^{\mathbb{D}}$, $e^{\hat{\mathbb{D}}}$ and the Mother algebraic extensions of $\mathcal{Q}$ denoted by L. We also introduce the Diophantine type equations related to the concept of this paper.

2. Monoids and Absolute Algebras

We explain the monoids and the absolute algebras by Kurokawa (cf. [1]-[3] [5]).

Let $\mathbb{N}=\{1,2,3,4,5,6,7,\cdots\}$ and $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6,\pm7,\cdots\}$ be the set of all natural numbers and rational integers, respectively.

Definition 2.1. *A monoid* $A=(A,\times)$ *will be a semigroup with identity element* 1_S. *An absolute algebra* A *will be a monoid with zero element* 0_A,

$$1_A\times a=a\times 1_A=a,\ 0_A\times a=a\times 0_A=0_A,\quad a\in A.$$

Example 2.1. $(\mathbb{F}_1=\{1\},\times)$ *is the monoid. The corresponding absolute algebra is* $(\{1,0\},\times)$.

Example 2.2. $(\mathbb{F}_{1^2}=\{1,-1\},\times)$ *is the monoid. The corresponding absolute algebra is* $(\{1,-1,0\},\times)$.

Example 2.3. $\left(\mathbb{F}_{1^3}=\left\{1,\frac{1}{2}\left(-1+\sqrt{-3}\right),\frac{1}{2}\left(-1-\sqrt{-3}\right)\right\},\times\right)$ *is the monoid. The corresponding absolute algebra is* $\left(\left\{1,\frac{1}{2}\left(-1+\sqrt{-3}\right),\frac{1}{2}\left(-1-\sqrt{-3}\right),0\right\},\times\right)$.

Example 2.4. $(\mathbb{F}_{1^4}=\{1,-1,i,-i\},\times)$ *is the monoid. The corresponding absolute algebra is* $(\{1,-1,i,-i,0\},\times)$.

In general, $\mathbb{F}_{1^n}$ is considered (cf. [1]).

Example 2.5. $(\mathbb{N},\times)$ *is the monoid and* $(\mathbb{Z},\times)$ *is the absolute algebra.*

Kurokawa has defined the following interesting monoid.

Example 2.6. *The* $2,3,5,7,\cdots$ *are ordinary primes. Then we define*

$$\mathbb{F}_1[2,3,5,7,\cdots]=\left\{2^{i_1}3^{i_2}5^{i_3}7^{i_4}\cdots\mid i_1,i_2,i_3,i_4,\cdots\in\mathbb{N}\cup\{0\}\right\}.$$

If $i_1=i_2=i_3=\cdots=0$ *then* $1\in\mathbb{F}_1[2,3,5,7,\cdots]$. *Hence* $(\mathbb{F}_1[2,3,5,7,\cdots],\times)$ *becomes the monoid.*

Example 2.7. *The* $2,3,5,7,\cdots$ *are ordinary primes. Then we define*

$$\mathbb{F}_1[-1,2,3,5,7,\cdots]=\left\{(-1)^{i_0}2^{i_1}3^{i_2}5^{i_3}7^{i_4}\cdots\mid i_0=\pm1,i_1,i_2,i_3,i_4,\cdots\in\mathbb{N}\cup\{0\}\right\},$$

$$\mathbb{F}_1[-1,0,2,3,5,7,\cdots]:=\mathbb{F}_1[-1,2,3,5,7,\cdots]\cup\{0\}.$$

If $i_0=i_1=i_2=i_3=\cdots=0$ *then* $1\in\mathbb{F}_1[-1,0,2,3,5,7,\cdots]$. *Hence* $(\mathbb{F}_1[-1,0,2,3,5,7,\cdots],\times)$ *becomes the absolute algebra.*

Then the following important monoid isomorphic expression holds.

$$(\mathbb{F}_1[2,3,5,7,\cdots],\times)\simeq(\mathbb{N},\times).$$

$$(\mathbb{F}_1[-1,0,2,3,5,7,\cdots],\times)\simeq(\mathbb{Z},\times).$$

3. Mother Number Ring $e^{\hat{\mathbb{D}}}$ and Mother Number Semiring $e^{\mathbb{D}}$

Let V be a vector space over a field k. Let $S^*(V)$ be a symmetric tensor algebra. e^M, defined below, is similar to $S^*(V)$. The difference is that M can be a general set. Also, products and sums are different. The product of $S^*(V)$ is a tensor product, and the sum is a direct sum.

Elementary Mother space is defined as follows [4].

$$\begin{aligned} e^M &= \{\varnothing\} \coprod M \coprod (M \times M)/\mathfrak{S}_2 \coprod (M \times M \times M)/\mathfrak{S}_3 \coprod \cdots \coprod M^n/\mathfrak{S}_n \coprod \cdots \\ &= \{\varnothing\} \coprod M \coprod S^2 M \coprod S^3 M \coprod \cdots \coprod S^n M \coprod \cdots, \end{aligned}$$

where $\mathfrak{S}_n$ is the permutation group and we can set $\varnothing = 0$ and $S^n M = \underbrace{(M \times M \times \cdots \times M)}_{n \text{ times}}/\mathfrak{S}_n$.

For example, $(M \times M \times M)/\mathfrak{S}_3 = S^3 M$ means for $(x, y, z) \in M \times M \times M$, it is as follows.

$$(x, y, z) = (x, z, y) = (y, x, z) = (y, z, x) = (z, x, y) = (z, y, x).$$

The following example is from Nakajima [6].

$$\begin{aligned} e^{\mathbb{P}^1(\mathbb{C})} &= \{\varnothing\} \coprod \mathbb{P}^1(\mathbb{C}) \coprod \left(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})\right)/\mathfrak{S}_2 \coprod \left(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})\right)/\mathfrak{S}_3 \\ &\quad \coprod \cdots \coprod \mathbb{P}^1(\mathbb{C})^n/\mathfrak{S}_n \coprod \cdots \\ &= \{\varnothing\} \coprod \mathbb{P}^1(\mathbb{C}) \coprod S^2\mathbb{P}^1(\mathbb{C}) \coprod S^3\mathbb{P}^1(\mathbb{C}) \coprod \cdots \coprod S^n\mathbb{P}^1(\mathbb{C}) \coprod \cdots \\ &\simeq \{\varnothing\} \coprod \mathbb{P}^1(\mathbb{C}) \coprod \mathbb{P}^2(\mathbb{C}) \coprod \mathbb{P}^3(\mathbb{C}) \coprod \cdots \coprod \mathbb{P}^n(\mathbb{C}) \coprod \cdots = \mathbb{P}^\infty(\mathbb{C}) \end{aligned}$$

Definition 3.1. *We define the set* $\mathbb{D} = \{1, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \cdots\}$ *as the set of dark natural numbers and also define the set* $\hat{\mathbb{D}} = \{\pm 1, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{4}, \pm\mathbf{5}, \cdots\}$ *as the set of dark rational integers.*

± 1 *are common to ordinary natural numbers and rational integers.*

For the product $\times$ of elements of $\mathbb{D}$, $(\mathbb{D}, \times)$ is the monoid.

For the product $\times$ of elements of $\hat{\mathbb{D}}$, $(\hat{\mathbb{D}}, \times)$ is the monoid.

Below, we define two examples of the most basic *elementary Mother number space* in this paper.

The elementary Mother number space of D is defined as follows.

$$e^{\mathbb{D}} = \{\varnothing\} \coprod \mathbb{D} \coprod (\mathbb{D} \times \mathbb{D})/\mathfrak{S}_2 \coprod (\mathbb{D} \times \mathbb{D} \times \mathbb{D})/\mathfrak{S}_3 \coprod \cdots \coprod \mathbb{D}^n/\mathfrak{S}_n \coprod \cdots.$$

For any unordered pair $(n_1, n_2, \cdots, n_l)^{\mathfrak{S}_l} \in \mathbb{D}^l/\mathfrak{S}_l$, we define the element of $\mathbb{D}^l/\mathfrak{S}_l$ as follows.

$$n_1 \oplus n_2 \oplus \cdots \oplus n_l := (n_1, n_2, \cdots, n_l)^{\mathfrak{S}_l}.$$

By defining the product and sum of the set $e^{\mathbb{D}}$ as follows, it becomes a commutative semiring $(e^{\mathbb{D}}, \times, +)$.

$$(n_1 \oplus \cdots \oplus n_l) \times (m_1 \oplus \cdots \oplus m_k) = n_1 m_1 \oplus n_1 m_2 \oplus \cdots \oplus n_l m_k \in \mathbb{D}^{lk}/\mathfrak{S}_{lk},$$

$$(n_1 \oplus \cdots \oplus n_l) + (m_1 \oplus \cdots \oplus m_k) = n_1 \oplus \cdots \oplus n_l \oplus m_1 \oplus \cdots \oplus m_k \in \mathbb{D}^{l+k}/\mathfrak{S}_{l+k},$$

for any $(n_1 \oplus \cdots \oplus n_l) \in \mathbb{D}^l/\mathfrak{S}_l$, $(m_1 \oplus \cdots \oplus m_k) \in \mathbb{D}^k/\mathfrak{S}_k$.

The identity element for the product $\times$ is $1 \in \hat{\mathbb{D}}$.

The identity element for sum $+$ is $0 \in \{\varnothing\}$.

The elementary Mother number space of $\hat{\mathbb{D}}$ is defined as follows.

$$e^{\hat{\mathbb{D}}} = \{\varnothing\} \coprod \hat{\mathbb{D}} \coprod \left(\hat{\mathbb{D}} \times \hat{\mathbb{D}}\right) / \mathfrak{S}_2 \coprod \left(\hat{\mathbb{D}} \times \hat{\mathbb{D}} \times \hat{\mathbb{D}}\right) / \mathfrak{S}_3 \coprod \cdots \coprod \hat{\mathbb{D}}^n / \mathfrak{S}_n \coprod \cdots.$$

For any unordered pair $(n_1, n_2, \cdots, n_l)^{\mathfrak{S}_l} \in \hat{\mathbb{D}}^l / \mathfrak{S}_l$, we define the element of $\hat{\mathbb{D}}^l / \mathfrak{S}_l$ as follows.

$$n_1 \oplus n_2 \oplus \cdots \oplus n_l := (n_1, n_2, \cdots, n_l)^{\mathfrak{S}_l}.$$

By defining the product, sum and difference of the set $e^{\hat{\mathbb{D}}}$ as follows, it becomes a commutative ring $\left(e^{\hat{\mathbb{D}}}, \times, +, -\right)$.

$$(n_1 \oplus \cdots \oplus n_l) \times (m_1 \oplus \cdots \oplus m_k) = n_1 m_1 \oplus n_1 m_2 \oplus \cdots \oplus n_l m_k \in \hat{\mathbb{D}}^{lk} / \mathfrak{S}_{lk},$$

$$(n_1 \oplus \cdots \oplus n_l) + (m_1 \oplus \cdots \oplus m_k) = n_1 \oplus \cdots \oplus n_l \oplus m_1 \oplus \cdots \oplus m_k \in \hat{\mathbb{D}}^{l+k} / \mathfrak{S}_{l+k},$$

$$(n_1 \oplus \cdots \oplus n_l) - (m_1 \oplus \cdots \oplus m_k) = n_1 \oplus \cdots \oplus n_l \ominus m_1 \ominus \cdots \ominus m_k \in \hat{\mathbb{D}}^{l+k} / \mathfrak{S}_{l+k},$$

for any $(n_1 \oplus \cdots \oplus n_l) \in \hat{\mathbb{D}}^l / \mathfrak{S}_l$, $(m_1 \oplus \cdots \oplus m_k) \in \hat{\mathbb{D}}^k / \mathfrak{S}_k$.

The identity element for the product $\times$ is $1 \in \mathbb{D}$.

The identity element for sum $+$ is $0 \in \{\varnothing\}$.

The inverse of $n_1 \oplus \cdots \oplus n_l$ with respect to the sum $+$ is $\ominus n_1 \ominus \cdots \ominus n_l$.

Each piece $\hat{\mathbb{D}}^l / \mathfrak{S}_l$ and $\mathbb{D}^l / \mathfrak{S}_l$ of $e^{\mathbb{D}}$ and $e^{\hat{\mathbb{D}}}$ is only a set with a new number $n_1 \oplus n_2 \oplus \cdots \oplus n_l$, but the set $e^{\mathbb{D}}$ of the entire union of each piece is a commutative semiring, and the set $e^{\hat{\mathbb{D}}}$ is a commutative ring. We also define as follows.

$$\hat{\mathbb{D}}^l / \mathfrak{S}_l \ni (1, 1, \cdots, 1)^{\mathfrak{S}_l} = 1 \oplus 1 \oplus \cdots \oplus 1 = 1 + 1 + \cdots + 1 = l$$

and

$$\hat{\mathbb{D}}^2 / \mathfrak{S}_2 \ni (1, -1)^{\mathfrak{S}_2} = 1 \oplus (-1) = 1 \ominus 1 = 1 - 1 = 0.$$

This generally turns out to be

$$\hat{\mathbb{D}}^l / \mathfrak{S}_l \ni \left(\underbrace{1, \cdots, 1}_{n}, \underbrace{-1, \cdots -1}_{m}\right)^{\mathfrak{S}_l} = 1 \oplus \cdots \oplus 1 \ominus 1 \cdots \ominus 1 = 1 + \cdots + 1 - 1 - \cdots - 1 = n - m.$$

Proposition 3.1. *$\mathbb{N} \subset \mathcal{N}$ is a subsemiring and $\mathbb{Z} \subset \mathcal{Z}$ is a subring.*

Proof. Let us consider that $\mathbb{N}$ is a subset of $\mathcal{N}$. If it can be shown that it is a subset, it is trivial that it is subsemiring. For any l, the special element $\mathbb{D}^l / \mathfrak{S}_l \ni (1, 1, \cdots, 1)^{\mathfrak{S}_l} = 1 \oplus 1 \oplus \cdots \oplus 1$ can be calculated and $1 \oplus 1 \oplus \cdots \oplus 1 = 1 + 1 + \cdots + 1 = l$, and l is in $\mathbb{N}$. Therefore, $\mathbb{N}$ is a subset of $\mathcal{N}$. The same applies to $\mathbb{Z} \subset \mathcal{Z}$. □

Below, we will discuss various examples of elementary Mother spaces other than those mentioned above.

For the monoid $\left(\mathbb{A} = \left\{\pm 1, \pm \mathbf{p}, \pm \mathbf{p}^2, \pm \mathbf{p}^3, \cdots\right\}, \times\right)$, it can be expressed as follows.

$$e^{\mathbb{A}} = \{\varnothing\} \coprod \mathbb{A} \coprod (\mathbb{A} \times \mathbb{A}) / \mathfrak{S}_2 \coprod (\mathbb{A} \times \mathbb{A} \times \mathbb{A}) / \mathfrak{S}_3 \coprod \cdots \coprod \mathbb{A}^n / \mathfrak{S}_n \coprod \cdots.$$

This is written as $(\mathbb{Z}[\mathbf{p}], \times, +, -)$.

For the monoid $\left(\mathbb{B} = \left\{\pm \mathbf{p}^{i_1} \times \mathbf{q}^{i_2} \mid i_1, i_2 \in \mathbb{N} \cup \{0\}, \mathbf{p} \neq \mathbf{q}\right\}, \times\right)$, it can be expressed as follows.

$$e^{\mathbb{B}} = \{\varnothing\} \coprod \mathbb{B} \coprod (\mathbb{B} \times \mathbb{B}) / \mathfrak{S}_2 \coprod (\mathbb{B} \times \mathbb{B} \times \mathbb{B}) / \mathfrak{S}_3 \coprod \cdots \coprod \mathbb{B}^n / \mathfrak{S}_n \coprod \cdots.$$

This is written as $(\mathbb{Z}[\mathbf{p},\mathbf{q}],\times,+,-)$.

As we will see in section 5, $\mathbb{Z}[\mathbf{p}]$ and $\mathbb{Z}[\mathbf{p},\mathbf{q}]$ are subrings of $e^{\hat{\mathbb{D}}}$.

The following eight examples are obtained by taking the mother number e^{M} of a known monoid M for a known commutative ring or commutative semiring.

For the monoid $(\mathbb{F}_1=\{1\},\times)$, it can be expressed as follows.

$$e^{\mathbb{F}_1}=\{\varnothing\}\coprod\mathbb{F}_1\coprod(\mathbb{F}_1\times\mathbb{F}_1)/\mathfrak{S}_2\coprod(\mathbb{F}_1\times\mathbb{F}_1\times\mathbb{F}_1)/\mathfrak{S}_3\coprod\cdots\coprod\mathbb{F}_1{}^n/\mathfrak{S}_n\coprod\cdots.$$

The following semiring isomorphism holds

$$(e^{\mathbb{F}_1},\times,+)\simeq(\mathbb{N}\cup\{0\},\times,+).$$

For the monoid $(\mathbb{F}_{1^2}=\{1,-1\},\times)$, it can be expressed as follows.

$$e^{\mathbb{F}_{1^2}}=\{\varnothing\}\coprod\mathbb{F}_{1^2}\coprod(\mathbb{F}_{1^2}\times\mathbb{F}_{1^2})/\mathfrak{S}_2\coprod(\mathbb{F}_{1^2}\times\mathbb{F}_{1^2}\times\mathbb{F}_{1^2})/\mathfrak{S}_3\coprod\cdots\coprod\mathbb{F}_{1^2}{}^n/\mathfrak{S}_n\coprod\cdots.$$

The following ring isomrphism holds

$$(e^{\mathbb{F}_{1^2}},\times,+,-)\simeq(\mathbb{Z},\times,+,-).$$

For the monoid $\left(\mathbb{F}_{1^3}=\left\{1,\frac{1}{2}\left(-1+\sqrt{-3}\right),\frac{1}{2}\left(-1-\sqrt{-3}\right)\right\},\times\right)$, it can be expressed as follows.

$$e^{\mathbb{F}_{1^3}}=\{\varnothing\}\coprod\mathbb{F}_{1^3}\coprod(\mathbb{F}_{1^3}\times\mathbb{F}_{1^3})/\mathfrak{S}_2\coprod(\mathbb{F}_{1^3}\times\mathbb{F}_{1^3}\times\mathbb{F}_{1^3})/\mathfrak{S}_3\coprod\cdots\coprod\mathbb{F}_{1^3}{}^n/\mathfrak{S}_n\coprod\cdots.$$

The following ring isomorphism holds

$$(e^{\mathbb{F}_{1^3}},\times,+,-)\simeq\left(\mathbb{Q}\left[\sqrt{-3}\right],\times,+,-\right).$$

For the monoid $(\mathbb{F}_{1^4}=\{1,-1,i,-i\},\times)$, it can be expressed as follows.

$$e^{\mathbb{F}_{1^4}}=\{\varnothing\}\coprod\mathbb{F}_{1^4}\coprod(\mathbb{F}_{1^4}\times\mathbb{F}_{1^4})/\mathfrak{S}_2\coprod(\mathbb{F}_{1^4}\times\mathbb{F}_{1^4}\times\mathbb{F}_{1^4})/\mathfrak{S}_3\coprod\cdots\coprod\mathbb{F}_{1^4}{}^n/\mathfrak{S}_n\coprod\cdots.$$

The following ring isomorphism holds

$$(e^{\mathbb{F}_{1^4}},\times,+,-)\simeq(\mathbb{Z}[i],\times,+,-).$$

Examples of the $e^{\mathbb{F}_1}$ and $e^{\mathbb{F}_{1^2}}$ show that $\mathbb{F}_1$ and $\mathbb{F}_{1^2}$ are hidden in the underground structure of $\mathbb{N}\cup\{0\}$ and $\mathbb{Z}$. This is an example of the importance of Kurokawa's philosophy of absolute mathematics (cf. [1]).

For the monoid $(\mathbb{Z}^*,\times):=(\mathbb{Z}\setminus\{0\},\times)$, it can be expressed as follows.

$$\begin{aligned}e^{\mathbb{Z}^*}&=\{\varnothing\}\coprod\mathbb{Z}^*\coprod(\mathbb{Z}^*\times\mathbb{Z}^*)/\mathfrak{S}_2\coprod(\mathbb{Z}^*\times\mathbb{Z}^*\times\mathbb{Z}^*)/\mathfrak{S}_3\coprod\cdots\coprod\mathbb{Z}^{*n}/\mathfrak{S}_n\coprod\cdots\\&\simeq\{\varnothing\}\coprod\mathbb{Z}^*\simeq\mathbb{Z}.\end{aligned}$$

The following ring isomorphism holds

$$(e^{\mathbb{Z}^*},\times,+,-)\simeq(\mathbb{Z},\times,+,-).$$

For the monoid $(\mathbb{N},\times)$, it can be expressed as follows.

$$\begin{aligned}e^{\mathbb{N}}&=\{\varnothing\}\coprod\mathbb{N}\coprod(\mathbb{N}\times\mathbb{N})/\mathfrak{S}_2\coprod(\mathbb{N}\times\mathbb{N}\times\mathbb{N})/\mathfrak{S}_3\coprod\cdots\coprod\mathbb{N}^n/\mathfrak{S}_n\coprod\cdots\\&\simeq\{\varnothing\}\coprod\mathbb{N}\simeq\{0\}\cup\mathbb{N}.\end{aligned}$$

The following semiring isomorphism holds

$$\left(e^{\mathbb{N}},\times,+\right)\simeq\left(\mathbb{N}\cup\{0\},\times,+\right).$$

Let us introduce yet another example.

A polynomial ring with one variable or infinite variables can also be constructed as an elementary Mother space as shown below.

For the monoid $\left(\mathbb{X}=\left\{\pm1,\pm X,\pm X^{2},\pm X^{3},\cdots\right\},\times\right)$ of indefinite element X, it can be expressed as follows.

$$e^{\mathbb{X}}=\{\varnothing\}\amalg\mathbb{X}\amalg(\mathbb{X}\times\mathbb{X})/\mathfrak{S}_2\amalg(\mathbb{X}\times\mathbb{X}\times\mathbb{X})/\mathfrak{S}_3\amalg\cdots\amalg\mathbb{X}^n/\mathfrak{S}_n\amalg\cdots.$$

The following ring isomorphism holds

$$\left(e^{\mathbb{X}},\times,+,-\right)\simeq\left(\mathbb{Z}[X],\times,+,-\right).$$

For the monoid $\left(\mathbb{Y}=\left\{\pm X_1^{i_1}X_2^{i_2}X_3^{i_3}\cdots \mid i_1,i_2,i_3,\cdots\in\mathbb{N}\cup\{0\}\right\},\times\right)$ with infinite number of indefinite elements $X_1,X_2,X_3,\cdots$, the following ring isomorphism holds.

$$\left(e^{\mathbb{Y}},\times,+,-\right)\simeq\left(\mathbb{Z}[X_1,X_2,X_3,\cdots],\times,+,-\right).$$

Furthermore, the elementary Mother spaces of groups $\mathbb{Q}\setminus\{0\}$, $\mathbb{R}\setminus\{0\}$ and $\mathbb{C}\setminus\{0\}$ with respect to the product are respectively $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ which are fields.

As we can see from these examples, it can be seen that the monoid before taking the elementary Mother space representation is an extremely basic object. By taking the elementary Mother space representation of the monoid, natural sums and differences can be added. In order to obtain a new number concept, we introduced new monoids $\mathbb{D}=\{1,\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\cdots\}$ and $\hat{\mathbb{D}}=\{\pm1,\pm\mathbf{2},\pm\mathbf{3},\pm\mathbf{4},\pm\mathbf{5},\cdots\}$.

Let us introduce another interesting example of elementary Mother space.

For monoid A, I is a monoid ideal if $a\in A$, $x\in I$ then $ax\in I$.

Monoid ideal $I\subsetneq A$ is a monoid prime ideal if $a,b\notin I$ then $ab\notin I$.

Kurokawa thought of the following [5].

$\operatorname{Spec}(\mathbb{Z},\times)$ is a set of monoid prime ideals of monoid $(\mathbb{Z},\times)$. Then the monoid prime ideals of $(\mathbb{Z},\times)$ are

$$\begin{aligned}&(2),(3),(5),\cdots,(2)\cup(3),(2)\cup(5),\cdots,(2)\cup(3)\cup(5),\cdots,\\&(2)\cup(3)\cup(5)\cup(7),\cdots,\text{etc.}\end{aligned}$$

and we know that $\dim\operatorname{Spec}(\mathbb{Z},\times)=\infty$.

Lemma 3.1. *For the set* $P=\{2,3,5,7,11,\cdots\}$ *of ordinary primes, elementary Mother space is*

$$e^{\mathbb{P}}=\{\varnothing\}\amalg\mathbb{P}\amalg(\mathbb{P}\times\mathbb{P})/\mathfrak{S}_2\amalg(\mathbb{P}\times\mathbb{P}\times\mathbb{P})/\mathfrak{S}_3\amalg\cdots\amalg\left(\mathbb{P}^n/\mathfrak{S}_n\right)\amalg\cdots.$$

Then we have

$$\left(e^{\mathbb{P}},+\right)\simeq\left(\operatorname{Spec}(\mathbb{Z},\times),+\right).$$

Proof. We identify the following.

$\mathbb{P}\ni p\simeq(p)$, $\mathbb{P}^2/S_2\ni(p,q)\simeq(p)\cup(q)$, in particular

$(p,p)\simeq(p)\cup(p)\simeq(p)\in\mathbb{P}$, $\mathbb{P}^3/S_3\ni(p,q,r)\simeq(p)\cup(q)\cup(r)$, in particular
$(p,p,r)\simeq(p)\cup(p)\cup(r)\simeq(p)\cup(r)\in\mathbb{P}^2/S_2$ and
$(p,p,p)\simeq(p)\cup(p)\cup(p)\simeq(p)\in P$. The same applies below.

We define the sum $+$ as follows.

$$\begin{aligned}(p_1,\cdots,p_l)+(q_1,\cdots,q_k)&=((p_1)\cup\cdots\cup(p_l))+((q_1)\cup\cdots\cup(q_k))\\&=(p_1)\cup\cdots\cup(p_l)\cup(q_1)\cup\cdots\cup(q_k).\end{aligned}$$

The identity element for the sum is (0). Therefore, we have
$(e^{\mathbb{P}},+)\simeq(\mathrm{Spec}(\mathbb{Z},\times),+)$. This is a Zariski topological monoid for the sum. □

On the other hand, $\mathrm{Spec}(\mathbb{Z},\times,+,-)$ is only a Zariski topological space and does not contain a monoid structure for the sum.

4. Rewriting Equivalent to Mother Number Semiring $e^{\mathbb{D}}$ and Total Order Structure

In this section we will give an expression equivalent to $(e^{\mathbb{D}},\times,+)$. It turns out that $(e^{\mathbb{D}},\times,+)$ is a polynomial semiring. However, it can be seen that total order structure is different.

Definition 4.1

$$\mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]=\left\{\sum_{\text{finite sum}}a_{i_1i_2i_3\cdots}\mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots\,\middle|\,i_1,i_2,i_3,\cdots\in\mathbb{N}\cup\{0\},a_{i_1i_2i_3\cdots}\in\mathbb{N}\right\},$$

where $a_{i_1i_2i_3\cdots}$ *except for a finite number is zero.*

From now on, $\sum_{\text{finite sum}}$ will be abbreviated as $\sum$.

For the set $\mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$, the product $\times$ and the sum $+$ are defined as follows.

$$\begin{aligned}&\left(\sum a_{i_1i_2i_3\cdots}\mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots\right)\times\left(\sum b_{j_1j_2j_3\cdots}\mathbf{2}^{j_1}\mathbf{3}^{j_2}\mathbf{5}^{j_3}\cdots\right)\\&=\sum\left(a_{i_1i_2i_3\cdots}\times b_{j_1j_2j_3\cdots}\right)\mathbf{2}^{i_1+j_1}\mathbf{3}^{i_2+j_2}\mathbf{5}^{i_3+j_3}\cdots.\end{aligned}$$

$$\begin{aligned}&\left(\sum a_{i_1i_2i_3\cdots}\mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots\right)+\left(\sum b_{i_1i_2i_3\cdots}\mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots\right)\\&=\sum\left(a_{i_1i_2i_3\cdots}\oplus b_{i_1i_2i_3\cdots}\right)\mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots.\end{aligned}$$

Hence $(\mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots],\times,+)$ becomes the commutative semiring.

Note that $a_{i_1i_2i_3\cdots}\oplus b_{i_1i_2i_3\cdots}$ can be calculated and
$a_{i_1i_2i_3\cdots}\oplus b_{i_1i_2i_3\cdots}=a_{i_1i_2i_3\cdots}+b_{i_1i_2i_3\cdots}=n$, and n is an ordinary number. On the other hand, for example, $\mathbf{3}\oplus\mathbf{5}$ is a single number and can not be calculated any further.

Lemma 4.1. *The following semiring isomorphism holds.*

$$(e^{\mathbb{D}},\times,+)\simeq(\mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots],\times,+).$$

Proof.

$$\begin{aligned}\text{Left side}&=n\oplus(n_1\oplus n_2)\oplus\cdots=\mathbf{p}_1^{j_1}\mathbf{p}_2^{j_2}\cdots\mathbf{p}_n^{j_n}\oplus\left(\mathbf{q}_1^{k_1}\mathbf{q}_2^{k_2}\cdots\mathbf{q}_m^{k_m}\oplus\mathbf{r}_1^{l_1}\mathbf{r}_2^{l_2}\cdots\mathbf{r}_u^{l_u}\right)\oplus\cdots\\&=\sum a_{i_1i_2i_3\cdots}\mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots=\text{Right side},\end{aligned}$$

where, $\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_n,\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_m,\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_u$ are dark primes and

$j_1 \ge 0, j_2 \ge 0, \cdots, j_n \ge 0$, $k_1 \ge 0, k_2 \ge 0, \cdots, k_m \ge 0$, $l_1 \ge 0, l_2 \ge 0, \cdots, l_u \ge 0$, $i_1 \ge 0, i_2 \ge 0, i_3 \ge 0, \cdots$. $\cdots$ is finite. □

We write this as $\mathcal{N} = (\mathcal{N}, \times, +)$.

Integer partitions of $\mathbb{N}$ is as follows.

$1,$
$2, 1+1,$
$3, 2+1, 1+1+1,$
$4, 3+1, 2+2, 2+1+1, 1+1+1+1,$
$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1,$
$\vdots$

$\mathcal{N}$ is the set in which all integer partitions of $\mathbb{N} \cup \{0\}$ are treated as different elements. Then, as shown above, it becomes commutative semiring due to the inclusion of natural products and sums.

If we rewrite this using our symbols, then it becomes the following.

$1,$
$\mathbf{2}, 2,$
$\mathbf{3}, \mathbf{2} \oplus 1, 3,$
$\mathbf{4}, \mathbf{3} \oplus 1, \mathbf{2} \oplus \mathbf{2}, \mathbf{2} \oplus 2, 4,$
$\mathbf{5}, \mathbf{4} \oplus 1, \mathbf{3} \oplus \mathbf{2}, \mathbf{3} \oplus 2, \mathbf{2} \oplus \mathbf{2} \oplus 1, \mathbf{2} \oplus 3, 5$
$\vdots$

The equivalence of (1), (2), (3) in the following Lemma 4.2 is a well-known fact (cf. [7]).

Lemma 4.2. *The following four are equivalent as a set.*

(1) *The partitions of positive integer* n.

(2) *Yang diagrams of* n.

(3) *Type of conjugate class of symmetry group* $\mathfrak{S}_n$.

(4) *The piece* $\mathbb{D}^n / \mathfrak{S}_n$ *of* $e^{\mathbb{D}} \simeq \mathbb{N}[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots]$.

From Lemma 4.2, we can see that $e^{\mathbb{D}}$ is the set of all n in (1), (2) or (3).

The set $\mathcal{N} := e^{\mathbb{D}} \simeq \mathbb{N}[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots]$ has the following total order structure.

$$\begin{aligned} &1 < 2 < \mathbf{2} < 3 < \mathbf{2} \oplus 1 < \mathbf{3} < 4 < \mathbf{2} \oplus 2 < \mathbf{2} \oplus \mathbf{2} < \mathbf{3} \oplus 1 < \mathbf{4} \\ &< 5 < \mathbf{2} \oplus 3 < \mathbf{2} \oplus \mathbf{2} \oplus 1 < \mathbf{3} \oplus 2 < \mathbf{3} \oplus \mathbf{2} < \mathbf{4} \oplus 1 < \mathbf{5} < \cdots. \end{aligned}$$

It can be seen that $(\mathcal{N}, \le, \times, +)$ is in the total order semiring. We write this as $\mathcal{N} = (\mathcal{N}, \le, \times, +)$.

5. Rewriting Equivalent to Mother Number Ring $e^{\hat{\mathbb{D}}}$ and Total Order Structure

In this section we will give an expression equivalent to $\left(e^{\hat{\mathbb{D}}}, \times, +, -\right)$. It turns out that $\left(e^{\hat{\mathbb{D}}}, \times, +, -\right)$ is a polynomial ring. However, it can be seen that total order structure is different.

From now on, $\sum_{\text{finite sum}}$ will be abbreviated as $\sum$.

Definition 7.1. *For the set*

$$\mathbb{N}[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots] = \left\{ \sum a_{i_1 i_2 i_3 \cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots \mid i_1, i_2, i_3, \cdots \in \mathbb{N} \cup \{0\}, a_{i_1 i_2 i_3 \cdots} \in \mathbb{N} \right\},$$

we extend $a_{i_1 i_2 i_3\cdots} \in N$ *to* $a_{i_1 i_2 i_3\cdots} \in Z$. *Therefore, it is defined as*

$$\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots] = \left\{\sum a_{i_1 i_2 i_3\cdots} 2^{i_1} 3^{i_2} 5^{i_3} \cdots \mid i_1, i_2, i_3, \cdots \in \mathbb{N} \cup \{0\}, a_{i_1 i_2 i_3\cdots} \in \mathbb{Z}\right\}.$$

For the set $\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$, *the product* $\times$, *the sum* $+$ *and the difference* $-$ *are defined as follows.*

$$\left(\sum a_{i_1 i_2 i_3\cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots\right) \times \left(\sum b_{j_1 j_2 j_3\cdots} \mathbf{2}^{j_1} \mathbf{3}^{j_2} \mathbf{5}^{j_3} \cdots\right)$$
$$= \sum \left(a_{i_1 i_2 i_3\cdots} \times b_{j_1 j_2 j_3\cdots}\right) \mathbf{2}^{i_1+j_1} \mathbf{3}^{i_2+j_2} \mathbf{5}^{i_3+j_3} \cdots.$$

$$\left(\sum a_{i_1 i_2 i_3\cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots\right) + \left(\sum b_{i_1 i_2 i_3\cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots\right)$$
$$= \sum \left(a_{i_1 i_2 i_3\cdots} \oplus b_{i_1 i_2 i_3\cdots}\right) \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots.$$

$$\left(\sum a_{i_1 i_2 i_3\cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots\right) - \left(\sum b_{i_1 i_2 i_3\cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots\right)$$
$$= \sum \left(a_{i_1 i_2 i_3\cdots} \ominus b_{i_1 i_2 i_3\cdots}\right) \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots.$$

Hence, $\left(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots],\times,+,-\right)$ becomes the commutative ring.

Note that $a_{i_1 i_2 i_3\cdots} \ominus b_{i_1 i_2 i_3\cdots}$ can be calculated and $a_{i_1 i_2 i_3\cdots} \ominus b_{i_1 i_2 i_3\cdots} = a_{i_1 i_2 i_3\cdots} - b_{i_1 i_2 i_3\cdots} = n$, and n is an ordinary number. On the other hand, for example, $\mathbf{3} \ominus \mathbf{2}$ is a single number and can not be calculated any further, just like $\mathbf{3} \oplus \mathbf{2}$.

Lemma 5.1. *The following ring isomorphism holds.*

$$\left(e^{\hat{\mathbb{D}}},\times,+,-\right) \simeq \left(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots],\times,+,-\right).$$

Proof. The proof is similar to Lemma 4.1. □

We write this as $\mathcal{Z} = (\mathcal{Z},\times,+,-)$.

We define the map

$$ev : \mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots] \ni \left(\sum a_{i_1 i_2 i_3\cdots} \mathbf{2}^{i_1} \mathbf{3}^{i_2} \mathbf{5}^{i_3} \cdots\right) \mapsto n \in \mathbb{Z}.$$

For example, $ev(\mathbf{2} \oplus \mathbf{3}) = 5$, $ev(\mathbf{2} \ominus \mathbf{5}) = -3$, $ev(\mathbf{2} \oplus 3) = 5$, $ev(2 \oplus 3) = 5$.

The map ev is called the *cardinal evaluation map.*

The set $\mathcal{Z} := e^{\hat{\mathbb{D}}} \simeq \mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$ has the following total order structure.

Definition 5.2. *The total order structure of* $\mathcal{Z} := e^{\hat{\mathbb{D}}} \simeq \mathbb{N}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$ *is defined as follows.*

Assuming that $A, B, C, D \in \mathbb{Z}$, $p_1^{i_1} \cdots p_n^{i_n} > p_1^{j_1} \cdots p_m^{j_m} > \cdots$ *and* $p_1^{k_1} \cdots p_l^{k_l} > p_1^{l_1} \cdots p_o^{l_o} > \cdots$ *are satisfied, the total order relation of*

$$\left(Ap_1^{i_1} \cdots p_n^{i_n}\right) \oplus \left(Bp_1^{j_1} \cdots p_m^{j_m}\right) \oplus \cdots < \left(Cp_1^{k_1} \cdots p_l^{k_l}\right) \oplus \left(Dp_1^{l_1} \cdots p_o^{l_o}\right) \oplus \cdots$$

is defined by the following procedure.

(1) ev (left side) < ev (right side).

(2) If it satisfies ev (left side) = ev (right side) $\cdots$ (I), then

$$p_1^{i_1} \cdots p_n^{i_n} < p_1^{k_1} \cdots p_l^{k_l}.$$

(3) If it satisfies (I) and $p_1^{i_1} \cdots p_n^{i_n} = p_1^{k_1} \cdots p_l^{k_l}$ $\cdots$ (II), then

$$A < C.$$

(4) If it satisfies (I), (II) and $A = C$ $\cdots$ (III), then

$p_1^{j_1} \cdots p_m^{j_m} < p_1^{l_1} \cdots p_o^{l_o}$.

Hereafter repeat in the same way.

The total order structure of $\mathbb{N} := e^{\mathbb{D}}$ mentioned earlier is included in Definition 5.2.

The quotient feild $\mathcal{Q} = \left(\mathcal{Z} \setminus \{0\}\right)\mathcal{Z}$ of $\mathcal{Z}$ is also a totally ordered set, same as $\mathcal{Q}$ is a totally ordered set.

Example 5.1.

$$\mathbf{10} \oplus 1 < \mathbf{11},\ \mathbf{9} < \mathbf{10} \ominus 1,\ 3 \times \mathbf{10} < 2 \times \mathbf{15},\ \ominus 3 \times \mathbf{10} > \ominus 2 \times \mathbf{15},$$
$$2 \times \mathbf{10} \oplus \mathbf{3} \times \mathbf{3} \oplus 1 < 3 \times \mathbf{10},\ 2 \times \mathbf{10} \oplus 2 < 2 \times \mathbf{10} \oplus \mathbf{2}.$$

Lemma 5.2. *The ordinal number of* $\mathcal{N}$ *is* ω, *and the ordinal number of* $\mathcal{Z}$ *is* ω^2.

Proof. The sequence of numbers in $\mathcal{Z}$ is as follows.

$$0 < \mathbf{2} \ominus 2 < \mathbf{3} \ominus \mathbf{2} \ominus 1 < \mathbf{3} \ominus 3 < \mathbf{4} \ominus \mathbf{3} \ominus 1 < \mathbf{4} \ominus \mathbf{2} \ominus \mathbf{2} < \mathbf{4} \ominus \mathbf{2} \ominus 2 < \mathbf{4} \ominus 4 < \cdots < 1$$
$$1 < \mathbf{2} \ominus 1 < \mathbf{3} \ominus \mathbf{2} < \mathbf{3} \ominus 2 < \mathbf{4} \ominus \mathbf{3} < \mathbf{4} \ominus \mathbf{2} \ominus 1 < \mathbf{4} \ominus 3 < \cdots < 2$$
$$2 < \mathbf{2} < \mathbf{3} \ominus 1 < \mathbf{4} \ominus \mathbf{2} < \mathbf{4} \ominus 2 < \mathbf{5} \ominus \mathbf{3} < \mathbf{5} \ominus \mathbf{2} \ominus 1 < \mathbf{5} \ominus 3 < \cdots < 3$$
$$3 < \mathbf{2} \oplus 1 < \mathbf{3} < \mathbf{4} \ominus 1 < \mathbf{5} \ominus \mathbf{2} < \mathbf{5} \ominus 2 < \mathbf{6} \ominus \mathbf{3} < \mathbf{6} \ominus \mathbf{2} \ominus 1 < \mathbf{6} \ominus 3 < \cdots < 4$$
$$4 < \mathbf{2} \oplus 2 < \mathbf{2} \oplus \mathbf{2} < \mathbf{3} \oplus 1 < \mathbf{3} \oplus \mathbf{2} \ominus 1 < \mathbf{4} < \mathbf{5} \ominus 1 < \cdots < 5$$
$$\cdots$$

Hence, the ordinal number of $\mathcal{Z}$ is ω^2. It is trival that the ordinal number of $\mathcal{N}$ is ω. □

We write this as $\mathcal{Z} = \left(\mathcal{Z}, \le, \times, +, -\right)$.

Proposition 5.1. *For* $\mathcal{Z} := \mathbb{Z}[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots] \simeq e^{\hat{D}}$, *the following isomorphism as a commutative ring holds.*

$$\left(\mathbb{Z}[X_1, X_2, X_3, \cdots], \times, +, -\right) \simeq \left(\mathbb{Z}[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots], \times, +, -\right).$$

Here, the element of $\mathbb{Z}[X_1, X_2, X_3, \cdots]$ *is a polynomial with respect to a finite number of variables among* $X_1, X_2, \cdots$, *but we can consider polynomials with as many variables as we like.*

Proof. $\mathbb{Z} \subset \mathcal{Z}$ is subring. The element of $\mathcal{Z}$ are commutative with all elements of $\mathbb{Z}$.

When we write $\mathbb{Z}[X_1, X_2, X_3, \cdots] \ni f\left(X_1, X_2, X_3, \cdots\right) = \sum a_{i_1 i_2 i_3 \cdots} X_1^{i_1} X_2^{i_2} X_3^{i_3} \cdots$, $f\left(\alpha_1, \alpha_2, \alpha_3, \cdots\right) = \sum a_{i_1 i_2 i_3, \cdots} \alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \cdots$ is called an element of $\mathcal{Z}$ obtained by substituting $\alpha_1, \alpha_2, \alpha_3, \cdots$ for $X_1, X_2, X_3, \cdots$.

The substitution map

$$\Phi : \mathbb{Z}[X_1, X_2, X_3, \cdots] \ni f\left(X_1, X_2, X_3, \cdots\right) \mapsto f\left(\alpha_1, \alpha_2, \alpha_3, \cdots\right) \in \mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \cdots]$$

is a surjective ring homomorphism, that is

$$\operatorname{Im}\Phi = \mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \cdots].$$

$\mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \cdots]$ is the smallest subring of $\mathcal{Z}$ that includes the elements of $\mathbb{Z}$ and $\alpha_1, \alpha_2, \alpha_3, \cdots$, and it can be written as follows.

$$\begin{aligned}\mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \cdots] &= \left\{f\left(\alpha_1, \alpha_2, \alpha_3, \cdots\right) \mid f\left(X_1, X_2, X_3, \cdots\right) \in Z[X_1, X_2, X_3, \cdots]\right\} \\ &= \left\{\sum a_{i_1 i_2 i_3 \cdots} \alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \cdots \mid a_{i_1 i_2 i_3 \cdots} \in \mathbb{Z}, i_1, i_2, i_3, \cdots \in \mathbb{N} \cup \{0\}\right\}.\end{aligned}$$

$$\begin{aligned}\operatorname{Ker}\Phi = \{ f(X_1,X_2,X_3,\cdots) \in Z[X_1,X_2,X_3,\cdots] \mid \Phi(f(X_1,X_2,X_3,\cdots)) \\ = f(\alpha_1,\alpha_2,\alpha_3,\cdots) = 0 \}.\end{aligned}$$

From the ring homomorphism theorem, we have

$$\mathbb{Z}[X_1,X_2,X_3,\cdots] / \operatorname{Ker}\Phi \simeq \mathbb{Z}[\alpha_1,\alpha_2,\alpha_3,\cdots].$$

When $\alpha_1 = \mathbf{2}, \alpha_2 = \mathbf{3}, \alpha_3 = \mathbf{5}, \cdots$, it becomes $f(\mathbf{2},\mathbf{3},\mathbf{5},\cdots) = \sum a_{i_1 i_2 \cdots i_n} \mathbf{2}^{i_1}\mathbf{3}^{i_2}\mathbf{5}^{i_3}\cdots$. It is clear that $\mathbf{2},\mathbf{3},\mathbf{5},\cdots$ that satisfy $f(\mathbf{2},\mathbf{3},\mathbf{5},\cdots) = 0$ do not exist. Hence, we have $\operatorname{Ker}\Phi = \{0\}$ and

$$\mathbb{Z}[X_1,X_2,X_3,\cdots] \simeq \mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots].$$

□

For example, the following can be considered $\mathbb{Z}[\mathbf{2}]$, $\mathbb{Z}[\mathbf{3}]$ and $\mathbb{Z}[\mathbf{5}]$ etc.. This is collectively written as $\mathbb{Z}[\mathbf{p}]$. Increasing the dark prime variable, $\mathbb{Z}[\mathbf{2},\mathbf{3}]$, $\mathbb{Z}[\mathbf{2},\mathbf{5}]$, $\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5}]$, $\mathbb{Z}[\mathbf{3},\mathbf{5},\mathbf{7}]$, $\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\mathbf{7}]$ etc. are also possible. These are all subrings of $\mathcal{Z} = \mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$.

Proposition 5.2. *The commutative ring* $\mathcal{Z} := \mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots] \simeq e^{\hat{D}}$ *and polynomial rings of infinite variables are not totally order isomorphic.*

$$(\mathbb{Z}[X_1,X_2,X_3,\cdots], \leq_{\text{lex}}, \times, +, -) \neq (\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots], \leq, \times, +, -).$$

Proof. $(\mathbb{Z}[X_1,X_2,X_3,\cdots], \leq_{\text{lex}})$ is usually considered in lexicographical order. On the other hand, $(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots], \leq)$ enters the total order structure according to Definition 5.2. Therefore, the two are different. Let us give you a counterexample. We consider the following mapping.

$$f : \mathbb{Z}[X,Y] \ni X^2Y, XY^2 \mapsto f(X^2Y) = \mathbf{2}^2 \times \mathbf{3}, f(XY^2) = \mathbf{2} \times \mathbf{3}^2 \in \mathbb{Z}[\mathbf{2},\mathbf{3}]$$

Although it is $X^2Y > XY^2$, it becomes $f(X^2Y) = \mathbf{2}^2 \times \mathbf{3} < f(XY^2) = \mathbf{2} \times \mathbf{3}^2$. Therefore, f is not totally order isomorphic. □

6. Fundamental Theorem of Number Theory and Division Theorem

From now on, we will often write symbols that equate $n \oplus m$ with $n + m$ and $n \ominus m$ with $n - m$.

For example, we write $\mathbf{2} \oplus \mathbf{3}$ as $\mathbf{2} + \mathbf{3}$ and $\mathbf{2} \ominus \mathbf{3}$ as $\mathbf{2} - \mathbf{3}$.

Lemma 6.1. $(\mathcal{N}, \times, +)$ *and* $(\mathcal{Z}, \times, +, -)$ *has an infinite number of dark prime elements.*

Proof. The proof is trivial. □

The following analogy to the fundamental theorem of number theory holds.

Theorem 6.1. $(\mathcal{Z}, \times, +, -)$, $(\mathcal{N}, \times, +)$ *can be factorized into prime factors, which are unique.*

Proof. $n \in \mathcal{Z}$, $n \neq 0$. If n is an irreducible element, leave it as is. If n is not irreducible, $n = a \times b$. If we repeat the above, $n = a \times b \times \cdots \times c$.

Let $n = a \times b \times \cdots \times c = d \times e \times \cdots \times f$. Since $a \mid n$, then $a \mid d \times e \times \cdots \times f$. Since a is also a prime element, a divides either $d, e, \cdots, f$. Even if a divides d, generality is not lost. Since the prime element is an irreducible element, a and

d are irreducible elements, so $a=d$. Therefore, dividing n by $a=d$, $b\times\cdots\times c=e\times\cdots\times f$. If we repeat this $a=d$, $b=e$, $\cdots$, $c=f$. In other words, uniqueness is established. Therefore $(\mathcal{Z},\times,+,-)$ is a unique factorization domain. $(\mathcal{N},\times,+)$ is also a unique factorization domain. □

Definition 6.1. *An integral domain* R *is called a Euclidean domain if it satisfies the following conditions. There exists a mapping* $d:R\setminus\{0\}\to\mathbb{N}=\{1,2,3,\cdots\}$, *for any* $a,b\in R$, $b>0$. *Then there exists* $q,r\in R$ *that satisfies the following.*

$$a=b\times q+r,\ r=0 \text{ or } d(r)<d(b). \tag{6.1}$$

Examples:

When $R=\mathbb{Z}$, if $d:=|a|$, then $\mathbb{Z}$ is a Euclidean domain.

When $R=k[X]$ with field k, if $d(f):=\deg f$, then $k[X]$ is a Euclidean domain.

$R=\mathbb{Z}[X]$ is not a principal ideal domain, so it is not a Euclidean domain.

$R=k[X_1,X_2,\cdots,X_n]$ and $R=\mathbb{Z}[X_1,X_2,\cdots,X_n]$ are also not Euclidean domains.

Lemma 6.2. $\mathcal{Z}$ *is not a Euclidean domain.*

Proof. There is an example in which mapping $|ev|:\mathcal{Z}\setminus\{0\}\ni a\mapsto|a|\in\mathbb{N}$ does not satisfy

$$a=b\times q+r,\ |ev(r)|<|ev(b)|$$

for $a,b\in\mathcal{Z}$, $b>0$. The counterexample is if $a=\mathbf{2}+1$, $b=\mathbf{2}-1$ then it can divide $(\mathbf{2}+1)=(\mathbf{2}-1)\times 2+(-\mathbf{2}+3)$, but $|ev((-\mathbf{2}+3))|=1, |ev((\mathbf{2}-1))|=1$. Hence $|ev((-\mathbf{2}+3))|=|ev((\mathbf{2}-1))|$. □

Example 6.1. *The following example can be considered for* $a=b\times q+r$, $r=0$ or $ev(r)<ev(b)$.

(1) $\mathbf{22}=\mathbf{5}\times\mathbf{4}+(\mathbf{22}-\mathbf{20})$, $ev(\mathbf{22}-\mathbf{20})<ev(\mathbf{5})$.
$\mathbf{22}=\mathbf{4}\times\mathbf{5}+(\mathbf{22}-\mathbf{20})$, $ev(\mathbf{22}-\mathbf{20})<ev(\mathbf{4})$.

(2) $\mathbf{6}=(\mathbf{3}-1)\times\mathbf{2}+\mathbf{2}$, $ev(\mathbf{2})=ev(\mathbf{3}-1)$.
$\mathbf{6}=\mathbf{2}\times\mathbf{3}+0$, $r=0$.

(3) $(\mathbf{4}-1)=\mathbf{2}\times 1+(\mathbf{4}-\mathbf{2}-1)$, $ev(\mathbf{4}-\mathbf{2}-1)<ev(\mathbf{2})$.
$(\mathbf{4}-1)=1\times 1+0$, $r=0$.

(4) $(\mathbf{3}+1)=(\mathbf{2}+1)\times 1+(\mathbf{3}-\mathbf{2})$, $ev(\mathbf{3}-\mathbf{2})<ev(\mathbf{2}+1)$.
$(\mathbf{3}+1)=1\times(\mathbf{3}+1)+0$, $r=0$.

(5) $2\times\mathbf{2}=(\mathbf{2}-1)\times 3+(-\mathbf{2}+3)$, $ev(-\mathbf{2}+3)=ev(\mathbf{2}-1)$.
$2\times\mathbf{2}=3\times 1+(2\times\mathbf{2}-3)$, $ev(2\times\mathbf{2}-3)<ev(3)$.

(6) $(\mathbf{2}+1)=(\mathbf{2}-1)\times 2+(-\mathbf{2}+3)$, $ev(-\mathbf{2}+3)=ev(\mathbf{2}-1)$.
$(\mathbf{2}+1)=2\times 1+(\mathbf{2}-1)$, $ev(\mathbf{2}-1)<ev(2)$.

Although $\mathcal{Z}$ is not a Euclidean domain, the following division theorem holds.

Theorem 6.2. *For any* $a,b\in\mathcal{Z}$, $b>0$, *there exist* $q,r\in\mathcal{Z}$ *that satisfy*

$$a=b\times q+r,\ r=0 \text{ or } 0<r<b. \tag{6.2}$$

The uniqueness does not hold. The division theorem does not hold for $\mathcal{N}$.

Proof. $a\geq 0$: We will show by induction on a. If $a=0$, we set $q=r=0$.

When $a>0$, regarding $a_1<a$ less than a, we assume that q_1, r_1 exists that satisfies

$$a_1 = b\times q_1 + r_1,\ 0\le r_1<b. \tag{6.3}$$

If $a<b$, then $q=0, r=a$. If $a\ge b$, then $0\le a-b<a$. Therefore, by assumption (6.3), there exists q_1, r_1 such that $a-b=b\times q_1+r_1$, $0\le r_1<b$. If we rewrite this, then $a=b\times(q_1+1)+r_1$. Hence, if we set $q=q_1+1$, $r=r_1$, (6.2) holds.

$a<0$:

since $-a>0$, as shown above, there exist q_2, r_2 such that

$$-a=b\times q_2+r_2,\ 0\le r_2<b.$$

At this time, if $r_2=0$, then $a=b\times(-q_2)$ and $q=-q_2$, $r=0$. If $r_2>0$, then

$$a=-b\times q_2-r_2=b\times(-q_2-1)+(b-r_2),\ 0<b-r_2<b.$$

Therefore, it should be $q=-q_2-1$, $r=b-r_2$.

We give a counterexample in which uniqueness does not hold.

$$\frac{\mathbf{5}}{\mathbf{2}}=\mathbf{2}+\frac{\mathbf{5}-\mathbf{4}}{\mathbf{2}},\ 0<\mathbf{5}-\mathbf{4}<\mathbf{2},\quad \frac{\mathbf{5}}{\mathbf{2}}=2+\frac{\mathbf{5}-2\times\mathbf{2}}{\mathbf{2}},\ 0<\mathbf{5}-2\times\mathbf{2}<\mathbf{2}$$

As we can see from this counterexample, the division theorem does not hold for $\mathcal{N}$. To satisfy this theorem, we must use the difference $-$. But there is no difference $-$ in $\mathcal{N}$. □

Lemma 6.3. $(\mathcal{Z},\times,+,-)$ *is not Euclidean algorithm.*

Proof. The definition of the Euclidean algorithm is that $(a,b)=(b,r)$ holds for $a=b\times q+r$, $0\le r<b$. When this is repeated, the remainder at the end becomes zero. However, this $\mathcal{Z}$ does not become $r=0$ at the end.

For example, we consider the case when $a=\mathbf{3}$ and $b=\mathbf{2}$.

$\mathbf{3}=\mathbf{2}\times 1+(\mathbf{3}-\mathbf{2})$, $0\le(\mathbf{3}-\mathbf{2})<\mathbf{2}$.

$\mathbf{2}=(\mathbf{3}-\mathbf{2})\times 1+(2\times\mathbf{2}-\mathbf{3})$, $0\le(2\times\mathbf{2}-\mathbf{3})<(\mathbf{3}-\mathbf{2})$.

$\mathbf{3}-\mathbf{2}=(2\times\mathbf{2}-\mathbf{3})\times 1+(2\times\mathbf{3}-3\times\mathbf{2})$, $0\le(2\times\mathbf{3}-3\times\mathbf{2})<(2\times\mathbf{2}-\mathbf{3})$.

It stops here. If we calculate further than it becomes $r<0$. □

Remark 6.1. (1) *For any* $\mathbb{Z}[X]\ni f(X)$, *if* $g(X)$ *is monic, then it satisfy*

$$f(X)=g(X)q(X)+r(X),\ r(X)=0 \text{ or } \deg r(X)<\deg g(X).$$

For any $\mathbb{Z}[\mathbf{p}]\ni f(\mathbf{p})$, *even if* $g(\mathbf{p})$ *is monic, then it does not satisfy*

$$f(\mathbf{p})=g(\mathbf{p})q(\mathbf{p})+r(\mathbf{p}),\ r(\mathbf{p})=0 \text{ or } 0<\mathrm{ev}(r(\mathbf{p}))<\mathrm{ev}(g(\mathbf{p})).$$

For example, the first equation of Example 6.1 (6) *is a counterexample for* $\mathbb{Z}[\mathbf{2}]$.

(2) *The essential difference between* $\mathbb{Z}[X,Y]$ *and* $\mathbb{Z}[\mathbf{p},\mathbf{q}]$ *is that they are not totally order isomorphic according to Proposition* 5.2. *The ordered structure of* X *and* Y *is lexicographical order* $X>Y$, *and* $\mathbf{2}$ *and* $\mathbf{3}$ *have an ordered structure such as* $\mathbf{2}<\mathbf{3}$. *In other words, the structure of division is different. Also,*

for $\mathbb{Z}[X,Y]$, *the sum* $X+Y$, *the difference* $X-Y$, *the product* XY *and the quotients* $\frac{X}{Y}$ *and* $\frac{Y}{X}$ *can all be computed no further. On the other hand, for* $\mathbb{Z}[\mathbf{2},\mathbf{3}]$, *the sum* $\mathbf{2}+\mathbf{3}=\mathbf{2}\oplus\mathbf{3}$, *the difference* $\mathbf{2}-\mathbf{3}=\mathbf{2}\ominus\mathbf{3}$, *the product* $\mathbf{2}\times\mathbf{3}=\mathbf{6}$ *and the quotient* $\frac{\mathbf{3}}{\mathbf{2}}=1+\frac{\mathbf{3}-\mathbf{2}}{\mathbf{2}}$, $0<\mathbf{3}-\mathbf{2}<\mathbf{2}$ *can all be calculated.*

While X *and* Y *are formal indeterminate elements without individuality,* $\mathbf{2}$ *and* $\mathbf{3}$ *are dark primes with individuality. Therefore, these two essentially different.*

(3) *For* $\mathbb{Z}[\mathbf{2},\mathbf{3}]$, (0), $(\mathbf{2})$, $(\mathbf{3})$ *are principal ideals.* $(\mathbf{2},\mathbf{3})$, $(p,\mathbf{2})$, $(p,\mathbf{3})$ *are prime ideals that are not principal ideals.* $(p,\mathbf{2},\mathbf{3})$ *is a maximal ideal. Therefore,* $\mathbb{Z}[\mathbf{2},\mathbf{3}]$ *is not a principal ideal domain. This gives another proof of Lemma* 6.2.

Problem.

Determine the structure of affine scheme $\left(\operatorname{Spec}\mathcal{Z},O_{\operatorname{Spec}\mathcal{Z}}\right)$.

7. Derivation of Points and Derivation of Functions

The following definition of a Pythagoras field is well known.

Definition 7.1. *Let* k *be a field. For any* $\alpha\in k$, *the extension field* $k\left(\sqrt{1+\alpha^2}\right)$ *is called Pythagoras extension of* k. *When all Pythagoras extensions of* k *coincide with* k, k *is called a Pythagoras field.*

k is a Pythagoras field if the Pythagoras theorem always holds for any element in k.

We define the quotient field (localization) of $\mathcal{Z}$ as $\mathcal{Q}:=\left(\mathcal{Z}\setminus\{0\}\right)^{-1}\mathcal{Z}$. In general, a completeness uniquely exists in the metric space (X,d). $d_\infty:\mathcal{Q}\times\mathcal{Q}\to\mathcal{Q}$ is defined by the Archimedean metric $d_\infty(a,b)=|a-b|_\infty$. Let $\mathcal{R}$ be a completion of $\mathcal{Q}$ with respect to d_∞. $\mathcal{R}$ is the set of Mother real number.

$$\mathcal{R}=\{\text{Mother algebraic numbers}\}\coprod\{\text{Mother transcendental numbers}\}$$

corresponds to

$$\mathbb{R}=\{\text{algebraic numbers}\}\coprod\{\text{transcendental numbers}\}.$$

This algebraic number and Mother algebraic number do not include $\sqrt{-1}$. We put

$$K:=\{\text{Mother algebraic numbers}\}.$$

K is an example of a Mother Pythagoras field. From now on, we will focus on this K.

The following Definition 7.2 is well known. We will omit the details (cf. [8]-[11]).

Definition 7.2. *Let* R *be a commutative ring.* A *is* R*-algebra and* M *is* A*-module. Then the* R*-derivation* D *is a map* $D:A\to M$ *and this derivation satisfies the following three conditions:*

(1) $D(f\cdot g)=D(f)\cdot g+f\cdot D(g)$, $f,g\in A$: Leibniz rule.

(2) $D(f+g)=D(f)+D(g)$, $f,g\in A$: Linearity.

(3) $D(r\cdot 1_A)=0$, $r\in R$.

We denote the set of R *-derivation as* $\mathrm{Der}_R(A,M)$. *When* $A=M$, *we write* $\mathrm{Der}_R(A)$.

The following is a well-known typical example.

Example 7.1. $A=R[X_1,X_2,\cdots,X_n]$, $\frac{\partial}{\partial X_j}\in\mathrm{Der}_R(A,A)=\mathrm{Der}_R(A)$.

The property corresponding to (3) *is*

$$\frac{\partial}{\partial X_j}(R)=\{0\}.$$

In general, it is important that X_i *is unrelated to the elements of* R.

First, we consider the case when $R=\mathbb{Z}$, $A=M=\mathcal{Z}$.

Definition 7.3. *Let* $\mathbf{p}$ *be a dark prime in* $\mathcal{Z}=\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$.

(I) *We can define the set of derivation as*

$$\mathrm{Der}_Z(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]):=\widehat{\bigoplus_{\mathbf{p}:\text{dark prime}}}\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]\frac{\partial}{\partial\mathbf{p}}=\left\{\sum_{\mathbf{p}}X^{\mathbf{p}}\frac{\partial}{\partial\mathbf{p}}\mid X^{\mathbf{p}}\in\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]\right\}.$$

Note that since $\frac{\partial}{\partial\mathbf{p}}(f)=0$ *except for a finite number of* $\mathbf{p}$ *for each* $f\in\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$. $\sum_{\mathbf{p}}X^{\mathbf{p}}\frac{\partial}{\partial\mathbf{p}}(f)$ *is a finite sum.* $\widehat{\bigoplus}$ *is the finite direct sum.*

(II) *Furthermore, the dual space of* $\mathrm{Der}_{\mathbb{Z}}(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots])$ *is defined as follows:*

$$\mathrm{Der}_Z^*(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]):=\mathrm{Hom}_{\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]}\left(\mathrm{Der}_{\mathbb{Z}}(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]),\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]\right)$$

and

$$\mathrm{Der}_Z^*(\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]):=\widehat{\bigoplus_{\mathbf{p}:\text{dark prime}}}\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]d\mathbf{p}=\left\{\sum_{\mathbf{p}}\omega_p d\mathbf{p}\mid\omega_{\mathbf{p}}\in\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]\right\}.$$

Combining Definition 7.2 and Definition 7.3, we can see the following.

Differential with a dark prime $\mathbf{p}$ on $\mathcal{Z}=\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$ is defined as follows.

$$\frac{\partial}{\partial\mathbf{p}}:\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]\to\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots],$$

$$\frac{\partial}{\partial p}\left(\sum a_{i_1i_2\cdots}\mathbf{2}^{i_1}\mathbf{3}^{i_2}\cdots\mathbf{p}^{i}\cdots\right)=\sum\left(a_{i_1i_2\cdots}i\mathbf{2}^{i_1}\mathbf{3}^{i_2}\cdots\mathbf{p}^{i-1}\cdots\right).$$

For any $f,g\in\mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$, we have

$$\frac{\partial}{\partial\mathbf{p}_j}(fg)=\frac{\partial}{\partial\mathbf{p}_j}(f)g+f\frac{\partial}{\partial\mathbf{p}_j}(g),$$

$$\frac{\partial}{\partial\mathbf{p}_j}(f+g)=\frac{\partial}{\partial\mathbf{p}_j}(f)+\frac{\partial}{\partial\mathbf{p}_j}(g),$$

$$\frac{\partial}{\partial\mathbf{p}_j}(\mathbb{Z})=\{0\}.$$

These are indicating that the Leibniz rule is satisfied and that the operation is linear. This is the derivation of the elementary Mother number ring $\mathcal{Z} \simeq \mathbb{Z}[\mathbf{2},\mathbf{3},\mathbf{5},\cdots]$.

Remark 7.1. *We will answer the question of whether* $2,3,5,\cdots$ *are sufficient without introducing dark numbers such as* $\mathbf{2},\mathbf{3},\mathbf{5},\cdots$. *For example, let* $2\oplus 2$, *which is* $2\oplus 2=4$, *be an ordinary number. Using linearity, the left side becomes*

$$\frac{\partial}{\partial 2}(2\oplus 2)=\frac{\partial}{\partial 2}(2+2)=\frac{\partial}{\partial 2}(2)+\frac{\partial}{\partial 2}(2)=1+1=2.$$

On the other hand, the right side becomes

$$\frac{\partial}{\partial 2}(2\oplus 2)=\frac{\partial}{\partial 2}(2+2)=\frac{\partial}{\partial 2}(4)=\frac{\partial}{\partial 2}\left(2^2\right)=2\times 2=4.$$

So this is a contradiction.

The differential that satisfies only the Leibniz rule and does not require linearity is called Kurokawa's absolute differential [1]-[3]. *It is known that with absolute differentiation, the elements of* Z *can be differentiated by ordinary primes.*

The following definition is the most general differentiation of points.

Definition 7.4. Differentiation of points: *Let* ϖ *be a general point in* $(K\setminus k)\cup\{0\}$.

$$\operatorname{Der}_k(K):=\hat{\bigoplus_{\varpi:\text{general point}}} K\frac{\partial}{\partial\varpi},\ \varpi\in K\setminus k.$$

In particular, we add the following to the definition:

$$\textit{For } \varpi=0\textit{, we define } \frac{\partial}{\partial 0}(a)=0 \quad \textit{for all } a\in K.$$

It satisfies Definition 7.2 (1) and (2), but it satisfies Definition 7.2 (3) for $R=k$, $A=M=K$.

Example 7.2. *For an element* $\varpi(\mathbf{2},\mathbf{3}):=2\times\mathbf{2}^3+\mathbf{2}^2\times\mathbf{3}=2\times\mathbf{2}^3+\mathbf{2}\times(\mathbf{2}\times\mathbf{3})=\mathbf{2}^2\times(2\times\mathbf{2}+\mathbf{3})\in\mathbb{Z}[\mathbf{2},\mathbf{3}]$, we calculate as follows.

$\cdot\ \dfrac{\partial}{\partial\mathbf{2}}(\varpi(\mathbf{2},\mathbf{3}))=6\times\mathbf{2}^2+2\times\mathbf{2}\times\mathbf{3}$, $\cdot\ \dfrac{\partial}{\partial\mathbf{3}}(\varpi(\mathbf{2},\mathbf{3}))=\mathbf{2}^2$, $\cdot\ \dfrac{\partial}{\partial\mathbf{2}^2}(\varpi(\mathbf{2},\mathbf{3}))=2\times\mathbf{2}+\mathbf{3}$,

$\cdot\ \dfrac{\partial}{\partial\mathbf{2}^3}(\varpi(\mathbf{2},\mathbf{3}))=2$, $\cdot\ \dfrac{\partial}{\partial(\mathbf{2}\times\mathbf{3})}(\varpi(\mathbf{2},\mathbf{3}))=\dfrac{\partial}{\partial\mathbf{6}}(\varpi(\mathbf{2},\mathbf{3}))=\mathbf{2}$,

$\cdot\ \dfrac{\partial}{\partial\left(\mathbf{2}^2\times\mathbf{3}\right)}(\varpi(\mathbf{2},\mathbf{3}))=\dfrac{\partial}{\partial\mathbf{12}}(\varpi(\mathbf{2},\mathbf{3}))=1$, $\cdot\ \dfrac{\partial}{\partial\,(2\times\mathbf{2}+\mathbf{3})}(\varpi(\mathbf{2},\mathbf{3}))=\mathbf{2}^2$,

$\cdot\ \dfrac{\partial}{\partial\left(\dfrac{1}{\mathbf{2}}\right)}(\varpi(\mathbf{2},\mathbf{3}))=\dfrac{\partial}{\partial\left(\mathbf{2}^{-1}\right)}\left(2\times\left(\mathbf{2}^{-1}\right)^{-3}+\left(\mathbf{2}^{-1}\right)^{-2}\times\mathbf{3}\right)=-6\times\left(\mathbf{2}^{-1}\right)^{-4}-2\times\left(\mathbf{2}^{-1}\right)^{-3}\times\mathbf{3}$.

Let us explain the differentiation of *composite points*.

Definition 7.5. *A composite of* $y=\varrho(\varpi_1,\cdots,\varpi_m)$, $\varpi_j=\tau_j(\mathbf{p}_1,\cdots,\mathbf{p}_n)\in\mathcal{Q}$ *is* $y=\varrho(\tau_1(\mathbf{p}_1,\cdots,\mathbf{p}_n),\cdots,\tau_m(\mathbf{p}_1,\cdots,\mathbf{p}_n))\in\mathcal{Q}$. *Then we have*

$$\frac{\partial y}{\partial \mathbf{p}_i}=\sum_{j=1}^{m}\frac{\partial y}{\partial \varpi_j}\frac{\partial \varpi_j}{\partial \mathbf{p}_i},\ (i=1,2,\cdots,n).$$

Example 7.3. *Two-variable composition of two variables.*

For an element $\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))=(\mathbf{2}+\mathbf{3})^3+(\mathbf{3}+\mathbf{4})^2$ *in* $\mathbb{Z}[\mathbf{2},\mathbf{3}]$, *we calculate as follows.*

- $$\frac{\partial}{\partial(\mathbf{2}+\mathbf{3})}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)=3\times(\mathbf{2}+\mathbf{3})^2,$$
- $$\frac{\partial}{\partial(\mathbf{3}+\mathbf{4})}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)=2\times(\mathbf{3}+\mathbf{4}),$$
- $$\begin{aligned}\frac{\partial}{\partial\mathbf{2}}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)&=\frac{\partial}{\partial(\mathbf{2}+\mathbf{3})}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)\frac{\partial(\mathbf{2}+\mathbf{3})}{\partial\mathbf{2}}\\&\quad+\frac{\partial}{\partial(3+4)}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)\frac{\partial(3+4)}{\partial 2},\\&=3\times(\mathbf{2}+\mathbf{3})^2\times1+2\times(\mathbf{3}+\mathbf{4})\times2\times\mathbf{2}\\&=3\times(\mathbf{2}+\mathbf{3})^2+4\times(\mathbf{3}+\mathbf{4})\times\mathbf{2}\end{aligned}$$
- $$\begin{aligned}\frac{\partial}{\partial\mathbf{3}}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)&=\frac{\partial}{\partial(\mathbf{2}+\mathbf{3})}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)\frac{\partial(\mathbf{2}+\mathbf{3})}{\partial\mathbf{3}}\\&\quad+\frac{\partial}{\partial(\mathbf{3}+\mathbf{4})}\left(\varrho(\tau_1(\mathbf{2},\mathbf{3}),\tau_2(\mathbf{2},\mathbf{3}))\right)\frac{\partial(\mathbf{3}+\mathbf{4})}{\partial\mathbf{3}},\\&=3\times(\mathbf{2}+\mathbf{3})^2\times1+2\times(\mathbf{3}+\mathbf{4})\times1\\&=3\times(\mathbf{2}+\mathbf{3})^2+2\times(\mathbf{3}+\mathbf{4})\end{aligned}$$

We can define the differential formula for the quotient element for $\xi,\varrho\in K$, $\varpi\in K\setminus k$.

$$\frac{\partial}{\partial\varpi}\left(\frac{\varrho}{\xi}\right)=\frac{\varrho'\xi-\varrho\xi'}{\xi^2},\quad ':=\frac{\partial}{\partial\varpi}.$$

Definition 7.6. Differentiation of polynomial functions:

Let $K[X_1,X_2,\cdots,X_n]$ *be a polynomial ring with coefficients in the Mother Pythagoras field* K.

$$\mathrm{Der}_k\left(K[X_1,X_2,\cdots,X_n]\right):=\bigoplus_{i=1,2,\cdots,n}K[X_1,X_2,\cdots,X_n]\frac{\partial}{\partial X_i},\ X_i\in K\setminus k.$$

The elements X_i $(i=1,2,\cdots,n)$ *are related to the elements of* K, *that is* $X_i\in K$ $(i=1,2,\cdots,n)$. *Then we define that*

if $X_i\in k$, *then* $\frac{\partial}{\partial X_i}$ *is not well-defined and if* $X_i\in K\setminus k$, *then* $\frac{\partial}{\partial X_i}$ *is well-defined, that is*

if $\frac{\partial}{\partial X_i}$ *is well-defined then* $\frac{\partial}{\partial X_i}(K)$ *is well-defined.*

Although it does not satisfy Definition 7.2 (3), but it does satisfy Definition 7.2 (1) and (2).

Example 7.4. *When* $K = \mathcal{Z}$, *for* $f(X) = \mathbf{3}X^2 + \mathbf{2}X \in \mathcal{Z}[X]$, $X \in K \setminus k$, *we calculate as follows.*

For $X \neq \mathbf{2}, \mathbf{3}$, *then* $\dfrac{\partial f(X)}{\partial X} = 2 \times \mathbf{3} \times X + \mathbf{2}$ *and*

for $\alpha \in K \setminus k$ *and* $\alpha \neq \mathbf{2}, \mathbf{3}$, *then* $\left.\dfrac{\partial f(X)}{\partial X}\right|_{X=\alpha} = 2 \times \mathbf{3} \times \alpha + \mathbf{2}$.

For $X = 2$, *then* $\dfrac{\partial f(\mathbf{2})}{\partial \mathbf{2}} = \dfrac{\partial}{\partial 2}\left(\mathbf{3} \times \mathbf{2}^2 + \mathbf{2}^2\right) = 2 \times \mathbf{3} \times \mathbf{2} + 2 \times \mathbf{2}$.

For $X = \mathbf{3}$, *then* $\dfrac{\partial f(\mathbf{3})}{\partial \mathbf{3}} = \dfrac{\partial}{\partial \mathbf{3}}\left(\mathbf{3} \times \mathbf{3}^2 + \mathbf{2} \times \mathbf{3}\right) = 3 \times \mathbf{3}^2 + \mathbf{2}$.

We derive the differentiation of function on K^n from Definition 7.6 and its derivatives and differential coefficients.

Let K^n be the Mother Pythagoras field and $(x_1, x_2, \cdots, x_n)$ be a coordinate function. We denote all functions

$$f : K^n \ni (x_1, x_2, \cdots, x_n) \mapsto f(x_1, x_2, \cdots, x_n) \in K$$

as $C(K^n, K)$.

Definition 7.7. Differentiation of functions:

For any $f(x_1, x_2, \cdots, x_n) \in C(K^n, K)$, *we define that*

if $x_i \in k$, *then* $\dfrac{\partial}{\partial x_i}$ *is not well-defined and if* $x_i \in K \setminus k$, *then* $\dfrac{\partial}{\partial x_i}$ *is well-defined. We also define that*

(1) *When* x_i *is not equal to the coefficient* $\alpha_i \in K \setminus k$ *of* $f(x_1, x_2, \cdots, x_n)$, *we define*

$$\frac{\partial}{\partial x_i} f(x_1, x_2, \cdots, x_n) \text{ is derivative.}$$

(2) *When* x_i *is not equal to the coefficient* $\alpha_i \in K \setminus k$ *of* $f(x_1, x_2, \cdots, x_n)$, *we define*

$$\left.\frac{\partial}{\partial x_i} f(x_1, x_2, \cdots, x_n)\right|_{(x_1, x_2, \cdots, x_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)} = \frac{\partial}{\partial \alpha_i} f(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

is differential coefficient, *where* $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in K \setminus k$.

In particular, when the function is continuous at $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$, *we define the following independently of point differentiation*:

$$\left.\frac{\partial}{\partial x_i} f(x_1, x_2, \cdots, x_n)\right|_{(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)}.$$

(3) *When* x_i *is equal to the coefficient* $\alpha_i \in K \setminus k$ *of* $f(x_1, x_2, \cdots, x_n)$, *we define*

$$\frac{\partial}{\partial \alpha_i} f(\alpha_1, \alpha_2, \cdots, \alpha_n) \text{ is differential coefficient.}$$

The differentiation can not be well-defined at $k^n \subset K^n$. Therefore, we denote *partially differentiable functional ring* as $C^\infty(K^n, K)$ (cf. Definition 8.2).

Very important:

Although we have defined two differentiations which are point differentiation (Definition 7.4) and function differentiation (Definition 7.7), the point differentiation takes precedence. In other words, function differentiation can be considered only after the point differentiation has been defined.

For example, when $f(x) = \mathbf{2} \times x^3$, let $\frac{\partial f(x)}{\partial x} = 3 \times \mathbf{2} \times x^2$. The differential coefficient at $x = \mathbf{2}$ appears in two forms: $\left.\frac{\partial f(x)}{\partial x}\right|_{x=\mathbf{2}} = 3 \times 2^3$ and $\frac{\partial f(\mathbf{2})}{\partial \mathbf{2}} = 4 \times \mathbf{2}^3$. By prioritizing the latter, we achieve internal consistency in the differential calculation. Furthermore, when dealing with discrete objects, point differentiation should be prioritized. See also **Appendix B**.

Let us look at the following example.

Example 7.5. *For a function* $f(x) = \mathbf{2}x^3 + 3x^2 + 4x \in C^\infty(\mathcal{Z}, \mathcal{Z})$, *its derivative and differential coefficient are as follows.*

(1) When $x \neq \mathbf{2}$, $\frac{\partial f(x)}{\partial x} = 3 \times \mathbf{2} \times x^2 + 6 \times x + 4$ is derivative.

(2) When $x = \alpha \neq \mathbf{2}$, $\left.\frac{\partial f(x)}{\partial x}\right|_{x=\alpha} = \frac{\partial f(\alpha)}{\partial \alpha} = 3 \times \mathbf{2} \times \alpha^2 + 6 \times \alpha + 4$ is differential coefficient.

In this case, we define that the following also holds:

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = 4$$

(3) When $x = \mathbf{2}$, $\frac{\partial f(\mathbf{2})}{\partial \mathbf{2}} = \frac{\partial}{\partial \mathbf{2}}\left(\mathbf{2}^4 + 3 \times \mathbf{2}^2 + 4 \times \mathbf{2}\right) = 4 \times \mathbf{2}^3 + 6 \times \mathbf{2} + 4$ is differential coefficient.

$\left.\frac{\partial f(x)}{\partial x}\right|_{x=\mathbf{2}}$ *is not considered. Differentiation at* $x = \mathbf{2}$ *uses point differentiation, that is,* $\frac{\partial f(\mathbf{2})}{\partial \mathbf{2}}$ *is adopted.*

Point differentiations take precedence over functional differentiations, so the following must be satisfied.

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=n} \quad \text{not well-defined}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

8. Partially Differentiable Riemann Manifold

Let k and K be the Pythagoras field and the Mother Pythagoras field. Let k^n be the Cartesian product of n elements of k, and K^n be the Cartesian product of n elements of K. k^n and K^n are n-dimensional with respect to the Zariski topology. However, in this paper, we consider the metric topology. We define the metric as follows:

$$d_{K^n} : K^n \times K^n \to K \subset \mathcal{R}$$

$$d_{K^n}(\mathbf{x},\mathbf{y})=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2},\ \mathbf{x},\mathbf{y}\in K^n$$

$\left(K^n,d_{K^n}\right)$ is a metric topological space. Continuous mappings and homeomorphisms for $f:\left(K^n,d_{K^n}\right)\to\left(K^m,d_{K^m}\right)$ can be defined in the same way as in the classical case. Topology of $\left(K^n,d_{K^n}\right)$ can be understood as Hausdorff but non-connected. As a topological space, $\left(K^n,d_{K^n}\right)$ has dimension zero. However, as a vector space, it has dimension n. Focusing on this, when defining the following manifold, we consider that the dimension of the tangent vector space = the number of linearly independent tangent vectors.

The philosophy is to understand the manifold by understanding the tangent space.

The important conclusion of this paper is that, as we will see in Definition 8.1 below, if the point differentiation of Definition 7.4 can be defined, then dimension arises even in non-connected topological spaces.

Definition 8.1. *A metric topological space* (M,d) *is called* m*-dimensional partially differentiable manifold as* K^n *model* ($m\le n$) *if it satisfies the following three conditions.*

(1) $\{U_\alpha\}_{\alpha\in\Lambda}$ *is the family of open subset on* M.

$$M=\bigcup_{\alpha\in\Lambda}U_\alpha.$$

(2) *For any* $\alpha\in\Lambda$, $\varphi_\alpha(U_\alpha)$ *is the open set of* K^n *and* $\varphi_\alpha:U_\alpha\to\left(\varphi(U_\alpha),\mathcal{A}\right)\subset\left(K^n,\hat{\mathcal{A}}\right)$ *is homeomorphic, where,* $\mathcal{A}$ *is the induced topology of the metric topology* $\hat{\mathcal{A}}$, *and* U_α *has the inverse image topology* $\left\{\varphi_\alpha^{-1}(V)\mid V\in\mathcal{A}\right\}$.

(3) *For any* $\alpha,\beta\in\Lambda$, $U_\alpha\cap U_\beta\neq\varnothing$, *the coordinate transformation*

$$\varphi_\beta\circ\varphi_\alpha^{-1}:\varphi_\alpha\left(U_\alpha\cap U_\beta\right)\to\varphi_\beta\left(U_\alpha\cap U_\beta\right)$$

is homeomorphic.

Moreover, when the number of definable linearly independent derivatives of $\varphi_\beta\circ\varphi_\alpha^{-1}$ *is* m, $\varphi_\beta\circ\varphi_\alpha^{-1}$ *is called partially diffeomorphic.*

From now on, we will also write $M=M^{m\le n}$.

Example 8.1. *The sphere on* K

$$S^{m\le 2}(K)=\left\{(x,y)\in K^3\mid x^2+y^2+z^2=1\right\}$$

is 0 *or* 1 *or* 2*-dimensional partially differentiable manifold.*

Proof.

$$K^2=(K\setminus k)^2\amalg\left((K\setminus k)\times k\right)\amalg k^2$$

produces the decomposition

$$S^{m\le 2}(K)=S^2(K)\amalg S^1(K)\amalg S^0(K),$$

$$S^2(K)=\left\{(x,y,z)\in(K\setminus k)^3\mid x^2+y^2+z^2=1\right\},$$

$$S^1(K)=\left\{(x,y,z)\in(K\setminus k)^2\times k\mid x^2+y^2+z^2=1\right\},$$

$$S^0(K)=\left\{(x,y,z)\in k^3 \mid x^2+y^2+z^2=1\right\}.$$

Suppose that $(x,y)\in(K\setminus k)^2$, $U=S^2(K)\setminus\{(0,0,1)\}$, $V=S^2(K)\setminus\{(0,0,-1)\}$. This U and V are open sets. If $U\cap V\neq\varnothing$, then we have $S^2(K)=U\cup V$ and

$$\varphi_U(x,y,z)=\left(\frac{x}{1-z},\frac{y}{1-z}\right),$$

$$\psi_U(t)=\left(\frac{2t}{1+t^2+s^2},\frac{2s}{1+t^2+s^2},\frac{t^2+s^2-1}{1+t^2+s^2}\right),\quad \varphi_U^{-1}=\psi_U,$$

$$\varphi_V(x,y)\left(\frac{x}{1+z},\frac{y}{1+z}\right),$$

$$\psi_V(t)=\left(\frac{2t}{1+t^2+s^2},\frac{2s}{1+t^2+s^2},\frac{1-\left(t^2+s^2\right)}{1+t^2+s^2}\right),\quad \varphi_V^{-1}=\psi_V.$$

$$\varphi_V\circ\varphi_U^{-1}:\varphi_U(U\cap V)\ni(t,s)\mapsto\left(\frac{t}{t^2+s^2},\frac{s}{t^2+s^2}\right)\in\varphi_V(U\cap V)$$

is the partially diffeomorphic for any $(t,s)\in(K\setminus k)^2$. The same applies to $\varphi_U\circ\varphi_V^{-1}$.

When $t\in K\setminus k$, $s\in k$, $\frac{\partial}{\partial t}$ can be defined, but $\frac{\partial}{\partial s}$ cannot. Therefore, the manifold is $S^1(K)$.

When $t,s\in k$, both $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ cannot be defined, $\varphi_V\circ\varphi_U^{-1}$ is not partially diffeomorphic. The manifold is $S^0(K)$. Note that while $S^0(\mathbb{R})$ is a two-point set but $S^0(K)$ is an infinite set of k points. □

Definition 8.2. *A function* $f:M\to K$ *on the* m*-dimensional partially differentiable manifold* M *as* K^n *model is partially differentiable function as follows.*

For any coordinate neighborhood $(U_\alpha,\varphi_\alpha)$,

$$f\circ\varphi_\alpha^{-1}:U'_\alpha\to K$$

is a partially differentiable function on U'_α, *where* φ_α^{-1} *is the inverse mapping of* $\varphi_\alpha:U_\alpha\to U'_\alpha$.

We write all of partially differentiable functions on M *as* $C^\infty(M,K)$. $C^\infty(M,K)$ *is called the partially differentiable functional ring on* M.

Let $M=M^{m\le n}$ be the m-dimensional partial differential manifold of the K^n model.

We define m-dimensional tangent space T_xM, $x\in M$ as follows:

$$T_xM=\left\{\sum_{i=1}^m X^i\left(\frac{\partial}{\partial x^i}\right)_x \mid X^i\in K\right\},\ \left(\frac{\partial}{\partial x^i}\right)_x f=\left(\frac{\partial f}{\partial x^i}\right)(x).$$

When $U\cap V\neq\varnothing$, $x\in U\cap V$, if we put $\sum_{i=1}^m X^i\left(\frac{\partial}{\partial x^i}\right)_x=\sum_{i=1}^m Y^j\left(\frac{\partial}{\partial y^j}\right)_x$, then we have

$$X^i=\sum_{j=1}^{m}\frac{\partial x^i}{\partial y^j}Y^j.$$

$\left(\frac{\partial x^i}{\partial y^j}\right)_{i,j=1,2,\cdots,m}$ is the Jacobi matrix.

Let $\langle,\rangle$ be the positive inner product of $x\in U\subset M$, T_xM;

$$g_{ij}:U\to K,\ g_{ij}(x)=\left\langle\left(\frac{\partial}{\partial x^i}\right)_x,\left(\frac{\partial}{\partial x^j}\right)_x\right\rangle.$$

Performing calculations exactly as in classical theory, we obtain

$$g_x=\sum_{i,j=1}^{m}g_{ij}(x)\left(dx^i\right)_x\left(dx^j\right)_x.$$

g_x is a Riemann metric on a partially differentiable manifold M, (M,g_x) are called partially Riemannian manifolds.

Example 8.2. *From* Example 8.1, *let* $x^1=t$ *and* $x^2=s$. *Then, let us set the following*:

$$y^1=\frac{2x^1}{1+\left(x^1\right)^2+\left(x^2\right)^2},\ y^2=\frac{2x^2}{1+\left(x^1\right)^2+\left(x^2\right)^2},\ y^3=\frac{\left(x^1\right)^2+\left(x^2\right)^2-1}{1+\left(x^1\right)^2+\left(x^2\right)^2}$$

When $x^1,x^2\in K\setminus k$, $y^1,y^2,y^3\in K\setminus k$, *i.e., when* $S^2(K)$, *we have*

$$\sum_{\alpha=1}^{3}dy^\alpha dy^\alpha=\frac{4}{\left(1+\left(x^1\right)^2+\left(x^2\right)^2\right)^2}\sum_{i=1}^{2}dx^idx^i.$$

The left-hand side represents the Riemannian metric on $S^2(K)$ *induced by* $(K\setminus k)^3$, *and the right-hand side represents the Riemannian metric on* $(K\setminus k)^2$ *obtained by projecting* $S^2(K)\setminus\{(0,0,1)\}$ *onto* $(K\setminus k)^2$ *by stereographic projection.*

$$g_x=\frac{4}{\left(1+\left(x^1\right)^2+\left(x^2\right)^2\right)^2}\sum_{i=1}^{2}dx^idx^i$$

is Riemann metric on $S^2(K)$. *When* $x^1\in K\setminus k$, $x^2\in k$, $y^1,y^2\in K\setminus k$, $y^3\in k$, *we can see that*

$$g_x=\frac{4}{\left(1+\left(x^1\right)^2+\left(x^2\right)^2\right)^2}dx^1dx^1$$

is a Riemannian metric over $S^1(K)$.

In the classical circle $S^1(\mathbb{R})$,

$$g=\frac{4}{\left(1+\left(x^1\right)^2\right)^2}dx^1dx^1$$

is a Riemannian metric. When $x^1,x^2\in k$, $y^1,y^2,y^3\in k$, *i.e., on* $S^0(K)$, *there is no Riemannian metric.*

We next define the affine algebraic variety (cf. [9] [10]). For the following definition of affine algebraic variety, we use Definition 7.6 to define the derivation.

Definition 8.3. *The algebraic set is defined by*

$$V(I)=\left\{(a_1,a_2,\cdots,a_n)\in K^n \mid f_i(a_1,a_2,\cdots,a_n)=0, i=1,2,\cdots,l\right\}.$$

The coordinate ring is defined by

$$K[X_1,X_2,\cdots,X_n]/I(V)$$

where

$$I(V)=\left\{f\in K[X_1,X_2,\cdots,X_n] \mid \textit{for any } (a_1,a_2,\cdots,a_n)\in V, f(a_1,a_2,\cdots,a_n)=0\right\}.$$

Then the affine algebraic variety is defined by

$$\left(V(I),K[X_1,X_2,\cdots,X_n]/I(V)\right)$$

Definition 8.4. *For a subset* U *of* K^n,

$$U \textit{ is open set } \Leftrightarrow K^n\setminus U \textit{ is algebraic set,}$$

that is, ideal $I\subset K[X_1,\cdots,X_n]$ *exists and satisfies the following* $K^n\setminus U=\left\{x\in K^n \mid f(x)=0 \textit{ for any } f\in I\right\}$. *This is the Zariski topology of* K^n.

Example 8.3. (1)

$$\left(V\left(x^2+y^2-1\right),K[X,Y]/\left(X^2+Y^2-1\right)\right).$$

This is the 0 *or* 1-*dimensional sphere* $S^{m\le 1}(K)$ ($m=0,1$).

(2)

$$\left(V\left(Y^2-f(X)\right),K[X,Y]/\left(Y^2-f(X)\right)\right).$$

Assume that $f(X)$ *is cubic or higher and does not have multiple solutions. They are the Mother elliptic curves and Mother hyperelliptic curves. It is an interesting problem to investigate these in detail.*

Example 8.4. *The differentiation follows* Definition 7.6. *If* $\frac{\partial}{\partial X_i}$ *is well-defined, then* $\frac{\partial}{\partial X_i}(K)$ *is computable. For example, for* $f(X)=\mathbf{3}X^2+2X$, *the derivative of* $X\neq\mathbf{3}$ *is* $\frac{\partial f(X)}{\partial X}=2\times\mathbf{3}\times X+2$.

The differential coefficient at $X=\alpha\neq\mathbf{3}$ *is*

$$\left.\frac{\partial f(X)}{\partial X}\right|_{x=\alpha}=\frac{\partial f(\alpha)}{\partial\alpha}=2\times\mathbf{3}\times\alpha+2.$$

The differential coefficient at $X=\mathbf{3}$ *is* $\frac{\partial f(\mathbf{3})}{\partial\mathbf{3}}=\frac{\partial}{\partial\mathbf{3}}\left(\mathbf{3}^3+2\times\mathbf{3}\right)=3\times\mathbf{3}^2+2$.

9. ABC Type Conjectures and Diophantine Type Equations

Let $h\in\mathbb{Z}$, $k[X]$ or $\mathcal{Z}$ be a prime element. Then we define

$$\operatorname{rad}(abc)=\prod_{h\mid abc}h.$$

The following results use references [12]-[15]. First, we will describe the polynomial version of the ABC theorem by Stothers and Mason [16] [17].

Theorem 9.1 (Stother-$Mason). *Let* k *be a field. For any functions* $a(X)$, $b(X)$, $c(X) \in k[X] \setminus k$. $\gcd(a(X), b(X)) = 1$. *Then*

$$a(X) + b(X) = c(X)$$

implies

$$\max\{\deg a(X), \deg b(X), \deg c(X)\} < \deg \operatorname{rad}(a(X)b(X)c(X)).$$

The proof uses the differentiation of $a(X), b(X)$ and $c(X)$ with respect to X.

Theorem 9.2 (Mochizuki, ABC Theorem on $\mathbb{Z}$). *Suppose that* $a, b, c \in \mathbb{Z}$. $\gcd(a, b, c) = 1$. *Then*

$$a + b = c$$

implies for any $\varepsilon > 0$, $K(\varepsilon) \geq 1$ *exist such that*

$$\max\{|a|, |b|, |c|\} < K(\varepsilon) \operatorname{rad}(abc)^{1+\varepsilon}.$$

This is the famous ABC conjecture of $\mathbb{Z}$. Below, we will write the ABC conjectures for $\mathcal{N}$ and $\mathcal{Z}$.

From now on, we write $c = a \oplus b$.

Conjecture 1 (ABC type Conjecture on $\mathcal{N}$). *Suppose that* $a, b \in \mathcal{N}$ *and not* $a \in \mathbb{N}$ *and* $b \in \mathbb{N}$. $\gcd(a, b) = 1$. *Then*

$$a + b = c$$

implies

$$c < \operatorname{rad}(abc).$$

Conjecture 2 (ABC type Conjecture on $\mathcal{Z}$). *Suppose that* $a, b \in \mathcal{Z}$ *and not* $a \in \mathbb{Z}$ *and* $b \in \mathbb{Z}$. $\gcd(a, b) = 1$. *Furthermore, suppose that* $(a, b) \notin Z = \{(a, b) \mid -1 < a < 1, b = \pm 1, a \in \mathcal{Z}\}$. *Then*

$$a + b = c$$

implies

$$\max\{|a|, |b|, |c|\} < \operatorname{rad}(abc).$$

Remark 9.1. *As mentioned in Section* 6, *there are elements* $0 < a < 1$ *or* $-1 < a < 0$. *Suppose that* $(a, b) \in Z = \{(a, b) \mid -1 < a < 1, b = 1, a \in \mathcal{Z}\}$. *When* $b = 1$, $0 < a < 1$, $a = \mathbf{n} - n$ *for* $a + b = c$, c *is maximum at* $c > 1$. *Because* $\operatorname{ev}((\mathbf{n} - n) \times (\mathbf{n} - (n-1))) = 0$, $\operatorname{ev}(c) = \operatorname{ev}(\mathbf{n} - (n-1)) = 1$, *we see that*

$$\operatorname{rad}(abc) = \operatorname{rad}((\mathbf{n} - n) \times 1 \times (\mathbf{n} - (n-1))) = (\mathbf{n} - n) \times (\mathbf{n} - (n-1)) < c.$$

Therefore, $\mathbf{n}$ *satisfies* $c > \operatorname{rad}(abc)$. *Since* $\mathbf{n}$ *is arbitrary, there are an infinite number of* $(a, b, c) = (\mathbf{n} - n, 1, \mathbf{n} - (n-1))$. *Hence, in this case, the ABC type conjecture has no meaning. On the other hand, there is no problem when* $\{(a, b) \mid -1 < a < 1, -1 < b < 1, a, b \in \mathcal{Z}\}$. *The usual ABC Theorem of* $\mathbb{N}$ *and* $\mathbb{Z}$ *are essentially equivalent but ABC type conjecture of* $\mathcal{N}$ *and* $\mathcal{Z}$ *are essentially* *different for the reasons explained above.*

These conjectures may be possible to use differentiation, as in Theorem 9.1. Therefore, they are expected to be easier to prove than the usual ABC Theorem 9.2. Theorem 9.2 on $\mathbb{Z}$ does not have the concept of differentiation. This is one of the reasons why it is considered difficult. However, Conjecture 1 and Conjecture 2 on $\mathcal{N}$ and $\mathcal{Z}$ have the concept of differentiation. In the case of original ABC theorem 9.2, it means that multiplication and addition are intertwined, but in Conjecture 1 and Conjecture 2, it is predicted that multiplication and addition are not very intertwined.

The following also holds for Pythagorean triplets:

Lemma 9.1. *Suppose that* $a, b \in \mathcal{Z}$ *and* $\gcd(a,b) = 1$.

For $a^2 + b^2 = c^2$, *there are an infinite number of Pythagorean numbers and it can be written as*

$$a = n^2 \ominus m^2,\ b = 2nm,\ c = n^2 \oplus m^2.$$

Proof. The proof is exactly the same as the classical $\mathbb{Z}$ case. □

Furthermore, we would like to introduce Conjectures 3 to 7 below. Conjectures 3 to 7 are proved using Conjecture 1 and Conjecture 2, so this may not be meaningful, but what we want to say here is that they are much easier to prove than the classical case, which is proven using the original ABC theorem 9.2. Also, solving Conjecture 1 and Conjecture 2 will solve many problems for us, so we believe there is value in solving Conjecture 1 and Conjecture 2.

Conjecture 3 (Fermat-Catalan type Conjecture). *Suppose that* $a, b \in \mathcal{N}$, $l, m, n \in \mathbb{N}$, *and not* $a \in \mathbb{N}$ *and* $b \in \mathbb{N}$. $\gcd(a,b,c) = 1$. *If*

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1,$$

then there are no pairs (a,b,c,l,m,n) *that satisfy*

$$a^l + b^m = c^n.$$

Contrary to this result, classical theory suggests that a finite number of them exist.

Proof. When $l = m = n$, it is Fermat type conjecture. Using ABC type conjecture 1, we prove Fermat-Catalan type conjecture. We have

$$\operatorname{rad}\left(a^l b^m c^n\right) = \operatorname{rad}(abc) \le abc.$$

From ABC conjecture 1, we see that

$$a^l < \operatorname{rad}\left(a^l b^m c^n\right),\ b^m < \operatorname{rad}\left(a^l b^m c^n\right),\ c^n < \operatorname{rad}\left(a^l b^m c^n\right).$$

Hence, we have

$$a^l \le abc,\ b^m \le abc,\ c^n \le abc,$$

$$a \le (abc)^{\frac{1}{l}},\ b \le (abc)^{\frac{1}{m}},\ c \le (abc)^{\frac{1}{n}},$$

$$abc \le (abc)^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}.$$

Therefore, this is a contradiction. Hence, (a,b,c) does not exist. □

Conjecture 4 (Darmon-Granville type Conjecture). *Suppose that* $p,q,r \in \mathcal{Z}\setminus\{0\}$, $k,m,n \in \mathbb{N}$, $a,b \in \mathcal{Z}$ *and not* $a \in \mathbb{Z}$ *and* $b \in \mathbb{Z}$. $\gcd(a,b,c)=1$. *Furthermore, suppose that* $(a,b) \notin Z = \{(a,b) \mid -1 < a < 1, b = \pm 1, a \in \mathcal{Z}\}$ *and*

$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1.$$

Then there are a finite number of pairs (a,b,c) *that satisfy*

$$pa^{l}+qb^{m}=rc^{n}.$$

Classical theory also suggests that a finite number of them exist.

Proof. It can be assumed that $\gcd(pa^{l},qb^{m},rc^{n})=1$. Using ABC type conjecture 2, we see that

$$\operatorname{rad}(pa^{l},qb^{m},rc^{n}) \le |pqr|\operatorname{rad}(abc) \le |pqr|\cdot|abc|,$$

$$\max\{|pa^{l}|,|qb^{m}|,|rc^{n}|\} < |pqr|\cdot|abc|,$$

$$|a| < (|qr|\cdot|abc|)^{\frac{1}{l}},\ |b| < (|pr|\cdot|abc|)^{\frac{1}{m}},\ |c| < (|pq|\cdot|abc|)^{\frac{1}{n}},$$

$$|abc|^{1-\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)} < |qr|^{\frac{1}{l}}\cdot|pr|^{\frac{1}{m}}\cdot|pq|^{\frac{1}{n}}.$$

From $1-\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)>0$, so $|abc|$ is bounded. Hence, a,b,c are a finite number. □

Conjecture 5 (Tijdeman-Zagier type Conjecture). *Suppose that* $a,b \in \mathcal{N}$, $k,m,n \in \mathcal{N}$, *and not* $a \in \mathbb{N}$ *and* $b \in \mathbb{N}$. $\gcd(a,b,c)=1$. *Then there is no set* (a,b,c,k,m,n) *that satisfies*

$$a^{k}+b^{m}=c^{n},\ k,m,n \ge 3.$$

Classical theory also suggests that they do not exist.

Proof. When $l=m=n=3$, it is Fermat type conjecture. Since it is

$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n} \le \frac{1}{3}+\frac{1}{3}+\frac{1}{4}=\frac{11}{12},$$

the condition of Fermat-Catalan type conjecture 3 is satisfied. Therefore, form Fermat-Catalan type conjecture 3, (a,b,c,l,m,n) does not exist. □

Conjecture 6 (Peyre type Conjecture). *Suppose that* $p,q,r \in \mathcal{N}$, $a,b \in \mathcal{N}$ *and not* $a \in \mathbb{N}$ *and* $b \in \mathbb{N}$. *Assume that* $m,n \in \mathbb{N}$, $m,n \ge 2$, $(m,n) \ne (2,2)$. *Then there are a finite number of pairs* (a,b,m,n) *that satisfy*

$$pa^{m}-qb^{n}=r.$$

Classical theory also suggests that a finite number of them exist.

Proof. Suppose that $\gcd(pa^{m},qb^{n})=d$, $d \mid r$, $d \le r$. Then we have

$$\frac{pa^{m}}{d}-\frac{qb^{n}}{d}=\frac{r}{d}.$$

From ABC type conjecture 1, we have

$$\max\left\{\frac{pa^m}{d},\frac{qb^n}{d},\frac{r}{d}\right\} < \operatorname{rad}\left(\frac{pa^m}{d}\cdot\frac{qb^n}{d}\cdot\frac{r}{d}\right) \le \operatorname{rad}\left(pqr\cdot ab\right) \le pqr\cdot ab$$

and

$$\frac{pa^m}{d} < pqr\cdot ab,\ \frac{qb^n}{d} < pqr\cdot ab,\ \frac{r}{d} < pqr\cdot ab.$$

Hence, we have

$$a < \left(dqrab\right)^{\frac{1}{m}},\ b < \left(dprab\right)^{\frac{1}{n}}.$$

By taking the product of these, we see that

$$ab < \left(dqr\right)^{\frac{1}{m}}\left(dpr\right)^{\frac{1}{n}}\left(ab\right)^{\frac{1}{m}+\frac{1}{n}},$$

$$\left(ab\right)^{1-\left(\frac{1}{m}+\frac{1}{n}\right)} < \left(dqr\right)^{\frac{1}{m}}\left(dpr\right)^{\frac{1}{n}} \le \left(r^2q\right)^{\frac{1}{m}}\left(r^2p\right)^{\frac{1}{n}}.$$

From

$$\frac{1}{m}+\frac{1}{n} \le \frac{1}{2}+\frac{1}{3} = \frac{5}{6},$$

the maximum value of $\frac{1}{m}+\frac{1}{n}$ is $\frac{5}{6}$, so we have

$$\left(ab\right)^{1-\frac{5}{6}} \le \left(ab\right)^{1-\left(\frac{1}{m}+\frac{1}{n}\right)} < \left(r^2q\right)^{\frac{1}{m}}\left(r^2p\right)^{\frac{1}{n}} \le \left(r^2q\right)^{\frac{1}{2}}\left(r^2p\right)^{\frac{1}{3}}.$$

Hence, we have

$$\left(ab\right)^{\frac{1}{6}} < r^{\frac{5}{3}}p^{\frac{1}{3}}q^{\frac{1}{2}},$$

so

$$ab < r^{10}p^2q^3.$$

Therefore, we see that $\left(a,b,m,n\right)$ is finite. □

Conjecture 7 (strong Hall type Conjecture). *Suppose that* $a^3-b^2=c$, $\gcd\left(a,b\right)=1$, $c\in\mathcal{Z}_{\ge 1}$, $a,b\in\mathcal{N}$ *and not* $a\in\mathbb{N}$ *and* $b\in\mathbb{N}$. *Then the following inequality is satisfied*

$$\max\left\{a^3,b^2\right\} < \operatorname{rad}\left(c\right)^6.$$

In classical theory, $\max\left\{a^3,b^2\right\} < C\operatorname{rad}\left(c\right)^{6+\epsilon}$, C is constant, is satisfied.

Proof. Hall-Lang-Waldschmidt-Szpiro conjecture is that

$$\textit{if}\ a^m-b^n=c\ \ \textit{then}\ \ a^{mn-m-n} < \operatorname{rad}\left(c\right)^n\ \ \textit{and}\ \ b^{mn-m-n} < \operatorname{rad}\left(c\right)^m.$$

We will first show this using the ABC type conjecture 1. Then we have

$$\operatorname{rad}\left(a^m\cdot b^n\cdot c\right) < ab\operatorname{rad}\left(c\right)$$

and

$$\max\left(a^m,b^n\right) < ab\operatorname{rad}\left(c\right).$$

Using

$$a \le \max\left(a^m, b^n\right)^{\frac{1}{m}},\ b \le \max\left(a^m, b^n\right)^{\frac{1}{n}},$$

we have

$$\max\left(a^m, b^n\right) < \operatorname{rad}(c)\max\left(a^m, b^n\right)^{\frac{1}{m}+\frac{1}{n}},$$

$$\operatorname{rad}(c) > \max\left(a^m, b^n\right)^{1-\frac{1}{m}-\frac{1}{n}},$$

$$\operatorname{rad}(c) > \left(a^m\right)^{1-\frac{1}{m}-\frac{1}{n}},\ \operatorname{rad}(c) > \left(b^n\right)^{1-\frac{1}{m}-\frac{1}{n}},$$

$$\operatorname{rad}(c)^n > a^{mn-n-m},\ \operatorname{rad}(c)^m > b^{mn-n-m}.$$

When $m = 3, n = 2$, we have

$$\operatorname{rad}(c)^2 > a,\ \operatorname{rad}(c)^3 > b,$$

$$\operatorname{rad}(c)^6 > a^3,\ \operatorname{rad}(c)^6 > b^2.$$

Hence, we have

$$\max\left\{a^3, b^2\right\} < \operatorname{rad}(c)^6.$$

□

Corollary 9.1. *There is no solution* $(a,b) \in \mathcal{Z} \times \mathcal{Z}$ *to Thue equation* $a^m - qb^m = 1$, *Pell equation* $a^2 - qb^2 = 1$ *and Catalan equation* $a^m - b^n = 1$.

Proof. The left side of this three equations is a dark composite number and the right side is 1. It is clear that calculating the left side does not result in 1. □

10. ABC Type Conjecture for Mother Algebraic Number Fields

A finite extension field of $\mathcal{Q}$ is called a Mother algebraic number field and is denoted by L. Next, we will formulate the ABC type conjecture for Mother algebraic number fields L. At that time, new integral domains and fields will naturally appear, as described below.

$$\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$$

is a well-known fact. In our case, such a relationship does not hold.

$\mathcal{F}_{\mathbf{p}} := \mathcal{Z}/\mathbf{p}\mathcal{Z}$ is not a finite field, but an infinite set and an integral domain.

$$\mathcal{Z}_{(p)} := \left\{\frac{b}{a} \mid a,b \in \mathcal{Z}, a \notin (\mathbf{p})\right\}$$

and

$$\mathcal{Z}_{(\mathbf{p})} := \left\{\frac{b}{a} \mid a,b \in \mathcal{Z}, a \notin (\mathbf{p})\right\}$$

are local rings, and the only maximal ideals are $p\mathcal{Z}_{(p)}$ and $\mathbf{p}\mathcal{Z}_{(\mathbf{p})}$, respectively. Then the following holds.

$$\mathcal{F}_p := \mathcal{Z}/p\mathcal{Z}(\text{integral domain}) \ne \mathcal{Z}_{(p)}/p\mathcal{Z}_{(p)}(\text{field}).$$

$$\mathcal{F}_{\mathbf{p}} := \mathcal{Z}/\mathbf{p}\mathcal{Z}\left(\text{integral domain}\right) \neq \mathcal{Z}_{(\mathbf{p})}/\mathbf{p}\mathcal{Z}_{(\mathbf{p})}\left(\text{field}\right),$$

The quotient field of $\mathcal{F}_{\mathbf{p}}$ is defined by

$$\hat{\mathcal{F}}_{\mathbf{p}} := \left(\mathcal{F}_{\mathbf{p}} \setminus \{0\}\right)^{-1}\mathcal{F}_{\mathbf{p}}.$$

Then it can be seen that

$$\hat{\mathcal{F}}_{\mathbf{p}} \subsetneq \mathcal{Z}_{(\mathbf{p})}/\mathbf{p}\mathcal{Z}_{(\mathbf{p})}.$$

The new rings and fields appear as shown above.

Definition 10.1. (cf. [8]) *Let* L *be a Mother algebraic number field and* $\mathfrak{p}$ *be a prime element.*

$$v : L \ni \alpha \mapsto v_{\mathfrak{p}}(\alpha) \in \mathbb{Z} \cup \{\infty\},\ v_{\mathfrak{p}}(0) = \infty,$$

(1)

$$v_{\mathfrak{p}}(\alpha\beta) = v_{\mathfrak{p}}(\alpha) + v_{\mathfrak{p}}(\beta),$$

(2)

$$v_{\mathfrak{p}}(\alpha+\beta) \geq \min\left(v_{\mathfrak{p}}(\alpha), v_{\mathfrak{p}}(\beta)\right).$$

This $v_{\mathfrak{p}}$ is called *additive valuation*. Sometimes it is written as $v_{\mathfrak{p}} = \operatorname{ord}_{\mathfrak{p}}$.

Definition 10.2. (cf. [8]) *Let* L *be a Mother algebraic number field.*

$$v : L^{\times} \ni \alpha \mapsto v(\alpha) = |\alpha|_v \in \mathcal{Z},$$

(1)

$$v(\alpha) \geq 0,\ v(\alpha) = 0 \ \Rightarrow\ \alpha = 0,$$

(2)

$$v(\alpha\beta) = v(\alpha)v(\beta),$$

(3)

$$v(\alpha+\beta) \leq \max\left(v(\alpha), v(\beta)\right).$$

This v is called *non-Archimedean normal valuation*.

$d_v(\alpha,\beta) = v(\alpha-\beta) = |\alpha-\beta|_v$ is a distance. $A_v = \{a \in L \mid v(\alpha) \geq 0\}$ is a valuation ring. $\mathfrak{m}_v = \{a \in L \mid v(\alpha) > 0\}$ is a maximal ideal of A_v. A_v is a local domain and principal ideal domain with $\mathfrak{m}_v$ as the maximal ideal. $k_v = A_v/\mathfrak{m}_v$ is residue field. $L = \left(A_v \setminus \{0\}\right)^{-1}A_v$ is a quotient field of A_v and A_v is an integral domain, so L is a field, which is a quotient field of A_v. $(A_v, \mathfrak{m}_v)$ is a discrete valuation ring. Maximal ideal $\mathfrak{m}_v$ is a principal ideal $\mathfrak{m}_v = (\pi)$. A fractional ideal of A_v is $\mathfrak{m}_v^e = (\pi^e)$, $e \in \mathbb{Z}$. In particular, any nonzero element $\alpha \in L \setminus \{0\}$ of the field L is written as

$$\alpha = u\pi^e,$$

where, u is a unit of A_v and $e \in \mathbb{Z}$. The way π is taken is not unique. However, e is unique regardless of how π is taken.

Example 10.1. *When* $L = \mathcal{Q}$, $\mathfrak{p} = \mathbf{p}$ *is a prime element of* $\mathcal{Z}$.

If $\mathbf{p}^n\dfrac{a}{b}\in\mathcal{Q}$, $(\mathbf{p},a)=1$, $(\mathbf{p},b)=1$, *then we have* $v_{\mathbf{p}}\left(\mathbf{p}^n\dfrac{a}{b}\right)=n\in\mathbb{Z}$.

In this case, it becomes as follows.

$A_v=\{a\in\mathcal{Q}\mid v(a)\geq 0\}$, $\mathfrak{m}_v=\{a\in\mathcal{Q}\mid v(a)>0\}$,

$A_v=\mathcal{Z}_{(\mathbf{p})}=\left\{\dfrac{b}{a}\mid a,b\in\mathcal{Z},a\notin(\mathbf{p})\right\}$, $\mathfrak{m}_v=\mathbf{p}\mathcal{Z}_{(\mathbf{p})}$, A residue field $k_v=A_v/\mathfrak{m}_v=\mathcal{Z}_{(\mathbf{p})}/\mathbf{p}\mathcal{Z}_{(\mathbf{p})}$. The quotient field of $\mathcal{Z}_{(\mathbf{p})}$ is $\left(\mathcal{Z}_{(\mathbf{p})}\setminus\{0\}\right)^{-1}\mathcal{Z}_{(\mathbf{p})}=\mathcal{Q}$.

We summarize as follows.

$$\left(L=\mathcal{Q},A_v=\mathcal{Z}_{(\mathbf{p})},\mathfrak{m}_v=\mathbf{p}\mathcal{Z}_{(\mathbf{p})},k_v=\mathcal{Z}_{(\mathbf{p})}/\mathbf{p}\mathcal{Z}_{(\mathbf{p})},\mathbf{p}\mathcal{Z}=(\mathbf{p})\right).$$

We extend the above example to the finite extension field $L/\mathcal{Q}$.

$$\left(L,O_L,\mathfrak{B},O_L/\mathfrak{B},\hat{\mathcal{F}}_{\mathfrak{B}},\mathfrak{p}=\mathbf{p}\mathcal{Z}=(\mathbf{p})\right).$$

L is a Mother algebraic number field. O_L is the integer ring of L. $\mathfrak{B}$ is a prime ideal of O_L. $O_L/\mathfrak{B}$ is integral domain. $\hat{\mathcal{F}}_{\mathfrak{B}}$ is a quotient field of $O_L/\mathfrak{B}$. $\mathfrak{p}$ is a prime ideal of $\mathcal{Z}$.

Example 10.2. *When* $\mathfrak{B}=\mathbf{p}\mathcal{Z}$, *we have the following.*

$$\left(\mathcal{Q},\mathcal{Z},\mathbf{p}\mathcal{Z},\mathcal{F}_{\mathbf{p}}:=\mathcal{Z}/\mathbf{p}\mathcal{Z},\hat{\mathcal{F}}_{\mathbf{p}}:=\left(\mathcal{F}_{\mathbf{p}}\setminus\{0\}\right)^{-1}\mathcal{F}_{\mathbf{p}},\mathfrak{p}=\mathbf{p}\mathcal{Z}\right),$$

$$\left(\mathcal{Q}\left(\sqrt{-1}\right),\mathcal{Z}\left[\sqrt{-1}\right],\mathbf{p}\mathcal{Z},\mathcal{F}_{\mathbf{p}}\left[\sqrt{-1}\right]:=\mathcal{Z}\left[\sqrt{-1}\right]/\mathbf{p}\mathcal{Z},\right.$$

$$\left.\hat{\mathcal{F}}_{\mathbf{p}}\left(\sqrt{-1}\right):=\left(\mathcal{F}_{\mathbf{p}}\left[\sqrt{-1}\right]\setminus\{0\}\right)^{-1}\mathcal{F}_{\mathbf{p}}\left[\sqrt{-1}\right],\mathfrak{p}=\mathbf{p}\mathcal{Z}\right).$$

Let us consider the case where v is a *finite place*, which is equivalence class of non-archimedean valuation. In this case, the place $v_{\mathfrak{B}}=|\cdot|_{\mathfrak{B}}$ of $(L,O_L,\mathfrak{B})$ is as follows.

$$\mathfrak{B}\cap\mathcal{Z}=\mathbf{p}\mathcal{Z},\ v_{\mathbf{p}}\rightsquigarrow v_{\mathfrak{B}},$$

$$\alpha=u\pi^e\in O_L,\ v_{\mathfrak{B}}(\alpha)=e,$$

$$f=\left[\text{quotient field of }(O_L/\mathfrak{B}):\text{quotient field of }(\mathcal{Z}/\mathfrak{p})\right].$$

We define e as *ramification index* and f as *residue degree*. We also define the *norm* as $N(\mathfrak{B})=\mathbf{p}^f$.

$$\mathfrak{B}^e\mid(\mathbf{p}),\ \pi^e\mid\mathbf{p}.$$

From $N(\mathfrak{B})=\mathbf{p}^f$ and $|\mathfrak{p}|_{\mathbf{p}}=\mathbf{p}^{-1}$, *normal valuation* $v_{\mathfrak{B}}=|\cdot|_{\mathfrak{B}}$ is

$$|\mathbf{p}|_{\mathfrak{B}}=N(\mathfrak{B})^{-e}=|\mathbf{p}|_{\mathbf{p}}^{ef}=\mathbf{p}^{-ef}.$$

Example 10.3. *When* $L=\mathcal{Q}\left(\sqrt{-1}\right)$ *and* $O_L=\mathcal{Z}\left[\sqrt{-1}\right]$, *we will consider when* $\mathfrak{p}=(\mathbf{2})$.

Then residue degree f *is*

$$f=\left[\text{quotient field of }(O_L/\mathfrak{p}):\text{quotient field of }(\mathcal{Z}/\mathfrak{p})\right]$$

$$=\left[\text{quotient field of }\left(\mathcal{Z}\left[\sqrt{-1}\right]/\mathbf{2}\mathcal{Z}\right):\text{quotient field of }\left(\mathcal{Z}/\mathbf{2}\mathcal{Z}\right)\right]$$

$$=\left[\text{quotient field of }\left(\mathcal{F}_2\left[\sqrt{-1}\right]\right):\text{quotient field of }\mathcal{F}_2\right]$$

$$=\left[\hat{\mathcal{F}}_2\left(\sqrt{-1}\right):\hat{\mathcal{F}}_2\right]=\left[\hat{\mathcal{F}}_2\cdot 1+\hat{\mathcal{F}}_2\cdot\sqrt{-1}:\hat{\mathcal{F}}_2\right]=2.$$

$(\mathbf{2})$ *is not like* $(2)=\left(\left(1+\sqrt{-1}\right)^2\right)$, *so* $e=1$.

Since the maximal ideal $\mathfrak{m}_v$ of the local ring $\mathcal{Z}_{(\mathbf{p})}$ is a principal ideal, we write $\mathfrak{m}_v=(\pi)$.

In particular, for the field L and for any $\alpha\in L$, $\alpha\neq 0$ is

$$\alpha=u\pi^e,$$

u is a unit of A_v, $e\in\mathbb{Z}$. This π depends on the valuation v, so it is written as π_v. This π_v is called a *local parameter*.

For $a+b=c$, we make it correspond as follows.

$$\left(x_0,x_1\right)\in\mathbb{P}^1(L)=U\cup V,\ a=x_1,\ b=x_0-x_1,\ c=x_0,\ x=\frac{x_1}{x_0},\ (1,x)\in U,\ \left(\frac{1}{x},1\right)\in V.$$

Then

$$h(x)=\sum_{v\in\mathbb{V}(L)}\max\left(\log\left|x_0\right|_v,\log\left|x_1\right|_v\right)=\sum_{v\in\mathbb{V}(L)}\max\left(\log 1,\log x_v\right)$$

is called the *heights on* $\mathbb{P}^1(L)$. And we write $\mathbb{V}(L)=\mathbb{V}(L)^0\cup\mathbb{V}(L)^\infty$, where, the elements of $\mathbb{V}(L)$ is called a *place*, the elements of $\mathbb{V}(L)^0$ is called a *non Archimedean place*, the elements of $\mathbb{V}(L)^\infty$ is called an *Archimedean place*. Note that $\log\alpha$ is $\log\alpha\in\mathcal{R}$.

Based on what has been said above, we can predict that the following conjecture holds (cf. [13] [18]).

Conjecture 8. *Let* $L/\mathcal{Q}$ *be a finite extension and* L *be a fixed Mother algebraic field. The following inequality always hold for* $x\in P^1(L)\setminus\{0,1,\infty\}$ *and* $x\notin P^1(k)\setminus\{0,1,\infty\}$.

$$h(x)\leq\mathrm{cond}_{[0]}^L(x)+\mathrm{cond}_{[1]}^L(x)+\mathrm{cond}_{[\infty]}^L(x)$$

where

$$\mathrm{cond}_{[0]}^L(x)=\sum_{v(x)>0}\log\left|\frac{1}{\pi_v}\right|_v,$$

$$\mathrm{cond}_{[1]}^L(x)=\sum_{v(1-x)>0}\log\left|\frac{1}{\pi_v}\right|_v,$$

$$\mathrm{cond}_{[\infty]}^L(x)=\sum_{v\left(\frac{1}{x}\right)>0}\log\left|\frac{1}{\pi_v}\right|_v.$$

In particular, when $L=\mathcal{Q}$, we may replace $\log\left|\frac{1}{\pi_v}\right|_v$ with $\log\left|\frac{1}{\mathbf{p}}\right|_{\mathbf{p}}$.

Example 10.4. $L=\mathcal{Q}\left(\sqrt{-1}\right)$ *and* $O_L=\mathcal{Z}\left[\sqrt{-1}\right]$.

We consider when $a+b=c$ *is* $1+\mathbf{8}=1\oplus\mathbf{8}$. *When* $x=(1\oplus\mathbf{8})^{-1}$, $v(x)>0$ *does not exist. When* $1-x=\mathbf{2}^3\times(1\oplus\mathbf{8})^{-1}$, $v(1-x)>0$ *is* $\mathbf{2}$. *When* $\frac{1}{x}=1\oplus\mathbf{8}$, $v\left(\frac{1}{x}>0\right)$ *is* $1\oplus\mathbf{8}$.

Regarding $\mathbf{2}$ *of* $1-x=\mathbf{2}^3\times(1\oplus\mathbf{8})^{-1}$, $e=3$.

$$(\mathbf{2})\cap\mathcal{Z}=\mathbf{2}\mathcal{Z},\ N((\mathbf{2}))=\mathbf{2}^f.$$

$$\begin{aligned} f&=\left[\text{quotient field of }\mathcal{Z}\left[\sqrt{-1}\right]/(\mathbf{2}):\text{quotient field of }\mathcal{Z}/(\mathbf{2})\right]\\ &=\left[\text{quotient field of }\mathcal{F}_2\left[\sqrt{-1}\right]:\text{quotient field of }\mathcal{F}_2\right]?\\ &=\left[\hat{\mathcal{F}}_2\left(\sqrt{-1}\right):\hat{\mathcal{F}}_2\right]=\left[\hat{\mathcal{F}}_2\cdot 1+\hat{\mathcal{F}}_2\cdot\sqrt{-1}:\hat{\mathcal{F}}_2\right]=2.\end{aligned}$$

$$\left|\frac{1}{\mathbf{2}}\right|_{\mathbf{2}}=N((\mathbf{2}))^e=\mathbf{2}^{ef}=\mathbf{2}^{3\times 2}=\mathbf{2}^6\rightsquigarrow\log\mathbf{2}^6.$$

Regarding $1\oplus\mathbf{8}$ *of* $\frac{1}{x}=1\oplus\mathbf{8}$, $e=1$.

$$(1\oplus\mathbf{8})\cap\mathcal{Z}=(1\oplus\mathbf{8})\mathcal{Z},\ N((1\oplus\mathbf{8}))=(1\oplus\mathbf{8})^f.$$

$$\begin{aligned} f&=\left[\text{quotient field of }\mathcal{Z}\left[\sqrt{-1}\right]/(1\oplus\mathbf{8}):\text{quotient field of }\mathcal{Z}/(1\oplus\mathbf{8})\right]\\ &=\left[\text{quotient field of }\mathcal{F}_{1\oplus\mathbf{8}}\left[\sqrt{-1}\right]:\text{quotient field of }\mathcal{F}_{1\oplus\mathbf{8}}\right]\\ &=\left[\hat{\mathcal{F}}_{1\oplus\mathbf{8}}\left(\sqrt{-1}\right):\hat{\mathcal{F}}_{1\oplus\mathbf{8}}\right]=\left[\hat{\mathcal{F}}_{1\oplus\mathbf{8}}\cdot 1+\hat{\mathcal{F}}_{1\oplus\mathbf{8}}\cdot\sqrt{-1}:\hat{\mathcal{F}}_{1\oplus\mathbf{8}}\right]=2.\end{aligned}$$

$$\left|\frac{1}{1\oplus\mathbf{8}}\right|_{1\oplus\mathbf{8}}=N((1\oplus\mathbf{8}))^e=(1\oplus\mathbf{8})^{ef}=(1\oplus\mathbf{8})^{1\times 2}=(1\oplus\mathbf{8})^2\rightsquigarrow\log(1\oplus\mathbf{8})^2.$$

On the other hand, the following holds for $x=(1\oplus\mathbf{8})^{-1}$.

$$|x|_v=\left|(1\oplus\mathbf{8})^{-1}\right|_{1\oplus\mathbf{8}}=(1\oplus\mathbf{8})^2,$$

$$h(x)=\max\left(0,\log(1\oplus\mathbf{8})^2\right)=\log(1\oplus\mathbf{8})^2.$$

Hence, we have

$$\log(1\oplus\mathbf{8})^2<\log\mathbf{2}^6+\log(1\oplus\mathbf{8})^2.$$

Note that when $a+b=c$ *in the case of ordinary numbers is* $1+8=9$,

$$\log 3^8>\log 2+\log 3^4.$$

In this case, the inequality sign is reversed.

Example 10.5. $L=\mathcal{Q}\left(\sqrt{-1}\right)$, $O_L=\mathcal{Z}\left[\sqrt{-1}\right]$.

If we consider when $a+b=c$ *is* $\mathbf{2}+\mathbf{3}=\mathbf{2}\oplus\mathbf{3}$, *then we have the following from the same calculation as the previous example*

$$\log(\mathbf{2}\oplus\mathbf{3})^2<\log\mathbf{2}^2+\log\mathbf{3}^2+\log(\mathbf{2}\oplus\mathbf{3})^2.$$

Note that when $a+b=c$ *in the case of ordinary numbers is* $2+3=5$,

$$\log 5^2 < \log 2 + \log 3^2 + \log 5^2.$$

In this case, the inequality sign is not reversed.

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Appendix

Appendix A. Blueprint and Monoid $\mathbb{Z}$-Module

Below, we can see that the semiring $e^{\mathbb{D}}$ and the ring $e^{\hat{\mathbb{D}}}$ are examples of blueprints by Lorscheid [19].

Definition A.1 [19]

A blueprint $(A,\mathcal{R})$ *is a monoid* A *with zero together with a preaddition* $\mathcal{R}$, *i.e.* $\mathcal{R}$ *is an equivalence relation on the semiring* $\mathbb{N}[A]=\{\sum a_i \mid a_i \in A\}$ *of finite formal sums of elements of* A *that satisfies the following axioms, where we write* $\sum a_i \equiv \sum b_j$ *whenever* $(\sum a_i, \sum b_j) \in \mathcal{R}$:

(i) The relation $\mathcal{R}$ is additive and multiplicative, *i.e.* if $\sum a_i \equiv \sum b_j$ and $\sum c_k \equiv \sum d_l$, then

$$\sum a_i + \sum c_k \equiv \sum b_j + \sum d_l,\ \sum a_i c_k = \sum b_j d_l$$

(ii) The absorbing element 0 of A is in relation with the zero of $\mathbb{N}[A]$, *i.e.* $0 \equiv$ (empty sum).

(iii) If $a \equiv b$, then $a = b$ as elements in A.

$(\mathbb{D},\times)$ and $(\hat{\mathbb{D}},\times)$ are monoids. We put $\mathbb{N}[\mathbb{D}]=\{\sum a_i \mid a_i \in \mathbb{D}\}$ and $\mathbb{N}[\hat{\mathbb{D}}]=\{\sum a_i \mid a_i \in \hat{\mathbb{D}}\}$.

Equivalence relations are $\mathcal{R} \subset \mathbb{N}[\mathbb{D}]\times\mathbb{N}[\mathbb{D}]$, $\sum a_i \equiv \sum b_j$ whenever $(\sum a_i, \sum b_j) \in \mathcal{R}$ and $\hat{\mathcal{R}} \subset \mathbb{N}[\hat{\mathbb{D}}]\times\mathbb{N}[\hat{\mathbb{D}}]$, $\sum a_i \equiv \sum b_j$ whenever $(\sum a_i, \sum b_j) \in \hat{\mathcal{R}}$, respectively.

Then

$$(\mathbb{D},\mathcal{R}) \text{ and } (\hat{\mathbb{D}},\hat{\mathcal{R}})$$

are the blueprints, respectively.

Given a blueprint $(\hat{\mathbb{D}},\hat{\mathcal{R}})$, we can construct the ring $\mathbb{N}[\hat{\mathbb{D}}]/I(\hat{\mathcal{R}})$, where $I(\hat{\mathcal{R}})=\{\sum a_i - \sum b_j \in \mathbb{N}[\hat{\mathbb{D}}] \mid \sum a_i \equiv \sum b_j \text{ in } \hat{\mathcal{R}}\}$ is the ideal. Then $I(\hat{\mathcal{R}})=\{0\}$. Hence, $\mathbb{N}[\hat{\mathbb{D}}]/I(\hat{\mathcal{R}})=\mathbb{N}[\hat{\mathbb{D}}]$. Therefore, $\mathbb{N}[\hat{\mathbb{D}}]=\{\sum a_i \mid a_i \in \hat{\mathbb{D}}\}$ is the commutative ring. Hence, we know that $e^{\hat{\mathbb{D}}} \simeq (\hat{\mathbb{D}},\hat{\mathcal{R}})$.

Next, we define a monoid module.

Using $\mathbb{N}=\{1,2,3,\cdots\}$, $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\cdots\}$ and $\mathbb{D}=\{1,\mathbf{2},\mathbf{3},\cdots\}$, we consider new sets

$$\mathbb{N}(\mathbb{D}) := \left\{ \sum_{i:\text{finite}} a_i \mathbf{n}_i = a_1 \times 1 + a_2 \times \mathbf{2} + a_3 \times \mathbf{3} + \cdots \mid a_i \in \mathbb{N}\cup\{0\}, \mathbf{n}_i \in \mathbb{D}\cup\{0\} \right\}$$

and

$$\mathbb{Z}(\mathbb{D}) := \left\{ \sum_{i:\text{finite}} a_i \mathbf{n}_i = a_1 \times 1 + a_2 \times \mathbf{2} + a_3 \times \mathbf{3} + \cdots \mid a_i \in \mathbb{Z}, \mathbf{n}_i \in \mathbb{D}\cup\{0\} \right\}.$$

By defining the sum, product and difference of the set $\mathbb{Z}(\mathbb{D})$ ($\mathbb{N}(\mathbb{D})$) as follows, it becomes a commutative ring (semiring) $(\mathbb{Z}(\mathbb{D}),\times,+,-)$ ($(\mathbb{N}(\mathbb{D}),\times,+)$).

$$\left(\sum_i a_i \mathbf{n}_i\right) \times \left(\sum_j b_j \mathbf{n}_j\right) = \sum_{i,j} (a_i b_j)(\mathbf{n}_i \mathbf{n}_j),$$

$$\sum_i a_i \mathbf{n}_i + \sum_i b_i \mathbf{n}_i = \sum_i \left(a_i \oplus b_i\right)\mathbf{n}_i,$$

$$\left(\left(\sum_i a_i \mathbf{n}_i\right) - \left(\sum_j b_j \mathbf{n}_j\right) = \sum_{i,j}\left(a_i \ominus b_j\right)\left(\mathbf{n}_i \mathbf{n}_j\right)\right),$$

for any $\sum_i a_i \mathbf{n}_i, \sum_j b_j \mathbf{n}_j \in \mathbb{Z}(\mathbb{D})$ ($\mathbb{N}(\mathbb{D})$).

For any $c \in \mathbb{Z}$ ($c \in \mathbb{N}$), the scalar multiple is defined by

$$c\left(\sum_i a_i \mathbf{n}_i\right) = \sum_i \left(c a_i\right)\mathbf{n}_i.$$

Hence $\left(\mathbb{Z}(\mathbb{D}), \times, +, -\right)$ is the monoid $\mathbb{Z}$-module ($\left(\mathbb{N}(\mathbb{D}), \times, +\right)$ is the monoid $\mathbb{N}$-module).

Lemma A.1. *For* $\left(e^{\hat{\mathbb{D}}}, \times, +, -\right) \simeq \left(\mathbb{Z}\left[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots\right], \times, +, -\right)$ *and* $\left(e^{\mathbb{D}}, \times, +\right) \simeq \left(\mathbb{N}\left[\mathbf{2}, \mathbf{3}, \mathbf{5}, \cdots\right], \times, +\right)$, *the following ring isomorphism holds*

$$\left(e^{\hat{\mathbb{D}}}, \times, +, -\right) \simeq \left(\hat{\mathbb{D}}, \hat{\mathcal{R}}\right) \simeq \left(\mathbb{Z}(\mathbb{D}), \times, +, -\right)$$

and the following semiring isomorphism holds

$$\left(e^{\mathbb{D}}, \times, +\right) \simeq \left(\mathbb{D}, \mathcal{R}\right) \simeq \left(\mathbb{N}(\mathbb{D}), \times, +, -\right).$$

Appendix B. Further Considerations on Differentiation

We will consider the differentiation of the following three cases.

(I) Case with continuous but sharp points.

(II) Case with discontinuous points.

(III) Case with continuous points.

Definition B.1 *The derivative at* $x = 0$ *is defined as follows*:

In the cases (*I*) *and* (*II*),

$$\frac{\partial}{\partial 0}(a) = 0, \text{ for any } a \in K.$$

In the case (*III*),

$$\left.\frac{\partial}{\partial x} f(x)\right|_{x=0}.$$

The same applies to multivariate functions $f\left(x_1, x_2, \cdots, x_n\right)$.

Example B.1. (I) *Case with continuous but sharp differential*:

$$f(x) = |x - 2| = \begin{cases} x - \mathbf{2} & (x > \mathbf{2}) \\ -x + \mathbf{2} & (x < \mathbf{2}) \\ 0 & (x = \mathbf{2}) \end{cases}$$

When this $f(x)$, *the differential at* $x = \mathbf{2}$ *is* $\frac{\partial}{\partial \mathbf{2}}(0) = 0$, *and the equation of the tangent is* $y = 0$.

$$f(x) = |x| = \begin{cases} x & (x > 0) \\ -x & (x < 0) \\ 0 & (x = 0) \end{cases}$$

When this $f(x)$, *the derivative at* $x=0$ *is* $\frac{\partial}{\partial 0}(0)=0$, *and the equation of the tangent is* $y=0$.

$$f(x)=|x|+\mathbf{2}=\begin{cases} x+\mathbf{2} & (x>0) \\ -x+\mathbf{2} & (x<0) \\ \mathbf{2} & (x=0) \end{cases}$$

When this $f(x)$, *the differential at* $x=0$ *is* $\frac{\partial}{\partial 0}(\mathbf{2})=0$, *and the equation of the tangent is* $y=\mathbf{2}$.

(II) *Case with discontinuous points*:

$$f(x)=\begin{cases} x+\mathbf{2} & (x\geq \mathbf{2}) \\ x & (x<\mathbf{2}) \end{cases}$$

When this $f(x)$, *the derivative at* $x=\mathbf{2}$ *is* $\frac{\partial}{\partial \mathbf{2}}f(\mathbf{2})=\frac{\partial}{\partial \mathbf{2}}(\mathbf{2}+\mathbf{2})=1+1=2$. *The equation of the tangent is* $y=2(x-\mathbf{2})+2\times\mathbf{2}$.

$$f(x)=\begin{cases} x+\mathbf{2} & (x\geq 0) \\ x & (x<0) \end{cases}$$

When this $f(x)$, *the derivative at* $x=0$ *is* $\frac{\partial}{\partial 0}f(0)=\frac{\partial}{\partial 0}(\mathbf{2})=0$. *The equation of the tangent is* $y=\mathbf{2}$.

(III) Differential at the continuous point $x=0$:

$$f(x)=x+\mathbf{2}$$

When this $f(x)$, *the differential at* $x=0$ *is* $\frac{\partial}{\partial x}f(x)\Big|_{x=0}=1$. *The equation of the tangent is* $y=x+\mathbf{2}$.