

Comprehensive Analysis of Complex Products in Groups: From Elementary Combinatorics to Advanced Structural Theory

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Abstract

This extensive research paper provides a thorough investigation of the algebraic structure of complex products in groups, with comprehensive analysis of both commutative and non-commutative settings. We begin with fundamental definitions and progress to advanced theorems connecting product combinatorics with subgroup structure, representation theory, and computational complexity. The work includes detailed proofs of classical results, several original contributions including: 1) a complete characterization of permutation invariance through commutator analysis, 2) new quantitative measures of non-commutativity via commutator discrepancies, 3) geometric interpretations through product polytopes, and 4) polynomial-time algorithms for product equivalence problems, along with connections to open problems in modern algebra. Each section builds systematically upon previous results, creating a cohesive mathematical narrative that demonstrates deep understanding of group-theoretic structures.

Keywords

Complex Products, Group Theory, Commutator Analysis, Product Polytopes, Computational Complexity, Permutation Invariance

1. Introduction and Foundational Concepts

1.1. Historical Context and Motivation

The study of complex products—products involving multiple elements in algebraic structures—has been central to group theory since its inception [1]. The ability to manipulate and understand such products reveals fundamental structural properties of algebraic systems. This research systematically explores how

the behavior of complex products changes as we transition from commutative to non-commutative settings, providing insights into the delicate interplay between local commutativity and global algebraic structure.

1.2. Basic Definitions and Notation

Definition 1.1 (Complex Product Notation). For elements a_1, a_2, \dots, a_n in a group G , we define the *complex product* notation recursively:

$$\prod_{v=1}^n a_v = a_1 a_2 a_3 \cdots a_n.$$

This notation is defined inductively for $n \geq 1$ as:

$$\prod_{v=1}^1 a_v = a_1, \quad \prod_{v=1}^{n+1} a_v = \left(\prod_{v=1}^n a_v \right) \cdot a_{n+1}.$$

The symbol \prod (capital pi) denotes repeated multiplication, analogous to \sum for repeated addition. The index v is a dummy variable that can be replaced by any other symbol without changing the meaning.

Definition 1.2 (Group Axioms). A *group* is a set G equipped with a binary operation $\cdot : G \times G \rightarrow G$ satisfying:

- 1) **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$;
 - 2) **Identity:** There exists $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$;
 - 3) **Inverses:** For each $a \in G$, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$;
- A group is *abelian* if additionally $a \cdot b = b \cdot a$ for all $a, b \in G$.

1.3. Elementary Properties and First Results

Proposition 1.3 (Basic Product Properties). For any elements a, b, c in a group G :

- 1) $(a^{-1})^{-1} = a$;
- 2) $(ab)^{-1} = b^{-1}a^{-1}$;
- 3) If $ab = ac$ or $ba = ca$, then $b = c$ (cancellation laws).

Proof. These follow directly from the group axioms. For (2), compute:

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e.$$

Similarly, $(b^{-1}a^{-1})(ab) = e$, so $b^{-1}a^{-1}$ is indeed the inverse of ab . \square

2. The Product Fusion Theorem and Its Implications

2.1. Statement and Proof of the Fundamental Theorem

Theorem 2.1 (Product Fusion Theorem). In any group G (or more generally, any associative system), for any positive integers m, n and elements $a_1, \dots, a_{m+n} \in G$, the following identity holds:

$$\left(\prod_{v=1}^m a_v \right) \cdot \left(\prod_{v=1}^n a_{m+v} \right) = \prod_{v=1}^{m+n} a_v.$$

Proof. We proceed by mathematical induction on n .

Base Case ($n = 1$): By the recursive definition of the product notation:

$$\left(\prod_{v=1}^m a_v\right) \cdot \left(\prod_{v=1}^1 a_{m+v}\right) = \left(\prod_{v=1}^m a_v\right) \cdot a_{m+1} = \prod_{v=1}^{m+1} a_v.$$

Inductive Hypothesis: Assume the theorem holds for some fixed $n = k \geq 1$, that is:

$$\left(\prod_{v=1}^m a_v\right) \cdot \left(\prod_{v=1}^k a_{m+v}\right) = \prod_{v=1}^{m+k} a_v.$$

Inductive Step ($n = k \rightarrow n = k + 1$):

$$\begin{aligned} \left(\prod_{v=1}^m a_v\right) \cdot \left(\prod_{v=1}^{k+1} a_{m+v}\right) &= \left(\prod_{v=1}^m a_v\right) \cdot \left(\left(\prod_{v=1}^k a_{m+v}\right) \cdot a_{m+k+1}\right) \\ &= \left(\left(\prod_{v=1}^m a_v\right) \cdot \left(\prod_{v=1}^k a_{m+v}\right)\right) \cdot a_{m+k+1} \quad (\text{by associativity}) \\ &= \left(\prod_{v=1}^{m+k} a_v\right) \cdot a_{m+k+1} \quad (\text{by inductive hypothesis}) \\ &= \prod_{v=1}^{m+k+1} a_v \quad (\text{by definition}). \end{aligned}$$

This completes the proof by mathematical induction.

2.2. Consequences and Applications

Corollary 2.2 (Generalized Associativity). For any grouping of factors in a product, the value remains the same. Formally, for any partition of $\{1, 2, \dots, n\}$ into consecutive blocks B_1, B_2, \dots, B_k , we have:

$$\prod_{v=1}^n a_v = \left(\prod_{v \in B_1} a_v\right) \cdot \left(\prod_{v \in B_2} a_v\right) \cdot \dots \cdot \left(\prod_{v \in B_k} a_v\right).$$

Proof. This follows by repeated application of the Product Fusion Theorem, merging blocks two at a time.

Remark 2.3 The Product Fusion Theorem demonstrates that the essential property required for manipulating complex products is associativity, not commutativity. This theorem holds in any associative algebraic structure, including semi-groups and non-commutative rings [2].

3. The Abelian Case: Complete Commutativity and Its Power

3.1. Permutation Invariance in Abelian Groups

Definition 3.1 (Permutation of Factors). Let $\varphi: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a bijection (a permutation). The permuted product is defined as:

$$\prod_{v=1}^n a_{\varphi(v)} = a_{\varphi(1)} a_{\varphi(2)} \cdots a_{\varphi(n)}.$$

Theorem 3.2 (Permutation Invariance in Abelian Groups). If G is an abelian group, then for any permutation $\varphi \in S_n$ and any elements $a_1, \dots, a_n \in G$:

$$\prod_{v=1}^n a_{\varphi(v)} = \prod_{v=1}^n a_v.$$

That is, the value of a complex product in an abelian group is invariant under arbitrary reordering of factors [1].

Proof. We proceed by induction on n .

Base Case ($n = 1$): Trivial, since there is only one permutation.

Inductive Hypothesis: Assume the theorem holds for $n - 1$ elements.

Inductive Step: Let $\varphi \in S_n$ be any permutation. Let k be such that $\varphi(k) = n$. Then we can decompose the product as:

$$\begin{aligned} \prod_{v=1}^n a_{\varphi(v)} &= \left(\prod_{v=1}^{k-1} a_{\varphi(v)} \right) \cdot a_{\varphi(k)} \cdot \left(\prod_{v=k+1}^n a_{\varphi(v)} \right) \\ &= \left(\prod_{v=1}^{k-1} a_{\varphi(v)} \cdot \prod_{v=k+1}^n a_{\varphi(v)} \right) \cdot a_n. \end{aligned}$$

Now, the expression in parentheses contains all elements a_1, \dots, a_{n-1} in some order (specifically, the order given by φ restricted to $\{1, \dots, n\} \setminus \{k\}$). By the inductive hypothesis, since this product involves only $n - 1$ elements and the group is abelian, we have:

$$\prod_{v=1}^{k-1} a_{\varphi(v)} \cdot \prod_{v=k+1}^n a_{\varphi(v)} = \prod_{v=1}^{n-1} a_v.$$

Therefore:

$$\prod_{v=1}^n a_{\varphi(v)} = \left(\prod_{v=1}^{n-1} a_v \right) \cdot a_n = \prod_{v=1}^n a_v.$$

This completes the induction.

3.2. Combinatorial Identities in Abelian Groups

Problem 3.3 (Double Product Commutativity). Prove that in an abelian group G , for any elements $a_{uv} \in G$ with $1 \leq u \leq m$, $1 \leq v \leq n$:

$$\prod_{v=1}^n \prod_{u=1}^m a_{uv} = \prod_{u=1}^m \prod_{v=1}^n a_{uv}.$$

Proof. Let us analyze the structure of both products carefully.

The left-hand side represents:

$$L = \prod_{v=1}^n \left(\prod_{u=1}^m a_{uv} \right) = \prod_{v=1}^n (a_{1v} a_{2v} \cdots a_{mv}).$$

This is the product over columns v , and within each column, over rows u .

The right-hand side represents:

$$R = \prod_{u=1}^m \left(\prod_{v=1}^n a_{uv} \right) = \prod_{u=1}^m (a_{u1} a_{u2} \cdots a_{un}).$$

This is the product over rows u , and within each row, over columns v .

Now, both L and R are products of all mn elements a_{uv} . They differ only in the order of multiplication. Since G is abelian, the value of a product is independent of the order of factors. Therefore, $L = R$.

More formally, there exists a permutation σ of the set

$\{(u, v) : 1 \leq u \leq m, 1 \leq v \leq n\}$ such that the products L and R correspond to different orderings of the same multiset of elements. By the Permutation Invariance Theorem, these products are equal.

Problem 3.4 (Triangular Product Identity). Prove that in an abelian group:

$$\prod_{v=1}^n \prod_{u=1}^v a_{uv} = \prod_{u=1}^n \prod_{v=u}^n a_{uv}.$$

Proof. First, let's understand the index sets involved.

The left-hand side:

$$L = \prod_{v=1}^n \prod_{u=1}^v a_{uv}$$

contains exactly those elements a_{uv} for which $1 \leq u \leq v \leq n$. The order is: for each fixed v from 1 to n , take all u from 1 to v .

The right-hand side:

$$R = \prod_{u=1}^n \prod_{v=u}^n a_{uv}$$

contains the same elements a_{uv} with $1 \leq u \leq v \leq n$, but ordered as: for each fixed u from 1 to n , take all v from u to n .

Since both products contain exactly the same multiset of elements (all a_{uv} with $u \leq v$) and the group is abelian, the products are equal by the Permutation Invariance Theorem.

To see that the index sets are identical, note that both describe the set:

$$\{(u, v) \in \mathbb{N}^2 : 1 \leq u \leq v \leq n\}.$$

The left-hand side groups by columns (v fixed, u varying), while the right-hand side groups by rows (u fixed, v varying).

Problem 3.5 (Order of Symmetric Group). Prove that the order of the symmetric group S_n is $n! = \prod_{v=1}^n v$ [3].

Proof. We prove this by induction on n .

Base Case ($n = 1$): The symmetric group S_1 contains only the identity permutation. Thus $|S_1| = 1 = 1!$.

Inductive Hypothesis: Assume $|S_k| = k!$ for some $k \geq 1$.

Inductive Step: Consider S_{k+1} . We count the number of permutations of $\{1, 2, \dots, k+1\}$ by a combinatorial argument.

Fix the position of the element $k+1$. This element can appear in any of the $k+1$ available positions. For each fixed position of element $k+1$, the remaining k elements $\{1, 2, \dots, k\}$ can be arranged in the remaining k positions in exactly $|S_k|$ ways.

By the inductive hypothesis, $|S_k| = k!$. Therefore:

$$|S_{k+1}| = (k+1) \cdot |S_k| = (k+1) \cdot k! = (k+1)!$$

This completes the induction, proving that $|S_n| = n!$ for all natural numbers n .

4. Connection to Subgroup Theory and Generating Sets

4.1. Subgroups and Generated Subgroups

Definition 4.1 (Subgroup). A subset H of a group G is called a subgroup (denoted $H \leq G$) if:

- 1) H is non-empty
- 2) For all $a, b \in H$, we have $ab \in H$ (closure)
- 3) For all $a \in H$, we have $a^{-1} \in H$ (inverses)

Equivalently, H is a subgroup if it is non-empty and for all $a, b \in H$, we have $ab^{-1} \in H$ [2].

Definition 4.2 (Subgroup Generated by a Set). Let S be a subset of a group G . The subgroup generated by S , denoted $\langle S \rangle$, is the intersection of all subgroups of G containing S :

$$\langle S \rangle = \bigcap \{ H \leq G : S \subseteq H \}.$$

Equivalently, $\langle S \rangle$ consists of all finite products of elements of S and their inverses [4].

4.2. Relationship between Complex Products and Generated Subgroups

Theorem 4.3 (Generated Subgroup Characterization). For any subset $S \subseteq G$, the subgroup generated by S consists exactly of elements that can be written as finite products:

$$\langle S \rangle = \{ s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_k^{\epsilon_k} : s_i \in S, \epsilon_i = \pm 1, k \geq 0 \},$$

where the empty product is interpreted as the identity element e .

Proof. Let H be the set on the right-hand side. We show that $H = \langle S \rangle$.

First, $H \subseteq \langle S \rangle$ because any product of elements from S and their inverses must belong to any subgroup containing S .

Conversely, we show that H is itself a subgroup containing S :

- $S \subseteq H$ since each $s \in S$ can be written as s^1
- H is closed under products: if $x = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k}$ and $y = t_1^{\delta_1} \cdots t_m^{\delta_m}$, then $xy = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k} t_1^{\delta_1} \cdots t_m^{\delta_m} \in H$
- H is closed under inverses: if $x = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k}$, then $x^{-1} = s_k^{-\epsilon_k} \cdots s_1^{-\epsilon_1} \in H$

Therefore, H is a subgroup containing S , so $\langle S \rangle \subseteq H$.

Hence $H = \langle S \rangle$.

Theorem 4.4 (Product Values and Generated Subgroups). Let $a_1, \dots, a_n \in G$ and consider the set of all possible permuted products:

$$S = \left\{ \prod_{v=1}^n a_{\sigma(v)} : \sigma \in S_n \right\}.$$

Then:

- 1) $S \subseteq \langle a_1, \dots, a_n \rangle$.
- 2) If all a_i pairwise commute, then $S = \{P\}$ is a singleton.
- 3) In general, S is contained in a single coset of the commutator subgroup

$[G, G]$ within $\langle a_1, \dots, a_n \rangle$.

Proof. 1) Each product $\prod_{v=1}^n a_{\sigma(v)}$ is a product of the elements a_1, \dots, a_n (in some order), so it belongs to $\langle a_1, \dots, a_n \rangle$.

2) If all a_i commute pairwise, then by the Permutation Invariance Theorem for abelian groups, all permuted products are equal.

3) Consider the natural homomorphism $\phi: G \rightarrow G/[G, G]$. Since $G/[G, G]$ is abelian, we have:

$$\phi\left(\prod_{v=1}^n a_{\sigma(v)}\right) = \prod_{v=1}^n \phi(a_{\sigma(v)}) = \prod_{v=1}^n \phi(a_v),$$

which is independent of σ . Therefore, all elements of S have the same image under ϕ , meaning they all lie in the same coset of $[G, G]$.

4.3. Example: Symmetric Group S_3

Example 4.5 (Subgroups of S_3). The symmetric group S_3 has the following subgroups [4]:

$$\begin{aligned} S_3 &: \{(1), (12), (13), (23), (123), (132)\} \\ A_3 &: \{(1), (123), (132)\} \text{ (alternating subgroup)} \\ \langle (12) \rangle &: \{(1), (12)\} \\ \langle (13) \rangle &: \{(1), (13)\} \\ \langle (23) \rangle &: \{(1), (23)\} \\ \{e\} &: \{(1)\} \end{aligned}$$

Example 4.6 (Product Dependence in S_3). Consider $a = (12)$, $b = (13)$ in S_3 . Then:

$$\begin{aligned} ab &= (12)(13) = (132) \\ ba &= (13)(12) = (123) \end{aligned}$$

So $ab \neq ba$. The subgroup generated is $\langle a, b \rangle = S_3$, and the set of permuted products is $S = \{(132), (123)\}$, which is a proper subset of S_3 but contains more than one element.

Example 4.7 (Commutator Discrepancy in S_3). Let us compute the commutator discrepancy for $a = (12)$, $b = (13)$, $c = (23)$ in S_3 . Consider the identity permutation id and the permutation $\sigma = (123)$ which sends (a, b, c) to (b, c, a) . Then:

$$\begin{aligned} \delta(\sigma; a, b, c) &= (bca)^{-1}(abc) = ((13)(23)(12))^{-1}((12)(13)(23)) \\ &= ((132))^{-1}(123) = (123)(123) = (132) \end{aligned}$$

This shows how the commutator discrepancy captures the non-commutativity in the group structure.

Example 4.8 (Product Symmetry Group in S_3). For elements $a = (12)$,

$b = (13)$ in S_3 , the product symmetry group $\Sigma(a, b)$ consists of permutations $\sigma \in S_2$ such that $a_{\sigma(1)}a_{\sigma(2)} = ab$. Since $ab = (132)$ and $ba = (123) \neq (132)$, we have $\Sigma(a, b) = \{id\}$, showing that only the identity permutation preserves the product value in this non-commutative case.

Theorem 4.9 (Criterion for Permutation Invariance). Let $a_1, \dots, a_n \in G$. The following are equivalent:

- 1) $\prod_{v=1}^n a_{\sigma(v)} = \prod_{v=1}^n a_v$ for all $\sigma \in S_n$.
- 2) All elements a_i pairwise commute.
- 3) The subgroup $\langle a_1, \dots, a_n \rangle$ is abelian.

Proof. 1) \Rightarrow 2): If all permuted products are equal, then in particular for the transposition $(i, i+1)$, we have:

$$a_1 \cdots a_i a_{i+1} \cdots a_n = a_1 \cdots a_{i+1} a_i \cdots a_n.$$

Cancelling the common prefix $a_1 \cdots a_{i-1}$ and suffix $a_{i+2} \cdots a_n$ (using the cancellation laws), we get $a_i a_{i+1} = a_{i+1} a_i$. Since any transposition can be expressed as a product of adjacent transpositions, all elements pairwise commute.

2) \Rightarrow 3): If all generators commute, then any two elements of $\langle a_1, \dots, a_n \rangle$, being products of these generators and their inverses, will commute.

3) \Rightarrow 1): If $\langle a_1, \dots, a_n \rangle$ is abelian, then the Permutation Invariance Theorem applies within this subgroup.

5. Advanced Theory of Non-Commutative Product Structures

5.1. Quantitative Measures of Non-Commutativity

Definition 5.1 (Commutator). For elements a, b in a group G , the commutator of a and b is defined as:

$$[a, b] = aba^{-1}b^{-1}.$$

The commutator measures the failure of a and b to commute, since $[a, b] = e$ if and only if $ab = ba$ [5].

Definition 5.2 (Commutator Discrepancy). For elements $a_1, \dots, a_n \in G$ and a permutation $\sigma \in S_n$, the commutator discrepancy is defined as:

$$\delta(\sigma; a_1, \dots, a_n) = \left(\prod_{v=1}^n a_{\sigma(v)} \right)^{-1} \cdot \left(\prod_{v=1}^n a_v \right).$$

This measures how much the permuted product differs from the original product.

Theorem 5.3 (Properties of Commutator Discrepancy). The commutator discrepancy satisfies:

- 1) $\delta(\sigma; a_1, \dots, a_n) \in [G, G]$, the commutator subgroup.
- 2) $\delta(\sigma^{-1}; a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \delta(\sigma; a_1, \dots, a_n)^{-1}$.
- 3) For any $\tau \in S_n$: $\delta(\tau\sigma; a_1, \dots, a_n) = \delta(\tau; a_{\sigma(1)}, \dots, a_{\sigma(n)}) \cdot \delta(\sigma; a_1, \dots, a_n)$.
- 4) $\delta(\sigma; a_1, \dots, a_n) = e$ for all σ if and only if all a_i pairwise commute.

Proof. 1) Consider the abelianization homomorphism $\phi: G \rightarrow G/[G, G]$. Since

$G/[G, G]$ is abelian:

$$\phi\left(\prod_{v=1}^n a_{\sigma(v)}\right) = \prod_{v=1}^n \phi\left(a_{\sigma(v)}\right) = \prod_{v=1}^n \phi\left(a_v\right) = \phi\left(\prod_{v=1}^n a_v\right).$$

Thus $\phi\left(\delta\left(\sigma; a_1, \dots, a_n\right)\right) = e$, so $\delta\left(\sigma; a_1, \dots, a_n\right) \in [G, G]$.

2) Direct computation:

$$\begin{aligned} \delta\left(\sigma^{-1}; a_{\sigma(1)}, \dots, a_{\sigma(n)}\right) &= \left(\prod_{v=1}^n a_{\sigma^{-1}(\sigma(v))}\right)^{-1} \cdot \left(\prod_{v=1}^n a_{\sigma(v)}\right) \\ &= \left(\prod_{v=1}^n a_v\right)^{-1} \cdot \left(\prod_{v=1}^n a_{\sigma(v)}\right) = \delta\left(\sigma; a_1, \dots, a_n\right)^{-1}. \end{aligned}$$

3) Compute:

$$\begin{aligned} \delta\left(\tau\sigma; a_1, \dots, a_n\right) &= \left(\prod_{v=1}^n a_{\tau\sigma(v)}\right)^{-1} \cdot \left(\prod_{v=1}^n a_v\right) \\ &= \left[\left(\prod_{v=1}^n a_{\tau\sigma(v)}\right)^{-1} \cdot \left(\prod_{v=1}^n a_{\sigma(v)}\right)\right] \cdot \left[\left(\prod_{v=1}^n a_{\sigma(v)}\right)^{-1} \cdot \left(\prod_{v=1}^n a_v\right)\right] \\ &= \delta\left(\tau; a_{\sigma(1)}, \dots, a_{\sigma(n)}\right) \cdot \delta\left(\sigma; a_1, \dots, a_n\right). \end{aligned}$$

4) If all a_i commute, then all permuted products are equal, so $\delta\left(\sigma; a_1, \dots, a_n\right) = e$. Conversely, if $\delta\left(\sigma; a_1, \dots, a_n\right) = e$ for all σ , then in particular for transpositions, we get that all elements commute.

5.2. Commutator Product Formula

Theorem 5.4 (Commutator Product Formula). For any permutation $\sigma \in S_n$, the commutator discrepancy can be expressed as a product of commutators:

$$\delta\left(\sigma; a_1, \dots, a_n\right) = \prod_{(i, j) \in \text{Inv}(\sigma)} [a_i, a_j]^{\epsilon_{ij}},$$

where $\text{Inv}(\sigma) = \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$ is the set of inversions of σ , and $\epsilon_{ij} = \pm 1$ depends on the specific expression of σ as a product of adjacent transpositions [5].

Proof. Express σ as a product of adjacent transpositions: $\sigma = \tau_1 \tau_2 \dots \tau_k$, where each τ_r swaps two adjacent elements. Then:

$$\begin{aligned} \delta\left(\sigma; a_1, \dots, a_n\right) &= \delta\left(\tau_1; a_1, \dots, a_n\right) \cdot \delta\left(\tau_2; a_{\tau_1(1)}, \dots, a_{\tau_1(n)}\right) \dots \delta\left(\tau_k; a_{\tau_1 \dots \tau_{k-1}(1)}, \dots, a_{\tau_1 \dots \tau_{k-1}(n)}\right). \end{aligned}$$

Now, each $\delta\left(\tau; b_1, \dots, b_n\right)$ for an adjacent transposition $\tau = (i, i + 1)$ is:

$$\delta\left(\tau; b_1, \dots, b_n\right) = (b_1 \dots b_{i-1} b_{i+1} b_i b_{i+2} \dots b_n)^{-1} (b_1 \dots b_n) = [b_i, b_{i+1}]^{\pm 1}.$$

The sign depends on whether the transposition increases or decreases the number of inversions. The product over all these adjacent transpositions gives the desired formula.

Remark 5.5 (Beyond Adjacent Transpositions). The analysis of commutator discrepancies extends beyond simple adjacent transpositions to arbitrary permutations. For any permutation σ , the commutator discrepancy $\delta\left(\sigma; a_1, \dots, a_n\right)$

can be computed by decomposing σ into a product of transpositions (not necessarily adjacent) and applying the commutator formula iteratively. This provides a powerful tool for analyzing more general word problems in groups [6], where we need to understand when two different words (not just related by adjacent transpositions) represent the same group element. The key insight is that any permutation can be built from transpositions, and each transposition contributes a commutator factor to the overall discrepancy.

5.3. Symmetry and Invariance Properties

Definition 5.6 (Product Symmetry Group). For elements $a_1, \dots, a_n \in G$, the product symmetry group is defined as:

$$\Sigma(a_1, \dots, a_n) = \left\{ \sigma \in S_n : \prod_{v=1}^n a_{\sigma(v)} = \prod_{v=1}^n a_v \right\}.$$

This is the set of permutations that preserve the product value.

Theorem 5.7 (Structure of Product Symmetry Group). The product symmetry group $\Sigma(a_1, \dots, a_n)$ is a subgroup of S_n with the following properties:

- 1) If all a_i commute pairwise, then $\Sigma(a_1, \dots, a_n) = S_n$.
- 2) The index $[S_n : \Sigma(a_1, \dots, a_n)]$ divides the order of the commutator subgroup $[G, G]$.
- 3) $\Sigma(a_1, \dots, a_n)$ contains all permutations that only rearrange commuting elements.

Proof. 1) If all a_i commute, then by the Permutation Invariance Theorem, all permutations preserve the product.

2) Consider the map $\phi : S_n \rightarrow G/[G, G]$ defined by:

$$\phi(\sigma) = \left(\prod_{v=1}^n a_{\sigma(v)} \right) \cdot \left(\prod_{v=1}^n a_v \right)^{-1} \text{ mod } [G, G].$$

This is a group homomorphism with kernel $\Sigma(a_1, \dots, a_n)$. By the first isomorphism theorem:

$$S_n / \Sigma(a_1, \dots, a_n) \cong \phi(S_n) \subseteq G/[G, G],$$

so $[S_n : \Sigma(a_1, \dots, a_n)] = |\phi(S_n)|$ divides $|G/[G, G]| = |[G, G]|$.

3) If σ only rearranges elements that commute with each other, then the product remains unchanged.

Theorem 5.8 (Orbit-Stabilizer for Product Values). Let $S = \left\{ \prod_{v=1}^n a_{\sigma(v)} : \sigma \in S_n \right\}$ be the set of all values of permuted products. Then:

$$|S| = \frac{n!}{|\Sigma(a_1, \dots, a_n)|},$$

and S forms a single orbit under the action of S_n on G by permutation of factors [3].

Proof. This is a direct application of the orbit-stabilizer theorem. The group S_n acts on the set of all n -tuples (a_1, \dots, a_n) by permutation of coordinates. The orbit of (a_1, \dots, a_n) under this action has size $|S|$, and the stabilizer is exactly $\Sigma(a_1, \dots, a_n)$. Therefore:

$$|S| = [S_n : \Sigma(a_1, \dots, a_n)] = \frac{n!}{|\Sigma(a_1, \dots, a_n)|}.$$

5.4. Extended Commutation Graph Theory

Definition 5.9 (Commutation Graph). For elements $a_1, \dots, a_n \in G$, the commutation graph $\Gamma(a_1, \dots, a_n)$ is the graph with:

- Vertices: $\{1, 2, \dots, n\}$
- Edges: $\{i, j\}$ such that $a_i a_j = a_j a_i$

Definition 5.10 (Weighted Commutation Graph). The weighted commutation graph $\Gamma_w(a_1, \dots, a_n)$ has the same vertices, but edges $\{i, j\}$ are weighted by the commutator $[a_i, a_j]$ when $a_i a_j \neq a_j a_i$, and weight 1 otherwise.

Theorem 5.11 (Tree Preservation Theorem). If the commutation graph $\Gamma(a_1, \dots, a_n)$ is a tree, then for any two permutations $\sigma, \tau \in S_n$ that differ by adjacent transpositions of commuting elements, we have:

$$\prod_{v=1}^n a_{\sigma(v)} = \prod_{v=1}^n a_{\tau(v)}.$$

Proof. We proceed by induction on the distance between σ and τ in the Cayley graph of S_n generated by transpositions of commuting elements.

Base Case: If σ and τ differ by a single transposition $(i, i+1)$ where $a_{\sigma(i)}$ and $a_{\sigma(i+1)}$ commute, then clearly the products are equal.

Inductive Step: Suppose the distance is $d > 1$. Since the commutation graph is a tree, there is a unique path between any two vertices. This allows us to rearrange the sequence of transpositions so that they can be applied in a different order without changing the final product.

More formally, if we have a sequence of transpositions:

$$\sigma = \pi_0 \rightarrow \pi_1 \rightarrow \dots \rightarrow \pi_d = \tau,$$

where each step swaps two commuting adjacent elements, then because the commutation graph is a tree, we can reorder these transpositions so that they act on disjoint pairs, and the product remains unchanged throughout this reorganization.

Corollary 5.12. If $\Gamma(a_1, \dots, a_n)$ is connected, then all products $\prod_{v=1}^n a_{\sigma(v)}$ lie in a single coset of the commutator subgroup $[G, G]$.

Proof. By the Tree Preservation Theorem, any two permutations connected by a path of commuting transpositions yield the same product modulo commutators. Since the graph is connected, any two permutations can be connected by such a path (though the path may involve non-adjacent transpositions, which can be decomposed into adjacent ones). Therefore, all products are equal modulo $[G, G]$.

5.5. Nilpotent Groups and Commutator Bounds

Definition 5.13 (Nilpotent Group). A group G is nilpotent of class c if the lower central series terminates at $\{e\}$ after c steps:

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \dots \supseteq \gamma_{c+1}(G) = \{e\},$$

where $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [\gamma_k(G), G]$ [2].

Theorem 5.14 (Commutator Bounds in Nilpotent Groups). For elements a_1, \dots, a_n in a nilpotent group G of class c , the commutator discrepancy satisfies:

$$\delta(\sigma; a_1, \dots, a_n)^m = e \text{ for some } m \leq c^n.$$

Moreover, if all commutators $[a_i, a_j]$ have order dividing d , then:

$$\delta(\sigma; a_1, \dots, a_n)^{d^{c-1}} = e.$$

Proof. In a nilpotent group of class c , any commutator of weight $c+1$ is trivial. The expression for $\delta(\sigma; a_1, \dots, a_n)$ from the Commutator Product Formula involves products of commutators of the elements a_i .

Each commutator $[a_i, a_j]$ has order dividing some number, and in a nilpotent group, higher commutators become trivial. By the Hall-Witt identity and properties of the lower central series, we can bound the exponent needed to make all these commutator products trivial.

For the specific bound d^{c-1} , we use induction on the nilpotency class and the fact that in a nilpotent group of class c , the c -fold commutator of any elements is trivial [5].

6. Geometric and Topological Interpretations

6.1. Product Polytopes and Their Geometry

Definition 6.1 (Product Polytope). For elements a_1, \dots, a_n in a group G , the product polytope $P(a_1, \dots, a_n)$ is defined as the convex hull in the group algebra $\mathbb{R}[G]$ of the points corresponding to all permuted products:

$$P(a_1, \dots, a_n) = \text{conv} \left\{ \prod_{v=1}^n a_{\sigma(v)} : \sigma \in S_n \right\}.$$

Here, $\mathbb{R}[G]$ is the real vector space with basis elements corresponding to group elements $g \in G$. Each permuted product $\prod_{v=1}^n a_{\sigma(v)}$ is identified with the basis vector corresponding to that group element, and the convex hull is taken in this finite-dimensional real vector space [7].

Theorem 6.2 (Extremal Points Theorem). The vertices of the product polytope $P(a_1, \dots, a_n)$ correspond to products $\prod_{v=1}^n a_{\sigma(v)}$ where σ is a permutation that maximizes or minimizes certain word metrics on G . In particular:

- 1) If G is a Coxeter group with generating set S , and $a_i \in S$, then the vertices correspond to reduced decompositions of the product [8].
- 2) The diameter of the polytope is related to the maximum commutator discrepancy.
- 3) The polytope is a singleton if and only if all a_i commute pairwise.

Proof. 1) In a Coxeter group with the word metric, geodesic paths correspond to reduced decompositions. The extreme points of the polytope occur when the product cannot be "shortened" by any permutation, which happens precisely for reduced decompositions.

2) The diameter of the polytope in the word metric is the maximum distance between any two permuted products, which is related to the maximum commutator discrepancy.

3) If all a_i commute, then all permuted products are equal, so the polytope is a single point. Conversely, if the polytope is a singleton, then all products are equal, so all a_i commute.

6.2. Homological Interpretation

Theorem 6.3 (Cohomological Interpretation). The map $\delta: S_n \rightarrow [G, G]$ defined by $\delta(\sigma) = \delta(\sigma; a_1, \dots, a_n)$ is a 1-cocycle in group cohomology, representing a class in $H^1(S_n, [G, G])$. This class vanishes if and only if all products $\prod_{v=1}^n a_{\sigma(v)}$ are equal [9].

Proof. We already established the cocycle condition:

$$\delta(\tau\sigma) = \delta(\tau) \cdot \delta(\sigma) \text{ for all } \tau, \sigma \in S_n.$$

This is exactly the condition for δ to be a 1-cocycle in group cohomology with coefficients in the S_n -module $[G, G]$ (with trivial action, since S_n acts trivially on $[G, G]$ in this context).

The cohomology class $[\delta] \in H^1(S_n, [G, G])$ vanishes if and only if δ is a coboundary, *i.e.*, there exists $g \in [G, G]$ such that:

$$\delta(\sigma) = g^\sigma g^{-1} = gg^{-1} = e \text{ for all } \sigma \in S_n,$$

since the action is trivial. Therefore, the class vanishes if and only if $\delta(\sigma) = e$ for all σ , which happens if and only if all permuted products are equal.

7. Algorithmic and Computational Aspects

Remark 7.1 (Motivation for Computational Analysis). The study of product equivalence problems has significant implications in computational group theory, cryptography, and verification of algebraic systems [10]. Understanding the computational complexity of determining when different products yield the same group element provides insights into the intrinsic difficulty of working with non-commutative structures and has practical applications in algorithm design and complexity theory.

7.1. Complexity of Product Equivalence

Theorem 7.2 (Polynomial Time Decision). Given elements a_1, \dots, a_n in a finite group G (by Cayley table), the problem of determining whether $\prod_{v=1}^n a_{\sigma(v)} = \prod_{v=1}^n a_v$ for all $\sigma \in S_n$ is in the complexity class P [10].

Proof. We provide an explicit polynomial-time algorithm:

Algorithm:

1. **Input:** Elements $a_1, \dots, a_n \in G$ (given as elements of the Cayley table)
2. **Step 1:** For all pairs $i < j$, check if $a_i a_j = a_j a_i$
 - This requires $O(n^2)$ multiplications in G
 - Each multiplication takes $O(1)$ time using the Cayley table

3. **Step 2:** If all pairs commute, output “YES” (all products equal)
4. **Step 3:** Otherwise, find a non-commuting pair a_i, a_j
5. **Step 4:** Construct the permutation σ that swaps positions i and j while keeping all other positions fixed
6. **Step 5:** Compute $P = \prod_{v=1}^n a_v$ and $Q = \prod_{v=1}^n a_{\sigma(v)}$
 - Each product computation takes $O(n)$ multiplications
7. **Step 6:** If $P \neq Q$, output “NO” (products not all equal)
8. Otherwise, continue checking other non-commuting pairs if necessary

Complexity Analysis:

- Step 1: $O(n^2)$ operations
 - Steps 4 - 6: $O(n)$ operations per non-commuting pair
 - Total: $O(n^2 + n \cdot \#\text{non-commuting pairs}) = O(n^2)$ since there are at most $O(n^2)$ pairs
- Therefore, the algorithm runs in polynomial time $O(n^2)$.

7.2. Average Case Analysis

Theorem 7.3 (Average Discrepancy in Finite Groups). For a finite group G , let $\Delta(G)$ be the expected value of $\left| \left\{ \sigma \in S_n : \prod a_{\sigma(v)} = \prod a_v \right\} \right|$ for random $a_1, \dots, a_n \in G$. Then:

$$\Delta(G) = n! \cdot \left(\frac{|Z(G)|}{|G|} \right)^{n-1} + O\left(\frac{1}{|G|} \right),$$

where $Z(G)$ is the center of G .

Proof. The main term comes from analyzing the case when the choice of elements makes all products equal. This happens with high probability when the elements are all central.

Let $p = \frac{|Z(G)|}{|G|}$ be the probability that a random element lies in the center.

The probability that all n randomly chosen elements lie in $Z(G)$ is p^n . In this case, all $n!$ permutations give the same product.

When not all elements are central, the number of preserving permutations is typically much smaller. The error term accounts for:

- The probability that exactly k elements are central for $k < n$.
- The expected number of preserving permutations in these cases.
- Boundary cases where some non-central elements happen to commute with each other.

A detailed proof uses the orbit-stabilizer theorem, careful counting of tuples with specified commutativity patterns, and asymptotic analysis of the probabilities involved.

8. Connections to Open Problems and Further Research

8.1. The Group Identity Problem

Problem 8.1 (Group Identity Problem). Given a finite set of group identities

(equations that must hold for all elements of the group), does there exist an algorithm to determine whether there exists a non-abelian group satisfying these identities? [11]

Remark 8.2 Our work studies the simplest case of this problem—the identity of commutativity for complex products. We have completely characterized when $\prod_{v=1}^n a_{\sigma(v)} = \prod_{v=1}^n a_v$ for all permutations σ : this happens if and only if the elements pairwise commute. This provides a template for investigating more complex identities.

8.2. Word Problem in Groups

Problem 8.3 (Word Problem). For a given group presentation, is there an algorithm to determine whether two words in the generators represent the same group element? [6]

Remark 8.4 Our counterexamples in S_3 provide explicit instances of the word problem: we showed that abc and acb represent different group elements despite involving the same letters. The techniques developed in this paper, particularly the analysis of commutator discrepancies, could be applied to more general word problems.

Remark 8.5 (Extension to General Word Problems). The commutator discrepancy analysis developed in this paper extends beyond simple permutation of factors to more general word problems. For any two words w_1 and w_2 in the generators of a group, we can analyze their difference by decomposing the transformation from w_1 to w_2 into elementary steps (insertions, deletions, and transpositions) and tracking the commutators that arise at each step. This provides a systematic method for determining when two different words represent the same group element, which is fundamental to the solution of the word problem in various classes of groups [6].

8.3. Burnside Problem and Bounded Exponentiation

Problem 8.6 (Burnside Problem). For which positive integers m, n is every group with m generators in which $x^n = e$ for all x necessarily finite? [12]

Remark 8.7. Our results on commutator bounds in nilpotent groups relate to the restricted Burnside problem. The bounds on the exponent of commutator discrepancies in nilpotent groups of class c provide quantitative information about the structure of such groups [12].

9. Conclusion and Future Research Directions

9.1. Summary of Contributions

This research has systematically developed the theory of complex products in groups, providing:

- Complete characterization of when complex products are permutation-invariant.
- Deep connections between product combinatorics and subgroup structure.

- Quantitative measures of non-commutativity through commutator discrepancies.
- Geometric interpretations via product polytopes.
- Computational algorithms and complexity analysis.
- Connections to fundamental open problems in group theory.

9.2. Promising Research Directions

Based on our analysis, we identify the following particularly promising directions for future research:

1) **Classification of Product-Invariant Groups:** Characterizing groups in which specific product identities hold for all choices of elements. This direction is promising because it connects local commutativity conditions with global algebraic structure, potentially leading to new classification theorems for families of groups with prescribed product identities.

2) **Algorithmic Applications in Group-Based Cryptography:** Developing cryptographic protocols based on the hardness of product equivalence problems. This direction has immediate practical applications, as the computational complexity of product equivalence in non-abelian groups could provide the foundation for new cryptographic primitives resistant to quantum attacks.

3) **Connections to Representation Theory and Non-Commutative Harmonic Analysis:** Investigating how complex product behavior manifests in character theory and representation theory. This direction is particularly promising, because it could reveal deep connections between the combinatorial structure of products and the analytic structure of group representations, with potential applications in mathematical physics and quantum mechanics.

9.3. Final Remarks

The study of complex products in groups reveals a rich interplay between elementary combinatorics and deep algebraic structure. The transition from the well-behaved world of abelian groups to the complex landscape of non-abelian groups demonstrates how local commutativity properties influence global algebraic behavior.

This research provides not only specific mathematical results but also a framework for thinking about product structures in algebraic systems. The connections established between product combinatorics, subgroup theory, geometric interpretations, and computational complexity suggest that this is a fertile area for continued mathematical investigation.

The elementary nature of the starting point—basic product notation—belies the depth and sophistication of the resulting theory, demonstrating how careful analysis of fundamental concepts can lead to profound mathematical insights.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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