Scientific Research Publishing

Advances in Pure Mathematics, 2025, 15(9), 563-569
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Note on the Intermediate Value Theorem (TVI)

Mohamed Bekkali

Department of Mathematics, Faculty of Sciences and Technology, Fez, Morocco
Email: mohamed.bekkali@usmba.ac.ma

How to cite this paper: Bekkali, M. (2025) Note on the Intermediate Value Theorem (TVI). *Advances in Pure Mathematics*, **15**, 563-569.
https://doi.org/10.4236/apm.2025.159030

Received: June 20, 2025
Accepted: August 29, 2025
Published: September 1, 2025

Abstract

The Intermediate Value Theorem (for short, TVI) is a real analysis concept that one encounters in her/his College first calculus course. It is stated for a real continuous function, f, on a closed non empty interval $I \subset \mathbb{R}$. These two hypothesis, on the function and the interval, ensure that all values in between $\min_I(f)$ and $\max_I(f)$ are taken by f. In this note, we consider any non constant real function, say g, on an interval I of the real line $\mathbb{R}$ so that g is continuous except on an *order-scattered* subset; under these conditions, the interior of $g(I)$ is not empty.

Keywords

Real Analysis, Scattered-Chains, Countable Boolean Algebras, 0-Dimensional Space, Stone Space

1. Introduction

The Intermediate Value Theorem (for short, TVI) is a popular concept in real analysis that freshmen learn in their College first calculus course. For recall that if $f : I = [a,b] \to \mathbb{R}$ is continuous ($a < b$), then $f([a,b]) = [\min_I(f), \max_I(f)]$; thus all values between $\min_I(f)$ and $\max_I(f)$ are reached by f. So, in order to elaborate on this, we start with the following definitions.

Definition 1.1 *Let* $(A, <_A)$ *and* $(B, <_B)$ *be two partially ordered sets. We say that* $(A, <_A)$ ***embeds in*** $(B, <_B)$ *whenever there is an injective function*

$$f : (A, <_A) \hookrightarrow (B, <_B) \text{ so that:}$$

$$x <_A y \text{ implies } fx <_A fy.$$

Definition 1.2 *A chain* $(C, <_C)$ *is called an* ***order-scattered*** *chain whenever* $(\mathbb{Q}, <)$ *does not embed in* $(C, <_C)$.

Next, the following examples, from general literature, shed light on what topological property should $Disc(f)$, the set of discontinuities of f, have to ensure TVI-property for f. This is stated in the main theorem of this note.

Example 1.3

There is a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *so that* $|Disc(f)|$ *is uncountable and the interior of* $f(I)$ *is empty.*

For, consider *Dirichlet's function* (rational indicator function):

$D(x) = 1$, if x is rational and 0, otherwise.

This is discontinuous everywhere.

Example 1.4

There is a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *so that* $Disc(f)$ *is a countable dense set; moreover, the interior of* $f(I)$ *is empty.*

For, consider *Thomae's function*, see [1]:

$T(x) = \frac{1}{q}$, if $x \in \mathbb{Q}$ and $x = \frac{p}{q}$ for $p \in \mathbb{Z}$, $q \in \mathbb{N}$ co-prime *i.e.*, their greatest common divisor is 1, $T(x) = 0$, otherwise.

This is continuous only at the irrational numbers.

Example 1.5

There is a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *so that* $Disc(f)$ *is an uncountable set:*

For, consider *Cantor indicator function*:

$C(x) = 1$, *if* x *is in the Cantor set,* $C(x) = 0$, *otherwise.*

This is continuous only at numbers not in the Cantor set.

This note is organized as follows:

In *Section* 2, we supply a concrete representation of elements of *Int*, the class of Boolean algebras over linearly ordered sets (Theorem 2.3); in addiction, Theorems 2.5 and 2.6 are proved. Next, scatterdness, in ordinal/topological sense, for linearly ordered sets, (Lemma 2.4 and Theorem 2.5), is established. In *section* 3, we state the main theorem (Theorem 2.8).

Recall that a Hausdorff space X (for short, T_2-*space*) is a topological space X in which two different points a, b are separated by two disjoint open sets in X (that is to say, there are two disjoint open sets U_a and U_b in X so that $a \in U_a$ and $b \in U_b$). Next, let $(\mathbb{Q}, <_{\mathbb{Q}})$ (respectively $(\mathbb{R}, <_{\mathbb{R}})$) be the set of rational numbers with its usual linear ordering (respectively $(\mathbb{R}, <_{\mathbb{R}})$ be the set of real numbers with its usual linear ordering $<_{\mathbb{R}}$ and endowed with the interval topology). Finally, denote by $Disc(f)$ the set of discontinuities of any function f.

A *linear ordering* $(L, <_L)$ is a partially ordering in which two different elements are comparable with respect to $<_L$ (linear ordering is used interchangeably with a *chain*). Now, if $(L, <_L)$ is a chain, set $b_t := \{u \in L : t \le u\}$ and consider the subalgebra, of the power set Boolean algebra $\wp(L)$, generated by $\langle b_t : t \in L \rangle$; it is called the *interval algebra over the chain* $(L, <_L)$ and denoted by $Int(L)$. Whenever, $(L, <_L)$ has a least element, the *Stone space associated with* $Int(L)$, $Ult(Int(L))$, is homeomorphic to $I(L, <_L)$, the set of *initial chains in* $(L, <_L)$, *that is to say, sets that are either empty or closed downwards* *in* $(L,<_L)$, endowed with Tychonoff's topology inherited from 2^L. Indeed, $\phi: Ult(Int(L)) \to I(L,<_L)$ defined by $\phi(U) := \{s \in L : b_s \in U\}$ is a homeomorphism, see [2].

2. Scattered Chains in $(\mathbb{R},<_R)$ versus Countable Ordinals

Recall the Cantor-Bendixon's derivation of a topological space. For, let X be a Hausdorff topological space and A is a non empty subset of X. We say that $a \in A$ is an *isolated point in* A, whenever there is an open set, U containing a, of X so that $A \cap U = \{a\}$. The set of isolated points of A is denoted by $Iso(A)$. Hence, for X a Hausdorff space, by induction on the ordinal α, define the α^{th}-Cantor derivative of X, as follows:

$$X^{(0)} = X \tag{2.1}$$

$$X^{(1)} = X \setminus Iso(X) \tag{2.2}$$

$$X^{(\alpha+1)} = \left(X^{(\alpha)}\right)^{(1)} \tag{2.3}$$

$$X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}, \text{ if } \lambda \text{ is a limit ordinal.} \tag{2.4}$$

Thus, $\left(X^{(\alpha)}\right)_\alpha$ is a decreasing sequence of closed subsets of X, which is actually a stationary sequence *i.e.*, there is a first ordinal δ, called the *rank* of X and denoted by $rk(X)$ so that $X^{(\delta)} = X^{(\delta+1)}$. Thus, either $X^{(\delta)} = \varnothing$, or $Iso\left(X^{(\delta)}\right) = \varnothing$.

Definition 2.1 *A Hausdorff space X is called **topologically scattered** space whenever every closed subspace has an isolated point.*

Proposition 2.2 *For any non empty Hausdorff compact and topologically scattered space X, the first δ so that $X^{(\delta)} = \varnothing$ is a successor ordinal i.e., $\delta = \beta + 1$ and thus $X^{(\beta)}$ is a finite set.*

Indeed, if δ is a limit ordinal and if $\bigcap_{\nu<\delta} X^{(\nu)} = \varnothing$, then, by compactness $\bigcap_{i<n} X^{\nu_i} = \varnothing \left(= X^{\max(\nu_i, i<n)}\right)$, for some $n < \omega$: Contradiction.

Next, recall that a Boolean algebra B is called *scattered* whenever its Stone space, $Ult(B)$, is a *scattered* topological space; sometime we use interchangeably the word *superatomic* Boolean algebras for *scattered* Boolean algebras. Moreover, if X denotes $Ult(B)$, then $X = \bigcup_{\alpha \le rk(X)} \left(X^{(\alpha)} \setminus X^{(\alpha+1)}\right)$. Thus, it follows that, whenever $Ult(B)$ is a scattered topological space, $|Ult(B)| \le |B|$, see [3], [Vol.1, Theorem.15.7., pp. 274].

In this section we give a characterization of countable topologically scattered compact spaces. We start with the following result.

Theorem 2.3

1) Any countable Boolean algebra B is generated by a chain.

2) Let B be a Boolean algebra generated by a chain $(C,\subseteq)$ in B. Then there is a chain $(L,\preceq)$ with a least element so that B is isomorphic to $Int(L)$. Moreover, the Stone space of B, denoted by $Ult(B)$, is homeomorphic to the set of initial chains $I(L,\preceq)$, endowed with Tychonoff's Topology inherited from

2^L.

The first part of this theorem appeared in [4]; here, we supply a proof. So, enumerate the whole algebra and construct, by induction, the ad hoc chain. Indeed, for let $\{x_n : n \in \mathbb{N}\}$ be a countable enumeration of $\mathcal{B}$. Denote by u' the complement of any u in B; then construct, by induction, a chain $\mathcal{C}$, so that $\mathcal{B} = \langle \mathcal{C} \rangle$.

Step 0. Let $x_0 \in \mathcal{B}$. Set $\mathcal{C}_0 = \{0, x_0, 1\}$ and $\mathcal{B}_0 = \langle \mathcal{C}_0 \rangle \subseteq \mathcal{B}$.

Step 1. If $x_1 \in \mathcal{B}_0$, we set $\mathcal{B}_0 = \langle \mathcal{C}_0 \rangle$ and $\mathcal{C}_1 = \mathcal{C}_0$. If $x_1 \notin \mathcal{B}_0$, we set

$$\mathcal{C}_1 = \{0, x_0 \wedge x_1, x_0, x_0 \vee x_1\}$$

Notice that $\mathcal{C}_1$ is a chain, then put $\mathcal{B}_1 = \langle \mathcal{C}_1 \rangle$, $\mathcal{B}_1 \supseteq \mathcal{B}_0$ and $\mathcal{C}_1 \supseteq \mathcal{C}_0$. Now, to check that x_1 is in $\mathcal{B}_1$, notice that

$$x_1 = \left(\left(\underbrace{x_0 \wedge x_1}_{\in \mathcal{C}_1} \right) \vee \left(\underbrace{x_0'}_{\in \mathcal{B}_0} \right) \right) \wedge \left(\underbrace{x_1 \vee x_0}_{\in \mathcal{C}_1} \right)$$

Step $n+1$. Suppose that chains $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n := \{a_0, \cdots, a_{\varphi(n)}\}$ are constructed so that: $0 = a_0 < a_1 < \cdots < a_{\varphi(n)} = 1$, and $\mathcal{B}_n = \langle \mathcal{C}_n \rangle$.

Now to finish up the $(n+1)^{th}$ step, assume that $x_0, x_1, \cdots, x_n \in \mathcal{B}_n$ and look at x_{n+1}.

Case 1. $x_{n+1} \in \mathcal{B}_n$. In this case we put $\mathcal{C}_{n+1} = \mathcal{C}_n$.

Case 2. $x_{n+1} \notin \mathcal{B}_n$. In this case, define a new sequence $(v_k)_k$ by $v_k := a_k \vee (x_{n+1} \wedge (a_{k+1} \wedge a_k'))$ for $k = 0, \cdots, \varphi(n) - 1$.

Note that

$$0 \le v_0 \le a_1 \le v_1 \le a_2 \le v_2 \le \cdots \le v_{\varphi(n)-1}.$$

Set

$$\mathcal{C}_{n+1} := \mathcal{C}_n \cup \{v_k, k = 0, \cdots, \varphi(n) - 1\}$$

Next for each k, $v_k \wedge a_k' = a_{k+1} \wedge a_k' \wedge x_{n+1}$ and, by induction,

$$\vee_{k=0}^{\varphi(n)-1} a_{k+1} \wedge a_k' = 1$$

Hence,

$$\begin{aligned} x_{n+1} &= 1 \wedge x_{n+1} = \left(\vee_{k=0}^{\varphi(n)-1} a_{k+1} \wedge a_k' \right) \wedge x_{n+1} \\ &= \vee_{k=0}^{\varphi(n)-1} \left(a_{k+1} \wedge a_k' \wedge x_{n+1} \right) = \vee_{k=0}^{\varphi(n)-1} \left(v_k \wedge a_k' \right) \in \mathcal{B}_{n+1} \end{aligned}$$

This finishes the induction step. Thus, $\mathcal{B} = \langle \cup \mathcal{C}_n \rangle$. which shows that $\mathcal{B}$ is generated by the chain $\mathcal{C} := \cup \mathcal{C}_n$.

2) Let $L := (C, \subseteq)$ where $\subseteq$ is the inclusion between subsets of $\mathcal{B} = \langle C \rangle$, $b_c := \{s \in C : c \subseteq s\}$ and define $\phi : \mathcal{B} = \langle C \rangle \to Int(L)$ by $\phi(c) = b_c$. Here, use Sikorski's extension criterion to extend ϕ to a homomorphism $\psi : \langle C \rangle \to Int(L)$. Then ψ is onto since ϕ is by construction. By, Sikorski's extension criterion again, the one-to-oness follows.

3) The function ϕ, defined by, $\phi(U) := \{u \in C : b_u = \{v \in C : u \subseteq v\} \in U\}$, is a homeomorphism from $Ult(\mathcal{B})$ onto the set of initial chains $I(C, \subseteq)$.

Lemma 2.4 *Let* $(C,<)$ *be a complete chain. If* $(C,<)$ *is a scattered topological space, then* $(\mathbb{Q},<_{\mathbb{Q}})$ *does not embed into* $(C,<)$ *as an order set. Notice that the assumption of completeness of* $(C,<)$ *is necessary.*

1) First, recall that $\overline{S}$ denotes the topological closure of S in $(C,<)$, where $(C,<)$, is endowed with the interval topology. Second, if $S \subseteq C$ is infinite, then $\overline{S} \setminus \mathrm{Isol}(\overline{S}) \neq \varnothing$. This follows since C is a compact. Now suppose that $S \subseteq C$ is a chain order isomorphic the $(\mathbb{Q},<_{\mathbb{Q}})$: We shall get a contradiction. Choose $x \in S' := \overline{S} \setminus \mathrm{Isol}(\overline{S})$, x isolated in C. Say, $u < x < v$; $(u,v) \cap C = \{x\}$.

2) There are s, $t \in S$ so that $u < s < t < v$, and $x \notin [s,t]$. In fact, since $x \notin \mathrm{Isol}(\overline{S})$, the set $(u,s) \cap \overline{S}$ is infinite. Hence there clearly exist $u < w_1 < w_2 < w_3 < v$ such that $(w_1,w_2) \neq \varnothing \neq (w_2,w_3)$ and $x \notin (w_1,w_2)$. Choose $s \in (w_1,w_2) \cap S$, $t \in (w_2,w_3) \cap S$; this proves 2). Now, taking s and t as in 2), put $S'' = (x,t) \cap S$. So S'' has type η. Clearly $\overline{S''} \setminus \mathrm{Isol}(\overline{S''}) \subseteq C$. Picking w in $\overline{S''} \setminus \mathrm{Isol}(\overline{S''})$ by 1), we obtain $w \in (u,v) \cap C \setminus \{x\}$, contradiction.

3) To see the necessity of completeness, let $Z := \mathbb{N} \cdot \mathbb{Q} := \sum_{r \in (\mathbb{Q},<_Q)} A_r$, where $A_r := (\mathbb{N},<_N)$ with its natural ordering. Now, for $u,v \in Z$, set:

$$u <_Z v$$

if, and only if

either ($u \in A_r$, $v \in A_s$, and $r <_Q s$) or ($u,v \in A_r$, for some $r \in \mathbb{Q}$ and $u <_N v$).

It follows that $(Z,<_Z)$ is a chain that is a counterexample.

Theorem 2.5

Let $(C,\prec)$ be a chain with a least element and $\subset$ be the inclusion of sets. Then, the following are equivalent:

1) $(\mathbb{Q},<_{\mathbb{Q}})$ does not embed into $(C,\prec)$;

2) $(\mathbb{Q},<_{\mathbb{Q}})$ does not embed into $(I(C),\subset)$; $(I(C),\subset)$ is considered as an order set;

3) $(\mathbb{Q},<_{\mathbb{Q}})$ does not embed into $(I(C),\subset)$; $(I(C),\subset)$ is considered as a topological space;

4) $(\mathbb{Q},<_{\mathbb{Q}})$ does not embed into $B(C)$ *i.e.*, $B(C)$ is a superatomic interval algebra.

3) and 4) are equivalent by the duality theory. 2) implies 1) since C embeds in $I(C)$. 1) implies 4) since a quotient of $B(C)$ is isomorphic to $B(C')$ for some subchain C' of C (see Theorem 15.22, p. 253 in [2]). Finally, 3) implies 2) by Lemma 2.4.

Next theorem characterizes countable topological scattered compact spaces.

Theorem 2.6 *Let* X *be a Hausdorff topologically scattered compact space. Then the following are equivalent statements.*

1) Each point of X has a countable basis,

2) X is homeomorphic to a countable ordinal,

3) X is a countable set.

2) implies 2) and 3) implies 2) are trivial. We only prove 1) implies 2) by induction

on $rk(X)$. For assume that $X^{(\delta)}$ is a finite set, for some ordinal δ. Hence, we may assume that the kernel of X, denoted by $K(X) := X^{rk(X)} = \{\infty\}$. Now, pick a countable strictly decreasing sequence, $(V_n)_{n\in\omega}$ of clopen sets of X so that $\bigcap_n V_n = \{\infty\}$. Set $W_n := V_n \setminus V_{n+1}$. It follows that $X = \bigcup_n W_n \cup \{\infty\}$, so that W_n is a clopen subset of X of rank less that δ. So by the induction hypothesis W_n is homeomorphic to a countable ordinal $\alpha_n + 1$ for some $n \in \omega$, where ω denotes the first infinite ordinal. Next, notice that X and $\left(\sum_{n<\omega} \alpha_n + 1\right) + 1$ are homeomorphic spaces.

Corollary 2.7 ([5]) *Any countable scattered Boolean algebra* B *is isomorphic to* $Int(\alpha)$, *for some countable ordinal* α.

By Theorem 2.6, $Ult(B)$ is homeomorphic to a countable ordinal, say $\alpha + 1$.

Theorem 2.8 (FTVI[1])

If a non constant real function f *is continuous on an interval* I *of the real line* $\mathbb{R}$ *except on an order-scattered set* $C \subseteq I$, *then the interior of* $f(I)$ *is not empty.*

The proof of the theorem follows from next two claims.

Claim 2.9 *Let* $(\mathbb{R}, <_{\mathbb{R}})$ *be the set of real numbers with its usual interval topology. Any order-scattered chain in* $(\mathbb{R}, <_{\mathbb{R}})$ *is at most countable.*

Assume the contrary. By Hausdorff's theorem on order-scattered chains that is to say, an uncountable scattered chain embeds a sub-chain ordered-isomorphic to the first uncountable ordinal ω_1, see [6], [Theorem 5.28, pp. 87]. Hence, $(\mathbb{R}, <_{\mathbb{R}})$ has an uncountable set of rational numbers, which is a contradiction.

Claim 2.10 *Let* $C \subseteq I \subseteq (\mathbb{R}, <_{\mathbb{R}})$ *be an order-scattered set as in the main theorem. Denote by* $(I(C), \subset)$ *the set of all initial chains of* $(C, <_{\mathbb{R}})$ *endowed with Tykonov's topology inherited from* 2^C. *Then, since* $(C, <_{\mathbb{R}})$ *is order-scattered,* $Int(C)$ *is a scattered Boolean algebra i.e.,* $Ult(Int(C))$ *is a topologically scattered space. Moreover, since* $(C, <_{\mathbb{R}})$ *is order-scattered and countable, it follows that* $Int(C)$ *is isomorphic to an interval algebra over a countable ordinal.*

It follows from Theorems 2.5 and 2.6, that $Int(C)$ and $Int(\alpha)$ are isomorphic Boolean algebras for some $\alpha < \omega_1$.

By, Claim 2.10, $(I(C), \subset)$, as a scattered topological space, has an isolated point (since it is homeomorphic to $\alpha + 1$), say $\downarrow s := \{t \in C : t \leq s\}$. Pick $u \in C$ so that $]s,u[\cap C = \varnothing$. So, $]s,u[\subseteq I \setminus C$. Next, f is continuous on $]s,u[$ and thus, $f(]s,u[)$ is an interval[2]: which, in turn, shows that $int(f(I)) \neq \varnothing$.

As a consequence, we get the usual TVI-theorem on the real line.

Corollary 2.11 (TVI) *If* f *is a continuous non constant real function on a closed interval* I, *then* $f(I) = \left[\min_I(f), \max_I(f)\right]$. *Therefore, the interior of* $f(I)$ *is not empty.*

Concluding remark 2.12 *Let* f *be a function defined on an interval* I.

---

[1] †, Fez-TVI (FTVI).

[2] Indeed, let $s < a < b < u$; assume, e.g., $fa < z < fb$. Now, construct two sequences $(a_n)_n$ $(b_n)_n$, so that: $fa_n < z < fb_n$, $[a,b] \supset [a_1,b_1] \supset \cdots [a_n,b_n]$ and diameter of $[a_n,b_n]$ converges to zero. So, $\bigcap_n [a_n,b_n] = \{c\}$. By continuity of f, $fc = z$. Thus, $f(]s,u[)$ is an interval.

Should the set of discontinuities be scattered, whenever the interior of $f(I)$ *is a non empty set, seems a very natural statement. For, recall Darboux theorem for TVI with respect to the derivative of a function: The derivative of a function always satisfies TVI regardless if this derivative is continuous function or not. So, one may ask: How big* $Disc(f')$, *would be for the derivative of a function* f ? *Actually, there is a characterization of the discontinuity set of a derivative and it is found in the following two references: Benedetto* [7] (*Chapter* 1.3.2, *Proposition*, 1.10, p. 30); *and Bruckner* [8] (*Chapter* 3, *Section* 2, *Theorem* 2.1, p. 34).

On the other hand, there is a well-known function called the *Conway's base*-13 *function* engineered by J. H. Conway as a counterexample to the converse of the intermediate value theorem. In other words, base-13 function is a function that satisfies the intermediate-value property on any interval but it is discontinuous at every point; since it is unbounded on every interval. Hence, this function answers the first question in this remark.

Finally, the choice of Boolean algebras, in this note, was naturally motivated by the notion of scattered chains in the real line.

Acknowledgements

The author thanks H. Si Kaddour for his suggestions and fruitful comments during the preparation of this note. Moreover, the author dedicates this note to Sami Bekkali.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Beanland, K., Roberts, J.W. and Stevenson, C. (2009) Modifications of Thomae's Function and Differentiability. *The American Mathematical Monthly*, **116**, 531-535. https://doi.org/10.4169/193009709X470425

[2] Bekkali, M. (2001) Pseudo Treealgebras. *Notre Dame Journal of Formal Logic*, **42**, 101-108. https://doi.org/10.1305/ndjfl/1054837936

[3] Koppelberg, S. (1989) Handbook of Boolean Algebras, Volume 1. North-Holland.

[4] Mostowski, A. and Tarski, A. (1939) Boolesche Ringe mit geordneter Basis. *Fundamenta Mathematicae*, **32**, 69-86. https://doi.org/10.4064/fm-32-1-69-86

[5] Baker, J. (1972) Compact Spaces Homeomorphic to a Ray of Ordinals. *Fundamenta Mathematicae*, **76**, 19-27. https://doi.org/10.4064/fm-76-1-19-27

[6] Rosenstein, J.G. (1982) Linear Orderings. Joseph G Rosenstein eBook.

[7] Benedetto, J.J. (1976) Real Variable and Integration: With Historical Notes. Vieweg + Teubner Verla, 278.

[8] Bruckner, A. (1994) Differentiation of Real Functions. 2nd Edition, American Mathematical Society. https://doi.org/10.1090/crmm/005