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Hodge Classes in Hodge Manifolds

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Abstract

The Hodge conjecture states that the Hodge group of Hodge classes is equal to the algebraic group generated by algebraic subvarieties on a Hodge manifold. To prove the conjecture, we introduce the Hodge structure, the Lefschetz decomposition, and polarization on the cohomology groups of the manifold, and we use mathematical induction on the degrees of the primitive cohomologies in the Lefschetz decomposition. We show that every Hodge class on a Hodge manifold is a rational linear combination of the cohomology classes of algebraic subvarieties of the manifold. The Lefschetz isomorphisms on the cohomology groups of a Hodge manifold are algebraic in the product space. As a consequence, we show that the inverses of the Lefschetz isomorphisms are also algebraic in the product space.

Keywords

Hodge Manifold, Hodge Decomposition. Lefschetz Decomposition, Hodge Class, Algebraic Cycle Class, Polarization, Hodge Conjecture, Lefschetz Standard Conjecture

1. Introduction

In 1951, W. Hodge [1] announced the Hodge conjecture. The early mathematicians studied the Hodge conjecture on the compact Kähler manifolds and the cohomology groups with the integer coefficients. Some of them are the followings.

Atiyah and Hirzebruch [2] constructed a torsion cohomology class which is Hodge but not algebraic. Kollár [3] found an example of a Hodge class which is not algebraic in the integral cohomology groups of a projective complex manifold. Mumford [4] constructed an abelian variety whose Hodge class is not generated by products of divisor classes, and Weil [5] generalized the example. Zucker [6] constructed a counterexample to the Hodge conjecture as complex tori with a Hodge class which is not algebraic. Voisin [7] proved that on Kähler varieties the

Chern classes of coherent sheaves give more Hodge class than the Chern classes of vector bundles, and the Chern classes of coherent sheaves are insufficient to generate the Hodge classes. Thus, the Hodge conjecture for Kähler varieties is not true.

The modern statement of the Hodge conjecture [8]-[10]: Let X be a smooth complex projective manifold. Then every Hodge class on X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

In mathematics, the solutions of differential equations give much information on the objects which we want to study [11]-[13]. The Hodge conjecture is to find the algebraic cycle class for each Hodge class. Many mathematicians [14]-[20] have studied the Hodge conjecture and its related areas.

Let X be a Hodge manifold of dimension n. The Hodge conjecture says the Hodge group $\mathrm{Hdg}^{2k}(X)$ is equal to the algebraic group $\mathrm{H}^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$ for $0 \le k \le n$. To prove the conjecture, we use the Hodge decomposition, the Lefschetz decomposition and polarization on the cohomology groups of X, and the induction on the degrees of the primitive cohomologies of X [9] [14] [21].

We introduce some definitions and results without proofs. In Section 2, we recall Kähler manifolds with fundamental form, and on cohomology groups Hodge decomposition, Lefschetz decomposition, and polarization. We review Hodge manifolds, Kodaira embedding and ample line bundle.

In Section 3, we introduce Hodge and algebraic classes of Hodge manifolds, the Hodge conjecture and some known results.

In Section 4, on Hodge manifolds, using the Lefschetz decomposition, polarization, and the induction on degrees, we want to show that every Hodge class of Hodge manifolds is algebraic. The morphisms between Hodge structures yield Hodge classes in the product space. The Lefschetz isomorphism and its inverse are morphisms on Hodge structures. Using above results, we show that they are algebraic. In this paper, the dimensions are the complex dimensions.

2. Hodge Manifold

We are familiar with the Kähler manifolds and every Hodge manifold is Kähler. We introduce the Kähler manifolds and their properties which we use in this paper.

2.1. Hodge Structure

Let (X,h) be a compact Kähler manifold of dimension n with a Kähler metric h and the associated fundamental 2-form Ω. Let $\ell = [\Omega]$ be the cohomology class of the 2-form Ω.

(2.1.1) In the holomorphic coordinates $(z_1,\cdots,z_n)$ of X, we can write them as follows:

$$h = \sum_{\mu,\nu} h_{\mu\nu} dz_\mu \otimes d\bar{z}_\nu = g + (-2i)\Omega, \quad h_{\mu\nu} = \bar{h}_{\nu\mu},$$

$$ {}^{t}\bar{u}\left(h_{\mu\nu}\right)u > 0 \text{ if } u \neq 0, $$

$$ \Omega = \frac{i}{2}\sum_{\mu,\nu} h_{\mu\nu} dz_{\mu} \wedge d\bar{z}_{\nu}, $$

and $d\Omega = 0$, Ω is real and type (1,1).

(2.1.2) If $\Delta = dd^{*} + d^{*}d$, $\square = \partial\partial^{*} + \partial^{*}$, $\bar{\square} = \bar{\partial}\bar{\partial}^{*} + \bar{\partial}^{*}\bar{\partial}$ are Laplacians, then

$$ \Delta = 2 = 2\bar{\square}, \ *\Delta = \Delta *, \ *\bar{\square} = \bar{\square}*, $$

where $*$ is the Hodge star operator. Every cohomology class is represented by a unique harmonic form.

(2.1.3) The cohomologies of X have the Hodge decompositions:

$$ H^{k}\left(X,\mathbb{C}\right) = \underset{p+q=k}{\oplus} H^{p,q}\left(X\right), \quad \bar{H}^{p,q}\left(X\right) = H^{q,p}\left(X\right). $$

(2.1.4) The Lefschetz decomposition:

$$ H^{k}\left(X,\mathbb{C}\right) = \underset{2r\leq k}{\oplus} \ell^{r} \cup H_{0}^{k-2r}\left(X,\mathbb{C}\right), $$

where $H_{0}^{k-2r}\left(X,\mathbb{C}\right) = \operatorname{Ker}\left[\ell^{n-k+2r+1}\cup : H^{k-2r}\left(X,\mathbb{C}\right) \to H^{2n-k+2r+2}\left(X,\mathbb{C}\right)\right]$ is the primitive cohomology of *X*. Also,

$$ H_{0}^{k}\left(X,\mathbb{C}\right) = \underset{p+q=k}{\oplus} H_{0}^{p,q}\left(X\right), $$

where $H_{0}^{p,q}\left(X\right) = \operatorname{Ker}\left[\ell^{n-k+1}\cup : H^{p,q}\left(X\right) \to H^{n-k+p+1,n-k+q+1}\left(X\right)\right]$.

(2.1.5) There is a polarization Q on $H^{k}\left(X,\mathbb{C}\right)$:

$$ Q : H^{k}\left(X,\mathbb{C}\right) \times H^{k}\left(X,\mathbb{C}\right) \to \mathbb{C} $$

defined by

$$ Q\left(\xi,\eta\right) = \sum_{2r\leq k} \left(-1\right)^{\frac{k(k+1)}{2}+r} \int_{X} \ell^{n-k+2r} \cup \xi_{r} \cup \eta_{r}, $$

where $\xi = \sum_{r} \ell^{r} \cup \xi_{r}$, $\eta = \sum_{r} \ell^{r} \cup \eta_{r} \in H^{k}\left(X,\mathbb{C}\right)$. Then the polarization Q on $H^{k}\left(X,\mathbb{C}\right)$ is orthogonal, nondegenerate, and positive definite on the Lefschetz decomposition, and satisfies the Hodge-Riemann bilinear relations [21].

2.2. Hodge Manifold

(2.2.1) A closed form φ on X is said to be integral if the cohomology class $[\varphi]$ is in the image of the natural map $H^{*}\left(X,\mathbb{Z}\right) \to H^{*}\left(X,\mathbb{C}\right)$. Let $\left(X,h\right)$ be a compact Kähler manifold and Ω be its associated fundamental form. If Ω is an integral form, then Ω is called a Hodge form of X and h is called a Hodge metric on X, and $\left(X,h\right)$ is called a Hodge manifold.

(2.2.2) Kodaira Embedding Theorem [22]: A compact Kähler manifold is Hodge if and only if it is a complex projective algebraic manifold.

i) Every complex submanifold of a complex projective space is Hodge.

ii) Let $\left\{w_{1},\cdots,w_{2n}\right\}$ be a $2n$ independent periods in the complex space $\mathbb{C}^{n}$ of dimension n. The complex torus $X := \mathbb{C}^{n}/\left[w_{1},\cdots,w_{2n}\right]$ is a compact Kähler manifold. However, X is a Hodge manifold if and only if the $n \times 2n$ matrix $\left[w_{1},\cdots,w_{2n}\right]$ is a Riemann matrix [21].

(2.2.3) A compact Kähler manifold X admits a positive line bundle $L \to X$ if and only if X is Hodge. Here a line bundle $L \to X$ is positive means that, if h is a metric on L, $\Omega = \bar{\partial}\partial \log h$ and its first Chern class $c_1(L) = \frac{i}{2\pi}[\Omega]$, then Ω is a positive definite Hermitian symmetric matrix.

(2.2.4) Let X be a Hodge manifold and $L \to X$ be a positive holomorphic line bundle. If for each point $x \in X$ there is a section $s \in \Gamma(X, L)$ such that $s(x) \neq 0$, then X is embedded in the projective space $\mathbb{P}^N$, where $N := \dim(\Gamma(X, L)) - 1$. In this case $L \to X$ is called very ample. A line bundle $L \to X$ is called ample if there is an $m \geq 1$ such that $L^{\otimes m} \to X$ is very ample. Thus every Hodge manifold admits an ample line bundle.

(2.2.5) Let X be an algebraic variety defined by the zero locus of k homogeneous polynomials on a projective space $\mathbb{P}^N$. Then X is a Hodge manifold of codimension k in $\mathbb{P}^N$. In particular, if $N = 4$, $k = 1$, and the homogeneous polynomial has degree 5, then the quintic threefold X is Hodge and Calabi-Yau since $c_1(TM) = 0$. The quintic threefold X plays a crucial role in mirror symmetry [11] and the Hodge conjecture holds on X by Subsection 3.5.

3. Hodge and Algebraic Classes

Let X be a Hodge manifold of dimension n, and $\ell = c_1(L) \in H^2(X, \mathbb{Q})$ be the first Chern class of an ample line bundle $L \to X$.

3.1. The Hard Lefschetz Theorem [23]

The cup product map

$$\ell^{n-k} \cup : H^k(X; \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q})$$

is an isomorphism of Hodge structures, and the pairing

$$(\alpha, \beta)_\ell = \int_X \ell^{n-k} \cup \alpha \cup \beta, \ \alpha, \beta \in H^k(X, \mathbb{Q})$$

is nondegenerate. On the Lefschetz decomposition:

$$H^{2k}(X, Q) = \bigoplus_{r=0}^{k} \ell^r \cup H_0^{2k-2r}(X, Q),$$

$$H_0^{k,k}(X) = \bigoplus_{r=0}^{k} \ell^r \cup H_0^{k-r,k-r}(X)$$

the pairing $(\ ,\ )_\ell$ is orthogonal and nondegenerate.

3.2. Hodge Group

For $0 \leq k \leq n$, the group

$$\mathrm{Hdg}^{2k}(X) := H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

is called the Hodge group of Hodge classes with degree $2k$, and the group

$$H^{2k}(X, \mathbb{Q})_{\mathrm{alg}} := \left\{ \sum \alpha_i [Z_i] : \text{a finite sum} \,\middle|\, \begin{array}{l} \alpha_i \in \mathbb{Q}, Z_i \text{ is a subvariety of} \\ \text{codimension } k \text{ in } X, \text{for each } i \end{array} \right\}$$

is called the algebraic group of X of algebraic classes with degree $2k$.

3.3. Algebraic Class

Let X be a Hodge manifold of dimension n. Then for $0 \le k \le n$, the Hodge decomposition is

$$H^{2k}(X,\mathbb{C}) = \bigoplus_{p+q=2k} H^{p,q}(X) \supset H^{k,k}(X)$$

For each subvariety Z of codimension k in X, if $\alpha \notin H^{n-k,n-k}(X)$, then $\int_Z \alpha = 0$ by type reason. Thus, the algebraic class $[Z]$ is in $\mathrm{Hdg}^{2k}(X)$. Therefore

$$H^{2k}(X,\mathbb{Q})_{\mathrm{alg}} \subset \mathrm{Hdg}^{2k}(X).$$

3.4. The Hodge Conjecture (1951)

Let X be a Hodge manifold of dimension n. Then

$$\mathrm{Hdg}^{2k}(X) = H^{2k}(X,\mathbb{Q})_{\mathrm{alg}},\ 0 \le k \le n.$$

That is, $\mathrm{Hdg}^{2k}(X) \subset H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$, $0 \le k \le n$.

3.5. Known Results [9]

Let X be a connected Hodge manifold of dimension n. Then

1) $H^0(X,\mathbb{Q}) = \mathrm{Hdg}^0(X) = H^0(X,\mathbb{Q})_{\mathrm{alg}} = \mathbb{Q}([X])$,
2) $H^{2n}(X,\mathbb{Q}) = \mathrm{Hdg}^{2n}(X) = H^{2n}(X,\mathbb{Q}) = \mathbb{Q}([\mathrm{pt}])$,
3) (Lefschetz)

$$\mathrm{Hdg}^2(X,\mathbb{Z}) = H^2(X,\mathbb{Z})_{\mathrm{alg}},$$

$$\mathrm{Hdg}^{2n-2}(X) = H^{2n-2}(X,\mathbb{Q})_{\mathrm{alg}}.$$

Here (3), (4) can be proved by the exponential exact sequence

$$0 \to \mathbb{Z} \to \Theta_X \to \Theta_X^* \to 1,$$

and Subsection 3.1 the hard Lefschetz theorem.

4. Hodge Classes Are Algebraic

Let X be a Hodge manifold of dimension n, and $L \to X$ be an ample line bundle with the Chern class $\ell := c_1(L) \in H^2(X,\mathbb{C})$.

4.1. Mathematical Induction

By Subsection 3.5 we have, for $k = 0,1,n-1,n$

$$\mathrm{Hdg}^{2k}(X) = H^{2k}(X,\mathbb{Q})_{\mathrm{alg}},$$

and by Subsection 3.3, for $0 \le k \le n$

$$H^{2k}(X,\mathbb{Q})_{\mathrm{alg}} \subset \mathrm{Hdg}^{2k}(X).$$

We want to prove that for $2 \le k \le n-2$,

$$\mathrm{Hdg}^{2k}(X) \subset H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$$

by the induction.

4.2. Hodge and Lefschetz Decompositions

The Hodge and Lefschetz decompositions are

$$H^{2k}(X,C) = \bigoplus_{p+q=2k} H^{p,q}(X) = \bigoplus_{0\le r\le k} \ell^r \cup H_0^{2k-2r}(X,C),$$
$$H^{k,k}(X) = \sum_{r=0}^{k} \ell^r \cup H_0^{k-r,k-r}(X),$$

where

$$H_0^{k-r,k-r} = \mathrm{Ker}\left[\ell^{n-2k+2r+1}\cup : H^{k-r,k-r}(X) \to H^{n-k+r+1,n-k+r+1}(X)\right]$$

is the primitive cohomology group of type $(k-r,k-r)$ of X. By (2.1.4), there is a polarization Q on $H^{2k}(X,\mathbb{C})$ such that the decompositions are orthogonal and nondegenerate.

4.3. Primitive Decomposition

Let $\alpha \in \mathrm{Hdg}^{2k}(X) = H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$ be a Hodge class with degree $2k$ of X. Then $\alpha \in H^{k,k}(X)$ has a type (k,k) with coefficients in $\mathbb{Q}$. Since

$$H^{k,k}(X) = \ell^0 \cup H_0^{k,k}(X) \oplus \ell^1 \cup H_0^{k-1,k-1}(X) \oplus \cdots \oplus \ell^k \cup H_0^{0,0}(X)$$

is orthogonal and nondegenerate, $\alpha \in H^{k,k}(X)$ is uniquely written as

$$\alpha = \sum_{r=0}^{k} \ell^r \cup \alpha_{k-r},\ \alpha_{k-r} \in H_0^{k-r,k-r}(X).$$

Choose generic sections σ_i of the ample bundle $L \to X$, $i = 1,2,\cdots,k$, and let $L_r = \bigcap_{i=1}^{r} \sigma_i^{-1}(0)$, and $L_0 = X$. Then $L_r \subset X$ is a subvariety of codimension r in X and the cycle class $[L_r] = \ell^r \in H^{2r}(X,\mathbb{Q}) \cap H^{r,r}(X)$.

4.4. Induction Hypothesis

Find a codimension $k-r$ subvariety A_{k-r} of X such that its cohomology cycle class $[A_{k-r}] = \alpha_{k-r} \in H_0^{k-r,k-r}(X)$. Then the Hodge class

$$\ell^r \cup \alpha_{k-r} = [L_r] \cup [A_{k-r}] = [L_r \cap A_{k-r}] \in \ell^r \cup H_0^{k-r,k-r}(X) \subset H^{k,k}(X)$$

is algebraic. Prove this by the mathematical induction on $k-r$. By Subsection 3.5 there are $A_0 = X$ and $A_1 \subset X$ such that

$$\alpha_0 = [X],\ \alpha_1 = [A_1]$$

and

$$\ell^k \cup \alpha_0 = [L_k \cap X] \in \ell^k \cup H_0^{0,0}(X),$$
$$\ell^{k-1} \cup \alpha_1 = [L_{k-1} \cap A_1] \in \ell^{k-1} \cup H_0^{1,1}(X).$$

4.5. Primitiveness

Assume that there are subvarieties $A_0, A_1, \cdots, A_{r-1}$ in X such that

$$[A_i] = \alpha_i,\ [L_{k-i} \cap A_i] = \ell^{k-i} \cup \alpha_i \in \ell^{k-i} \cup H_0^{i,i}(X).$$

Since $\alpha_{k-r} \in H_0^{k-r,k-r}(X)$, the cup map

$$\ell^{n-2k+2r+1} \cup : H^{k-r,k-r}(X) \to H^{n-k+r+1,n-k+r+1}(X)$$

satisfies $\ell^{n-2k+2r+1} \cup \alpha_{k-r} = 0$. Locally we can write

$$\ell^{n-2k+2r+1} := \left[\sum_{|J|=n-2k+2r+1} c_J dz_J \wedge d\bar{z}_J \right],$$

and

$$\alpha_{k-r} := \left[\sum_{|I|=k-r} d_I dz_I \wedge d\bar{z}_I \right].$$

Then $\ell^{n-2k+2r+1} \cup \alpha_{k-r} = 0$ if and only if for each I, J, $I \cap J \neq \varnothing$, $I, J \subset \{1, 2, \cdots, n\}$.

4.6. Analytic Continuation

While ℓ is locally represented, in holomorphic coordinates

$$\ell = \left[\frac{i}{2} \sum_{\mu,\nu} h_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu \right]$$

and ℓ is hermitian symmetric. Since $I \cap J \neq \varnothing$ for each I, J in $\ell^{n-2k+2r+1}$ and α_{k-r} there is a 2-form ℓ' in $\ell^{n-2k+2r+1}$ and α_{k-r} if necessary, by the change of coordinates. By the analytic continuation [24], the 2-form is globally defined on X such that $\ell^{n-2k+2r} \cup [\ell'] = \ell^{n-2k+2r+1}$ and $[\ell'] = \ell$. A generic section $s : X \to L$ has the zero locus $s^{-1}(0)$, and its cycle class

$$[s^{-1}(0)] = [L_1] = [\ell'] = \ell = c_1(L).$$

The Lefschetz isomorphism

$$\ell^{n-2(k-r)} \cup : H^{2(k-r)}(X) \to H^{2n-2(k-r)}(X)$$

implies that $\ell^{n-2(k-r)} \cup \alpha_{k-r} \neq 0$ if $\alpha_{k-r} \neq 0$. However $\ell^{n-2(k-r)+1} \cup \alpha_{k-r} = \left(\ell^{n-2(k-r)} \cup \alpha_{k-r}\right) \cup \ell = 0$.

4.7. Interior and Exterior Products

The 2-form $\ell \in H^{1,1}(X)$, the orientation, the metric on X and the polarization Q on $H^{2k}(X, \mathbb{C})$ define the interior and exterior products, for the definitions see [8]. Let

$$i : H^{k-r,k-r}(X) \to H^{k-r-1,k-r-1}(X),$$
$$e : H^{k-r-1,k-r-1}(X) \to H^{k-r,k-r}(X)$$

be the interior and exterior products.

Then for each $\ell \in H^{1,1}(X)$, $\alpha \in H^{k-r,k-r}(X)$, $\beta \in H^{k-r-1,k-r-1}(X)$,

$$\begin{aligned} Q(i(l)\alpha,\beta) &= (-1)^{\left[\frac{(2k-2r-2)(2k-2r-1)}{2}+(r+1)\right]} \int_X \ell^{n-2k+2r+2} \cup i(\alpha) \cup \beta \\ &= (-1)^k \int_X \ell^{n-2k+2r+1} \cup \alpha \cup \beta \\ &= (-1)^k \int_X \ell^{n-2k+2r} \cup \alpha \cup e(l)\beta \\ &= (-1)^{\left[\frac{(2k-2r)(2k-2r+1)}{2}+r\right]} \int_X \ell^{n-2k+2r} \cup \alpha \cup e(l)\beta \\ &= Q(\alpha, e(l)\beta). \end{aligned}$$

Thus, we have

$$Q(i(\ell)\alpha,\beta) = Q(\alpha, e(\ell)\beta).$$

Then there is a unique class $\beta_{k-r-1} \in H^{k-r-1,k-r-1}(X)$ such that

$$i(\ell)\cdot\alpha_{k-r} = \beta_{k-r-1} \text{ and } \alpha_{k-r} = e(\ell)\beta_{k-r-1}.$$

Also

$$Q(\beta_{k-r-1},\beta_{k-r-1}) = Q(i(\ell)\alpha_{k-r}, i(\ell)\alpha_{k-r}) = Q(\alpha_{k-r}, e(\ell)i(\ell)\alpha_{k-r}) = Q(\alpha_{k-r},\alpha_{k-r}) \neq 0,$$

since $\alpha_{k-r} \neq 0$. Thus the $\beta_{k-r-1} \neq 0$. Moreover,

$$\begin{aligned} \ell^{n-2k+2(r+1)+1} \cup \beta_{k-r-1} &= \ell^{n-2k+2r+1} \cup (l \cup \beta_{k-r-1}) \cup l \\ &= (\ell^{n-2k+2r+1} \cup \alpha_{k-r}) \cup l \\ &= 0 \cup l = 0. \end{aligned}$$

Thus $\beta_{k-r-1} \in H_0^{k-r-1,k-r-1}(X)$ is primitive. While the degree of β_{k-r-1} is

$$\deg(\beta_{k-r-1}) = 2(k-r-1) < 2(k-r)$$

for $0 \le r \le k$. By our induction hypothesis of the degree k, there is a subvariety B_{k-r-1} of codimension $k-r-1$ in X such that $[B_{k-r-1}] = \beta_{k-r-1}$.

4.8. Induction

Let $A_{k-r} = B_{k-r-1} \cap L_1$. Then A_{k-r} is a subvariety of codimension $k-r$ in X. The corresponding cycle class is

$$[A_{k-r}] = [B_{k-r-1} \cap L_1] = [B_{k-r-1}] \cup [L_1] = \beta_{k-r-1} \cup \ell = \alpha_{k-r}.$$

The Lefschetz components of the Hodge class α

$$\ell^r \cup \alpha_{k-r} = [L_r \cap A_{k-r}] \in H^{2k}(X,\mathbb{Q})_{\text{alg}}$$

are algebraic cycle classes. Thus, the Hodge class

$$\alpha = \sum_{r=0}^{k} \ell^r \cup \alpha_{k-r} = \sum_{r=0}^{k} [L_r \cap A_{k-r}] \in H^{2k}(X,Q)_{\text{alg}}$$

is an algebraic cycle class, that is,

$$\mathrm{Hdg}^{2k}(X) \subset H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}.$$

4.9. Theorem 1

Let X be a Hodge manifold of dimension n. Then the Hodge group $\mathrm{Hdg}^{2k}(X)$ is equal to the algebraic group $H^{2k}(X,\mathbb{Q})_{\mathrm{alg}}$ for $k=0,1,\cdots,n$. ■

4.10. Morphism of Hodge Structures

Let H_1 and H_2 be Hodge structures of weight k_1 and k_2, respectively and $k_2-k_1=2r$. Then the morphisms $\phi: H_1 \to H_2$ are Hodge classes in $\mathrm{Hdg}^{2r}(\mathrm{Hom}(H_1,H_2))$. For details see [9].

4.11. The Algebraicity of Lefschetz Isomorphism

Let X be a Hodge manifold of dimension n and ℓ be the first Chern class of an ample line bundle $L \to X$. Then the Lefschetz theorem states that the cup map

$$\ell^{n-k}\cup: H^k(X,\mathbb{Q}) \to H^{2n-k}(X,\mathbb{Q})$$

is an isomorphism of Hodge structures for each k. By Subsection 4.10,

$$\begin{aligned}\left[\ell^{n-k}\cup\right] &\in \mathrm{Hdg}^{**}\left[Hom\left(H^k(X,Q),H^{2n-k}(X,Q)\right)\right]\\ &= \mathrm{Hdg}^{**}\left[H^k(X,\mathbb{Q})^* \otimes H^{2n-k}(X,Q)\right]\\ &= \mathrm{Hdg}^{**}\left[H^{2n-k}(X,Q)\otimes H^{2n-k}(X,Q)\right]\\ &\subset \mathrm{Hdg}^{**}\left[H^{4n-2k}(X\times X,Q)\right],\end{aligned}$$

where $**=(2n-k)+(2n-k)=4n-2k$. By (4.9), the Hodge class $\left[\ell^{n-k}\cup\right]$ is algebraic in $X\times X$.

In fact, let $\sigma_i: X \to L$, $i=1,2,\cdots,n-k$, be generic sections, $Z=\bigcap_{i=1}^{n-k}\sigma_i^{-1}(0)\subset X$, and $W=i_\Delta(Z)$ where $i_\Delta: X\to X\times X$ is the diagonal map, then Z has codimension $n-k$ in X, and W has codimension $2n-k$ in $X\times X$. Thus $\left[\ell^{n-k}\cup\right]=[W]\in H^{4n-2k}(X\times X,\mathbb{Q})$ is algebraic.

4.12. The Inverse of Lefschetz Isomorphism

The inverse

$$\left(\ell^{n-k}\cup\right)^{-1}: H^{2n-k}(X,\mathbb{Q}) \to H^k(X,\mathbb{Q})$$

of the Lefschetz isomorphism is an isomorphism of Hodge structures. As in Subsection 4.11,

$$\begin{aligned}\left[\left(\ell^{n-k}\cup\right)^{-1}\right] &\in \mathrm{Hdg}^{2k}\left(Hom\left(H^{2n-k}(X,Q),H^k(X,Q)\right)\right)\\ &= \mathrm{Hdg}^{2k}\left(H^k(X,\mathbb{Q})\otimes H^k(X,\mathbb{Q})\right)\subset \mathrm{Hdg}^{2k}\left(H^{2k}(X\times X,\mathbb{Q})\right)\end{aligned}$$

is Hodge and algebraic by Subsection 4.9 since the product space $X\times X$ is a Hodge manifold of dimension $2n$.

These contents are called the Lefschetz standard conjecture.

4.13. Theorem 2

Let X be a Hodge manifold of dimension n and ℓ be the first Chern class of an ample line bundle $L \to X$. Then there is a codimension k subvariety Z in $X \times X$ such that

$$[Z] = \left[\left(\ell^{n-k} \cup \right)^{-1} \right] \in H^{2k}\left(X \times X, \mathbb{Q} \right),$$

that is, the inverse of the Lefschetz isomorphism is algebraic. ■

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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