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Some Hermite-Hadamard-Mercer Inequalities for General Fractional Convex on the Coordinates

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Abstract

In this paper, we establish new Hermite-Hadamard-Mercer type inequalities for functions that are convex on the coordinates by employing the framework of generalized fractional integrals. Several integral identities are derived to obtain midpoint, trapezoidal, and Simpson-type inequalities as special cases. The presented results unify and extend numerous existing inequalities in convex analysis and fractional calculus, thereby providing a more general framework for further applications in numerical analysis, approximation theory, and optimization.

Keywords

Hermite-Hadamard-Mercer Inequalities, Fractional Integrals, Generalized Convexity, Coordinate-Wise Convex Functions, Fractional Calculus

1. Introduction

The classical Hermite-Hadamard inequality, initially introduced by Hermite and Hadamard, offers fundamental bounds for the integral mean of convex functions. For a real-valued convex function $f:[a,b]\to\mathbb{R}$ is a convex then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le\frac{f(a)+f(b)}{2}.$$

This inequality has inspired numerous extensions and generalizations across various domains, including operator theory, approximation theory, and fractional calculus.

In 2003, Mercer [1] introduced a refined version of the classical Jensen inequality, now known as the Jensen-Mercer inequality, stated as follows.

For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequality holds for each $x_j\in[a,b]$:

$$f\left(a+b-\sum_{j=1}^{n}u_jx_j\right)\le f(a)+f(b)-\sum_{j=1}^{n}u_jf(x_j),$$

where $u_j\in[0,1]$ and $\sum_{j=1}^{n}u_j=1$.

The Jensen-Mercer inequality prompted significant developments in the analysis of convexity in higher dimensions. More recently, attention has been devoted to the study of coordinate-wise convex functions, *i.e.*, functions convex in each variable separately, which naturally arise in the context of bivariate and multivariate analysis.

Alongside these developments, the framework of fractional calculus has emerged as a powerful tool for extending classical integral inequalities. The incorporation of Riemann-Liouville, Hadamard, and Katugampola-type fractional operators has led to refined inequalities that capture nonlocal and memory-dependent behavior of convex functions. Sarikaya *et al.*, Katugampola, and others have contributed to this area by developing fractional analogues of Hermite-Hadamard-type inequalities. Coordinate convex functions naturally arise in optimization problems, economic modeling, and approximation theory, where convexity along each coordinate direction plays a crucial role in ensuring stability and tractability of solutions.

Motivated by these advances, Toseef *et al.* [2] recently proposed a Hermite-Hadamard-Mercer-type inequality on the coordinates, which integrates the ideas of Mercer convexity and fractional integrals:

For a convex mapping $f:[a,b]\times[c,d]\to\mathbb{R}$, the following inequality holds for each $x_j\in[a,b]$ and $y_j\in[c,d]$:

$$\begin{aligned}&f\left(a+b-\sum_{j=1}^{n}u_jx_j,c+d-\sum_{j=1}^{n}w_jy_j\right)\\&\le f(a,c)+f(a,d)+f(b,c)+f(b,d)-\sum_{j=1}^{n}u_jw_jf(x_j,y_j),\end{aligned}$$

where $u_j,w_j\in[0,1]$ and $\sum_{j=1}^{n}u_j=1$ and $\sum_{j=1}^{n}w_j=1$.

In 2013, Kian and Mosleshian [3] used the new Jensen-Mercer inequality and established the following new version of the Hermite-Hadamard inequality:

For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold for all $x,y\in[a,b]$ and $x<y$:

$$\begin{aligned}f\left(a+b-\frac{x+y}{2}\right)&\le f(a)+f(b)-\frac{1}{y-x}\int_x^y f(u)\mathrm{d}u\\&\le f(a)+f(b)-f\left(\frac{x+y}{2}\right)\end{aligned}$$

and

$$f\left(a+b-\frac{x+y}{2}\right)\le\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(u)\mathrm{d}u$$

$$\leq \frac{f(a+b-x)+f(a+b-y)}{2}$$
$$\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right).$$

Building on this foundation, the present paper aims to extend and generalize such inequalities within the framework of generalized fractional integrals, particularly those involving kernels that unify and extend classical operators.

The Riemann-Liouville fractional integral operators are defined as:

For an integrable function f on $[a,b]$, the left and the right Riemann-Liouville fractional integrals are defined as:

$$J_{a^+}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1} f(u)\mathrm{d}u,$$

$$J_{b^-}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (u-x)^{\alpha-1} f(u)\mathrm{d}u,$$

where Γ represents the Gamma function function and is defined as
$\Gamma(\alpha)=\int_0^{\infty} \mathrm{e}^{-\tau}\tau^{\alpha-1}\mathrm{d}\tau$.

Sarikaya *et al.* [4] used the Riemann-Liouville fractional integrals and derived the following version of the Hermite-Hadamard-type inequality for convex functions.

For a convex function $f:[a,b]\to\mathbb{R}$, the following inequality holds:

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right]\leq\frac{f(a)+f(b)}{2}.$$

For a convex function $f:[a,b]\to\mathbb{R}$, the following inequality holds for all $x,y\in[a,b]$ and $x<y$:

$$f\left(a+b-\frac{x+y}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{(a+b-y)^+}^{\alpha} f(a+b-x)+J_{(a+b-x)^-}^{\alpha} f(a+b-y)\right]$$
$$\leq\frac{f(a+b-x)+f(a+b-y)}{2}$$
$$\leq f(a)+f(b)-\frac{f(x)+f(y)}{2}.$$

For an integrable function f on $[a,b]$, the left and the right κ-Riemann-Liouville fractional integrals are defined as:

$$J_{a^+}^{\alpha,\kappa} f(x)=\frac{1}{\kappa\Gamma_{\kappa}(\alpha)}\int_a^x (x-u)^{\frac{\alpha}{\kappa}-1} f(u)\mathrm{d}u,$$

$$J_{b^-}^{\alpha} f(x)=\frac{1}{\kappa\Gamma_{\kappa}(\alpha)}\int_x^b (u-x)^{\frac{\alpha}{\kappa}-1} f(u)\mathrm{d}u,$$

where $\Gamma_{\kappa}(\alpha)$ is the κ-Gamma function, and is defined as

$\Gamma_{\kappa}(\alpha)=\int_0^{\infty} \mathrm{e}^{-\frac{\tau^{\kappa}}{\kappa}}\tau^{\alpha-1}\mathrm{d}\tau$.

Sarikaya and Ata [5] presented the generalized variant of the Hermite-Hadamard inequality bu using the beta function:

For a convex function $f:[a,b]\to\mathbb{R}$, the following inequality holds:

$$f\left(\frac{a+b}{2}\right)\beta(m,n)\le\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Xi(u)f(u)\mathrm{d}u\le\frac{f(a)+f(b)}{2}\beta(m,n)$$

where $\beta(m,n)=\int_0^1\tau^{m-1}(1-\tau)^{n-1}\mathrm{d}\tau$ is the beta function and

$$\Xi(u)=(b-u)^{m-1}(u-a)^{n-1}+(b-u)^{n-1}(u-a)^{m-1}.$$

Ali, Zhang and Fečkan [6] extended the generalized variant of the Hermite-Hadamard-Mercer inequality by using the beta function:

For a convex function $f:[a,b]\to\mathbb{R}$, the following inequality holds:

$$\begin{aligned}&f\left(a+b-\frac{x+y}{2}\right)\beta(m,n)\\&\le[f(a)+f(b)]\beta(m,n)-\frac{1}{2(y-x)^{m+n-1}}\int_x^y\Xi(u)f(u)\mathrm{d}u\\&\le\left[f(a)+f(b)-\frac{f(a)+f(b)}{2}\right]\beta(m,n),\end{aligned}$$

where $\beta(m,n)=\int_0^1\tau^{m-1}(1-\tau)^{n-1}\mathrm{d}\tau$ is the beta function and

$$\Xi(u)=(y-u)^{m-1}(u-x)^{n-1}+(y-u)^{n-1}(u-x)^{m-1}$$

and $x,y\in[a,b]$.

For a convex function $f:[a,b]\to\mathbb{R}$, the following inequality holds:

$$\begin{aligned}&f\left(a+b-\frac{x+y}{2}\right)\beta(m,n)\\&\le[f(a)+f(b)]\beta(m,n)-\frac{1}{2(y-x)^{m+n-1}}\int_{a+b-y}^{a+b-x}\Xi(u)f(u)\mathrm{d}u\\&\le\frac{f(a+b-x)+f(a+b-y)}{2}\beta(m,n)\\&\le\left[f(a)+f(b)-\frac{f(a)+f(b)}{2}\right]\beta(m,n),\end{aligned}$$

where $\beta(m,n)$ is the beta function and

$$\begin{aligned}\Xi(u)=&[(a+b-x)-u]^{m-1}[u-(a+b-y)]^{n-1}\\&+[(a+b-x)-u]^{n-1}[u-(a+b-y)]^{m-1}\end{aligned}$$

and $x,y\in[a,b]$.

In [7], Katugampola introduced a new fractional which generalizes the Riemann-Liouville and the Hadamard fractional integrals into a single form as follow.

Let $[a,b]\subset\mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integralsod order $\alpha>0$ of $f\in X_c^p(a,b)$ are defined by

$${}^{\rho}I_{a^+}^{\alpha}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}f(t)\mathrm{d}t$$

and

$$ {}^{\rho}I_{b^-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{\left(t^{\rho} - x^{\rho}\right)^{1-\alpha}} f(t) \mathrm{d}t $$

where $a < x < b$ and $\rho > 0$, if the integral exists.

Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,

1) $\lim_{\rho \to 1} {}^{\rho}I_{a^+}^{\alpha} f(x) = J_{a^+}^{\alpha} f(x)$,

2) $\lim_{\rho \to 0^+} {}^{\rho}I_{a^+}^{\alpha} f(x) = H_{a^+}^{\alpha} f(x)$.

Similar results also hold for right-sided operators.

In [8], Sarikaya and Ertuğral gave the definition of generalized fractional integrals (GFIs) as following:

The left-sided and right-sided GFIs are denoted by ${}_{a^+}I_{\varphi}$ and ${}_{b^-}I_{\varphi}$ as followings:

$$ {}_{a^+}I_{\varphi} f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} \mathrm{d}t, \quad x > a, $$

and

$$ {}_{b^-}I_{\varphi} f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} \mathrm{d}t, \quad x < b, $$

where a function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the condition $\int_0^1 \frac{\varphi(t)}{t} \mathrm{d}t < \infty$.

The most important feature of generalized fractional integrals is that they generalize some type of fractional integrals such as the Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional, and Hadamard feractional integrals. These important special cases of integral operator are mentioned below.

(1) If we choose $\varphi(x) = x$, the operators ${}_{a^+}I_{\varphi} f(x)$ and ${}_{b^-}I_{\varphi} f(x)$ are reduce to the Riemann integral.

(2) Considering $\varphi(x) = \frac{x^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, the operators ${}_{a^+}I_{\varphi} f(x)$ and ${}_{b^-}I_{\varphi} f(x)$ are reduce to the Riemann-Liouville fractioal integrals $I_{a^+}^{\alpha} f(x)$ and $I_{b^-}^{\alpha} f(x)$, respectively. Here, Γ is a gamma function.

(3) For $\varphi(x) = \frac{1}{k\Gamma(\alpha)} x^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators ${}_{a^+}I_{\varphi} f(x)$ and ${}_{b^-}I_{\varphi} f(x)$ are reduce to the k-Riemann-Liouville fractional integrals $I_{a^+,k}^{\alpha} f(x)$ and $I_{b^-,k}^{\alpha} f(x)$, respectively. Here, Γ_k is a k-gamma function.

Further more, for more results in the field if coordinated convex we refer the interested to see [8].

For a mapping $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ is a convex on the coordinates, the following inequality holds:

$$ \begin{aligned} & f\left(ta + (1-t)b, sc + (1-s)d\right) \\ & \leq tsf(a,c) + t(1-s) f(a,d) + (1-t)sf(b,c) + (1-t)(1-s) f(b,d) \end{aligned} $$

for $t, s \in [0,1]$.

The following Hermite-Hadamard type inequality for coordinated convex functions on the retangle from the plane $\mathbb{R}^2$ was proved in [4], that is:

Suppose that a function $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is a convex on coordinates. Then one has the inequalities:

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\mathrm{d}x+\frac{1}{b-a}\int_a^b f(x,d)\mathrm{d}x\right.\\
&\quad\left.+\frac{1}{d-c}\int_c^d f(a,y)\mathrm{d}y+\frac{1}{d-c}\int_c^d f(b,y)\mathrm{d}y\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

In 2024, The Hermite-Hadamard-Mercer type inequalities for coordinated convex functions was recently established by Toseef *et al.* in [2], which is stated as:

For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequality holds for each $x_j\in[a,b]$, $y_j\in[c,d]$:

$$\begin{aligned}
&f\left(a+b-\sum_{j=1}^{n}u_jx_j,\,c+d-\sum_{j=1}^{n}w_jy_j\right)\\
&\le f(a,c)+f(a,d)+f(b,c)+f(b,d)-\sum_{i=1}^{n}u_jw_jf\left(x_j,y_y\right),
\end{aligned}$$

where $u_j,w_j\in[0,1]$ and $\sum_{j=1}^{n}u_j=1$, $\sum_{j=1}^{n}w_j=1$.

Assume that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is a convex on coordinates. Then one has the inequalities:

$$\begin{aligned}
&f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\\
&\le\frac{1}{2}\left[\frac{1}{m-n}\int_{a+b-n}^{a+b-m}f\left(x,c+d-\frac{k+l}{2}\right)\mathrm{d}x+\frac{1}{l-k}\int_{c+d-l}^{c+d-k}f\left(a+b-\frac{m+n}{2},y\right)\mathrm{d}y\right]\\
&\le\frac{1}{(m-n)(l-k)}\int_{a+b-n}^{a+b-m}\int_{c+d-l}^{c+d-k}f(x,y)\,\mathrm{d}y\mathrm{d}x\\
&\le\frac{1}{2}\left[\frac{1}{m-n}\int_{a+b-n}^{a+b-m}f(x,c+d-l)\mathrm{d}x+\frac{1}{m-n}\int_{a+b-n}^{a+b-m}f(x,c+d-k)\mathrm{d}x\right.\\
&\quad\left.+\frac{1}{l-k}\int_{c+d-l}^{c+d-k}f(a+b-n,y)\mathrm{d}y+\frac{1}{l-k}\int_{c+d-l}^{c+d-k}f(a+b-m,y)\mathrm{d}y\right]\\
&\quad-\frac{1}{4}\left[\frac{1}{m-n}\int_{a+b-n}^{a+b-m}f(x,l)\mathrm{d}x+\frac{1}{m-n}\int_{a+b-n}^{a+b-m}f(x,k)\mathrm{d}x\right.\\
&\quad\left.+\frac{1}{l-k}\int_{c+d-l}^{c+d-k}f(n,y)\mathrm{d}y+\frac{1}{l-k}\int_{c+d-l}^{c+d-k}f(m,y)\mathrm{d}y\right]\\
&\le\frac{1}{4}\left[f(a+b-m,c+d-k)+f(a+b-m,c+d-l)\right.
\end{aligned}$$

$$+f(a+b-n,c+d-k)+f(a+b-n,c+d-l)\Big]$$
$$\le f(a,c)+f(a,d)+f(b,c)+f(b,d)-\left[\frac{f(m,k)+f(m,l)+f(n,k)+f(n,l)}{4}\right],$$

where $m,n\in[a,b]$ and $k,l\in[c,d]$.

We introduce a new class of Hermite-Hadamard-Mercer-type inequalities involving generalized fractional integrals for functions that are convex on the coordinates.

$${}_{a^+,c^+}I_\varphi f(x,y)=\int_c^y\int_a^x\frac{\varphi(x-t)\varphi(y-s)}{(x-t)(y-s)}f(t,s)\mathrm{d}t\mathrm{d}s,$$

$${}_{a^+,d^-}I_\varphi f(x,y)=\int_y^d\int_a^x\frac{\varphi(x-t)\varphi(s-y)}{(x-t)(s-y)}f(t,s)\mathrm{d}t\mathrm{d}s,$$

$${}_{b^-,c^+}I_\varphi f(x,y)=\int_c^y\int_x^b\frac{\varphi(t-x)\varphi(y-s)}{(t-x)(y-s)}f(t,s)\mathrm{d}t\mathrm{d}s,$$

and

$${}_{b^-,d^-}I_\varphi f(x,y)=\int_y^d\int_x^b\frac{\varphi(t-x)\varphi(s-y)}{(t-x)(s-y)}f(t,s)\mathrm{d}t\mathrm{d}s.$$

We derive integral identities that enable us to establish midpoint, trapezoidal inequalities in this generalized setting.

We demonstrate that our results encapsulate several known inequalities as special cases, thus unifying prior literature under a broader and more flexible framework.

2. Main Results

Firstly, we establish an integral identity for differential function via general fractional integrals to dirive main results.

Lemma 2.1

Assume that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is continuous and integrable, the following fractional equality holds:

$$\Big[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)+f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\Big]$$
$$-\frac{1}{\Lambda(1)}\Big[{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$
$$-\frac{1}{\Delta(1)}\Big[{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$
$$+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$

$$
=\frac{(n-m)(l-k)}{\Lambda(1)\Delta(1)}\int_0^1\int_0^1\Omega(t,s)\left[\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\right)\mathrm{d}s\mathrm{d}t\right.
$$

$$
+\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-(sk+(1-s)l)\right)\mathrm{d}s\mathrm{d}t
$$

$$
+\frac{\partial^2}{\partial t\partial s}f\left(a+b-(tm+(1-t)n),c+d-((1-s)k+sl)\right)\mathrm{d}s\mathrm{d}t
$$

$$
\left.+\frac{\partial^2}{\partial t\partial s}f\left(a+b-(tm+(1-t)n),c+d-(sk+(1-s)l)\right)\mathrm{d}s\mathrm{d}t\right],
$$

where

$$
\Omega(t,s)=\int_0^t\int_0^s\frac{\varphi((n-m)u)\varphi((l-k)\lambda)}{u\lambda}\mathrm{d}\lambda\mathrm{d}u
$$

and

$$
\Lambda(t)=\int_0^t\frac{\varphi((n-m)u)}{u}\mathrm{d}u \quad \text{and} \quad \Delta(s)=\int_0^s\frac{\varphi((l-k)\lambda)}{\lambda}\mathrm{d}\lambda .
$$

Proof.

It suffices to note that

$$
S_1=\int_0^1\int_0^1\Omega(t,s)\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\right)\mathrm{d}s\mathrm{d}t
$$

$$
S_2=\int_0^1\int_0^1\Omega(t,s)\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\right)\mathrm{d}s\mathrm{d}t
$$

$$
S_3=\int_0^1\int_0^1\Omega(t,s)\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\right)\mathrm{d}s\mathrm{d}t
$$

and

$$
S_4=\int_0^1\int_0^1\Omega(t,s)\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\right)\mathrm{d}s\mathrm{d}t.
$$

If we add from $S_1-S_2-S_3+S_4$ and multiply by $\frac{(n-m)(l-k)}{\Lambda(1)\Delta(1)}$, we obtain the proof.

Theorem 2.2

Assume that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is continuous and integrable and $\left|\frac{\partial^2}{\partial t\partial s}f\right|$ satisfies the Mercer's ineaqality, the following fractional inequality holds:

$$
\left|\frac{\left[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)+f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\right]}{4}\right.
$$

$$
-\frac{1}{4\Lambda(1)}\left[{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\right.
$$

$$
\left.+{}_{(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\right]
$$

$$
-\frac{1}{4\Delta(1)}\left[{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-k)\right.
$$

$$+ {}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-l) + {}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$

$$+\frac{1}{4\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l) + {}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$

$$+ {}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(a+b-m,c+d-l) + {}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Big|$$

$$\le \frac{(n-m)(l-k)}{\Lambda(1)\Delta(1)}\int_0^1\int_0^1|\Omega(t,s)|\Big[\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|$$

$$-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Big]\mathrm{d}s\mathrm{d}t.$$

Proof:

From the Lemma 2.1, we have

$$\Big|\big[f(a+b-n,c+d-l) + f(a+b-n,c+d-k) + f(a+b-m,c+d-l) + f(a+b-m,c+d-k)\big]$$

$$-\frac{1}{\Lambda(1)}\Big[{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l) + {}_{(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$

$$+ {}_{(c+d-k)^-}I_\varphi f(a+b-m,c+d-l) + {}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$

$$-\frac{1}{\Delta(1)}\Big[{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l) + {}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-k)$$

$$+ {}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-l) + {}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$

$$+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l) + {}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$

$$+ {}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(a+b-m,c+d-l) + {}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Big|$$

$$\le \frac{(n-m)(l-k)}{\Lambda(1)\Delta(1)}\int_0^1\int_0^1|\Omega(t,s)|\Big[\Big|\frac{\partial^2}{\partial t\partial s}f\big(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\big)\mathrm{d}s\mathrm{d}t$$

$$+\frac{\partial^2}{\partial t\partial s}f\big(a+b-((1-t)m+tn),c+d-(sk+(1-s)l)\big)\mathrm{d}s\mathrm{d}t$$

$$+\frac{\partial^2}{\partial t\partial s}f\big(a+b-(tm+(1-t)n),c+d-((1-s)k+sl)\big)\mathrm{d}s\mathrm{d}t$$

$$\frac{\partial^2}{\partial t\partial s}f\big(a+b-(tm+(1-t)n),c+d-(sk+(1-s)l)\big)\mathrm{d}s\mathrm{d}t\Big|\Big]$$

$$\le \frac{4(n-m)(l-k)}{\Lambda(1)\Delta(1)}\int_0^1\int_0^1|\Omega(t,s)|\Big[\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right| + \left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|$$

$$+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right| - \left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Big]\mathrm{d}s\mathrm{d}t.$$

Remark 2.3

If the kernel function is chosen as $\varphi(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$ with $\alpha > 0$, then the generalized fractional integral reduces to the classical Riemann-Liouville operator. In this case, Theorem 2.2 recovers the Hermite-Hadamard-Mercer type inequality obtained by Sarikaya and Yildirim [4].

Theorem 2.4

Suppose that tha assumptions of Lemma 2.1 are hold. If $\left|\frac{\partial^2}{\partial t\partial s} f(t,s)\right|^\beta$ is co-ordinated convex function on $[a,b]\times[c,d]$, then we have the inequality

$$\begin{aligned}
&\Bigg|\frac{\left[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)+f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\right]}{4}\\
&-\frac{1}{4\Lambda(1)}\Big[{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\\
&+{}_{(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\\
&-\frac{1}{4\Delta(1)}\Big[{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-k)\\
&+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)\Big]\\
&+\frac{1}{4\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\\
&+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Bigg|\\
&\le\frac{4(n-m)(l-k)}{\Lambda(1)\Delta(1)}\left(\int_0^1\int_0^1|\Omega(t,s)|^\alpha\,\mathrm{dsdt}\right)^{\frac{1}{\alpha}}\times\Bigg[\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^\beta+\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^\beta+\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^\beta\\
&+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^\beta\Bigg]^{\frac{1}{\beta}}
\end{aligned}$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$, $\beta>1$.

Proof.

From the Lemma 2.1 and and Jensen-Mercer inequality by using the Hölder inequality and the convexity of $\left|\frac{\partial^2}{\partial t\partial s} f(t,s)\right|^\beta$, we obtain

$$\begin{aligned}
&\Big|\left[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)+f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\right]\\
&-\frac{1}{\Lambda(1)}\Big[{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\\
&+{}_{(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\\
&-\frac{1}{\Delta(1)}\Big[{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-k)\\
&+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)\Big]\\
&+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\\
&+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Big|
\end{aligned}$$

$$\leq \frac{(n-m)(l-k)}{\Lambda(1)\Delta(1)}\int_0^1\int_0^1|\Omega(t,s)|\left[\left|\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\right)\mathrm{dsd}t\right.\right.$$
$$+\frac{\partial^2}{\partial t\partial s}f\left(a+b-((1-t)m+tn),c+d-(sk+(1-s)l)\right)\mathrm{dsd}t$$
$$+\frac{\partial^2}{\partial t\partial s}f\left(a+b-(tm+(1-t)n),c+d-((1-s)k+sl)\right)\mathrm{dsd}t$$
$$\left.\left.+\frac{\partial^2}{\partial t\partial s}f\left(a+b-(tm+(1-t)n),c+d-(sk+(1-s)l)\right)\mathrm{dsd}t\right|\right]$$
$$\leq \frac{4(n-m)(l-k)}{\Lambda(1)\Delta(1)}\left(\int_0^1\int_0^1|\Omega(t,s)|^{\alpha}\,\mathrm{dsd}t\right)^{\frac{1}{\alpha}}\times\left[\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{\beta}+\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{\beta}+\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{\beta}\right.$$
$$\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{\beta}\right]^{\frac{1}{\beta}}.$$

Theorem 2.5

Suppose that tha assumptions of Lemma 2.1 are hold. If $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^{r}$ is co-ordinated convex function on $[a,b]\times[c,d]$ for $r\geq 1$, then we have the inequality

$$\left|\frac{\left[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)+f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\right]}{4}\right.$$
$$-\frac{1}{4\Lambda(1)}\left[{}_{(c+d-k)^-}I_{\varphi}f(a+b-n,c+d-l)+{}_{(c+d-l)^+}I_{\varphi}f(a+b-n,c+d-k)\right.$$
$$\left.+{}_{(c+d-k)^-}I_{\varphi}f(a+b-m,c+d-l)+{}_{(c+d-l)^+}I_{\varphi}f(a+b-m,c+d-k)\right]$$
$$-\frac{1}{4\Delta(1)}\left[{}_{(a+b-m)^-}I_{\varphi}f(a+b-n,c+d-l)+{}_{(a+b-m)^-}I_{\varphi}f(a+b-n,c+d-k)\right.$$
$$\left.+{}_{(a+b-n)^+}I_{\varphi}f(a+b-m,c+d-l)+{}_{(a+b-n)^+}I_{\varphi}f(a+b-m,c+d-k)\right]$$
$$+\frac{1}{4\Lambda(1)\Delta(1)}\left[{}_{(a+b-m)^-,(c+d-k)^-}I_{\varphi}f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_{\varphi}f(a+b-n,c+d-k)\right.$$
$$\left.\left.+{}_{(a+b-n)^+,(c+d-k)^-}I_{\varphi}f(a+b-m,c+d-l)+{}_{(a+b-n)^+,(c+d-l)^+}I_{\varphi}f(a+b-m,c+d-k)\right]\right|$$
$$\leq\frac{4(n-m)(l-k)}{\Lambda(1)\Delta(1)}\left(\int_0^1\int_0^1|\Omega(t,s)|\,\mathrm{dsd}t\right)\times\left[\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{r}+\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{r}+\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{r}\right.$$
$$\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{r}\right]^{\frac{1}{r}}.$$

Proof.

From the Lemma 2.1 and and Jensen-Mercer inequality by using the Power mean inequality and the convexity of $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^{r}$, we obtain

$$\Big|\big[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)+f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\big]$$
$$-\frac{1}{\Lambda(1)}\Big[{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$
$$-\frac{1}{\Delta(1)}\Big[{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$
$$+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(a+b-m,c+d-l)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Big|$$
$$\le\frac{(n-m)(l-k)}{\Lambda(1)\Delta(1)}\int_0^1\int_0^1|\Omega(t,s)|\Big[\Big|\frac{\partial^2}{\partial t\partial s}f\big(a+b-((1-t)m+tn),c+d-((1-s)k+sl)\big)\mathrm{d}s\mathrm{d}t$$
$$+\frac{\partial^2}{\partial t\partial s}f\big(a+b-((1-t)m+tn),c+d-(sk+(1-s)l)\big)\mathrm{d}s\mathrm{d}t$$
$$+\frac{\partial^2}{\partial t\partial s}f\big(a+b-(tm+(1-t)n),c+d-((1-s)k+sl)\big)\mathrm{d}s\mathrm{d}t$$
$$+\frac{\partial^2}{\partial t\partial s}f\big(a+b-(tm+(1-t)n),c+d-(sk+(1-s)l)\big)\mathrm{d}s\mathrm{d}t\Big|\Big]$$
$$\le\frac{4(n-m)(l-k)}{\Lambda(1)\Delta(1)}\left(\int_0^1\int_0^1|\Omega(t,s)|^\alpha\,\mathrm{d}s\mathrm{d}t\right)\times\Big[\Big|\frac{\partial^2}{\partial t\partial s}f(a,c)\Big|^r+\Big|\frac{\partial^2}{\partial t\partial s}f(a,d)\Big|^r+\Big|\frac{\partial^2}{\partial t\partial s}f(b,c)\Big|^r$$
$$+\Big|\frac{\partial^2}{\partial t\partial s}f(b,d)\Big|^r-\Big|\frac{\partial^2}{\partial t\partial s}f(m,k)\Big|^r-\Big|\frac{\partial^2}{\partial t\partial s}f(m,l)\Big|^r-\Big|\frac{\partial^2}{\partial t\partial s}f(n,k)\Big|^r-\Big|\frac{\partial^2}{\partial t\partial s}f(n,l)\Big|^r\Big]^{\frac{1}{r}}.$$

Lemma 2.6

Assume that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is continuous and integrable, the following fractional equality holds:

$$\Big[4f\Big(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\Big)-2f\Big(a+b-n,c+d-\frac{k+l}{2}\Big)-2f\Big(a+b-m,c+d-\frac{k+l}{2}\Big)$$
$$-2f\Big(a+b-\frac{m+n}{2},c+d-l\Big)-f\Big(a+b-\frac{m+n}{2},c+d-k\Big)+2f(a+b-n,c+d-l)$$
$$+2f(a+b-n,c+d-k)+2f(a+b-m,c+d-l)+2f(a+b-m,c+d-k)\Big]$$
$$-\frac{2}{\Delta(1)}\Big[{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)\Big]$$
$$-\frac{2}{\Lambda(1)}\Big[{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)\Big]$$
$$+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$
$$+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(b,c)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]$$

$$= \frac{1}{\Lambda(1)\Delta(1)} \sum_{k=1}^{16} J_k$$

where

$$J_1 = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_2 = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} -\Lambda_1(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_3 = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} -\Lambda_1(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_4 = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_5 = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 -\Lambda_2(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_6 = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 -\Lambda_2(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_7 = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \Lambda_2(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_8 = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 -\Lambda_2(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_9 = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} -\Lambda_3(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{10} = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Lambda_3(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{11} = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Lambda_3(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{12} = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} -\Lambda_3(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{13} = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Lambda_4(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{14} = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 -\Lambda_4(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left((1-s)k+sl\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{15} = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 -\Lambda_4(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left((1-t)m+tn\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

$$J_{16} = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Lambda_4(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-\left(tm+(1-t)n\right), c+d-\left(sk+(1-s)l\right)\right) \mathrm{d}t\mathrm{d}s,$$

and

$$\Lambda_1(t,s) = \int_0^t \int_0^s \frac{\varphi\left((n-m)u\right)\varphi\left((l-k)\lambda\right)}{u\lambda} \mathrm{d}\lambda\mathrm{d}u,$$

$$\Lambda_2(t,s) = \int_0^t \int_s^1 \frac{\varphi\left((b-a)u\right)\varphi\left((d-c)\lambda\right)}{u\lambda} \mathrm{d}\lambda\mathrm{d}u,$$

$$\Lambda_3(t,s)=\int_t^1\int_0^s \frac{\varphi((b-a)u)\varphi((d-c)\lambda)}{u\lambda}\mathrm{d}\lambda\mathrm{d}u,$$
$$\Lambda_4(t,s)=\int_t^1\int_s^1 \frac{\varphi((b-a)u)\varphi((d-c)\lambda)}{u\lambda}\mathrm{d}\lambda\mathrm{d}u.$$

where $\Lambda(t)=\int_0^t \frac{\varphi((n-m)u)}{u}\mathrm{d}u$ and $\Delta(s)=\int_0^s \frac{\varphi((l-k)\lambda)}{\lambda}\mathrm{d}\lambda$.

Proof.

Here, we apply integration by parts, then we completes the proof.

Theorem 2.7

If assumptions of Lemma 2.6 hold and $\left|\frac{\partial^2}{\partial t\partial s}f\right|$ satisfies the Mercer's ineaqality, then the following inequality holds:

$$\begin{aligned}&\Big|4f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)-2f\left(a+b-n,c+d-\frac{k+l}{2}\right)-2f\left(a+b-m,c+d-\frac{k+l}{2}\right)\\&-2f\left(a+b-\frac{m+n}{2},c+d-l\right)-f\left(a+b-\frac{m+n}{2},c+d-k\right)+2f(a+b-n,c+d-l)\\&+2f(a+b-n,c+d-k)+2f(a+b-m,c+d-l)+2f(a+b-m,c+d-k)\\&-\frac{2}{\Delta(1)}\Big[{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)\Big]\\&-\frac{2}{\Lambda(1)}\Big[{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)\Big]\\&+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\\&+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(b,c)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Big|\\&\le\frac{1}{\Lambda(1)\Delta(1)}\Bigg\{\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right]\\&\times\left[\int_0^{\frac12}\int_0^{\frac12}|\Lambda_1(t,s)|\mathrm{d}t\mathrm{d}s+\int_0^{\frac12}\int_{\frac12}^1|\Lambda_2(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_0^{\frac12}|\Lambda_3(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_{\frac12}^1|\Lambda_4(t,s)|\mathrm{d}t\mathrm{d}s\right]\\&-\left[\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\\&\times\left[\int_0^{\frac12}\int_0^{\frac12}|\Lambda_1(t,s)|\mathrm{d}t\mathrm{d}s+\int_0^{\frac12}\int_{\frac12}^1|\Lambda_2(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_0^{\frac12}|\Lambda_3(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_{\frac12}^1|\Lambda_4(t,s)|\mathrm{d}t\mathrm{d}s\right]\Bigg\}.\end{aligned}$$

Proof.

From the Lemma 2.6, we have

$$\begin{aligned}&\Big|4f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)-2f\left(a+b-n,c+d-\frac{k+l}{2}\right)-2f\left(a+b-m,c+d-\frac{k+l}{2}\right)\\&-2f\left(a+b-\frac{m+n}{2},c+d-l\right)-f\left(a+b-\frac{m+n}{2},c+d-k\right)+2f(a+b-n,c+d-l)\\&+2f(a+b-n,c+d-k)+2f(a+b-m,c+d-l)+2f(a+b-m,c+d-k)\end{aligned}$$

$$-\frac{2}{\Delta(1)}\Big[{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)\Big]$$

$$-\frac{2}{\Lambda(1)}\Big[{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)\Big]$$

$$+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)$$

$$+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(b,c)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\Big]\Big|$$

$$\le\frac{1}{\Lambda(1)\Delta(1)}\Bigg\{\int_0^{\frac12}\int_0^{\frac12}|\Lambda_1(t,s)|\Bigg[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|$$

$$-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Bigg]\mathrm{d}t\mathrm{d}s$$

$$+\int_0^{\frac12}\int_{\frac12}^1|\Lambda_2(t,s)|\Bigg[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|$$

$$-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Bigg]\mathrm{d}t\mathrm{d}s$$

$$+\int_{\frac12}^1\int_0^{\frac12}|\Lambda_3(t,s)|\Bigg[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|$$

$$-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Bigg]\mathrm{d}t\mathrm{d}s$$

$$+\int_{\frac12}^1\int_{\frac12}^1|\Lambda_4(t,s)|\Bigg[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|$$

$$-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Bigg]\mathrm{d}t\mathrm{d}s\Bigg\}$$

$$=\frac{1}{\Lambda(1)\Delta(1)}\Bigg\{\Bigg[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\Bigg]$$

$$\times\left[\int_0^{\frac12}\int_0^{\frac12}|\Lambda_1(t,s)|\mathrm{d}t\mathrm{d}s+\int_0^{\frac12}\int_{\frac12}^1|\Lambda_2(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_0^{\frac12}|\Lambda_3(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_{\frac12}^1|\Lambda_4(t,s)|\mathrm{d}t\mathrm{d}s\right]$$

$$-\Bigg[\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\Bigg]$$

$$\times\left[\int_0^{\frac12}\int_0^{\frac12}|\Lambda_1(t,s)|\mathrm{d}t\mathrm{d}s+\int_0^{\frac12}\int_{\frac12}^1|\Lambda_2(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_0^{\frac12}|\Lambda_3(t,s)|\mathrm{d}t\mathrm{d}s+\int_{\frac12}^1\int_{\frac12}^1|\Lambda_4(t,s)|\mathrm{d}t\mathrm{d}s\right]\Bigg\}.$$

Theorem 2.8

Suppose that tha assumptions of Lemma 2.6 are hold. If $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^\beta$ is co-ordinated convex function on $[a,b]\times[c,d]$, then we have the inequality

$$\Bigg|4f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)-2f\left(a+b-n,c+d-\frac{k+l}{2}\right)-2f\left(a+b-m,c+d-\frac{k+l}{2}\right)$$

$$-2f\left(a+b-\frac{m+n}{2},c+d-l\right)-f\left(a+b-\frac{m+n}{2},c+d-k\right)+2f\left(a+b-n,c+d-l\right)$$
$$+2f\left(a+b-n,c+d-k\right)+2f\left(a+b-m,c+d-l\right)+2f\left(a+b-m,c+d-k\right)$$
$$-\frac{2}{\Delta(1)}\left[{}_{(c+d-l)^+}I_\varphi f\left(a+b-m,c+d-k\right)+{}_{(c+d-k)^-}I_\varphi f\left(a+b-n,c+d-l\right)\right]$$
$$-\frac{2}{\Lambda(1)}\left[{}_{(a+b-n)^+}I_\varphi f\left(a+b-m,c+d-k\right)+{}_{(a+b-m)^-}I_\varphi f\left(a+b-n,c+d-l\right)\right]$$
$$+\frac{1}{\Lambda(1)\Delta(1)}\left[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f\left(a+b-n,c+d-l\right)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f\left(a+b-n,c+d-k\right)\right.$$
$$\left.+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f\left(b,c\right)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f\left(a+b-m,c+d-k\right)\right]\Bigg|$$
$$\le\frac{1}{\Lambda(1)\Delta(1)}\left\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|^\alpha \mathrm{d}t\mathrm{d}s\right)^\alpha+\left(\int_{\frac{1}{2}}^{\frac{1}{2}}\int_0^1\left|\Lambda_2(t,s)\right|^\alpha \mathrm{d}t\mathrm{d}s\right)^\alpha\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|^\alpha \mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left|\Lambda_4(t,s)\right|^\alpha \mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}\right\}$$
$$\times\left[4^\beta\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^\beta+4^\beta\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^\beta+4^\beta\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^\beta+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^\beta\right.$$
$$\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^\beta-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^\beta\right]^{\frac{1}{\beta}}.$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$, $\beta>1$.

Proof.

From the Lemma 2.6 and and Jensen-Mercer inequality by using the Hölder inequality and the convexity of $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^\beta$, we obtain

$$\Bigg|4f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)-2f\left(a+b-n,c+d-\frac{k+l}{2}\right)-2f\left(a+b-m,c+d-\frac{k+l}{2}\right)$$
$$-2f\left(a+b-\frac{m+n}{2},c+d-l\right)-f\left(a+b-\frac{m+n}{2},c+d-k\right)+2f\left(a+b-n,c+d-l\right)$$
$$+2f\left(a+b-n,c+d-k\right)+2f\left(a+b-m,c+d-l\right)+2f\left(a+b-m,c+d-k\right)$$
$$-\frac{2}{\Delta(1)}\left[{}_{(c+d-l)^+}I_\varphi f\left(a+b-m,c+d-k\right)+{}_{(c+d-k)^-}I_\varphi f\left(a+b-n,c+d-l\right)\right]$$
$$-\frac{2}{\Lambda(1)}\left[{}_{(a+b-n)^+}I_\varphi f\left(a+b-m,c+d-k\right)+{}_{(a+b-m)^-}I_\varphi f\left(a+b-n,c+d-l\right)\right]$$
$$+\frac{1}{\Lambda(1)\Delta(1)}\left[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f\left(a+b-n,c+d-l\right)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f\left(a+b-n,c+d-k\right)\right.$$
$$\left.+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f\left(b,c\right)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f\left(a+b-m,c+d-k\right)\right]\Bigg|$$

$$
\begin{aligned}
&\le \frac{1}{\Lambda(1)\Delta(1)}\left\{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.\right.\\
&\quad\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s\\
&\quad+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|\Lambda_2(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.\\
&\quad\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s\\
&\quad+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.\\
&\quad\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s\\
&\quad+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left|\Lambda_4(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.\\
&\quad\left.\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Lambda(1)\Delta(1)}\left\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}\times\left[4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{\beta}\right.\right.\\
&\quad\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{\beta}\right]^{\frac{1}{\beta}}\\
&\quad+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|\Lambda_2(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}\times\left[4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{\beta}\right.\\
&\quad\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{\beta}\right]^{\frac{1}{\beta}}\\
&\quad+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}\times\left[4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{\beta}\right.\\
&\quad\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{\beta}\right]^{\frac{1}{\beta}}\\
&\quad+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left|\Lambda_4(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}\times\left[4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{\beta}\right.\\
&\quad\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{\beta}\right]^{\frac{1}{\beta}}
\end{aligned}
$$

$$
=\frac{1}{\Lambda(1)\Delta(1)}\left\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\alpha}+\left(\int_{\frac{1}{2}}^{\frac{1}{2}}\int_0^1\left|\Lambda_2(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\alpha}\right.
$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\Lambda_4(t,s)\right|^{\alpha}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{\alpha}}\Bigg\}$$

$$\times\left[4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{\beta}+4^{\beta}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{\beta}+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{\beta}\right.$$

$$\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{\beta}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{\beta}\right]^{\frac{1}{\beta}}.$$

Theorem 2.9

Suppose that tha assumptions of Lemma 2.6 are hold. If $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^{r}$ is co-ordinated convex function on $[a,b]\times[c,d]$, then we have the inequality

$$\Bigg|4f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)-2f\left(a+b-n,c+d-\frac{k+l}{2}\right)-2f\left(a+b-m,c+d-\frac{k+l}{2}\right)$$

$$-2f\left(a+b-\frac{m+n}{2},c+d-l\right)-f\left(a+b-\frac{m+n}{2},c+d-k\right)+2f(a+b-n,c+d-l)$$

$$+2f(a+b-n,c+d-k)+2f(a+b-m,c+d-l)+2f(a+b-m,c+d-k)$$

$$-\frac{2}{\Delta(1)}\left[{}_{(c+d-l)^+}I_{\varphi}f(a+b-m,c+d-k)+{}_{(c+d-k)^-}I_{\varphi}f(a+b-n,c+d-l)\right]$$

$$-\frac{2}{\Lambda(1)}\left[{}_{(a+b-n)^+}I_{\varphi}f(a+b-m,c+d-k)+{}_{(a+b-m)^-}I_{\varphi}f(a+b-n,c+d-l)\right]$$

$$+\frac{1}{\Lambda(1)\Delta(1)}\Big[{}_{(a+b-m)^-,(c+d-k)^-}I_{\varphi}f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_{\varphi}f(a+b-n,c+d-k)$$

$$+{}_{(a+b-n)^+,(c+d-k)^-}I_{\varphi}f(b,c)+{}_{(a+b-n)^+,(c+d-l)^+}I_{\varphi}f(a+b-m,c+d-k)\Big]\Bigg|$$

$$\leq\frac{1}{\Lambda(1)\Delta(1)}\Bigg\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|\mathrm{d}t\mathrm{d}s\right)\times\left[4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{r}+4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{r}+4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{r}\right.$$

$$\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{r}\right]^{\frac{1}{r}}$$

$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|\Lambda_2(t,s)\right|\mathrm{d}t\mathrm{d}s\right)\times\left[4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{r}+4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{r}+4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{r}\right.$$

$$\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{r}\right]^{\frac{1}{r}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|\mathrm{d}t\mathrm{d}s\right)\times\left[4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^{r}+4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^{r}+4^{r}\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^{r}\right.$$

$$\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^{r}-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^{r}\right]^{\frac{1}{r}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\Lambda_4(t,s)\right|\mathrm{d}t\mathrm{d}s\right)\times\left[4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^r\right.$$

$$\left.+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^r\right]^{\frac{1}{r}}$$

Proof.

From the Lemma 2.6 and and Jensen-Mercer inequality by using the Power mean inequality and the convexity of $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^r$, we obtain

$$\left|4f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)-2f\left(a+b-n,c+d-\frac{k+l}{2}\right)-2f\left(a+b-m,c+d-\frac{k+l}{2}\right)\right.$$

$$-2f\left(a+b-\frac{m+n}{2},c+d-l\right)-f\left(a+b-\frac{m+n}{2},c+d-k\right)+2f(a+b-n,c+d-l)$$

$$+2f(a+b-n,c+d-k)+2f(a+b-m,c+d-l)+2f(a+b-m,c+d-k)$$

$$-\frac{2}{\Delta(1)}\left[{}_{(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)\right]$$

$$-\frac{2}{\Lambda(1)}\left[{}_{(a+b-n)^+}I_\varphi f(a+b-m,c+d-k)+{}_{(a+b-m)^-}I_\varphi f(a+b-n,c+d-l)\right]$$

$$+\frac{1}{\Lambda(1)\Delta(1)}\left[{}_{(a+b-m)^-,(c+d-k)^-}I_\varphi f(a+b-n,c+d-l)+{}_{(a+b-m)^-,(c+d-l)^+}I_\varphi f(a+b-n,c+d-k)\right.$$

$$\left.\left.+{}_{(a+b-n)^+,(c+d-k)^-}I_\varphi f(b,c)+{}_{(a+b-n)^+,(c+d-l)^+}I_\varphi f(a+b-m,c+d-k)\right]\right|$$

$$\le\frac{1}{\Lambda(1)\Delta(1)}\left\{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.\right.$$

$$\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s$$

$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|\Lambda_2(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.$$

$$\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s$$

$$+\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.$$

$$\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s$$

$$+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\Lambda_4(t,s)\right|\left[4\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|+4\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right.$$

$$\left.\left.-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|\right]\mathrm{d}t\mathrm{d}s\right\}$$

$$\leq \frac{1}{\Lambda(1)\Delta(1)}\Bigg\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\Lambda_1(t,s)\right|\text{d}t\text{d}s\right)\times\Bigg[4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^r$$

$$+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^r\Bigg]^{\frac{1}{r}}$$

$$+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|\Lambda_2(t,s)\right|\text{d}t\text{d}s\right)\times\Bigg[4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^r$$

$$+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^r\Bigg]^{\frac{1}{r}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left|\Lambda_3(t,s)\right|\text{d}t\text{d}s\right)\times\Bigg[4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^r$$

$$+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^r\Bigg]^{\frac{1}{r}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\Lambda_4(t,s)\right|\text{d}t\text{d}s\right)\times\Bigg[4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^r+4^r\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^r$$

$$+\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(m,l)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,k)\right|^r-\left|\frac{\partial^2}{\partial t\partial s}f(n,l)\right|^r\Bigg]^{\frac{1}{r}}$$

3. Comparison with Existing Results

The inequalities established in this paper extend and unify several known results. For instance, by selecting $\varphi(x)=\frac{x^{\alpha}}{\Gamma(\alpha)}$, our inequalities reduce to the Riemann-Liouville fractional versions in [9]. Similarly, with $\varphi(x)=\frac{1}{k\Gamma(\alpha)}x^{\frac{\alpha}{k}}$, the results coincide with the k-Riemann-Liouville variants discussed in [10]. Therefore, the present framework highlights the advantages of generalized fractional integrals, which provide a unified setting accommodating multiple fractional operators within a single structure.

4. Conclusions

In this study, we have developed new variants of Hermite-Hadamard-Mercer type inequalities for functions that are convex on the coordinates, by employing a general class of fractional integral operators. Our approach unifies and extends a wide range of previously known results involving Riemann-Liouville, Katugampola, and generalized fractional integrals. Through the introduction of integral identities involving two-variable convex functions and generalized kernels, we derived midpoint, trapezoidal, and Simpson-type inequalities in a broader fractional context.

The novelty of our work lies in the incorporation of generalized fractional integrals with kernel flexibility, enabling the modeling of a rich family of fractional behaviors under a unified analytical structure. This framework captures more refined integral bounds than classical methods and provides tools for analyzing fractional convexity in multi-dimensional domains.

Moreover, the inequalities obtained herein are not only theoretically significant but also pave the way for future applications in fractional differential equations, numerical approximation, and optimization problems where coordinate-wise convexity plays a central role. Potential directions for further research include the extension to higher-dimensional settings, incorporation of stochastic or fuzzy fractional operators, and applications to systems governed by nonlocal memory effects.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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