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Levely Multiplicative Functions on N

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Abstract

In this paper, as a result of dividing the set of natural numbers into disjoint levels, we have studied levely multiplicative functions that are defined from $\mathbf{N}$ to $\mathbf{N}$ and which keep numbers at the same level when restricted on levels, *i.e.* $f_{|L_i}: L_i \to L_i$. We have defined the level multiplicative function and explained how to generate it using the restriction function $f_{|L_1}$. Furthermore, if $f_{|L_1}$ is bijective, then all $f_{|L_i}$ and f are bijective functions. Moreover, the levely multiplicative function preserves the unique factorization for any number $n \in \mathbf{N}$ and maintains divisibility properties.

Keywords

Prime-Power Decomposition, Levels of $\mathbf{N}$, Upward Closed Set, Levely Multiplicative Function

1. Introduction

Let $\mathbf{N}$ be the set of natural numbers. A number a divides a number b (written $a \mid b$), if there exists a number c such that $b = ca$ (in this case a is said to be a factor or divisor of b). The greatest divisor of a and b (written $(a,b) = d$) is a number d such that $d \mid a, d \mid b$, and for any number c such that $c \mid a, c \mid b$, we have $c \le d$. A number a is said to be a prime number if and only if $a > 1$ and is divisible by 1 and itself. We denote the set of prime numbers by $\mathbf{P}$. Two numbers a and b are said to be relatively prime if $(a,b) = 1$ [1], [2].

Lemma 1.1 [3]: *Every* $n \in \mathbf{N}, n > 1$*, can be written as a product of prime numbers.*

Theorem 1.1 [3]: ***The Unique Factorization Theorem****. Any natural number grater than one can be written as a product of primes in one and only one way.*

i.e. for any $n \in \mathbf{N}, n > 1$, can be written exactly in one way in the form

$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where $e_i \geq 0, i = 1,2,\cdots,k$, each p_i is prime, and $p_i \neq p_j$. We call this representation the prime-power decomposition of n.

For any $a \in \mathbf{N}$, the set of all multiple numbers of a is $a\uparrow = \{na : n \in \mathbf{N}\}$. For any $A \subseteq \mathbf{N}$ the set of all multiple of elements of A is $A\uparrow = \{n \in \mathbf{N} : \exists a \in A, a \mid n\}$. We say that A is an upward closed subset of $\mathbf{N}$ if $A = A\uparrow$. The set of upward closed subsets of $\mathbf{N}$ is denoted by $\mu = \{A \subseteq \mathbf{N} : A = A\uparrow\}$ [4].

Lemma 1.2 [4]: *For any* $A \subseteq \mathbf{N}$, $A\uparrow = \bigcup_{a \in A} a\uparrow$.

Let $\mathbf{P}$ be the set of prime numbers and let $L_0 = \{1\}$, $L_1 = \{a : a \in \mathbf{P}\}$, $L_2 = \{a_1 a_2 : a_1, a_2 \in L_1\}$, $\cdots$, $L_n = \{a_1 a_2 \cdots a_n : a_1, a_2, \cdots, a_n \in L_1\}$, $\cdots$. Then we say that L_i, where $i \geq 0$ are the levels of $\mathbf{N}$. A number $a \in \mathbf{N}$ is in the level L_n if $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $p_i \in \mathbf{P}$, $0 \leq e_i \leq n$, $i = 1,2,\cdots,k$, and $e_1 + e_2 + \cdots + e_k = n$ [5].

Lemma 1.3 [5]: (a) $\mathbf{N} = \bigcup_{i=0}^{\infty} L_i$

(b) $L_i \cap L_j = \phi$ where $i \neq j$ $\left(\bigcap_{i=0}^{\infty} L_i = \phi\right)$

If $f : \mathbf{N} \to \mathbf{N}$ is a function $A \subseteq \mathbf{N}$, then the restriction function of f on A is a function $f_{|A} : A \to \mathbf{N}$ such that $f(a) = f_{|A}(a)$ for any $a \in A$. A function $f : \mathbf{N} \to \mathbf{N}$ is injective if $a_1 = a_2$ whenever $f(a_1) = f(a_2)$. It is surjective if for any $b \in \mathbf{N}$, there exists $a \in \mathbf{N}$ such that $f(a) = b$. A function that is both injective and surjective is called bijective. If $f : \mathbf{N} \to \mathbf{N}$ is bijective, then $f^{-1} : \mathbf{N} \to \mathbf{N}$ is also a bijective function, and it is called the inverse function of f. A function $f : \mathbf{N} \to \mathbf{N}$ is called increasing if and only if $f(a) \leq f(b)$ whenever $a \leq b$ [6] [7].

Lemma 1.4 *If* $f : \mathbf{N} \to \mathbf{N}$ *is a function. Then*

(a) $f_{|L_i} \cap f_{|L_j} = \phi$, $\forall\ i \neq j$

(b) $f = \bigcup_{i=0}^{\infty} f_{|L_i}$

Proof: (a) By (Lemma 1.3 (b)), $L_i \cap L_j = \phi$ for $i \neq j$.

Therefore, $f_{|L_i}(L_i) \cap f_{|L_j}(L_j) = \phi$, which implies $f_{|L_i} \cap f_{|L_j} = \phi$.

(b) Let $(a, f(a)) \in f$. Since $\mathbf{N} = \bigcup_{i=0}^{\infty} L_i$, we have $a \in L_i, f(a) = f_{|L_i}(a)$ for some i. Therefore $(a, f_{|L_i}(a)) \in f_{|L_i}$, which implies $f \subseteq \bigcup_{i=0}^{\infty} f_{|L_i}$.

On the other hand, since $f_{|L_i} \subset f$ $\forall\ i = 0,1,2,\cdots$, we have $\bigcup_{i=0}^{\infty} f_{|L_i} \subseteq f$. Thus, $f = \bigcup_{i=0}^{\infty} f_{|L_i}$. ■

2. Levely Multiplicative Functions on N

As a consequence of dividing $\mathbf{N}$ into infinitely many disjoint levels, we can divide any function $f : \mathbf{N} \to \mathbf{N}$ into infinitely many disjoint functions, which can be obtained by restricting f to L_i for all $i = 0,1,2,\cdots$. If each function $f_{|L_i}$ sends numbers to the same level, and the image of any number in level L_i under the effect of $f_{|L_i}$ is a product of images of its prime decomposition under the function $f_{|L_1}$, then $f_{|L_1}$ generates the function f, and it is called a levely multiplicative function.

Definition 2.1: *A function* $f : \mathbf{N} \to \mathbf{N}$*, is said to be levely function if and only if* $f(a) \in L_i$ *for all* $a \in L_i, i = 0,1,2,\cdots$.

i.e. $f_{|L_i}(L_i) \subseteq L_i$, $f_{|L_0}(1) = 1$

Example 2.1: *If* $f:\mathbf{N}\to\mathbf{N}$ *is defined by* $f(n) = 2^i, n \in L_i, i = 2j$ *, and* $f(n) = 3^i, n \in L_i, i = 2j+1, j = 0,1,2,\cdots$, then f is a levely function.

Definition 2.2: *A levely function* $f:\mathbf{N}\to\mathbf{N}$ *is said to be a multiplicative function or a function that is generated by* $f_{|L_1} = \{(a, f_{|L_1}(a)) : a \in L_1\}$, *if and only if* $f_{|L_i}(a) = f_{|L_i}(a_1 a_2 \cdots a_i) = f_{|L_1}(a_1) f_{|L_1}(a_2) \cdots f_{|L_1}(a_i)$ *for any* $f_{|L_i}, i = 2,3,\cdots$, *and for any* $a = a_1 a_2 \cdots a_i \in L_i$, *where* $a_1, a_2, \cdots, a_i \in L_1$.

i.e. $$f_{|L_i} = \left\{ \left(a_1 a_2 \cdots a_i, f_{|L_1}(a_1) f_{|L_1}(a_2) \cdots f_{|L_1}(a_i) \right) : a_1, a_2, \cdots, a_i \in L_1, \; f_{|L_1}(a_1), f_{|L_1}(a_2), \cdots, f_{|L_1}(a_i) \in f_{|L_1}(L_1) \right\},$$

$$f = \bigcup_{i=0}^{\infty} f_{|L_i}$$

When $f:\mathbf{N}\to\mathbf{N}$ is a levely multiplicative function, we will write $f|_{L_i} : L_i \to L_i$ for all $i = 0,1,2,\cdots$.

And

$$f_{|L_i}(L_i) = \left\{ f_{|L_1}(a_1) f_{|L_1}(a_2) \cdots f_{|L_1}(a_i) : f_{|L_1}(a_1), f_{|L_1}(a_2), \cdots, f_{|L_1}(a_i) \in f_{|L_1}(L_1) \right\}.$$

for all $i \geq 1$.

Example 2.2: *Let* $f_{|L_1} : L_1 \to L_1$ *be defined by* $f_{|L_1}(n) = 2 \;\; \forall\; n \in L_1$.

Then, $f_{|L_1}$ generates all the functions $f_{|L_i}$, where $i \geq 2$ as follows:

$$\begin{aligned} f_{|L_i}(n) &= f_{|L_i}(p_1 p_2 \cdots p_i) \\ &= f_{|L_1}(p_1) \cdot f_{|L_1}(p_2) \cdots f_{|L_1}(p_i) \\ &= 2 \cdot 2 \cdots 2 = 2^i \end{aligned}$$

And the levely multiplicative function $f:\mathbf{N}\to\mathbf{N}$ is defined by $f(n) = 2^i$ where $n \in L_i$.

Theorem 2.1: *Every function* $f_{|L_1} : L_1 \to L_1$ *generates a unique levely multiplicative function* $f:\mathbf{N}\to\mathbf{N}$, $f = \bigcup_{i=0}^{\infty} f_{|L_i}$.

Proof: By (Definition 2.1) and (Lemma 1.4 (b)) $f = \bigcup_{i=0}^{\infty} f_{|L_i}$ is generated by $f_{|L_1}$. To prove that f is unique, let f_1 and f_2 be levely multiplicative functions generated by $f_{|L_1}$. Then, for any $a = a_1 a_2 \cdots a_i \in L_i, i \geq 1$, we have

$$\begin{aligned} f_1(a) &= f_{|L_i}(a) \\ &= f_{|L_i}(a_1 a_2 \cdots a_i) \\ &= f_{|L_1}(a_1) f_{|L_1}(a_2) \cdots f_{|L_1}(a_i) \end{aligned}$$

and

$$\begin{aligned} f_2(a) &= f_{|L_i}(a) \\ &= f_{|L_i}(a_1 a_2 \cdots a_i) \\ &= f_{|L_1}(a_1) f_{|L_1}(a_2) \cdots f_{|L_1}(a_i) \end{aligned}$$

Therefore, $f_1 = \bigcup_{i=0}^{\infty} f_{|L_i} = f_2$. Hence, f is unique. ■

Lemma 2.1: *If* $f:\mathbf{N}\to\mathbf{N}$ *is a levely multiplicative function, then*

(a) $f_{|L_i}(a) f_{|L_j}(b) = f_{|L_{i+j}}(ab) \;\; \forall\; a \in L_i, b \in L_j$

(b) $f(ab) = f(a) f(b) \;\; \forall\; a, b \in \mathbf{N}$

Proof: (a) Let $a = a_1 a_2 \cdots a_i \in L_i, b = b_1 b_2 \cdots b_j \in L_j$. Then,

$$\begin{aligned} f_{|L_i}(a) f_{|L_j}(b) &= f_{|L_i}(a_1 a_2 \cdots a_i) f_{|L_j}(b_1 b_2 \cdots b_j) \\ &= f_{|L_1}(a_1) f_{|L_1}(a_2) \cdots f_{|L_i}(a_i) f_{|L_1}(b_1) f_{|L_1}(b_2) \cdots f_{|L_1}(b_j) \\ &= f_{|L_{i+j}}(a_1 a_2 \cdots a_i b_1 b_2 \cdots b_j) \\ &= f_{|L_{i+j}}(ab) \end{aligned}$$

(b) Let $a, b \in \mathbf{N}$, then there are some i, j such that $a = a_1 a_2 \cdots a_i \in L_i$, $b = b_1 b_2 \cdots b_j \in L_j$, so $ab = a_1 a_2 \cdots a_i b_1 b_2 \cdots b_j \in L_{i+j}$. By (a) we have

$$\begin{aligned} f(ab) &= f_{|L_{i+j}}(ab) \\ &= f_{|L_i}(a) f_{|L_j}(b) \\ &= f(a) f(b) \end{aligned}$$

■

Corollary 2.1: *If* $f : \mathbf{N} \to \mathbf{N}$ *is a levely multiplicative function, then*:

(a) $f_{|L_1}(a_1) f_{|L_2}(a_2) \cdots f_{|L_n}(a_n) = f_{|L_{1+2+\cdots+n}}(a_1 a_2 \cdots a_n)$

(b) $f_{|L_i}(a_i) f_{|L_{i+1}}(a_{i+1}) \cdots f_{|L_j}(a_j) = f_{|L_{i+(i+1)+\cdots+j}}(a_i a_{i+1} \cdots a_j)$, where $i < j$, $\forall$ $a_i \in L_i, a_{i+1} \in L_{i+1}, \cdots, a_j \in L_j$

(c) $f(a_1 a_2 \cdots a_n) = f(a_1) f(a_2) \cdots f(a_n)$ $\forall$ $a_1, a_2, \cdots, a_n \in \mathbf{N}$

(d) $f(a^i) = (f(a))^i \in f(L_i)$ $\forall$ $a^i \in L_i$

Proof: (a) We will use mathematical induction.

The result is obvious when $n = 1$.

By (Lemma 2.1 (a)) when $n = 2$, we have

$f_{|L_1}(a_1) f_{|L_2}(a_2) = f_{|L_{1+2}}(a_1 a_2)$.

Suppose that it is true for $n = k - 1$,

$f_{|L_1}(a_1) f_{|L_2}(a_2) \cdots f_{|L_{k-1}}(a_{k-1}) = f_{|L_{1+2+\cdots+(k-1)}}(a_1 a_2 \cdots a_{k-1})$

We will show that this implies it is true for $n = k$

$$\begin{aligned} f_{|L_1}(a_1) f_{|L_2}(a_2) \cdots f_{|L_k}(a_k) &= \left(f_{|L_1}(a_1) f_{|L_2}(a_2) \cdots f_{|L_{k-1}}(a_{k_1})\right)\left(f_{|L_k}(a_k)\right) \\ &= f_{|L_{1+2+\cdots+(k-1)}}(a_1 a_2 \cdots a_{k-1}) f_{|L_k}(a_k) \\ &= f_{|L_{1+2+\cdots+k}}(a_1 a_2 \cdots a_k) \end{aligned}$$

(b) Similar to (a) by induction. However, we will start with the number i instead of 1.

If $j = i$ it is obvious.

By (Lemma 2.1 (a)), in case of $j = i + 1$, we have

$f_{|L_i}(a_i) f_{|L_{i+1}}(a_{i+1}) = f_{|L_{i+(i+1)}}(a_i a_{i+1})$.

If we suppose that

$f_{|L_i}(a_i) f_{|L_{i+1}}(a_{i+1}) \cdots f_{|L_{j-1}}(a_{j-1}) = f_{|L_{i+(i+1)+\cdots+(j-1)}}(a_i a_{i+1} \cdots a_{j-1})$.

then

$$\begin{aligned} f_{|L_i}(a_i) f_{|L_{i+1}}(a_{i+1}) \cdots f_{|L_j}(a_j) &= \left(f_{|L_i}(a_i) f_{|L_{i+1}}(a_{i+1}) \cdots f_{|L_{j-1}}(a_{j-1})\right)\left(f_{|L_j}(a_j)\right) \\ &= f_{|L_{i+(i+1)+\cdots+(j-1)}}(a_i a_{i+1} \cdots a_{j-1}) f_{|L_j}(a_j) \\ &= f_{|L_{i+(i+1)+\cdots+(j-1)+j}}(a_i a_{i+1} \cdots a_{j-1} a_j) \end{aligned}$$

(c) Similar to (a)

(d) Let $a^i \in L_i$. Then, by (b) we have

$$f\left(a^i\right)=f\left(a\cdots a\right)$$
$$=f\left(a\right)\cdots f\left(a\right)$$
$$=\left(f\left(a\right)\right)^i\in f\left(L_i\right)$$

■

Theorem 2.2: *If* $f:\mathbf{N}\to\mathbf{N}$ *is a levely multiplicative function, then for any* $a=a_1^{e_1}a_2^{e_2}\cdots a_n^{e_n}\in L_n$, *where* $0\le e_j\le n$, $j=1,2,\cdots,n$, $e_1+e_2+\cdots+e_n=n$, *and* $a_1,a_2,\cdots,a_n\in\mathbf{P}$ *in the unique decomposition of prime powers, we have* $f\left(a\right)=\left(f\left(a_1\right)\right)^{e_1}\left(f\left(a_2\right)\right)^{e_2}\cdots\left(f\left(a_n\right)\right)^{e_n}\in f\left(L_n\right)$ *in the unique decomposition of the prime powers.*

Proof: Let, $a=a_1^{e_1}a_2^{e_2}\cdots a_n^{e_n}\in L_n$, where $0\le e_j\le n$, $j=1,2,\cdots,n$, $e_1+e_2+\cdots+e_n=n$, and $a_1,a_2,\cdots,a_n\in\mathbf{P}$. Then, by (Corollary 2.1 (b), (c)), we have

$$f\left(a\right)=f\left(a_1^{e_1}a_2^{e_2}\cdots a_n^{e_n}\right)$$
$$=f\left(a_1^{e_1}\right)f\left(a_2^{e_2}\right)\cdots f\left(a_n^{e_n}\right)$$
$$=\left(f\left(a_1\right)\right)^{e_1}\left(f\left(a_2\right)\right)^{e_2}\cdots\left(f\left(a_n\right)\right)^{e_n}\in f\left(L_n\right)$$

Now, by (Theorem 1.1), since $a=a_1^{e_1}a_2^{e_2}\cdots a_n^{e_n}$ is unique,

$$f\left(a\right)=\left(f\left(a_1\right)\right)^{e_1}\left(f\left(a_2\right)\right)^{e_2}\cdots\left(f\left(a_n\right)\right)^{e_n}$$

is also unique. ■

Furthermore, one of the characteristics that distinguishes the levely multiplicative function $f:\mathbf{N}\to\mathbf{N}$, which is generated by $f_{|L_1}$, is that if $f_{|L_1}$ is a bijective function, then all $f_{|L_i}, i\ge 2$, and f are bijective.

Theorem 2.3: *If* $f:\mathbf{N}\to\mathbf{N}$ *is a levely multiplicative function and* $f_{|L_1}:L_1\to L_1$ *is injective, then*:

(a) $f_{|L_i}:L_1\to L_1$ is bijective, $\forall\ i=1,2,3,\cdots$

(b) f is bijective.

Proof: (a) First, we will show that $f_{|L_i}:L_i\to L_i$ is injective, $\forall\ i=2,3,\cdots$

Let $a,b\in L_i, a\ne b$, where $a=a_1a_2\cdots a_i, b=b_1b_2\cdots b_i$, $a_1,a_2,\cdots,a_i,b_1,b_2,\cdots,b_i\in L_1$. Since, $a\ne b$, there exists $a_n\in L_1, 1\le n\le i$, and $b_m\in L_i, 1\le m\le i$, such that $a_n\ne b_m$. Since $f_{|L_1}$ is injective, $f_{|L_1}\left(a_n\right)\ne f_{|L_1}\left(b_m\right)$. By (Theorem 2.2), we have

$$f_{|L_i}\left(a\right)=f_{|L_1}\left(a_1\right)f_{|L_1}\left(a_2\right)\cdots f_{|L_1}\left(a_n\right)\cdots f_{|L_1}\left(a_i\right)$$
$$\ne f_{|L_1}\left(b_1\right)f_{|L_1}\left(b_2\right)\cdots f_{|L_1}\left(b_m\right)\cdots f_{|L_1}\left(b_i\right)$$
$$=f_{|L_i}\left(b\right)$$

Hence, $f_{|L_i}$ is injective for all $i=2,3,\cdots$

To prove that $f_{|L_i}$ is surjective for all $i=2,3,\cdots$, let $b=b_1b_2\cdots b_i\in L_i$, so $b_1,b_2,\cdots,b_i\in L_1$. Since $f_{|L_1}$ is surjective, there exist $a_1,a_2,\cdots,a_i\in L_1$ such that $f_{|L_1}\left(a_1\right)=b_1, f_{|L_1}\left(a_2\right)=b_2,\cdots,f_{|L_1}\left(a_i\right)=b_i$.

Therefore, $f_{|L_1}\left(a_1\right)f_{|L_1}\left(a_2\right)\cdots f_{|L_1}\left(a_i\right)=b_1b_2\cdots b_i$,

and $f_{|L_i}\left(a_1a_2\cdots a_i\right)=b$, where $a_1a_2\cdots a_i\in L_i$.

So $f_{|L_i}$ is surjective. Hence, $f_{|L_i}$ is bijective for all $i=1,2,\cdots$

(b) First, we show that f is injective.

i) If $a,b\in\mathbf{N}, a\neq b$, and $a\in L_i, b\in L_j, i\neq j$, then
$f(a)=f_{|L_i}(a)\in f_{|L_i}(L_i)$, $f(b)=f_{|L_j}(b)\in f_{|L_j}(L_j)$,
and since $f_{|L_i}(L_i)\cap f_{|L_j}(L_j)=\phi$, we have $f(a)\neq f(b)$

ii) If $a,b\in\mathbf{N}, a\neq b$, and $a,b\in L_i$, then by (a), we have

$$f(a)=f_{|L_i}(a)\neq f_{|L_i}(b)=f(b).$$

Hence, f is injective.

Now, by (a) $f_{|L_i}(L_i)=L_i, \forall i=0,1,2,\cdots$. Therefore

$$\begin{aligned} f(\mathbf{N}) &= f\left(\bigcup_{i=0}^{\infty} L_i\right) \\ &= \bigcup_{i=0}^{\infty} f(L_i) \\ &= \bigcup_{i=0}^{\infty} f_{|L_i}(L_i) \\ &= \bigcup_{i=0}^{\infty} L_i = \mathbf{N} \end{aligned}$$

Hence, f is surjective and therefore bijective. ■

Corollary 2.2 *If* $f:\mathbf{N}\to\mathbf{N}$ *is a levely multiplicative and a bijective function, then* $f^{-1}:\mathbf{N}\to\mathbf{N}$ *is levely multiplicative and bijective.* ■

3. Levely Multiplicative Functions with Divisibility

Let $f:\mathbf{N}\to\mathbf{N}$ be a levely multiplicative function. If $a\mid b$, then $f(a)\mid f(b)$ is necessary, but $f(a)\mid f(b)$ is not sufficient for $a\mid b$. For example, in (Example 2.2) $2=f(2)\mid f(5)=2$, but $2\nmid 5$.

For $f(a)\mid f(b)$ to be necessary and sufficient for $a\mid b$, f must be bijective.

Theorem 3.1 *Let* $f:\mathbf{N}\to\mathbf{N}$ *be a levely multiplicative function.*

(a) If $a\mid b$, then $f(a)\mid f(b)$.

(b) If f is bijective, then $a\mid b$ if and only if $f(a)\mid f(b)$.

Proof: (a) Let $a,b\in\mathbf{N}, a\mid b$. Then there exists $c\in\mathbf{N}$ such that $b=ca$. Therefore $f(b)=f(ca)=f(c)f(a)$. Thus, $f(a)\mid f(b)$.

(b) ($\Rightarrow$) By (a).

($\Leftarrow$) Let $a,b\in\mathbf{N}$, $f(a)\mid f(b)$. Then there exists $c\in\mathbf{N}$ such that $f(b)=cf(a)$. Since f is surjective, there exist $d\in\mathbf{N}$ such that $c=f(d)$. Therefore, $f(b)=f(d)f(a)=f(da)$. Since f is injective, $b=da$. Hence, $a\mid b$. ■

As a result of Theorem 3.1, the next theorem and corollary show that a bijective, levely multiplicative, and increasing function preserves the greatest common divisor for any two numbers in $\mathbf{N}$ and the relative prime numbers.

Theorem 3.2 *Let* $f:\mathbf{N}\to\mathbf{N}$ *be a levely multiplicative, bijective, and increasing function, then* $(a,b)=d$ *if and only if* $(f(a),f(b))=f(d)$.

Proof: ($\Rightarrow$) Let $(a,b)=d$ .Then $d\mid a$ and $d\mid b$. By (Theorem 3.1), we have $f(d)\mid f(a)$ and $f(d)\mid f(b)$. If $f(c)\mid f(a)$ and $f(c)\mid f(b)$, and we suppose that $f(c) \geq f(d)$, then by (Theorem 3.1), we have $c \mid a$, and $c \mid b$' Therefore $c \leq d$. But f is an increasing function, so we have a contradiction. Hence $f(d) \leq f(c)$, and $(f(a), f(b)) = f(d)$.

($\Leftarrow$) Let $(f(a), f(b)) = f(d)$. Then $f(d) \mid f(a)$ and $f(d) \mid f(b)$. By (Theorem 3.1) we have $d \mid a, d \mid b$. If we suppose $c \mid a$ and $c \mid b$, and we suppose $c \geq d$, then by (Theorem 3.1), we have $f(c) \mid f(a)$ and $f(c) \mid f(b)$. Therefore $f(c) \leq f(d)$. But f is an increasing function, so we have a contradiction. Hence, $d \leq c$. Thus, $(a,b) = d$. ■

Corollary 3.1: *Let* $f : \mathbf{N} \to \mathbf{N}$ *be a levely multiplicative, bijective and increasing function, then* $(a,b) = 1$ *if and only if* $(f(a), f(b)) = 1$.

Proof: In (Theorem 3.2) when $d = 1$ ■

Now, when $f : \mathbf{N} \to \mathbf{N}$ is a levely multiplicative function, in case f is not bijective function, it is not necessary that $f(a\uparrow) = f(a)\uparrow$. For instance, from (Example 2.2) we have

$$\begin{aligned} f(2\uparrow) &= f(\{2n : n \in \mathbf{N}\}) \\ &= \{2^n : n \in \mathbf{N}\} \end{aligned}$$

but

$$\begin{aligned} f(2)\uparrow &= 2\uparrow \\ &= \{2n : n \in \mathbf{N}\} \end{aligned}$$

Theorem 3.3: *If* $f : \mathbf{N} \to \mathbf{N}$ *is a levely multiplicative and bijective function, then*:

(a) $f(a\uparrow) = f(a)\uparrow$

(b) $f(A\uparrow) = f(A)\uparrow$

Proof: (a) Let $a\uparrow = \{na : n \in \mathbf{N}\}$.

Since f is bijective, we get

$$\begin{aligned} f(a\uparrow) &= \{f(na) : n \in \mathbf{N}\} \\ &= \{f(n) f(a) : n \in \mathbf{N}\} \\ &= f(a)\uparrow \end{aligned}$$

(b) Let $A \subseteq \mathbf{N}$. By (Lemma 1.2), we have $A\uparrow = \bigcup_{a \in A} a\uparrow$. By (a), we have.

$$\begin{aligned} f(A\uparrow) &= f\left(\bigcup_{a \in A} a\uparrow\right) \\ &= \bigcup_{f(a) \in f(A)} f(a\uparrow) \\ &= \bigcup_{f(a) \in f(A)} f(a)\uparrow \\ &= f(A)\uparrow \end{aligned}$$

■

Corollary 3.2: *If* $f : \mathbf{N} \to \mathbf{N}$ *is a levely multiplicative and bijective function and* $A \subseteq \mathbf{N}$ *is upward closed, then* $f(A) = f(A)\uparrow$

Proof: Let $A \subseteq \mathbf{N}$, $A = A\uparrow$. By (Theorem 3.3 (b)) we have

$$\begin{aligned} f(A) &= f(A\uparrow) \\ &= f(A)\uparrow \end{aligned}$$

Thus, $f(A)$ is upward closed. ■

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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