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On the Connections between Goldbach Conjecture and Prime Number Theorem

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Abstract

By our suggested definition, an even number L_n is called the largest strong Goldbach number generated by the n-th prime P_n if every even number from 4 to L_n is the sum of two primes not greater than P_n but $L_n + 2$ is not such a sum. We discovered the existence of step-type distribution for L_n arising from observed fact that $L_n \le L_{n+1}$ and we proved that $L_n \le L_{n+1}$ for all $n > 0$. Every such step is called a Goldbach step whose width is $(n2 + 1) - n1$, where $n1$ is the starting point and $n2$ is the finishing point for the step. We proved that if Goldbach conjecture is true then there are infinitely many Goldbach steps. It is expected that distribution of Goldbach steps is asymptotically expressed as $Q(n) \sim n/\log n$ same as prime number theorem, where $Q(n)$ is the number of Goldbach steps. It means Goldbach steps have like-prime nature, thus, all $n1$ can be called like-primes and $g_i = n1\text{-}(i + 1) - n1\text{-}i$ is defined as gap between the i-th and the $(i + 1)$-th like-primes. We proved that if there are infinitely many like-prime gaps whose length k is uncertain but bounded by a finite integer $N > 1$, then Goldbach conjecture is true. Considering $k = 1$ for twin like-primes, it is conjectured that there are infinitely many like-primes $n1$ such that $n1 + 1$ is also like-prime to imply Goldbach conjecture and it is expected that distribution of twin like-primes is asymptotically expressed as $Q_2(n) \sim 2C_2n/(\log n)^2$ akin to prime number theorem and same as a special case of the first Hardy-Littlewood conjecture, where $Q_2(n)$ is the number of twin like-primes and C_2 is twin prime constant. We also studied distributions of triplet like-primes and quadruplet like-primes to imply Goldbach conjecture. We presented there are bounds of $L_n/2$ such that $n\log n + n\log\log n - n < L_n/2 < n\log n + n\log\log n$ for $n \ge 20542$, and in this paper, the bounds have been verified up to $n = 4000000000$. If it can be proven that bounds of prime, $n\log n + n\log\log n - n < P_n < n\log n + n\log\log n$ for $n \ge 6$, can be used as bounds of $L_n/2$ for $n \ge 20542$, then Goldbach conjecture is true. Further, we proved that if there is a bounded integer $k > 20541$ such that bounds of prime can be used as bounds of $L_n/2$ for $n \ge k$ then Goldbach conjecture is true, where bounded integer $k > 20541$ means value of k is uncertain but there exists upper bound $N > 20542$ for k.

Keywords

Largest Strong Goldbach Number, Goldbach Step, Like-Prime, Polignac's Conjecture, The First Hardy-Littlewood Conjecture, Like-Prime Gap, Twin Like-Prime Conjecture, Triplet and Quadruplet Like-Primes, Bounds of the Largest Strong Goldbach Number, Goldbach Conjecture, Prime Number Theorem

1. Introduction

As is well known, Goldbach conjecture is one of the most famous unsolved problems in mathematics and the conjecture states that every even number greater than 2 is the sum of two primes. An even number is called a Goldbach number if the even number is a sum of two primes. It is traditional definition of Goldbach number and brings mathematicians an approach to the conjecture. If it can be proven that every even number greater than 2 is Goldbach number by studying exceptional set of Goldbach numbers then Goldbach conjecture is true [1]-[7]. We suggested another definition of Goldbach number, that is, an even number is called a Goldbach number generated by a given prime P_n if the even number is the sum of two primes not greater than P_n [8]-[11]. It is a stronger concept than traditional Goldbach number, which would limit the number of prime pairs to form an even number. It is such limit makes $2P_n$ have definite bounds. We further suggested definition of the largest strong Goldbach number, that is, an even number L_n is called the largest strong Goldbach number generated by the n-th prime P_n if every even number from 4 to L_n is the sum of two primes not greater than P_n but $L_n + 2$ is not such a sum. Therefore, all even numbers from 4 to L_n must be Goldbach numbers generated by the prime. It means that if L_n approaches infinity as n grows without bound then all even numbers greater than 2 will be Goldbach numbers and Goldbach conjecture is true.

We discovered existence of step-type distribution for L_n based on observed data to imply $L_n \le L_{n+1}$, further, we proved that $L_n \le L_{n+1}$ for all $n > 0$. Thus every such step is called a Goldbach step whose width is $(n2 + 1) - n1$ to denote the number of largest strong Goldbach numbers to form the step, where $n1$ is the starting point and $n2$ is the finishing point for the step. We proved that if Goldbach conjecture is true then there are infinitely many Goldbach steps and can expect that distribution of Goldbach steps is asymptotically expressed as $Q(n) \sim n/\log n$ same as prime number theorem, where $Q(n)$ is the number of Goldbach steps. From it we see all Goldbach steps have like-prime nature, therefore, all $n1$ also have such like-prime nature and can be called like-primes. Let $n1\text{-}i$ denote the i-th like-prime. Then it is clear that $n1\text{-}i < n1\text{-}(i + 1)$ for all $i > 0$. Therefore, we can use all like-primes to represent the existence of all Goldbach steps so that $g_i = n1\text{-}(i + 1) - n1\text{-}i$ is defined as gap between the i-th and the $(i+1)$-th like-primes.

By introducing like-prime gap, some useful basic concepts are established such as twin like-prime, triplet like-prime, quadruplet like-prime, and so on. If it can be proven that there are infinitely many like-primes then Goldbach conjecture is true. The conjecture seems to be supported by some conjectures corresponding to the first Hardy-Littlewood conjecture and Polignac's conjecture.

We also discover Goldbach conjecture to be implied by existence of a pair of bounds for L_n or $L_n/2$. We proposed there are bounds such that $2n\log n + 2n\log\log n - 2n < L_n < 2n\log n + 2n\log\log n$ or $n\log n + n\log\log n - n < L_n/2 < n\log n + n\log\log n$ for $n \geq 20542$, and the bounds have been verified up to $n = 4000000000$ in this paper. If it can be proven that bounds of prime, $n\log n + n\log\log n - n < P_n < n\log n + n\log\log n$ for $n \geq 6$, can be used as bounds of $L_n/2$ for $n \geq 20542$, then Goldbach conjecture is true.

2. Like-Prime Nature of Goldbach Steps

2.1. Prime Number Theorem

Let x be positive integer and prime counting function $\pi(x)$ denote the number of primes not greater than x. Then prime number theorem states that there is a limit such that

$$\lim_{x\to\infty}\frac{\pi(x)}{\left(\dfrac{x}{\log x}\right)} = 1. \tag{2.1}$$

This limit means the relative error between $\pi(x)$ and $x/\log x$ approaches 0 as x grows without bound, that is, $\pi(x)$ is asymptotically expressed as

$$\pi(x) \sim \frac{x}{\log x}. \tag{2.2}$$

It is asymptotic distribution law of primes.

The prime number theorem can also be written as an approximation for $\pi(x)$ as follows

$$\pi(x) \approx Li(x), \tag{2.3}$$

where

$$Li(x) = \int_2^x \frac{\mathrm{d}t}{\log t} \tag{2.4}$$

is the logarithmic integral and it has an equivalent asymptotic series such that

$$\begin{aligned} Li(x) &\approx \frac{x}{\log x}\sum_{k=0}^{\infty}\frac{k!}{(\log x)^k} \\ &= \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{(\log x)^2} + \cdots\right). \end{aligned} \tag{2.5}$$

2.2. The Largest Strong Goldbach Number Generated by Prime

Definition 2.1 Let P_n denote the n-th prime. Then $P_i + P_k$ is called a *Goldbach num-*

ber generated by P_n if $i \le n$, $k \le n$. L_n is called *the largest strong Goldbach number generated by* P_n if L_n is an even number such that every even number from 4 to L_n is the sum of two primes not greater than P_n but $L_n + 2$ is not such a sum. Every even number from 4 to L_n is called a *strong Goldbach number generated by* P_n.

By Definition 2.1, we can find all Goldbach numbers generated by a given prime, for example, all Goldbach numbers generated by $P_{11} = 31$ are 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 58, 60, 62. But 56 = 54 + 2 is not Goldbach number generated by $P_{11} = 31$. The smallest Goldbach number is 4 and the largest Goldbach number is 62 but every even number from 4 to 54 is strong Goldbach number generated by $P_{11} = 31$ and 54 is the largest strong Goldbach number generated by $P_{11} = 31$ in the example. The first 50 largest strong Goldbach numbers are listed as follows

4, 6, 10, 14, 18, 26, 30, 38, 42, 42, 54, 62, 74, 74, 90, 90, 90, 108, 114, 114, 134, 134, 146, 162, 172, 180, 186, 186, 218, 222, 230, 240, 240, 254, 258, 270, 270, 290, 290, 290, 330, 348, 348, 366, 366, 366, 398, 398, 410, 410.

More largest strong Goldbach numbers generated by primes less than 10^7 can be found in [12]. **Figure 1** in [11] shows distributions of P_n and L_n for 1000000 ≤ $n \le 100000000$.

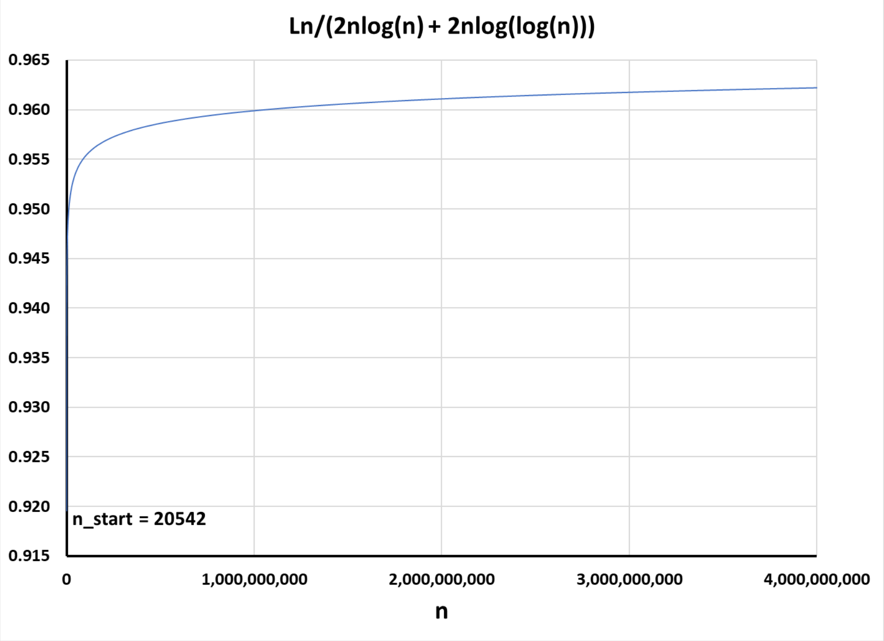

Figure 1. Distribution of $L_n/B_{\mathrm{up}}(n)$ for $20542 \le n \le 4000000000$.

2.3. Step-Type Distribution of Largest Strong Goldbach Numbers

Numerical evidence for the first 50 largest strong Goldbach numbers presents that there is an observed fact that $L_n \le L_{n+1}$, and such distribution characteristic can be seen more clearly from figure 3 in [9] to show distribution of L_n for $n \le 200$. The figure makes us realize that distribution of L_n is a step-type curve with upward trend. We have the following theorem to show the rationality of general existence for such steps.

Theorem 2.2 $L_n \le L_{n+1}$ *for every* $n > 0$.

Proof. Let L_n be the largest strong Goldbach number generated by P_n. Then L_n must be a strong Goldbach number generated by P_{n+1} by Definition 2.1. Therefore, it is impossible that $L_n > L_{n+1}$ for any $n > 0$. Suppose $L_n = L_{n+1}$ for all $n > 0$. Then if there is a counterevidence to this hypothesis then this hypothesis does not hold. Since there are many counterevidences to this hypothesis such as $L_{21722} = 491586 < L_{21723} = 491612$. Hence this hypothesis does not hold. Suppose $L_n < L_{n+1}$ for all $n > 0$. Then if there is a counterevidence to this hypothesis then this hypothesis does not hold. Since there are many counterevidences to this hypothesis such as $L_{21726} = L_{21727} = 491612$. Hence this hypothesis does not hold. Thus $L_n \le L_{n+1}$ for every $n > 0$ and the theorem holds.

Although there are infinitely many largest strong Goldbach numbers generated by infinitely many primes because L_n one-to-one corresponds to P_n, we can not confirm if there are infinitely many steps arising from $L_n \le L_{n+1}$. In order to consider whether there are infinitely many such steps, we have the following definitions.

Definition 2.3 Every step in distribution curve of L_n is called a *Goldbach step*.

Definition 2.4 For a given Goldbach step, W is called *width* of the Goldbach step if $W = (n2 + 1) - n1$, where $n1$ is n-value at the starting point of the Goldbach step and $n2$ is n-value at the finishing point of the Goldbach step but $n2 + 1$ is n-value at the starting point of next Goldbach step. H is called *height* of the Goldbach step if $H = L_{n1}$ for the Goldbach step, where $n1$ is n-value at the starting point of the Goldbach step.

By Definition 2.3 and Definition 2.4, $W \ge 1$ for all $n > 0$ and every Goldbach step must be formed by one largest strong Goldbach number or more consecutive largest strong Goldbach numbers, thus, every L_n for $n > 0$ must belong to a Goldbach step. For example, $L_{12869} = 275466 < L_{12870} = 275706 < L_{12871} = 276132$ means there is a Goldbach step $L_{12870} = 275706$ whose height is $L_{12870} = 275706$ and width is $(n2 + 1) - n1 = (12870 + 1) - 12870 = 1$. Another example is $L_{21722} = 491586 < L_{21723} = L_{21724} = L_{21725} = L_{21726} = L_{21727} = 491612 < L_{21728} = 491624$. It means there is a Goldbach step $L_{21723} = L_{21724} = L_{21725} = L_{21726} = L_{21727} = 491612$ whose height is $L_{21723} = 491612$ and width is $(n2 + 1) - n1 = (21727 + 1) - 21723 = 5$.

Remark 2.5 Theorem 2.2 implies general existence of Goldbach steps, that is, there is no a largest strong Goldbach number generated by prime not to belong to a Goldbach step. It means the prime sequence $\{P_n; P_n < P_{n+1}, n = 1, 2, 3, \cdots\}$ to be an infinite sequence will generate the largest strong Goldbach number sequence $\{L_n; L_n \le L_{n+1}, n = 1, 2, 3, \cdots\}$ to be also an infinite sequence and show general existence of Goldbach steps. But we have not known if there are infinitely many Goldbach steps based on existence of the largest strong Goldbach number sequence $\{L_n; L_n \le L_{n+1}, n = 1, 2, 3, \ldots\}$.

2.4. Infinitude of Goldbach Steps and Their Like-Prime Nature

Lemma 2.6 *If* L_n *approaches infinity as n grows without bound, then Goldbach conjecture is true.*

Proof. By Definition 2.1, every even number from 4 to L_n for a given prime P_n is a strong Goldbach number generated by P_n, thus, all even numbers from 4 to L_n also must be Goldbach numbers generated by P_n. It means if L_n approaches infinity as n grows without bound then all even numbers greater than 2 will be Goldbach numbers and Goldbach conjecture is true.

Theorem 2.7 *If Goldbach conjecture is true, then there are infinitely many Goldbach steps.*

Proof. If there is a finite number of Goldbach steps, then there must exist a Goldbach step to be the last Goldbach step which has infinite width to prevent the occurrence of more Goldbach steps, and the Goldbach step with infinite width must also stop the increase of value of largest strong Goldbach number generated by prime so that there is no asymptotic result such that L_n approaches infinity as n grows without bound to imply Goldbach conjecture as Lemma 2.6 describes, thus, Goldbach conjecture is not true. Since this conclusion contradicts condition in this theorem. Hence if Goldbach conjecture is true then there is no the last Goldbach step whose width is infinite but there are infinitely many Goldbach steps and the theorem holds.

Let $Q(n)$ be Goldbach step counting function and denote the number of Goldbach steps among the first n largest strong Goldbach numbers. Then we can get the observed value of $Q(n)$ for $1 \le n \le 4000000000$ in this paper. We had proposed an approximation $Q'(n)$ for $Q(n)$ as follows [11]

$$Q'(n) = Li(n) + \frac{n}{\log n \log\log n}, \tag{2.6}$$

where

$$Li(n) = \int_2^n \frac{\mathrm{d}t}{\log t},$$

and the asymptotic series for $Li(n)$ is that

$$Li(n) \approx \frac{n}{\log n} \sum_{k=0}^{\infty} \frac{k!}{(\log n)^k} = \frac{n}{\log n}\left(1 + \frac{1}{\log n} + \frac{2}{(\log n)^2} + \cdots\right). \tag{2.7}$$

Taking the first three terms in (2.7) [13], (2.6) becomes

$$Q'(n) \approx \frac{n}{\log n}\left(1 + \frac{1}{\log n} + \frac{2}{(\log n)^2} + \frac{1}{\log\log n}\right). \tag{2.8}$$

In this paper, (2.6) is improved as

$$Q'(n) = Li(n) + \frac{n}{\log n}\left(\frac{1}{\log\log n} + \frac{1}{4(\log\log n)^2}\right). \tag{2.9}$$

Taking the first three terms in (2.7), above approximation becomes

$$Q'(n) \approx \frac{n}{\log n}\left(1 + \frac{1}{\log n} + \frac{2}{(\log n)^2} + \frac{1}{\log\log n} + \frac{1}{4(\log\log n)^2}\right). \tag{2.10}$$

We can give relative error between $Q(n)$ and $Q'(n)$ in **Table 1**, where $Q(n)$ is Goldbach step counting function, $Q'(n)$ is the value predicted by (2.10) and $\pi(n)$ is prime counting function.

Table 1. The relative error between $Q(n)$ and $Q'(n)$.

n	$\pi(n)$	$Q(n)$	$Q'(n)$	$Q(n)/\pi(n)$	relative error
100	25	55	45	2.2000000	0.18181818
1000	168	277	256	1.6488095	0.07581227
10000	1229	1868	1771	1.5199349	0.05192719
100000	9592	13693	13472	1.4275437	0.01613963
1000000	78498	109565	108429	1.3957680	0.01036827
10000000	664579	912224	905814	1.3726344	0.00702678
100000000	5761455	7819295	7768442	1.3571736	0.00650352
1000000000	50847534	68459494	67942959	1.3463680	0.00754511

Based on above numerical evidence, one can expect the relative error between $Q(n)$ and $Q'(n)$ will approach 0 as n grows without bound. It means that there is a limit as follows

$$\lim_{n\to\infty}\frac{Q(n)}{Q'(n)}=1. \tag{2.11}$$

By (2.9), the limit becomes

$$\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\frac{n}{\log n}\left(\frac{1}{\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1, \tag{2.12}$$

and we have the following theorem.

Theorem 2.8 *If* $\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\frac{n}{\log n}\left(\frac{1}{\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1$, *then* $\lim_{n\to\infty}\frac{Q(n)}{\left(\frac{n}{\log n}\right)}=1$.

Proof. Using (2.7), $Li(n)+\frac{n}{\log n}\left(\frac{1}{\log\log n}+\frac{1}{4(\log\log n)^2}\right)$ can be written as

$$\frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}+\frac{n}{\log n}\left(\frac{1}{\log\log n}+\frac{1}{4(\log\log n)^2}\right). \tag{2.13}$$

Considering asymptotic series in (2.13), we see that the k-th term approaches higher order infinity than the $(k+1)$-th term as n grows without bound in the asymptotic series because there is the following limit.

$$\lim_{n\to\infty}\frac{\dfrac{(k+1)!n}{(\log n)^{k+2}}}{\dfrac{k!n}{(\log n)^{k+1}}}=\lim_{n\to\infty}\frac{k+1}{\log n}=0. \tag{2.14}$$

Since the first term in the asymptotic series for $Li(n)$ is $n/\log n$, there are two limits for two additional terms as follows

$$\lim_{n\to\infty}\frac{\dfrac{n}{\log n}\dfrac{1}{\log\log n}}{\dfrac{n}{\log n}}=\lim_{n\to\infty}\frac{1}{\log\log n}=0, \tag{2.15}$$

$$\lim_{n\to\infty}\frac{\dfrac{n}{\log n}\dfrac{1}{4(\log\log n)^2}}{\dfrac{n}{\log n}}=\lim_{n\to\infty}\frac{1}{4(\log\log n)^2}=0. \tag{2.16}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\dfrac{n}{\log n}\left(\dfrac{1}{\log\log n}+\dfrac{1}{4(\log\log n)^2}\right)}=1,$$

by (2.13), (2.14), (2.15) and (2.16) we have

$$\lim_{n\to\infty}\frac{Q(n)}{\left(\dfrac{n}{\log n}\right)}=1. \tag{2.17}$$

Hence the theorem holds.

Corollary 2.9 *If* $\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\dfrac{n}{\log n}\left(\dfrac{1}{\log\log n}+\dfrac{1}{4(\log\log n)^2}\right)}=1$, *then Goldbach conjecture is true.*

Proof. Suppose $\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\dfrac{n}{\log n}\left(\dfrac{1}{\log\log n}+\dfrac{1}{4(\log\log n)^2}\right)}=1$. Then, by Theorem 2.8 we have the following limit.

$$\lim_{n\to\infty}\frac{Q(n)}{\left(\dfrac{n}{\log n}\right)}=1. \tag{2.18}$$

It means $Q(n)$ is asymptotically expressed as

$$Q(n)\sim\frac{n}{\log n}. \tag{2.19}$$

From (2.19) we see that $Q(n)$ approaches infinity as n grows without bound if $n/\log n$ approaches infinity as n grows without bound. Since it is obvious that

$n/\log n$ approaches infinity as n grows without bound. Hence $Q(n)$ approaches infinity as n grows without bound. It means that there are infinitely many Goldbach steps. Since height $H_i = L_{n1-i}$ of the i-th Goldbach step must be smaller than height $H_{i+1} = L_{n1-(i+1)}$ of the $(i + 1)$-th Goldbach step by Theorem 2.2 and definition 2.4. Hence $L_{n1-i} < L_{n1-(i+1)}$ for all $i > 0$ so that L_{n1-i} approaches infinity as i grows without bound. Since the sequence $\left\{L_{n1-i}; L_{n1-i} < L_{n1-(i+1)}, i = 1,2,3,\cdots\right\}$ is a subsequence of the largest strong Goldbach number sequence $\{L_n; L_n \le L_{n+1}, n = 1, 2, 3, \ldots\}$. Hence the result that L_{n1-i} approaches infinity as i grows without bound must lead to the result that L_n approaches infinity as n grows without bound. By Lemma 2.6, Goldbach conjecture is true and the corollary holds.

Limit (2.18) and asymptotic expression (2.19) mean that the relative error between $Q(n)$ and $n/\log n$ approaches 0 as n grows without bound, thus, both the two results are same as statement of the prime number theorem as follows

$$\lim_{n\to\infty} \frac{\pi(n)}{\left(\dfrac{n}{\log n}\right)} = 1, \tag{2.20}$$

and

$$\pi(n) \sim \frac{n}{\log n}. \tag{2.21}$$

Formulas (2.18), (2.19), (2.20) and (2.21) mean asymptotic distribution law of Goldbach steps is same as asymptotic distribution law of primes described by prime number theorem if Goldbach conjecture is true so that Goldbach steps are infinite by Theorem 2.7. Therefore, we say that Goldbach steps have like-prime nature and the nature will bring us a chance to study a kind of special natural numbers which seem to have such like-prime nature to represent the existence of Goldbach steps.

3. Gap between Like-Primes

3.1. Like-Prime

We have seen that it is clear that Goldbach steps have like-prime nature if there are infinitely many Goldbach steps because asymptotic distribution law of Goldbach steps is same as asymptotic distribution law of primes described by prime number theorem. Therefore, it is necessary that we should find a kind of special natural numbers for representing existence of all Goldbach steps and these numbers can well embody like-prime nature of Goldbach steps.

Definition 3.1 Let P_n denote the n-th prime. Then natural number $n = n1$ is called a *like-prime* if $n = n1$ is n-value at the starting point of a Goldbach step.

Remark 3.2 By Definition 3.1, the first like-prime is 1 and every like-prime greater than 1 satisfies $L_{n1-1} < L_{n1}$ but the first like-prime 1 does not satisfy $L_{n1-1} < L_{n1}$ because there is no $L_{n-1} = L_0$. By Definition 3.1, we also see that if a natural number n is not like-prime then the natural number must satisfy $L_{n-1} = L_n$, which means if $L_{n-1} = L_n$ then L_n must be on a Goldbach step whose width is greater than

1 but $n > n1$ for the step.

By Definition 3.1 we can find known like-primes. Let $n1\text{-}i$ denote the i-th like-prime. Then the largest like-prime is known as $n1\text{-}68459494 = 999999987$ for $n \le 10^9$ and the first 50 like-primes are listed as follows

1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 18, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86.

More like-primes can be found in [12]. $Q(n)$ in **Table 1** gives the number of like-primes for $n \le 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$.

Lemma 3.3 $n1\text{-}i < n1\text{-}(i + 1)$ *for all* $i > 0$.

Proof. Since $n1\text{-}i$ is n-value at the starting point of the i-th Goldbach step and $n1\text{-}(i + 1)$ is n-value at the starting point of the $(i + 1)$-th Goldbach step by Definition 2.4. Hence $n1\text{-}i < n1\text{-}(i + 1)$ for all $i > 0$ by Theorem 2.2 and the lemma holds.

Remark 3.4 Definition 3.1 and Lemma 3.3 mean that the existence of all Goldbach steps is represented by the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ but we do not know if the sequence is an infinite sequence.

Theorem 3.5 *If Goldbach conjecture is true, then the like-prime sequence* $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ *is an infinite sequence.*

Proof. By Theorem 2.7, If Goldbach conjecture is true then there are infinitely many Goldbach steps. Since every Goldbach step is represented by a like-prime $n1$ and $n1\text{-}i < n1\text{-}(i + 1)$ for all $i > 0$ by Lemma 3.3. Hence if Goldbach conjecture is true then there are infinitely many like-primes $n1$ such that $n1\text{-}i < n1\text{-}(i + 1)$ for all $i > 0$, thus, the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ is an infinite sequence and the theorem holds.

Note there is a converse theorem of Theorem 3.5.

Theorem 3.6 *If the like-prime sequence* $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3,\ldots\}$ *is an infinite sequence, then Goldbach conjecture is true.*

Proof. Suppose the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ is an infinite sequence. Then there are infinitely many Goldbach steps because every $n1$ denotes a Goldbach step by Definition 3.1. Since $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ for all $i > 0$ by Definition 2.4. Hence $L_{n1\text{-}i}$ approaches infinity as i grows without bound to lead to the result that L_n approaches infinity as n grows without bound. By Lemma 2.6 Goldbach conjecture is true and the theorem holds.

Remark 3.7 Theorem 3.6 means if it can be proven that the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ is an infinite sequence, then Goldbach conjecture is true.

3.2. Largest Strong Goldbach Numbers with Distinct Values

Definition 3.8 Largest strong Goldbach number L_n is called a *largest strong Goldbach number with distinct value* if $n = n1$.

Remark 3.9 Definition 3.8 means all L_{n1}, which are heights of Goldbach steps, are a kind of special largest strong Goldbach numbers to satisfy $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ for all $i >$

0, where $L_{n1\text{-}i}$ is the i-th largest strong Goldbach number with distinct value. Thus, there is the sequence of largest strong Goldbach numbers with distinct values $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \dots\}$. By Definition 3.8, approximate and asymptotic distribution laws of Goldbach steps (2.9) and (2.19) are also approximate and asymptotic distribution laws of largest strong Goldbach numbers with distinct values.

By Definition 3.8 we can find known largest strong Goldbach numbers with distinct values. The largest $L_{n1\text{-}i}$ is $L_{n1\text{-}68459494} = L_{999999987} = 45603524304$ for $n \le 10^9$ and the first 50 largest strong Goldbach numbers with distinct values are listed as follows

4, 6, 10, 14, 18, 26, 30, 38, 42, 54, 62, 74, 90, 108, 114, 134, 146, 162, 172, 180, 186, 218, 222, 230, 240, 254, 258, 270, 290, 330, 348, 366, 398, 410, 434, 440, 474, 522, 528, 566, 570, 614, 630, 634, 650, 680, 686, 722, 794, 822.

More largest strong Goldbach numbers with distinct values can be found in [12] and $Q(n)$ in Table 1 gives the number of largest strong Goldbach numbers with distinct values for $n \le 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$.

Lemma 3.10 $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ *for all* $i > 0$.

Proof. Since $L_{n1\text{-}i}$ is height of the i-th Goldbach step and $L_{n1\text{-}(i+1)}$ is height of the $(i + 1)$-th Goldbach step by Definition 2.4. Hence $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ for all $i > 0$ and the lemma holds.

Remark 3.11 Definition 3.8 and Lemma 3.10 mean that the existence of all Goldbach steps is represented by the sequence of largest strong Goldbach numbers with distinct values $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \dots\}$ but we do not know if the sequence is an infinite sequence.

Theorem 3.12 *If Goldbach conjecture is true, then the sequence of largest strong Goldbach numbers with distinct values* $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \dots\}$ *is an infinite sequence.*

Proof. By Theorem 2.7, If Goldbach conjecture is true then there are infinitely many Goldbach steps. Since every Goldbach step is represented by a largest strong Goldbach number with distinct value and $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ for all $i > 0$ by Lemma 3.10. Hence if Goldbach conjecture is true then there are infinitely many largest strong Goldbach numbers with distinct values such that $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ for all $i > 0$, that is, the sequence of largest strong Goldbach numbers with distinct values $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \dots\}$ is an infinite sequence and the theorem holds.

Note there is a converse theorem of Theorem 3.12.

Theorem 3.13 *If the sequence of largest strong Goldbach numbers with distinct values* $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \dots\}$ *is an infinite sequence, then Goldbach conjecture is true.*

Proof. Suppose the sequence of largest strong Goldbach numbers with distinct values $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \dots\}$ is an infinite sequence. Then there are infinitely many Goldbach steps because L_{n1} denotes height of Goldbach step by Definition 2.4 and Definition 3.8. Since $L_{n1\text{-}i} < L_{n1\text{-}(i+1)}$ for all $i > 0$ by Lemma 3.10. Hence $L_{n1\text{-}i}$ approaches infinity as i grows without bound to lead to the result that L_n approaches infinity as n grows without bound. By Lemma 2.6 Goldbach conjecture is true and the theorem holds.

Remark 3.14 Theorem 3.13 means if it can be proven that the sequence of largest strong Goldbach numbers with distinct values $\{L_{n1\text{-}i};\ L_{n1\text{-}i} < L_{n1\text{-}(i+1)},\ i = 1, 2, 3, \ldots\}$ is an infinite sequence, then Goldbach conjecture is true. The sequence of largest strong Goldbach numbers with distinct values seems to be similar to the like-prime sequence.

3.3. Like-Prime Gap

Because $n1\text{-}i < n1\text{-}(i + 1)$ for all $i > 0$ by Lemma 3.3, there must exist gap between like-primes as gap between primes does. Thus, we have the following definition.

Definition 3.15 g_i is called *the i-th like-prime gap* if $g_i = n1\text{-}(i + 1) - n1\text{-}i$.

By Definition 3.15, the first 100 like-prime gaps are listed as follows

1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 3, 1, 2, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2, 3, 1, 2, 3, 2, 2, 1, 2, 5, 2, 2, 1, 3, 2, 1, 1, 1, 5, 4, 2, 3, 2, 3, 2, 2, 1, 5, 4, 1, 9, 4, 2, 4, 5, 2, 1, 2, 3, 2, 3, 3, 5, 1, 1, 2, 6, 1, 8, 1, 6, 1, 9, 2, 5, 10, 2, 3, 6, 1, 3, 3, 2, 4, 3, 5, 3, 3, 1, 1, 1, 1, 8.

There are some natural numbers to become length of gap between like-primes such as 1, 2, 3, 4, 5, 6, 8, 9, 10 among the first 100 like-prime gaps. More like-prime gaps can be found in [12]. By Definition 3.1 we see that if there is a natural number n greater than 1 such that $L_{n-1} = L_n$ then the natural number n is not a like-prime, thus, there are $k - 1$ consecutive natural numbers not to be like-prime between $n1\text{-}i$ and $n1\text{-}(i + 1)$ if $g_i = k$ because like-prime gap is defined as $g_i = n1\text{-}(i + 1) - n1\text{-}i$. Considering $k = 1$, there is no natural number between $n1\text{-}i$ and $n1\text{-}(i + 1)$. Thus, we conjecture the number of consecutive natural numbers between $n1\text{-}i$ and $n1\text{-}(i + 1)$ can be arbitrarily large, that is, length of gap between like-primes can be arbitrarily large. If Goldbach conjecture is true, by Theorem 3.5 there are infinitely many like-primes to satisfy $n1\text{-}i < n1\text{-}(i + 1)$. Therefore, by Definition 3.15 there are infinitely many like-prime gaps and we have the following result.

$$n = 1 + \sum_{i=1}^{\infty} g_i \ .$$

3.4. Twin Prime Conjecture and Its Strong Form

Let N be a natural number greater than 1. Then there is a sequence including $N - 1$ consecutive composite numbers $N! + 2, N! + 3, \ldots, N! + N$ because $N! + 2 = 2(N!/2 + 1)$, $N! + 3 = 3(N!/3 + 1)$, …, $N! + N = N(N!/N + 1)$ all are composite numbers. Therefore, gap between primes can be arbitrarily large so that every positive even number can become length of a prime gap. Let p denote prime. Then Polignac's conjecture states that there are infinitely many primes p such that $p + 2k$ is also prime for every natural number k [14], and specially, there are infinitely many primes p such that $p + 2k$ is also prime for $k = 1$, which is twin prime conjecture. There is a strong form of twin prime conjecture, that is, an asymptotic distribution law of twin primes akin to the prime number theorem is a special case of the first Hardy-Littlewood conjecture [15].

Let $\pi_2(x)$ denote the number of primes $p \le x$ such that $p + 2$ is also prime. Define twin prime constant C_2 as [16]

$$C_2 = \prod_{p\geq 3}\left(1-\frac{1}{(p-1)^2}\right) = \prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2} \approx 0.660161815\cdots. \tag{3.1}$$

Then there is a special case of the first Hardy-Littlewood conjecture such that

$$\pi_2(x) \approx 2C_2\int_2^x \frac{\mathrm{d}t}{(\log t)^2},$$

$$\pi_2(x) \sim 2C_2\frac{x}{(\log x)^2}, \tag{3.2}$$

in the sense that the quotient of the two expressions approaches 1 as x grows without bound [17]. Although (3.2) to be a strong form of twin prime conjecture has not been proven, the conjecture seems certain to be true. Obviously, if (3.2) holds then twin prime conjecture is true. Table 2 gives the number of twin primes and the values predicted by the Hardy-Littlewood formula [15].

Table 2. Counted and predicted values for the number of twin primes.

x	$\pi_2(x)$	Hardy-Littlewood	ratio	relative error
100000	1224	1249	1.0204248	0.0200160
1000000	8169	8248	1.0096707	0.0095780
10000000	58980	58754	0.9961681	0.0038318
100000000	440312	440368	1.0001271	0.0001271

3.5. Twin Like-Prime Conjecture and Its Strong Form

Definition 3.16 A like-prime $n1$ is called a *twin like-prime* if $n1 + 1$ is also like-prime.

By Definition 3.16, we can find known pairs of twin like-primes and the first 50 pairs of twin like-primes are listed as follows

(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (11, 12), (12, 13), (18, 19), (23, 24), (24, 25), (25, 26), (26, 27), (29, 30), (30, 31), (31, 32), (34, 35), (35, 36), (41, 42), (51, 52), (63, 64), (69, 70), (70, 71), (71, 72), (95, 96), (105, 106), (129, 130), (148, 149), (149, 150), (159, 160), (168, 169), (175, 176), (213, 214), (240, 241), (241, 242), (242, 243), (243, 244), (267, 268), (268, 269), (269, 270), (280, 281), (321, 322), (322, 323), (333, 334), (334, 335), (345, 346), (368, 369), (369, 370).

More twin like-primes can be found in [12] and $Q_2(n)$ in Table 3 gives the number of twin like-primes for $n \leq 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$.

Conjecture 3.17 *There are infinitely many like-primes* $n1$ *such that* $n1 + 1$ *is also like-prime.*

Corollary 3.18 *If Conjecture 3.17 is true, then Goldbach conjecture is true.*

Proof. Since every pair of twin like-primes ($n1$, $n1 + 1$) corresponds to a Goldbach step with width to be 1, the infinitude of twin like-prime pairs means that there are infinitely many Goldbach steps with width to be 1 to lead to the existence of infinitely many Goldbach steps. By $L_{n1-(i+1)} > L_{n1-t}$ for all $i > 0$, L_{n1-t} approaches infinity as i grows without bound to lead to the result that L_n approaches infinity as n grows without bound. The result implies Goldbach conjecture by Lemma 2.6 and the corollary holds.

Table 3. The relative error between $Q_2(n)$ and $Q_2'(n)$.

n	$\pi_2(n)$	$Q_2(n)$	$Q_2(n)/\pi_2(n)$	$Q_2'(n)$	relative error
100	8	27	3.37500000	21	0.22222222
1000	35	77	2.20000000	65	0.15584415
10000	205	356	1.73658536	306	0.14044943
100000	1224	1947	1.59068627	1790	0.08063687
1000000	8169	12146	1.48684049	11760	0.03178000
10000000	58980	84339	1.42995930	83077	0.01496342
100000000	440312	619480	1.40691146	616587	0.00467004
1000000000	3424506	4741957	1.38471271	4751900	0.00209242

Remark 3.19 Conjecture 3.17 can be called twin like-prime conjecture to correspond to twin prime conjecture. As there is a strong form of twin prime conjecture, which is a special case of the first Hardy-Littlewood conjecture, there is also a strong form of twin like-prime conjecture as the following discussion does.

Let $Q_2(n)$ denote the number of twin like-primes among the first n positive integers to number the first n primes for generating largest strong Goldbach numbers and be also called twin like-prime counting function. Then we propose that there is an approximation $Q_2'(n)$ for $Q_2(n)$ such that

$$Q_2(n) \approx Q_2'(n),$$

$$Q_2'(n) = 2C_2\int_2^n \frac{\mathrm{d}t}{(\log t)^2} + 2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n} + \frac{1}{(\log\log n)^2}\right), \tag{3.3}$$

where C_2 is twin prime constant (3.1).

Since $\int_2^n \frac{\mathrm{d}t}{(\log t)^2} = Li(n) + \frac{2}{\log 2} - \frac{n}{\log n}$ [15], (3.3) becomes

$$\begin{aligned} Q_2'(n) &= 2C_2\left(Li(n) - \frac{n}{\log n}\right) + 2C_2\frac{2}{\log 2} \\ &\quad + 2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n} + \frac{1}{(\log\log n)^2}\right). \end{aligned} \tag{3.4}$$

There is the asymptotic series for $Li(n)$ such that

$$\begin{aligned} Li(n) &\approx \frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k} \\ &= \frac{n}{\log n}\left(1 + \frac{1}{\log n} + \frac{2}{(\log n)^2} + \frac{6}{(\log n)^3} + \cdots\right). \end{aligned} \tag{3.5}$$

Taking the first four terms in (3.5), by (3.4) we get

$$Q_2'(n) \approx 2C_2 \frac{n}{(\log n)^2}\left(1+\frac{2}{\log n}+\frac{6}{(\log n)^2}+\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right) + 2C_2\frac{2}{\log 2}. \tag{3.6}$$

We can give relative error between $Q_2(n)$ and $Q_2'(n)$ as **Table 3** shows. In the table, $Q_2(n)$ is the number of twin like-primes, $Q_2'(n)$ is the values predicted by formula (3.6) and $\pi_2(n)$ is the number of twin primes.

Theorem 3.20 *If*

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\int_2^n \frac{\mathrm{d}t}{(\log t)^2}+2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right)}=1, \text{ then}$$

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\frac{n}{(\log n)^2}}=1.$$

Proof. As we know, there are two results such that

$$\int_2^n \frac{\mathrm{d}t}{(\log t)^2} = Li(n)+\frac{2}{\log 2}-\frac{n}{\log n}, \tag{3.7}$$

$$Li(n) \approx \frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k} = \frac{n}{\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{6}{(\log n)^3}+\cdots\right). \tag{3.8}$$

Let $Q_2'(n) = 2C_2\int_2^n \frac{\mathrm{d}t}{(\log t)^2}+2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right)$. By (3.7) and (3.8) we have

$$Q_2'(n) \approx 2C_2\frac{n}{\log n}\sum_{k=1}^{\infty}\frac{k!}{(\log n)^k}+2C_2\frac{2}{\log 2} + 2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right). \tag{3.9}$$

Considering asymptotic series in (3.9), we see that the k-th term approaches higher order infinity than the $(k+1)$-th term as n grows without bound because of the existence of the following limit.

$$\lim_{n\to\infty}\frac{2C_2\frac{(k+1)!n}{(\log n)^{k+2}}}{2C_2\frac{k!n}{(\log n)^{k+1}}} = \lim_{n\to\infty}\frac{k+1}{\log n} = 0. \tag{3.10}$$

Since the first term of asymptotic series in (3.9) is

$$2C_2\frac{n}{(\log n)^2},$$

there is a limit for constant term in (3.9) as follows

$$\lim_{n\to\infty}\frac{2C_2\dfrac{2}{\log 2}}{2C_2\dfrac{n}{(\log n)^2}}=\frac{2}{\log 2}\lim_{n\to\infty}\frac{(\log n)^2}{n}=0\,. \tag{3.11}$$

For two additional terms in (3.9), we have the following limits.

$$\lim_{n\to\infty}\frac{2C_2\dfrac{n}{(\log n)^2(\log\log n)}}{2C_2\dfrac{n}{(\log n)^2}}=\lim_{n\to\infty}\frac{1}{\log\log n}=0\,, \tag{3.12}$$

$$\lim_{n\to\infty}\frac{2C_2\dfrac{n}{(\log n)^2(\log\log n)^2}}{2C_2\dfrac{n}{(\log n)^2}}=\lim_{n\to\infty}\frac{1}{(\log\log n)^2}=0\,. \tag{3.13}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\displaystyle\int_2^n\frac{\mathrm{d}t}{(\log t)^2}+2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right)}=1\,,$$

by (3.7), (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13) we get

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\dfrac{n}{(\log n)^2}}=1\,.$$

Hence the theorem holds.

Corollary 3.21 *If*

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\displaystyle\int_2^n\frac{\mathrm{d}t}{(\log t)^2}+2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right)}=1\,,$$ *then Goldbach conjecture is true.*

Proof. Suppose $\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\int_2^n\frac{\mathrm{d}t}{(\log t)^2}+2C_2\frac{n}{(\log n)^2}\left(\frac{1}{\log\log n}+\frac{1}{(\log\log n)^2}\right)}=1\,.$

Then we have the following limit by Theorem 3.20.

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\dfrac{n}{(\log n)^2}}=1\,. \tag{3.14}$$

It means there is an asymptotic expression as follows

$$Q_2(n)\sim 2C_2\frac{n}{(\log n)^2}\,. \tag{3.15}$$

Formula (3.14) means the relative error between $Q_2(n)$ and $2C_2n/(\log n)^2$ approaches 0 as n grows without bound, and formula (3.15) means that $Q_2(n)$ is asymptotically equal to $2C_2n/(\log n)^2$. It is obvious that result (3.15) means that there are infinitely many pairs of twin like-primes since $2C_2n/(\log n)^2$ approaches infinity as n grows without bound. By Conjecture 3.17 and Corollary 3.18, Goldbach conjecture is true and the corollary holds.

3.6. Triplet Prime Conjecture

There are two forms of prime 3-tuplet such that $(p, p + 2, p + 6)$ and $(p, p + 4, p + 6)$, and also called prime triplet. It is conjectured that there are infinitely many primes p such that $p + 2$ and $p + 6$ are also primes. It is also conjectured that there are infinitely many primes p such that $p + 4$ and $p + 6$ are also primes. The conjectures are called triplet prime conjecture. There is a strong form of triplet prime conjecture to be a special case of the first Hardy-Littlewood conjecture as the following discussion does.

Let $\pi_3(x)$ denote the number of primes $p \le x$ such that $p + 2$ and $p + 6$ or $p + 4$ and $p + 6$ are also primes. Define a constant C_3 as [15]

$$C_3 = \frac{9}{2}\prod_{p\ge 5}\frac{p^2(p-3)}{(p-1)^3} \approx 2.858248596\cdots. \tag{3.16}$$

Then there is a special case of the first Hardy-Littlewood conjecture such that

$$\pi_3(x) \approx C_3\int_2^x \frac{\mathrm{d}t}{(\log t)^3},$$

$$\pi_3(x) \sim C_3\frac{x}{(\log x)^3}, \tag{3.17}$$

in the sense that the quotient of the two expressions approaches 1 as x grows without bound. Obviously, if (3.17) holds then triplet prime conjecture is true and twin prime conjecture is also true because every prime triplet must include a pair of twin primes. There are two tables, **Table 4** and **Table 5**, to give the number of prime triplets and the values predicted by the Hardy-Littlewood formula [15].

Table 4. Counted and predicted numbers of prime triplets of the form $(p, p + 2, p + 6)$.

x	$\pi_3(x)$	Hardy-Littlewood	ratio	relative error
100000	259	279	1.0772200	0.0716845
1000000	1393	1446	1.0380473	0.0366528
10000000	8543	8591	1.0056186	0.0055872
100000000	55600	55491	0.9980395	0.0019604

Table 5. Counted and predicted numbers of prime triplets of the form $(p, p + 4, p + 6)$.

x	$\pi_3(x)$	Hardy-Littlewood	ratio	relative error
100000	248	279	1.1250000	0.1111111
1000000	1444	1446	1.0013850	0.0013831
10000000	8677	8591	0.9900887	0.0099112
100000000	55556	55491	0.9988300	0.0011699

3.7. Triplet Like-Prime Conjecture and Its Strong Form

Because there are two forms, $(p, p + 2, p + 6)$ and $(p, p + 4, p + 6)$, to represent existence of prime triplets. Correspondingly, we have the following two definitions.

Definition 3.22 A like-prime $n1$ is called a *triplet like-prime* if $n1 + 1$ and $n1 + 3$ are also like-primes but $n1 + 2$ is not a like-prime.

By Definition 3.22, we can find known like-prime triplets of this form and the first 50 like-prime triplets of this form are listed as follows

(8, 9, 11), (12, 13, 15), (18, 19, 21), (26, 27, 29), (31, 32, 34), (35, 36, 38), (41, 42, 44), (51. 52, 54), (129, 130, 132), (269, 270, 272), (394, 395, 397), (397, 398, 400), (437, 438, 440), (472, 473, 475), (543, 544, 546), (666, 667, 669), (1192, 1193, 1195), (1381, 1382, 1384), (1434, 1435, 1437), (1874, 1875, 1877), (2036, 2037, 2039), (2612, 2613, 2625), (2847, 2848, 2850), (2998 2999, 3001), (3489, 3490, 3492), (3492, 3493, 3495), (3561, 3562, 3564), (3803, 3804, 3806), (4071, 4072, 4074), (4150, 4151, 4153), (4311, 4312, 4314), (4405, 4406, 4408), (4619, 4620, 4622), (4808, 4809, 4811), (4831, 4832, 4834), (5269, 5270, 5272), (5297, 5298, 5300), (5394, 5395, 5397), (5459, 5460, 5462), (5510, 5511, 5513), (5630, 5631, 5633), (5647, 5648, 5650), (5761, 5762, 5764), (5876, 5877, 5879), (5879, 5880, 5882), (5958, 5959, 5961), (6014, 6015, 6017), (6388, 6389, 6391), (6983, 6984, 6986), (7345, 7346, 7348).

More like-prime triplets of this form can be found in [12] and $Q_3(n)$ in Table 6 gives the number of like-prime triplets of the form $(n1, n1 + 1, n1 + 3)$ with $n1 + 2$ not to be like-prime for $n \le 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$.

Definition 3.23 A like-prime $n1$ is called a *triplet like-prime* if $n1 + 2$ and $n1 + 3$ are also like-primes but $n1 + 1$ is not a like-prime.

By Definition 3.23, we can find known like-prime triplets of this form and the first 50 like-prime triplets of this form are listed as follows

(9, 11, 12), (21, 23, 24), (27, 29, 30), (32, 34, 35), (49, 51, 52), (61, 63, 64), (67, 69, 70), (93, 95, 96), (127, 129, 130), (319, 321, 322), (331, 333, 334), (395, 397, 398), (400, 402, 403), (537, 539, 340), (662, 664, 665), (667, 669, 670), (1163, 1165, 1166), (1172, 1174, 1175), (1190, 1192, 1193), (1379, 1381, 1382), (1384, 1386, 1387), (1432, 1434, 1435), (1646, 1648, 1649), (1880, 1882, 1883), (2037, 2039, 2040), (2101, 2103, 2104), (2179, 2181, 2182), (2532, 2534, 2535), (2570, 2572, 2573), (2617, 2619, 2620), (3184, 3186, 3187), (3487, 3489, 3490), (3490, 3492, 3493), (3631, 2633, 3634), (3643, 3645, 3646), (4031, 4033, 4034), (4069, 4071, 4072), (4147, 4149, 4150), (4406, 4408, 4409), (4448, 4450, 4451), (4832, 4834, 4835), (5270, 5272, 5273), (5325, 5327, 5328), (5392, 5394, 5395), (5416, 5418, 5419), (5642, 5644, 5645), (5677, 5679, 5680), (5759, 5761, 5762), (5877, 5879, 5880), (5959, 5961, 5962).

More like-prime triplets of this form can be found in [12] and $Q_3(n)$ in Table 7 gives the number of like-prime triplets of the form $(n1, n1 + 2, n1 + 3)$ with $n1 + 1$ not to be like-prime for $n \le 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$.

Conjecture 3.24 *There are infinitely many triplet like-primes* $n1$ *defined by*

Definition 3.22.

Corollary 3.25 *If Conjecture* 3.24 *is true, then Goldbach conjecture is true.*

Proof. Since every like-prime triplet ($n1$, $n1 + 1$, $n1 + 3$) with $n1 + 2$ not to be like-prime must include a pair of twin like-primes to correspond to existence of a Goldbach step with width to be 1, the infinitude of such like-prime triplets means that there are infinitely many Goldbach steps with width to be 1 to lead to the existence of infinitely many Goldbach steps. By $L_{n1\text{-}(i+1)} > L_{n1\text{-}t}$ for all $i > 0$, $L_{n1\text{-}t}$ approaches infinity as i grows without bound to lead to a result that L_n approaches infinity as n grows without bound. The result implies Goldbach conjecture by Lemma 2.6 and the corollary holds.

Conjecture 3.26 *There are infinitely many triplet like-primes* $n1$ *defined by Definition* 3.23.

Corollary 3.27 *If Conjecture* 3.26 *is true, then Goldbach conjecture is true.*

Proof. Since every like-prime triplet ($n1$, $n1 + 2$, $n1 + 3$) with $n1 + 1$ not to be like-prime must include a pair of twin like-primes to correspond to existence of a Goldbach step with width to be 1, the infinitude of such like-prime triplets means that there are infinitely many Goldbach steps with width to be 1 to lead to the existence of infinitely many Goldbach steps. By $L_{n1\text{-}(i+1)} > L_{n1\text{-}t}$ for all $i > 0$, $L_{n1\text{-}t}$ approaches infinity as i grows without bound to lead to a result that L_n approaches infinity as n grows without bound. The result implies Goldbach conjecture by Lemma 2.6 and the corollary holds.

Remark 3.28 Triplet like-primes defined by Definition 3.22 can be called the first kind of triplet like-primes and triplet like-primes defined by Definition 3.23 can be called the second kind of triplet like-primes. Both Conjecture 3.24 and Conjecture 3.26 can be called triplet like-prime conjecture to correspond to triplet prime conjecture. As there is a strong form of triplet prime conjecture to be a special case of the first Hardy-Littlewood conjecture, there is also a strong form of triplet like-prime conjecture.

Let $Q_3(n)$ denote the number of like-prime triplets among the first n positive integers to number the first n primes for generating largest strong Goldbach numbers. Then we propose that there is an approximation $Q_3'(n)$ for $Q_3(n)$ such that

$$Q_3(n) \approx Q_3'(n),$$

$$Q_3'(n) = C_3 \int_2^n \frac{\mathrm{d}t}{(\log t)^3} - C_3 \frac{n}{(\log n)^3} \left(\frac{1}{3 \log \log n} + \frac{1}{3 (\log \log n)^2} \right), \tag{3.18}$$

where C_3 is defined as a constant as follows

$$C_3 = \frac{9}{2} \prod_{p \geq 5} \frac{p^2 (p-3)}{(p-1)^3} \approx 2.858248596 \cdots. \tag{3.19}$$

This constant is just constant (3.16).

By $\int_2^n \frac{\mathrm{d}t}{(\log t)^3} = \frac{1}{2} Li(n) - \frac{n}{2(\log n)^2} - \frac{n}{2 \log n} + \frac{1}{\log 2} + \frac{1}{(\log 2)^2}$ [15], (3.18) becomes

$$\begin{aligned}Q_3'(n) &= C_3\left(\frac{1}{2}Li(n)-\frac{n}{2(\log n)^2}-\frac{n}{2\log n}\right)+C_3\left(\frac{1}{\log 2}+\frac{1}{(\log 2)^2}\right)\\ &\quad -C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right).\end{aligned}\tag{3.20}$$

There is the asymptotic series for $Li(n)/2$ such that

$$\begin{aligned}\frac{1}{2}Li(n) &\approx \frac{n}{2\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}\\ &= \frac{n}{2\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{6}{(\log n)^3}+\frac{24}{(\log n)^4}+\cdots\right).\end{aligned}\tag{3.21}$$

Taking the first five terms in the asymptotic series, (3.20) becomes

$$\begin{aligned}Q_3'(n) &\approx C_3\frac{n}{(\log n)^3}\left(1+\frac{3}{\log n}+\frac{12}{(\log n)^2}-\frac{1}{3\log\log n}-\frac{1}{3(\log\log n)^2}\right)\\ &\quad +C_3\left(\frac{1}{\log 2}+\frac{1}{(\log 2)^2}\right).\end{aligned}\tag{3.22}$$

We can give the relative error between $Q_3(n)$ and $Q_3'(n)$ as **Table 6** shows. In the table, $Q_3(n)$ is the number of the first kind of like-prime triplets and $Q_3'(n)$ is the values predicted by formula (3.22) and $\pi_3(n)$ is the number of prime triplets of the form $(p, p + 2, p + 6)$.

Table 6. The relative error between $Q_3(n)$ and $Q_3'(n)$ for the first kind of like-prime triplets.

n	$\pi_3(n)$	$Q_3(n)$	$Q_3(n)/\pi_3(n)$	$Q_3'(n)$	relative error
100	4	8	2.00000000	15	0.46666666
1000	15	16	1.06666666	22	0.27272727
10000	55	59	1.07272727	56	0.05084745
100000	259	238	0.91891891	227	0.04621848
1000000	1393	1238	0.88872936	1208	0.02423263
10000000	8543	7497	0.87756057	7314	0.02440976
100000000	55600	48173	0.86642086	47749	0.00880161
1000000000	379508	323262	0.85179232	329517	0.01898232

Table 7. The relative error between $Q_3(n)$ and $Q_3'(n)$ for the second kind of like-prime triplets.

n	$\pi_3(n)$	$Q_3(n)$	$Q_3(n)/\pi_3(n)$	$Q_3'(n)$	relative error
100	4	8	2.00000000	15	0.46666666
1000	15	17	1.13333333	22	0.22727272
10000	57	70	1.22807017	56	0.20000000
100000	248	284	1.14516129	227	0.20070422

Continued

1000000	1444	1354	0.93767313	1208	0.10782865
10000000	8677	7601	0.87599400	7314	0.03775818
100000000	55556	48155	0.86678306	47749	0.00843110
1000000000	379748	324149	0.85358974	329517	0.01629051

We can give the relative error between $Q_3(n)$ and $Q_3'(n)$ as Table 7 shows. In the table, $Q_3(n)$ is the number of the second kind of like-prime triplets, $Q_3'(n)$ is the values predicted by formula (3.22) and $\pi_3(n)$ is the number of prime triplets of the form $(p, p + 4, p + 6)$.

From Table 6 and Table 7 we see both counted numbers of the first kind of like-prime triplets and the second kind of like-prime triplets seem to be close to the value predicted by formula (3.22), which means that it is reasonable to study Goldbach problem by introducing like-prime and like-prime gap.

Theorem 3.29 *If*

$$\lim_{n\to\infty}\frac{Q_3(n)}{C_3\int_2^n\frac{\mathrm{d}t}{(\log t)^3}-C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right)}=1,\ \textit{then}$$

$$\lim_{n\to\infty}\frac{Q_3(n)}{C_3\frac{n}{(\log n)^3}}=1.$$

Proof. As we know, there are two results such that

$$\int_2^n\frac{\mathrm{d}t}{(\log t)^3}=\frac{1}{2}Li(n)-\frac{n}{2(\log n)^2}-\frac{n}{2\log n}+\frac{1}{\log 2}+\frac{1}{(\log 2)^2},\tag{3.23}$$

$$\begin{aligned}Li(n)&\approx\frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}\\&=\frac{n}{\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{6}{(\log n)^3}+\cdots\right).\end{aligned}\tag{3.24}$$

Let $Q_3'(n)=C_3\int_2^n\frac{\mathrm{d}t}{(\log t)^3}-C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right)$. By (3.23) and (3.24) we have an expression for $Q_3'(n)$ as follows

$$\begin{aligned}Q_3'(n)\approx{}&\frac{C_3}{2}\frac{n}{\log n}\sum_{k=2}^{\infty}\frac{k!}{(\log n)^k}+C_3\left(\frac{1}{\log 2}+\frac{1}{(\log 2)^2}\right)\\&-C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right).\end{aligned}\tag{3.25}$$

Considering above asymptotic series, we see the k-th term approaches higher order infinity than the $(k + 1)$-th term as n grows without bound because there is a limit such that

$$\lim_{n\to\infty}\frac{\frac{C_3}{2}\frac{(k+1)!n}{(\log n)^{k+2}}}{\frac{C_3}{2}\frac{k!n}{(\log n)^{k+1}}}=\lim_{n\to\infty}\frac{k+1}{\log n}=0. \tag{3.26}$$

Since the first term in above asymptotic series is

$$C_3\frac{n}{(\log n)^3},$$

there is a limit for constant term as follows

$$\lim_{n\to\infty}\frac{C_3\left(\frac{1}{\log 2}+\frac{1}{(\log 2)^2}\right)}{C_3\frac{n}{(\log n)^3}}=\left(\frac{1}{\log 2}+\frac{1}{(\log 2)^2}\right)\lim_{n\to\infty}\frac{(\log n)^3}{n}=0. \tag{3.27}$$

For above two additional terms, there are the following limits.

$$\lim_{n\to\infty}\frac{C_3\frac{n}{3(\log n)^3(\log\log n)}}{C_3\frac{n}{(\log n)^3}}=\lim_{n\to\infty}\frac{1}{3\log\log n}=0, \tag{3.28}$$

$$\lim_{n\to\infty}\frac{C_3\frac{n}{3(\log n)^3(\log\log n)^2}}{C_3\frac{n}{(\log n)^3}}=\lim_{n\to\infty}\frac{1}{3(\log\log n)^2}=0. \tag{3.29}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{Q_3(n)}{C_3\int_2^n\frac{\mathrm{d}t}{(\log t)^3}-C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right)}=1. \tag{3.30}$$

By (3.23), (3.24), (3.25), (3.26), (3.27), (3.28),(3.29) and (3.30), we get

$$\lim_{n\to\infty}\frac{Q_3(n)}{C_3\frac{n}{(\log n)^3}}=1.$$

Hence the theorem holds.

Corollary 3.30 *If*

$$\lim_{n\to\infty}\frac{Q_3(n)}{C_3\int_2^n\frac{\mathrm{d}t}{(\log t)^3}-C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right)}=1,$$

then Goldbach conjecture is true.

Proof. Suppose $\lim_{n\to\infty}\frac{Q_3(n)}{C_3\int_2^n\frac{\mathrm{d}t}{(\log t)^3}-C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n}+\frac{1}{3(\log\log n)^2}\right)}=1$.

Then we have the following limit by Theorem 3.29.

$$\lim_{n\to\infty}\frac{Q_3(n)}{C_3\frac{n}{(\log n)^3}}=1. \tag{3.31}$$

It means there is an asymptotic expression as follows

$$Q_3(n)\sim C_3\frac{n}{(\log n)^3}. \tag{3.32}$$

Formula (3.31) means the relative error between $Q_3(n)$ and $C_3 n/(\log n)^3$ approaches 0 as n grows without bound, and formula (3.32) means that $Q_3(n)$ is asymptotically equal to $C_3 n/(\log n)^3$. It is obvious that $C_3 n/(\log n)^3$ approaches infinity as n grows without bound to lead to a result such that there are infinitely many triplet like-primes. Since every like-prime triplet must include a pair of twin like-primes. Hence there are infinitely many pairs of twin like-primes. By Conjecture 3.17 and Corollary 3.18, Goldbach conjecture is true and the corollary holds.

3.8. Quadruplet Prime Conjecture

There is a form of prime 4-tuplet such that $(p, p+2, p+6, p+8)$, and also called a prime quadruplet. It can be conjectured that there are infinitely many primes p such that $p+2$, $p+6$, $p+8$ are also primes. There is a strong form of the conjecture to be a special case of the first Hardy-Littlewood conjecture as the following discussion does.

Let $\pi_4(x)$ denote the number of primes $p \le x$ such that $p+2$, $p+6$ and $p+8$ are also primes. Define a constant C_4 as [15]

$$C_4=\frac{27}{2}\prod_{p\ge 5}\frac{p^3(p-4)}{(p-1)^4}\approx 4.151180864\cdots. \tag{3.33}$$

Then there is a special case of the first Hardy-Littlewood conjecture such that

$$\pi_4(x)\approx C_4\int_2^x\frac{\mathrm{d}t}{(\log t)^4},$$

$$\pi_4(x)\sim C_4\frac{x}{(\log x)^4}, \tag{3.34}$$

in the sense that the quotient of the two expressions approaches 1 as x grows without bound. Obviously, if (3.34) holds then quadruplet prime conjecture is true and twin prime conjecture is also true because every prime quadruplet must include two pairs of twin primes. Table 8 gives the number of prime quadruplets of the form $(p, p+2, p+6, p+8)$ and the values predicted by the Hardy-Littlewood formula [15].

Table 8. Counted and predicted numbers of prime quadruplets.

x	$\pi_4(x)$	Hardy-Littlewood	ratio	relative error
100000	38	53	1.3947368	0.2830188
1000000	166	184	1.1084337	0.0978260
10000000	899	863	0.9599555	0.0400444
100000000	4768	4735	0.9930788	0.0069211

3.9. Quadruplet Like-Prime Conjecture and Its Strong Form

Definition 3.31 A like-prime $n1$ is called a *quadruplet like-prime* if $n1 + 1$, $n1 + 3$ and $n1 + 4$ are also like-primes but $n1 + 2$ is not like-prime.

By Definition 3.31, we can find known like-prime quadruplets and the first 50 like-prime quadruplets are listed as follows

(8, 9, 11, 12), (26, 27, 29, 30), (31, 32, 34, 35), (394, 395, 397, 398), (666, 667, 669, 670), (2036, 2037, 2039, 2040), (3489, 3490, 3492, 3493), (4405, 4406, 4408, 4409), (4831, 4832, 4834, 4835), (5269, 5270, 5272, 5273), (5876, 5877, 5879, 5880), (5958, 5959, 5961, 5962), (8091, 8092, 8094, 8095), (10497, 10498, 10500, 10501), (10733, 10734, 10736, 10737), (11131, 11132, 11134, 11135), (12997, 12998, 13000, 13001), (13127, 13128, 13130, 13131), (14213, 14214, 14216, 14217), (17904, 17905, 17907, 17908), (27625, 27626, 27628, 27629), (30245, 30246, 30248, 30249), (31632, 31633, 31635, 31636), (39646, 39647, 39649, 30650), (41557, 41558, 41560, 41561), (41994, 41995, 41997, 41998), (43936, 43937, 43939, 43940), (44993, 44994, 44996, 44997), (46776, 46777, 46779, 46780), (46878, 46879, 46881, 46882), (46881, 46882, 46884, 46885), (64189, 64190, 64192, 64193), (66571, 66572, 66574, 66575), (69408, 69409, 69411, 69412), (74943, 74944, 74946, 74947), (75532, 75533, 75535, 75536), (83170, 83171, 83173, 83174), (91873, 91874, 91876, 91877), (111024, 111025, 111027, 111028), (115356, 115357, 115359, 115360), (120590, 120591, 120593, 120594), (123186, 123187, 123189, 123190), (127922, 127923, 127925, 127926), (129456, 129457, 129459, 129460), (130140, 130141, 130143, 130144), (137394, 137395, 137397, 137398), (138142, 138143, 138145, 138146), (139466, 139467, 139469, 139470), (141529, 141530, 141532, 141533), (143319, 143320, 143322, 143323).

More like-prime quadruplets can be found in [12] and $Q_4(n)$ in **Table 9** gives the number of like-prime quadruplets of the form ($n1$, $n1 + 1$, $n1 + 3$, $n1 + 4$) with $n1 + 2$ not to be like-prime for $n \le 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$.

Conjecture 3.32 *There are infinitely many like-primes* $n1$ *such that* $n1 + 1$, $n1 + 3$ *and* $n1 + 4$ *are also like-primes but* $n1 + 2$ *is not like-prime.*

Corollary 3.33 *If Conjecture* 3.32 *is true, then Goldbach conjecture is true.*

Proof. Since every like-prime quadruplet ($n1$, $n1 + 1$, $n1 + 3$, $n1 + 4$) with $n1 + 2$ not to be like-prime must include two pairs of twin like-primes to correspond to two Goldbach steps with width to be 1, the infinitude of like-prime quadruplets means that there are infinitely many pairs of twin like-primes. By Conjecture 3.17 and Corollary 3.18, Goldbach conjecture is true and the corollary holds.

Remark 3.34 Conjecture 3.32 can be called quadruplet like-prime conjecture to correspond to quadruplet prime conjecture. As there is a strong form of quadruplet prime conjecture, which is a special case of the first Hardy-Littlewood conjecture, there is also a strong form of quadruplet like-prime conjecture as the following discussion does.

Let $Q_4(n)$ denote the number of quadruplet like-primes among the first n positive integers to number the first n primes for generating largest strong Goldbach numbers. Then we propose that there is an approximation $Q_4'(n)$ for $Q_4(n)$ such that

$$Q_4(n) \approx Q_4'(n),$$

$$Q_4'(n)=C_4\int_2^n\frac{\mathrm{d}t}{(\log t)^4}-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right),\tag{3.35}$$

where C_4 is defined as a constant as follows

$$C_4=\frac{27}{2}\prod_{p\ge 5}\frac{p^3(p-4)}{(p-1)^4}\approx 4.151180864\cdots.\tag{3.36}$$

This constant is just constant (3.33). By

$$\int_2^n\frac{\mathrm{d}t}{(\log t)^4}=\frac{1}{6}Li(n)-\frac{n}{3(\log n)^3}-\frac{n}{6(\log n)^2}-\frac{n}{6\log n}+\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2}$$

[15], (3.35) becomes

$$\begin{aligned}Q_4'(n)&=C_4\left(\frac{1}{6}Li(n)-\frac{n}{3(\log n)^3}-\frac{n}{6(\log n)^2}-\frac{n}{6\log n}\right)\\&\quad+C_4\left(\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2}\right)\\&\quad-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right).\end{aligned}\tag{3.37}$$

There is the asymptotic series for $Li(n)/6$ such that

$$\begin{aligned}\frac{1}{6}Li(n)&\approx\frac{n}{6\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}\\&=\frac{n}{6\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{6}{(\log n)^3}+\frac{24}{(\log n)^4}+\frac{120}{(\log n)^5}+\cdots\right).\end{aligned}\tag{3.38}$$

Taking the first six terms in the asymptotic series, (3.37) becomes

$$\begin{aligned}Q_4'(n)&\approx C_4\frac{n}{(\log n)^4}\left(1+\frac{4}{\log n}+\frac{20}{(\log n)^2}-\frac{1}{4\log\log n}-\frac{1}{4(\log\log n)^2}\right)\\&\quad+C_4\left(\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2}\right).\end{aligned}\tag{3.39}$$

We can give the relative error between $Q_4(n)$ and $Q_4'(n)$ as **Table 9** shows. In the table, $Q_4(n)$ is the number of like-prime quadruplets, $Q_4'(n)$ is the values predicted by formula (3.39) and $\pi_4(n)$ is the number of prime quadruplets.

Table 9. The relative error between $Q_4(n)$ and $Q_4'(n)$.

n	$\pi_4(n)$	$Q_4(n)$	$Q_4(n)/\pi_4(n)$	$Q_4'(n)$	relative error
100	2	3	1.50000000	15	0.80000000
1000	5	5	1.00000000	16	0.68750000
10000	12	13	1.08333333	22	0.40909090
100000	38	38	1.00000000	45	0.15555555

Continued

1000000	166	155	0.93373493	157	0.01273885
10000000	899	784	0.87208008	753	0.03954081
100000000	4768	4095	0.85885067	4199	0.02476780
1000000000	28388	24198	0.85240242	25447	0.04908240

Theorem 3.35 *If*

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\int_2^n\frac{\mathrm{d}t}{(\log t)^4}-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1\text{, } \textit{then}$$

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\frac{n}{(\log n)^4}}=1.$$

Proof. As we know, there are two results such that

$$\int_2^n\frac{\mathrm{d}t}{(\log t)^4}=\frac{1}{6}Li(n)-\frac{n}{3(\log n)^3}-\frac{n}{6(\log n)^2}-\frac{n}{6\log n}+\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2},$$

$$\begin{aligned}Li(n)&\approx\frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}\\&=\frac{n}{\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{6}{(\log n)^3}+\cdots\right).\end{aligned}\tag{3.40}$$

Let $Q_4'(n)=C_4\int_2^n\frac{\mathrm{d}t}{(\log t)^4}-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right)$. By (3.40) we have

$$\begin{aligned}Q_4'(n)\approx{}&\frac{C_4}{6}\frac{n}{\log n}\sum_{k=3}^{\infty}\frac{k!}{(\log n)^k}+C_4\left(\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2}\right)\\&-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right).\end{aligned}\tag{3.41}$$

Considering the asymptotic series in (3.41), we see the *k*-th term approaches higher order infinity than the (*k* + 1)-th term as *n* grows without bound. Therefore, we have the following limit.

$$\lim_{n\to\infty}\frac{\frac{C_4}{6}\frac{(k+1)!n}{(\log n)^{k+2}}}{\frac{C_4}{6}\frac{k!n}{(\log n)^{k+1}}}=\lim_{n\to\infty}\frac{k+1}{\log n}=0.\tag{3.42}$$

Since the first term in the asymptotic series in (3.41) is

$$C_4\frac{n}{(\log n)^4},$$

there is a limit for above constant term as follows

$$\lim_{n\to\infty}\frac{C_4\left(\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2}\right)}{C_4\frac{n}{(\log n)^4}} = \left(\frac{2}{3(\log 2)^3}+\frac{1}{3(\log 2)^2}+\frac{1}{3\log 2}\right)\lim_{n\to\infty}\frac{(\log n)^4}{n}=0. \tag{3.43}$$

For two additional terms in (3.41), there are the following limits.

$$\lim_{n\to\infty}\frac{C_4\frac{n}{4(\log n)^4(\log\log n)}}{C_4\frac{n}{(\log n)^4}}=\lim_{n\to\infty}\frac{1}{4\log\log n}=0, \tag{3.44}$$

$$\lim_{n\to\infty}\frac{C_4\frac{n}{4(\log n)^4(\log\log n)^2}}{C_4\frac{n}{(\log n)^4}}=\lim_{n\to\infty}\frac{1}{4(\log\log n)^2}=0. \tag{3.45}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\int_2^n\frac{\mathrm{d}t}{(\log t)^4}-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1, \tag{3.46}$$

by (3.40), (3.41), (3.42), (3.43), (3.44), (3.45) and (3.46) we get

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\frac{n}{(\log n)^4}}=1.$$

Hence the theorem holds.

Corollary 3.36 *If*

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\int_2^n\frac{\mathrm{d}t}{(\log t)^4}-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1$$

, then Goldbach conjecture is true.

Proof. Suppose

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\int_2^n\frac{\mathrm{d}t}{(\log t)^4}-C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1.$$

Then we have the following limit by Theorem 3.35.

$$\lim_{n\to\infty}\frac{Q_4(n)}{C_4\frac{n}{(\log n)^4}}=1. \tag{3.47}$$

It means there is an asymptotic expression as follows

$$Q_4(n) \sim C_4 \frac{n}{(\log n)^4}. \tag{3.48}$$

Formula (3.47) means the relative error between $Q_4(n)$ and $C_4n/(\log n)^4$ approaches 0 as n grows without bound, and formula (3.48) means that $Q_4(n)$ is asymptotically equal to $C_4n/(\log n)^4$. It is obvious that $C_4n/(\log n)^4$ approaches infinity as n grows without bound to lead $Q_4(n)$ to approach infinity as n grows without bound, thus, there are infinitely many pairs of twin like-primes since every like-prime quadruplet must include two pairs of twin like-primes. By Conjecture 3.17 and Corollary 3.18, Goldbach conjecture is true and the corollary holds.

3.10. General Like-Prime Gap Conjecture

Suppose length of like-prime gap can be arbitrarily large. Then every natural number can become length of a like-prime gap and we can make the following conjecture.

Conjecture 3.37 *There are infinitely many like-primes* $n1$ *such that* $n1 + k$ *is also like-prime for every natural number* k.

Remark 3.38 Conjecture 3.27 can be called general like-prime gap conjecture and the conjecture corresponds to Polignac's conjecture. Specially, it is conjectured that there are infinitely many like-primes $n1$ such that $n1 + k$ is also like-prime for $k = 1$, which is just our proposed twin like-prime conjecture.

Conjecture 3.39 *There are infinitely many like-primes* $n1$ *such that* $n1 + k$ *is also like-prime for a special natural number* k.

Conjecture 3.40 *There are infinitely many like-prime gaps whose length* k *is uncertain but bounded by a finite integer* $N > 1$.

Remark 3.41 Conjecture 3.39 is made for any special natural number, for example, $k = 100$ or $k = 10000$, but Conjecture 3.40 is made for k to be uncertain but bounded by a finite integer $N > 1$, that is, value of k is uncertain but there exists upper bound $N > 1$ for k. Thus Conjecture 3.40 is weaker than Conjecture 3.39. However, if it is proven that $N = 2$ then $k = 1$ to be a special natural number and twin like-prime conjecture is true.

Corollary 3.42. *If Conjecture* 3.37 *is true, then Goldbach conjecture is true.*

Proof. Suppose there are infinitely many like-primes $n1$ such that $n1 + k$ is also like-prime for every natural number k. Then the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ is an infinite sequence because natural numbers are infinite. By Theorem 3.6 Goldbach conjecture is true and the corollary holds.

Corollary 3.43 *If Conjecture* 3.39 *is true, then Goldbach conjecture is true.*

Proof. Suppose there are infinitely many like-primes $n1$ such that $n1 + k$ is also like-prime for a special natural number k. Then there are infinitely many like-primes because the set of all like-primes $n1$ such that $n1 + k$ is also like-prime for a special natural number k is a subset of the set of all like-primes. Hence the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i + 1),\ i = 1, 2, 3, \ldots\}$ is an infinite sequence. By Theorem 3.6 Goldbach conjecture is true and the corollary holds.

Corollary 3.44 *If Conjecture* 3.40 *is true, then Goldbach conjecture is true.*

Proof. Suppose there are infinitely many like-prime gaps whose length k is uncertain but bounded by a finite integer $N > 1$. Then there are infinitely many like-primes because the set of all like-primes $n1$ such that $n1 + k$ is also like-prime for k to be uncertain but bounded by a finite integer $N > 1$ is a subset of the set of all like-primes. Hence the like-prime sequence $\{n1\text{-}i;\ n1\text{-}i < n1\text{-}(i+1),\ i = 1, 2, 3, \dots\}$ is an infinite sequence. By Theorem 3.6 Goldbach conjecture is true and the corollary holds.

3.11. Prime, Almost Prime and Like-Prime

By definition of prime, that is, a prime is a natural number which has exactly two natural number divisors: 1 and itself, the fundamental theorem of arithmetic states that every natural number greater than 1 can be written as a product of distinct prime factors as follows

$$n = \prod_{j=1}^{r} p_j^{a_j},$$

where p_j is the j-th distinct prime factor, a_j is exponent of p_j and r is the number of distinct prime factors. Let k denote the number of all prime factors of n. Then

$$k = \sum_{j=1}^{r} a_j .$$

Thus the natural number n greater than 1 is called a k-almost prime. It is clear that every natural number greater than 1 must be a k-almost prime and all primes are 1-almost primes. By studying almost prime, it was proven that every large even number can be represented as the sum of a prime and the product of at most two primes [18] [19], it was proven that there are infinitely many primes p such that $p + 2$ has at most two prime factors [18] [20], it was proven that a number P which is either a prime or a semiprime (2-almost prime) does always satisfy a condition such that there always exists a prime P between n^2 and $(n + 1)^2$ [21]. We see that 1 is not an almost prime but 1 is a like-prime. As every natural number greater than 1 must be a k-almost prime which may be a prime, an odd composite number or an even number, every like-prime greater than 1 must be also a k-almost prime which may be a prime, an odd composite number or an even number. It is not true that every natural number greater than 1 is a like-prime but it is true that every like-prime greater than 1 is an almost prime. The smallest gap between 1-almost primes $g_n = P_{n+1} - P_n$ is 2 for $n > 1$ but the smallest gap between like-primes $g_i = n1\text{-}(i + 1) - n1\text{-}i$ is 1 for $i \geq 1$ and such a pair of like-primes is called a pair of twin like-primes to correspond to a pair of twin primes so that distribution of twin like-primes can be described by an approximation similar to a special case of the first Hardy-Littlewood conjecture and can be asymptotically expressed as $Q_2(n) \sim 2C_2n/(\log n)^2$. By our definitions about two kinds of triplet like-primes to correspond to two forms of triplet primes, distribution of triplet like-primes can be described by an approximation similar to a special case of the first Hardy-Littlewood conjecture and can be asymptotically expressed as $Q_3(n) \sim C_3n/(\log n)^3$. By our definition about quadruplet like-prime to correspond to quadruplet prime, distribution of quadruplet like-primes is described by an approximation similar to a special case of the first Hardy-Littlewood conjecture and can be asymptotically expressed as $Q_4(n) \sim C_4 n/(\log n)^4$. Such studies on distributions of twin like-primes, triplet like-primes and quadruplet like-primes make us more clearly understand like-prime nature of Goldbach steps generated by primes because like-prime always represents existence of Goldbach step. Thus, a basic link between primes and like-primes not only has been established in studying on distribution of like-primes but also established in studying on distribution of twin like-primes, triplet like-primes and quadruplet like-primes. These links seem to be conducive to finding more possible approaches to prove Goldbach conjecture.

4. Bounds of the Largest Strong Goldbach Number

4.1. Bounds of P_n and Bounds of $2P_n$

There is an equivalent statement of prime number theorem, that is,

$$\lim_{n\to\infty}\frac{P_n}{n\log n}=1, \tag{4.1}$$

and an asymptotic expression is as follows

$$P_n \sim n\log n. \tag{4.2}$$

It is asymptotic form of prime.

Rosser proved that [22]

$$P_n > n\log n \quad \text{for} \quad n\ge 1, \tag{4.3}$$

but the theorem does not mean $n\log n$ is lower bound of prime P_n. Cesàro gave a better approximation for P_n in 1894 as follows

$$\begin{aligned}\frac{P_n}{n} &= \log n+\log\log n-1+\frac{\log\log n-2}{\log n}\\ &\quad -\frac{(\log\log n)^2-6\log\log n+11}{2(\log n)^2}+o\left(\frac{1}{(\log n)^2}\right),\end{aligned} \tag{4.4}$$

and it was proven that there are non-asymptotic bounds of P_n such that [23] [24]

$$n\log n+n\log\log n-n < P_n < n\log n+n\log\log n \quad \text{for} \quad n\ge 6. \tag{4.5}$$

By (4.5) it is clear that both upper and lower bounds of P_n are definite number for $n \ge 6$. Let $A_{\text{up}}(n)$ denote the upper bound of P_n and $A_{\text{low}}(n)$ denote the lower bound of P_n. Then we have four examples to verify the definiteness of bounds for prime as Table 10 shows.

It could be easy understood that $2P_n$ are a kind of special even numbers among all even numbers because $2P_n$ is able to be written as a sum of two known primes, $2P_n = P_n + P_n$, thus, $2P_n$ can be thought as a known Goldbach number according to traditional definition of Goldbach number and also a known Goldbach number generated by P_n according to our suggested definition of Goldbach number. It means every $2P_n$ must be a Goldbach number formed by a pair of known primes

(P_n, P_n). Therefore, there is a method for calculating bounds of $2P_n$. Let $B_{\text{up}}(n)$ denote the upper bound of $2P_n$ and $B_{\text{low}}(n)$ denote the lower bound of $2P_n$. By $2P_n = P_n + P_n$ we have

Table 10. Examples verifying definiteness of bounds for prime.

n	P_n	$A_{\text{up}}(n)$	$A_{\text{low}}(n)$
100000	1299709	1395639	1295639
1000000	15485863	16441302	15441302
10000000	179424673	188980382	178980382
100000000	2038074743	2133415472	2033415472

$$\begin{aligned} B_{\text{up}}(n) &= A_{\text{up}}(n) + A_{\text{up}}(n) \\ &= (n\log n + n\log\log n) + (n\log n + n\log\log n) \\ &= 2n\log n + 2n\log\log n, \end{aligned} \tag{4.6}$$

$$\begin{aligned} B_{\text{low}}(n) &= A_{\text{low}}(n) + A_{\text{low}}(n) \\ &= (n\log n + n\log\log n - n) + (n\log n + n\log\log n - n) \\ &= 2n\log n + 2n\log\log n - 2n, \end{aligned} \tag{4.7}$$

and bounds of $2P_n$ can be expressed as

$$2n\log n + 2n\log\log n - 2n < 2P_n < 2n\log n + 2n\log\log n \quad \text{for} \quad n \geq 6. \tag{4.8}$$

However, if $2P_n$ is thought as a Goldbach number based on traditional definition of Goldbach number then (P_n, P_n) is not the only prime pair to form $2P_n$. Suppose Goldbach conjecture is true. Then there is an integer $a > 0$ such that $P_n - a$ is a prime P_i less than P_n and $P_n + a$ is a prime P_k greater than P_n so that there is another prime pair (P_i, P_k) to form $2P_n$, that is, $2P_n = P_i + P_k$. So, there is another pair of bounds for $2P_n$ as follows

$$\begin{aligned} B_{\text{up}}(n) &= A_{\text{up}}(i) + A_{\text{up}}(k) \\ &= (i\log i + i\log\log i) + (k\log k + k\log\log k), \end{aligned} \tag{4.9}$$

$$\begin{aligned} B_{\text{low}}(n) &= A_{\text{low}}(i) + A_{\text{low}}(k) \\ &= (i\log i + i\log\log i - i) + (k\log k + k\log\log k - k). \end{aligned} \tag{4.10}$$

It is obvious that (4.9) is not equal to (4.6) and (4.10) is also not equal to (4.7). In fact, it has been known that there is a tendency such that the higher the value of an even number is, the larger the number of prime pairs to form the even number is. Thus, especially for large n, there are many different prime pairs to form a given $2P_n$ so that there are so many different pairs of bounds for $2P_n$. It means that bounds of $2P_n$ are indefinite for a given $2P_n$, and the result is different from the definiteness of bounds for prime P_n. In other words, the definiteness of bounds for prime P_n does not support the definiteness of bounds for $2P_n$ if $2P_n$ is thought as a Goldbach number based on traditional definition of Goldbach number. However, if $2P_n$ is thought as a Goldbach number generated by P_n then (P_n, P_n) must be the only prime pair to form $2P_n$ because every Goldbach number formed by prime pair (P_i, P_k) for $i \le n$ and $k < n$ must be smaller than $2P_n$ and $2P_n$ is the largest Goldbach number generated by P_n. Therefore, we say that (P_n, P_n) is the only prime pair to form $2P_n$ according to our suggested definition of Goldbach number so that bounds of $2P_n$ are definite and formula (4.8) can show the definiteness of bounds for $2P_n$ (see Theorem 3.5 in [11]). In other words, the definiteness of bounds for prime P_n supports the definiteness of bounds for $2P_n$ if $2P_n$ is thought as a Goldbach number generated by P_n. It means our suggested definition of Goldbach number strongly limits the number of prime pairs to form $2P_n$ and it is such limit that makes $2P_n$ have definite bounds. **Table 11** gives some examples to verify the definiteness of bounds for $2P_n$.

Table 11. Examples verifying definiteness of bounds for $2P_n$.

n	P_n	$2P_n$	$B_{\text{up}}(n)$	$B_{\text{low}}(n)$
10000000	179424673	358849346	377960764	357960764
20000000	373587883	747175766	785331628	745331628
30000000	573259391	1146518782	1203755294	1143755294
40000000	776531401	1553062802	1629347336	1549347336
50000000	982451653	1964903306	2060265255	1960265255
60000000	1190494759	2380989518	2495424749	2375424749
70000000	1400305337	2800610674	2934109797	2794109797
80000000	1611623773	3223247546	3375811752	3215811752
90000000	1824261409	3648522818	3820150459	3640150459
100000000	2038074743	4076149486	4266830945	4066830945

4.2. The Relative Error between L_n and $2P_n$

According to Definition 2.1, $2P_n$ is the largest Goldbach number generated by P_n and is also the largest possible value of L_n generated by P_n. So, there must be $L_n \le 2P_n$ for a given P_n. However, it is obvious that density of primes will be smaller and smaller with growth of n because average gap between primes is about $\log n$ and will be larger and larger with growth of n as prime number theorem describes. Thus one can expect that there is an integer $k > 0$ such that there may be some examples for $L_n = 2P_n$ for $1 \le n \le k$ but $L_n < 2P_n$ for all $n > k$. It is clear that prime number theorem supports the expectation, and value of k has been found. After checking all largest strong Goldbach numbers generated by primes less than 10^7 [12], we discovered that there exist seven examples for $L_n = 2P_n$ for $1 \le n \le 29$ such that $L_n = 2P_n$ for $n = 1, 2, 3, 4, 6, 8, 29$ (see **Table 2** in [11]) but $L_n < 2P_n$ for $n > 29$ among all largest strong Goldbach numbers generated by primes less than 10^7. The observed fact strongly supports above expectation, that is, there is an integer $k = 29$ such that there are seven examples for $L_n = 2P_n$ for $1 \le n \le 29$ but $L_n < 2P_n$ for $30 \le n \le 664579$ because there is no example for $L_n = 2P_n$ for $30 \le n \le 664579$ ($P_{664579} = 9999991$ is the last prime less than 10^7) and density of primes will be smaller and smaller with growth of n for $n > 664579$ by prime number theorem. So, we have the following proposition.

Proposition 4.1 *$L_n < 2P_n$ for all $n > 29$.*

After checking all L_n for $30 \le n \le 4000000000$, we have verified Proposition 4.1 for $30 \le n \le 4000000000$ and the proposition seems to be true. Based on general existence of $L_n < 2P_n$ for $n > 29$, A noteworthy fact is that the relative error between L_n and $2P_n$, $\delta(n) = (2P_n - L_n)/2P_n$, is smaller and smaller with growth of n as Table 12 shows. By the general trend one can expect that the relative error between L_n and $2P_n$ approaches 0 as n grows without bound. However, we also discovered the existence of some large local fluctuations for decreasing trend of $\delta(n)$ as Table 13 shows. On the other hand, there must exist continuous small upturns for the relative error between L_n and $2P_n$ on every Goldbach step whose width is greater than 1. Since the relative error between L_n and $2P_n$ can be written as

$$\delta(n) = (2P_n - L_n)/2P_n = 1 - L_n/2P_n\ , \tag{4.11}$$

there must be continuous small upturns of $\delta(n)$ on a Goldbach step with width greater than 1 as the following theorem shows.

Table 12. The relative error between L_n and $2P_n$.

n	P_n	L_n	$\delta(n)$
100	541	966	0.107208872
1000	7919	15522	0.019952014
10000	104729	208926	0.002539888
100000	1299709	2598332	0.000417785
1000000	15485863	30970934	0.000025571
10000000	179424673	358847082	0.000006309
100000000	2038074743	4076147580	0.000000467
1000000000	22801763489	45603524304	0.000000058

Theorem 4.2 *For every largest strong Goldbach number on a given Goldbach step with width greater than* 1, *the relative error between L_n and $2P_n$ is smaller than the relative error between L_{n+1} and $2P_{n+1}$.*

Proof. Let $\delta(n) = 1 - L_n/2P_n$ denote the relative error between L_n and $2P_n$. Since L_n remains unchanged, that is, $L_{n+1} = L_n$, but $2P_n$ would increase, that is, $2P_{n+1} > 2P_n$ for every largest strong Goldbach number on a given Goldbach step with width greater than 1. Hence $L_n/2P_n > L_{n+1}/2P_{n+1}$ to lead to $1 - L_n/2P_n < 1 - L_{n+1}/2P_{n+1}$ so that $\delta(n) < \delta(n + 1)$ for the Goldbach step and the theorem holds.

Table 13. Some large local fluctuations for decreasing trend of $\delta(n)$.

n	P_n	L_n	$\delta(n)$
10000000	179424673	358847082	0.000006309
20000000	373587883	747172802	0.000003966

Continued

30000000	573259391	1146516350	0.000002121
40000000	776531401	1553057954	0.000003121
50000000	982451653	1964900462	0.000001447
60000000	1190494759	2380986204	0.000001391
70000000	1400305337	2800608066	0.000000931
80000000	1611623773	3223242669	0.000001513
90000000	1824261409	3648518604	0.000001154

By Theorem 4.2, as an example, there is a Goldbach step $L_{664300} = L_{664301} = L_{664302} = L_{664303} = L_{664304} = L_{664305} = L_{664306} = 19989300$ (since $L_{664299} = 19989090 < 19989300$ and $L_{664307} = 19989602 > 19989300$) and the relative error between L_n and $2P_n$ on the Goldbach step has been calculated as **Table 14** shows. Thus we see that there are continuous small upturns for relative error between L_n and $2P_n$ on the Goldbach step such that $\delta(664300) < \delta(664301) < \delta(664302) < \delta(664303) < \delta(664304) < \delta(664305) < \delta(664306)$ in the table.

Table 14. A verification for Theorem 4.2.

n	P_n	L_n	$\delta(n)$
664300	9995413	19989300	0.00007633
664301	9995437	19989300	0.00007873
664302	9995477	19989300	0.00008273
664303	9995483	19989300	0.00008333
664304	9995497	19989300	0.00008473
664305	9995519	19989300	0.00008693
664306	9995527	19989300	0.00008773

Above numerical evidences mean that there are complex fluctuations for relative error between L_n and $2P_n$ including large upturns for the relative error between L_n and $2P_n$ as **Table 13** shows and continuous small upturns for the relative error between L_n and $2P_n$ on a Goldbach step with width greater than 1 as **Table 14** shows. Despite of existence of such complex fluctuations for the relative error between L_n and $2P_n$, the total developing tendency of the relative error between L_n and $2P_n$ is obviously smaller and smaller with growth of n as **Table 12** shows. By the total decreasing tendency for the relative error between L_n and $2P_n$, it seems to be reasonable that we can conjecture that, considering the general trend, the relative error between L_n and $2P_n$ will be smaller and smaller for $n > 29$ so that the relative error between L_n and $2P_n$ will approach 0 as n grows without bound. It means that there is a limit

$$\lim_{n\to\infty} \delta(n) = 0\,. \tag{4.12}$$

According to (4.11), above limit becomes

$$\lim_{n\to\infty}\frac{L_n}{2P_n}=1. \tag{4.13}$$

By means of (4.2), we get

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1, \tag{4.14}$$

and there is an asymptotic expression as follows

$$L_n\sim 2n\log n. \tag{4.15}$$

It is obvious that asymptotic expression (4.15) means that L_n approaches infinity as n grows without bound to imply Goldbach conjecture. However, there is a problem to be considered before making above conjecture. The point of this problem is that, even for a finite range of n, we are unable to confirm the relative error between L_n and $2P_n$ to decrease for every n within the range though there is a total decreasing tendency of the relative error within the range. In other words, we are unable to find numerical evidence to describe decreasing process of the relative error between L_n and $2P_n$ by checking every relative error value caused by every n-value within a given range of n, because there are complex fluctuations for the relative error between L_n and $2P_n$. Therefore, we should establish a rigorous criterion to replace checking the relative error between L_n and $2P_n$ for every n-value and the criterion will arise from study on bounds of L_n and bounds of $2P_n$.

4.3. Bounds of L_n and Bounds of $2P_n$

As $2P_n$ are a kind of special even numbers to be able to be written as a sum of two known primes among all even numbers, L_n are also a kind of special even numbers to be able to be written as a sum of two known primes among all even numbers because every L_n must be the sum of two primes not greater than P_n, which can be thought as a Goldbach number generated by P_n according to our suggested definition of Goldbach number and also a Goldbach number according to traditional definition of Goldbach number. It is different from $2P_n$ that our definition of Goldbach number can limit the number of prime pairs to form $2P_n$ and lead (P_n, P_n) to become the only prime pair to form $2P_n$ but there may be many prime pairs to form L_n in general case. Even under our definition, if Goldbach conjecture is true then there could exist an integer $a > 0$ such that $L_n/2 - a$ is a prime P_i less than $L_n/2$ but $L_n/2 + a$ is a prime P_k greater than $L_n/2$ so that (P_i, P_k) is a prime pair to form L_n and there may exist many such prime pairs to form a given L_n. Let $C_{\text{up}}(n)$ denote the upper bound of L_n and $C_{\text{low}}(n)$ denote the lower bound of L_n. Then we have upper and lower bounds of L_n as follows

$$C_{\text{up}}(n)=A_{\text{up}}(i)+A_{\text{up}}(k), \tag{4.16}$$

$$C_{\text{low}}(n)=A_{\text{low}}(i)+A_{\text{low}}(k). \tag{4.17}$$

However, it is obvious that both $C_{\text{up}}(n)$ and $C_{\text{low}}(n)$ are not a number but a set for a given L_n. In other words, both upper and lower bounds of L_n are indefinite for a given L_n. For example, there are three prime pairs (P_{17}, P_{11}), (P_{16}, P_{12}), (P_{15}, P_{14}) to form $L_{17} = 90$ because $P_{17} = 59$ and $P_{11} = 31$, $P_{16} = 53$ and $P_{12} = 37$, $P_{15} = 47$ and $P_{14} = 43$, therefore, by (4.16) there are three values 102, 101, 106 for the upper bound of L_{17} but by (4.17) there are three values 74, 73, 77 for the lower bound of L_{17}. It means that we are unable to construct two functions of n to describe bounds of L_n though we have known that bounds of $2P_n$ can be described by (4.8).

By calculating the relative error between L_n and $2P_n$ for $30 \le n \le 4000000000$, we see there are two characteristics for the relative error between L_n and $2P_n$ such that $L_n < 2P_n$ and the relative error between L_n and $2P_n$ is smaller and smaller with growth of n in total developing trend for $30 \le n \le 4000000000$. But we do not know if it is reasonable that bounds of $2P_n$ can be thought as bounds of L_n at least for $30 \le n \le 4000000000$. As we know, upper bound of L_n formed by every prime pair will be smaller than upper bound of $2P_n$ as Theorem 3.10 and Corollary 3.11 in [11] show, and lower bound of L_n formed by every prime pair will be also smaller than lower bound of $2P_n$ as Theorem 3.14 and Corollary 3.15 in [11] show. If these results are generalized to infinite range for $n \ge 30$ then we have the following preliminary estimates. First, it is reasonable that the upper bound of $2P_n$ can be used as the upper bound of L_n for $n \ge 30$ because upper bound of L_n formed by every prime pair will be smaller than upper bound of $2P_n$ so that every L_n must be smaller than the upper bound of $2P_n$ for $n \ge 30$. We have verified that the expectation holds for $20542 \le n \le 4000000000$ in this paper as **Figure 1** shows, and we have verified that $L_n/B_{\text{up}}(n) = 461882/502284 = 0.9195634342 < 1$ for $n = 20542$ and $L_n/B_{\text{up}}(n) = 194023368524/201644562409 = 0.9622048132 < 1$ for $n = 4000000000$. Second, if lower bound of $2P_n$ is used as lower bound of L_n then we are unable to confirm whether every L_n is larger than lower bound of $2P_n$ for $n \ge 30$. Value of ratio $L_n/B_{\text{low}}(n)$ is called a normal event for $L_n/B_{\text{low}}(n)$ if $L_n > 2n\log n + 2n\log\log n - 2n$ for the n-value, that is, $L_n/B_{\text{low}}(n) > 1$ for the n-value. But value of ratio $L_n/B_{\text{low}}(n)$ is called an abnormal event for $L_n/B_{\text{low}}(n)$ if $L_n < 2n\log n + 2n\log\log n - 2n$ for the n-value, that is, $L_n/B_{\text{low}}(n) < 1$ for the n-value. In fact, by checking every value of ratio $L_n/B_{\text{low}}(n)$, we discovered that there are 5225 abnormal events for $L_n/B_{\text{low}}(n)$ for $n \le 20541$, that is, the case for $L_n < 2n\log n + 2n\log\log n - 2n$ has appeared 5225 times for $n \le 20541$. The last seven abnormal events for $L_n/B_{\text{low}}(n)$ are listed in **Table 15**. These results correspond to the last seven largest strong Goldbach numbers on a long Goldbach step whose width is 33 ($L_{20509} = L_{20510} = L_{20511} = \ldots = L_{20539} = L_{20540} = L_{20541} = 461024$). But there is no checked abnormal event for $L_n/B_{\text{low}}(n)$ for $20542 \le n \le 400000000$ in our previous work [11], which means $L_n > 2n\log n + 2n\log\log n - 2n$ for every n for $20542 \le n \le 400000000$. Figure 3 in [11] shows the ratio of L_n to lower bound of $2P_n$ for $20542 \le n \le 300000000$. Now this verification has been developed up to $n = 4000000000$ and we have known there is no abnormal event for $L_n/B_{\text{low}}(n)$ for $20542 \le n \le 4000000000$ by checking every value of ratio $L_n/B_{\text{low}}(n)$ for $20542 \le n \le 4000000000$, that is, we have verified that $L_n > 2n\log n + 2n\log\log n - 2n$ for $20542 \le n \le 4000000000$ as **Figure 2** shows and it has been verified that $L_n/B_{\text{low}}(n) = 194023368524/193644562409 = 1.0019561928 > 1$ for $n = 4000000000$. Of course, $L_{20542}/B_{\text{low}}(20542) =$

461882/461200 = 1.0014787510 > 1 is the first normal event for $L_n/B_{\text{low}}(n)$ for 20542 ≤ n ≤ 4000000000 and all values of $L_n/B_{\text{low}}(n)$ for 20542 ≤ n ≤ 4000000000 are normal events for $L_n/B_{\text{low}}(n)$ in this paper.

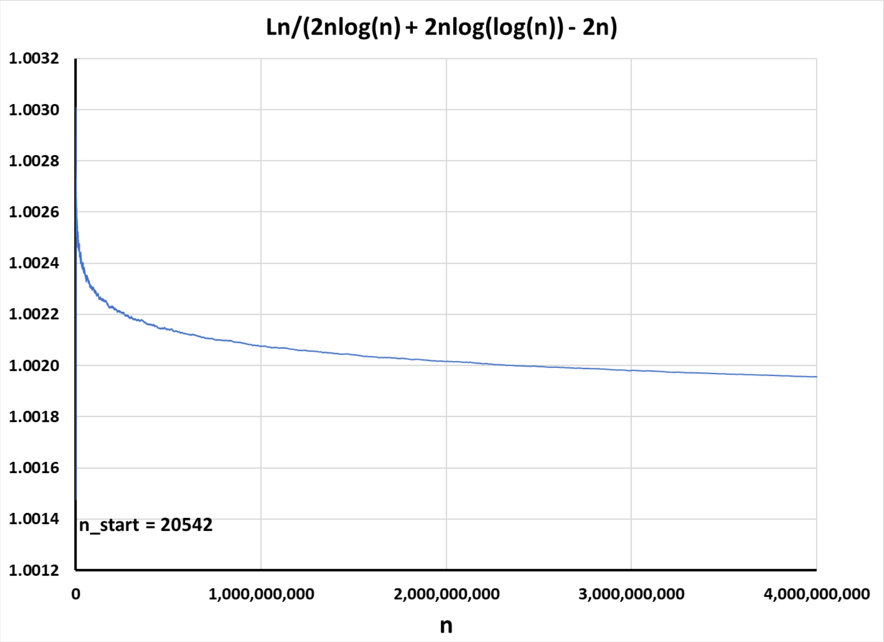

Figure 2. Distribution of $L_n/B_{\text{low}}(n)$ for 20542 ≤ n ≤ 4000000000.

Proposition 4.3 *$L_n/B_{low}(n) > 1$ for $n \geq 20542$.*

Proposition 4.3 supports bounds of $2P_n$ to be used as bounds of L_n for $n \geq 20542$ and it was proven that if bounds of $2P_n$ can be used as bounds of L_n for $n \geq 20542$ then there is a limit such that

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1, \tag{4.18}$$

and Goldbach conjecture is true (see Theorem 3.19 and Corollary 3.20 in [11]). We see (4.18) is just (4.14). Using $L_n/B_{\text{low}}(n) = L_n/(2n\log n + 2n\log\log n - 2n)$ to replace $\delta(n) = 1 - L_n/2P_n$, we have the following theorem to correspond to Theorem 4.2.

Theorem 4.4 *For every largest strong Goldbach number on a given Goldbach step with width greater than* 1, *$L_n/B_{low}(n)$ is greater than $L_{n+1}/B_{low}(n+1)$.*

Proof. Let $L_n/B_{\text{low}}(n) = L_n/(2n\log n + 2n\log\log n - 2n)$ denote the ratio of L_n to lower bound of $2P_n$. Since L_n remains unchanged, that is, $L_{n+1} = L_n$, but $B_{\text{low}}(n)$ would increase, that is, $B_{\text{low}}(n+1) > B_{\text{low}}(n)$ for every largest strong Goldbach number on a given Goldbach step with width greater than 1. Hence $L_{n+1}/B_{\text{low}}(n+1) < L_n/B_{\text{low}}(n)$ and the theorem holds.

We can change Table 14 as Table 16 to give a verification for Theorem 4.4.

Remark 4.5 Theorem 4.4 means that, although there are continuous small upturns of $\delta(n)$ on a Goldbach step with width greater than 1 by Theorem 4.2, every value of $L_n/B_{\text{low}}(n)$ remains greater than 1 if $L_{n2}/B_{\text{low}}(n2) > 1$, where $n2$ is the finishing point of the Goldbach step. Considering $n1 = n2$ for a Goldbach step with width to be 1, we can expect that if Goldbach conjecture is true then $L_{n1-1}/B_{\text{low}}(n1 - 1) > 1$ for every $n1 > 20542$ so that $L_n/B_{\text{low}}(n) > 1$ for every $n \geq 20542$ by Theorem 4.4. The expectation means that general existence of continuous small upturns of $\delta(n)$ on a Goldbach step with width greater than 1 would not lead to appearing of $L_n/B_{\text{low}}(n) < 1$ if $L_{n2}/B_{\text{low}}(n2) > 1$ for the Goldbach step. It means that if $L_{n2}/B_{\text{low}}(n2) > 1$ for $n \geq 20542$ then Goldbach conjecture is true.

Table 15. The last seven abnormal events for $L_n/B_{\text{low}}(n)$.

n	P_n	L_n	$L_n/B_{\text{low}}(n)$
20535	231223	461024	0.9999913237
20536	231241	461024	0.9999392693
20537	231269	461024	0.9998850517
20538	231271	461024	0.9998330083
20539	231277	461024	0.9997744660
20540	231289	461024	0.9997246021
20541	231293	461024	0.9996725754

Table 16. A verification for Theorem 4.4.

n	P_n	L_n	$L_n/B_{\text{low}}(n)$
664300	9995413	19989300	1.00287683
664301	9995437	19989300	1.00287522
664302	9995477	19989300	1.00287356
664303	9995483	19989300	1.00287200
664304	9995497	19989300	1.00287034
664305	9995519	19989300	1.00286873
664306	9995527	19989300	1.00286712

Theorem 4.6 *If there is a bounded integer* $k > 20541$ *such that bounds of* $2P_n$ *can be used as bounds of* L_n *for* $n \geq k$*, then Goldbach conjecture is true.*

Proof. By our suggested definition of Goldbach number, (P_n, P_n) is the only prime pair to form $2P_n$ and bounds of $2P_n$ can be expressed as follows

$$2n\log n + 2n\log\log n - 2n < 2P_n < 2n\log n + 2n\log\log n \quad \text{for} \quad n \geq 6. \tag{4.19}$$

Suppose there is a bounded integer $k > 20541$ such that bounds of $2P_n$ can be used as bounds of L_n for $n \geq k$ though value of k is uncertain. Let $C_{\text{up}}(n)$ denote upper bound of L_n and $C_{\text{low}}(n)$ denote lower bound of L_n. Then we have the following results.

$$C_{\text{up}}(n) = 2n\log n + 2n\log\log n \quad \text{for} \quad n \geq k, \tag{4.20}$$

$$C_{\text{low}}(n) = 2n\log n + 2n\log\log n - 2n \quad \text{for} \quad n \geq k. \tag{4.21}$$

By (4.20) we obtain

$$\lim_{n\to\infty}\frac{L_n}{C_{\text{up}}(n)}=\lim_{n\to\infty}\frac{L_n}{2n\log n+2n\log\log n}, \tag{4.22}$$

and by (4.21) we obtain

$$\lim_{n\to\infty}\frac{L_n}{C_{\text{low}}(n)}=\lim_{n\to\infty}\frac{L_n}{2n\log n+2n\log\log n-2n}. \tag{4.23}$$

Since $C_{\text{up}}(n)$ denotes upper bound of L_n and $C_{\text{low}}(n)$ denotes lower bound of L_n, we get

$$\lim_{n\to\infty}\frac{L_n}{C_{\text{up}}(n)}=1, \tag{4.24}$$

$$\lim_{n\to\infty}\frac{L_n}{C_{\text{low}}(n)}=1. \tag{4.25}$$

By (4.22) we have

$$\lim_{n\to\infty}\frac{L_n}{2n\log n+2n\log\log n}=1, \tag{4.26}$$

and by (4.23) we have

$$\lim_{n\to\infty}\frac{L_n}{2n\log n+2n\log\log n-2n}=1. \tag{4.27}$$

Considering

$$\lim_{n\to\infty}\frac{2n\log\log n}{2n\log n}=\lim_{n\to\infty}\frac{\log\log n}{\log n}=0, \tag{4.28}$$

by (4.26) we obtain

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1. \tag{4.29}$$

Considering

$$\lim_{n\to\infty}\frac{2n\log\log n-2n}{2n\log n}=\lim_{n\to\infty}\frac{\log\log n}{\log n}-\lim_{n\to\infty}\frac{1}{\log n}=0, \tag{4.30}$$

by (4.27) we obtain

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1. \tag{4.31}$$

Formulas (4.29) and (4.31) mean that there is the only result as follows

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1. \tag{4.32}$$

The limit means L_n is asymptotically expressed as

$$L_n\sim 2n\log n. \tag{4.33}$$

Obviously, result (4.33) implies Goldbach conjecture by Lemma 2.6 and the theorem holds.

Note bounded integer $k > 20541$ in Theorem 4.6 means value of k is uncertain but there exists upper bound $N > 20542$ for k. However, if it is proven that $N =$

20543 then $k = 20542$ to be a certain value, which is the strongest form of Theorem 4.6.

4.4. Bounds of $L_n/2$ and Bounds of P_n

As we know, it was proven that there are non-asymptotic bounds of P_n such that

$$n\log n + n\log\log n - n < P_n < n\log n + n\log\log n \quad \text{for} \quad n \geq 6, \tag{4.34}$$

and it is clear that bounds of prime are definite. L_n is an even number for any $n > 0$ so that $L_n/2$ must be an integer which may be a prime less than P_n, an odd composite number or an even number for $n > 29$. It is obvious that there is no a direct method to discuss bounds of $L_n/2$ for establishing a link between bounds of $L_n/2$ and bounds of prime. Thus we have the following definition.

Definition 4.7 $D_{\text{up}}(n)$ is called *upper bound of* $L_n/2$ if $D_{\text{up}}(n) = C_{\text{up}}(n)/2$ and $D_{\text{low}}(n)$ is called *lower bound of* $L_n/2$ if $D_{\text{low}}(n) = C_{\text{low}}(n)/2$.

According to Definition 4.7, both $D_{\text{up}}(n)$ and $D_{\text{low}}(n)$ are indefinite because $C_{\text{up}}(n)$ and $C_{\text{low}}(n)$ are indefinite as we have known. However, It has been verified that bounds of $2P_n$ can be used as bounds of L_n for $20542 \leq n \leq 4000000000$ because $L_n/B_{\text{low}}(n) > 1$ for $20542 \leq n \leq 4000000000$ so that it can be conjectured that bounds of $2P_n$ can be used as bounds of L_n for $n \geq 20542$. By Definition 4.7 it has also been verified that bounds of P_n can be used as bounds of $L_n/2$ for $20542 \leq n \leq 4000000000$ as **Figure 1** and **Figure 2** show because $(L_n/2)/A_{\text{up}}(n) = L_n/B_{\text{up}}(n)$ and $(L_n/2)/A_{\text{low}}(n) = L_n/B_{\text{low}}(n)$. Equivalently, it has been verified that $(L_n/2)/A_{\text{up}}(n) = 0.9195634342 < 1$ and $(L_n/2)/A_{\text{low}}(n) = 1.0014787510 > 1$ for $n = 20542$ but $(L_n/2)/A_{\text{up}}(n) = 0.9622048132 < 1$ and $(L_n/2)/A_{\text{low}}(n) = 1.0019561928 > 1$ for $n = 4000000000$. Thus it can also be conjectured that bounds of P_n can be used as bounds of $L_n/2$ for $n \geq 20542$. So, there is an approximation for $L_n/2$ for $n \geq 20542$ such that

$$\begin{aligned}\frac{\left(\frac{L_n}{2}\right)}{n} &\approx \log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} \\ &- \frac{(\log\log n)^2 - 6\log\log n + 11}{2(\log n)^2} + o\left(\frac{1}{(\log n)^2}\right),\end{aligned} \tag{4.35}$$

and we have the following bounds of $L_n/2$ for $n \geq 20542$.

$$n\log n + n\log\log n - n < L_n/2 < n\log n + n\log\log n\,. \tag{4.36}$$

It had been proven that if bounds of P_n can be used as bounds of $L_n/2$ for $n \geq 20542$ then Goldbach conjecture is true (see Theorem 4.15 and Corollary 4.16 in [11]).

Theorem 4.8 *If there is a bounded integer* $k > 20541$ *such that bounds of* P_n *can be used as bounds of* $L_n/2$ *for* $n \geq k$, *then Goldbach conjecture is true.*

Proof. As we know, it was proven that bounds of prime P_n are expressed as

$$n\log n + n\log\log n - n < P_n < n\log n + n\log\log n \quad \text{for} \quad n \geq 6\,. \tag{4.37}$$

Suppose there is a bounded integer $k > 20541$ such that bounds of P_n can be used as bounds of $L_n/2$ for $n \geq k$ though value of k is uncertain. Let $D_{\text{up}}(n)$ denote upper bound of $L_n/2$ and $D_{\text{low}}(n)$ denote lower bound of $L_n/2$. Then we have the following results.

$$D_{\text{up}}(n) = n\log n + n\log\log n \quad \text{for} \quad n \geq k\ , \tag{4.38}$$

$$D_{\text{low}}(n) = n\log n + n\log\log n - n \quad \text{for} \quad n \geq k\ . \tag{4.39}$$

By (4.38) we obtain

$$\lim_{n\to\infty}\frac{L_n/2}{D_{\text{up}}(n)} = \lim_{n\to\infty}\frac{L_n/2}{n\log n + n\log\log n}, \tag{4.40}$$

and by (4.39) we obtain

$$\lim_{n\to\infty}\frac{L_n/2}{D_{\text{low}}(n)} = \lim_{n\to\infty}\frac{L_n/2}{n\log n + n\log\log n - n}. \tag{4.41}$$

Since $D_{\text{up}}(n)$ and $D_{\text{low}}(n)$ denote upper and lower bound of $L_n/2$, we get

$$\lim_{n\to\infty}\frac{L_n/2}{D_{\text{up}}(n)} = 1, \tag{4.42}$$

$$\lim_{n\to\infty}\frac{L_n/2}{D_{\text{low}}(n)} = 1. \tag{4.43}$$

By (4.40) and (4.42) we have

$$\lim_{n\to\infty}\frac{L_n/2}{n\log n + n\log\log n} = 1, \tag{4.44}$$

by (4.41) and (4.43) we have

$$\lim_{n\to\infty}\frac{L_n/2}{n\log n + n\log\log n - n} = 1. \tag{4.45}$$

Considering

$$\lim_{n\to\infty}\frac{n\log\log n}{n\log n} = \lim_{n\to\infty}\frac{\log\log n}{\log n} = 0, \tag{4.46}$$

by (4.44) we obtain

$$\lim_{n\to\infty}\frac{L_n/2}{n\log n} = 1. \tag{4.47}$$

Considering

$$\lim_{n\to\infty}\frac{n\log\log n - n}{n\log n} = \lim_{n\to\infty}\frac{\log\log n}{\log n} - \lim_{n\to\infty}\frac{1}{\log n} = 0, \tag{4.48}$$

by (4.45) we obtain

$$\lim_{n\to\infty}\frac{L_n/2}{n\log n} = 1. \tag{4.49}$$

Formulas (4.47) and (4.49) mean that there is the only result as follows

$$\lim_{n\to\infty}\frac{L_n/2}{n\log n} = 1. \tag{4.50}$$

The limit means $L_n/2$ is asymptotically expressed as

$$L_n/2 \sim n\log n\,. \tag{4.51}$$

Obviously, result (4.51) implies Goldbach conjecture by Lemma 2.6 and the theorem holds.

Note bounded integer $k > 20541$ in Theorem 4.8 means value of k is uncertain but there exists upper bound $N > 20542$ for k. However, if it is proven that $N = 20543$ then $k = 20542$ to be a certain value, which is the strongest form of Theorem 4.8.

5. What Propositions Will Imply Goldbach Conjecture?

Based on our above discussions, if any of the following propositions is proven then Goldbach conjecture is true.

Proposition 5.1 *L_n approaches infinity as n grows without bound.*

Proposition 5.2 *There are infinitely many Goldbach steps.*

Proposition 5.3 *There is a limit such that*

$$\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\dfrac{n}{\log n}\left(\dfrac{1}{\log\log n}+\dfrac{1}{4(\log\log n)^2}\right)}=1,$$

where $Q(n)$ is the number of Goldbach steps, $Li(n)$ is logarithmic integral

$$Li(n)=\int_2^n\frac{\mathrm{d}t}{\log t}$$

and $Li(n)$ has an asymptotic series as follows

$$Li(n)\approx\frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}\,.$$

Proposition 5.4 *There are infinitely many like-primes.*

Proposition 5.5 *There are infinitely many largest strong Goldbach numbers with distinct values.*

Proposition 5.6 *There are infinitely many like-primes n1 such that $n1 + 1$ is also like-prime.*

Note Proposition 5.6 is equivalent to the following Proposition 5.7.

Proposition 5.7 *There are infinitely many pairs of twin like-primes.*

Proposition 5.8 *There are infinitely many triplet like-primes.*

Proposition 5.9 *There are infinitely many quadruplet like-primes.*

Proposition 5.10 *There is a limit such that*

$$\lim_{n\to\infty}\frac{Q_2(n)}{2C_2\int_2^n\dfrac{\mathrm{d}t}{(\log t)^2}+2C_2\dfrac{n}{(\log n)^2}\left(\dfrac{1}{\log\log n}+\dfrac{1}{(\log\log n)^2}\right)}=1,$$

where $Q_2(n)$ is the number of twin like-primes, C_2 is defined as a constant

$$C_2=\prod_{p\ge3}\left(1-\frac{1}{(p-1)^2}\right)=\prod_{p\ge3}\frac{p(p-2)}{(p-1)^2}\approx0.660161815\cdots,$$

and

$$\int_2^n \frac{\mathrm{d}t}{(\log t)^2} = Li(n) + \frac{2}{\log 2} - \frac{n}{\log n}.$$

Proposition 5.11 *There is a limit such that*

$$\lim_{n\to\infty} \frac{Q_3(n)}{C_3\int_2^n \frac{\mathrm{d}t}{(\log t)^3} - C_3\frac{n}{(\log n)^3}\left(\frac{1}{3\log\log n} + \frac{1}{3(\log\log n)^2}\right)} = 1,$$

where $Q_3(n)$ *is the number of triplet like-primes,* C_3 *is defined as a constant*

$$C_3 = \frac{9}{2}\prod_{p\ge 5}\frac{p^2(p-3)}{(p-1)^3} \approx 2.858248596\cdots,$$

and

$$\int_2^n \frac{\mathrm{d}t}{(\log t)^3} = \frac{1}{2}Li(n) - \frac{n}{2(\log n)^2} - \frac{n}{2\log n} + \frac{1}{\log 2} + \frac{1}{(\log 2)^2}.$$

Proposition 5.12 *There is a limit such that*

$$\lim_{n\to\infty} \frac{Q_4(n)}{C_4\int_2^n \frac{\mathrm{d}t}{(\log t)^4} - C_4\frac{n}{(\log n)^4}\left(\frac{1}{4\log\log n} + \frac{1}{4(\log\log n)^2}\right)} = 1,$$

where $Q_4(n)$ *is the number of quadruplet like-primes,* C_4 *is defined a constant*

$$C_4 = \frac{27}{2}\prod_{p\ge 5}\frac{p^3(p-4)}{(p-1)^4} \approx 4.151180864\cdots,$$

and

$$\int_2^n \frac{\mathrm{d}t}{(\log t)^4} = \frac{1}{6}Li(n) - \frac{n}{3(\log n)^3} - \frac{n}{6(\log n)^2} - \frac{n}{6\log n} + \frac{2}{3(\log 2)^3} + \frac{1}{3(\log 2)^2} + \frac{1}{3\log 2}.$$

Proposition 5.13 *There are infinitely many like-primes* $n1$ *such that* $n1 + k$ *is also like-prime for every natural number k.*

Proposition 5.14 *There are infinitely many like-primes* $n1$ *such that* $n1 + k$ *is also like-prime for a special natural number k.*

Proposition 5.15 *There are infinitely many like-prime gaps whose length k is uncertain but bounded by a finite integer* $N > 1$.

Proposition 5.16 *There are bounds of* L_n *such that* $2nlogn + 2nloglogn - 2n < L_n < 2nlogn + 2nloglogn$ *for* $n \ge 20542$.

Proposition 5.17 *There is a bounded integer* $k > 20541$ *such that* $2nlogn + 2nloglogn - 2n < L_n < 2nlogn + 2nloglogn$ *for all* $n \ge k$*, where bounded integer* $k > 20541$ *means value of k is uncertain but there exists upper bound* $N > 20542$ *for k.*

Proposition 5.18 *There are bounds of* $L_n/2$ *such that* $nlogn + nloglogn - n < L_n/2 < nlogn + nloglogn$ *for* $n \ge 20542$.

Proposition 5.19 *There is a bounded integer* $k > 20541$ *such that nlogn + nlog-*

logn – $n < L_n/2 < n\log n + n\log\log n$ *for all* $n \geq k$*, where bounded integer* $k > 20541$ *means value of k is uncertain but there exists upper bound* $N > 20542$ *for k.*

Note Proposition 5.16 is equivalent to the following Proposition 5.20.

Proposition 5.20 *Bounds of* $2P_n$ *can be used as bounds of* L_n *for* $n \geq 20542$.

Note Proposition 5.18 is equivalent to the following Proposition 5.21.

Proposition 5.21 *Bounds of* P_n *can be used as bounds of* $L_n/2$ *for* $n \geq 20542$.

Note Proposition 5.17 is equivalent to the following Proposition 5.22.

Proposition 5.22 *There is a bounded integer* $k > 20541$ *such that bounds of* $2P_n$ *can be used as bounds of* L_n *for all* $n \geq k$*, where bounded integer* $k > 20541$ *means value of k is uncertain but there exists upper bound* $N > 20542$ *for k.*

Note Proposition 5.19 is equivalent to the following Proposition 5.23.

Proposition 5.23 *There is a bounded integer* $k > 20541$ *such that bounds of* P_n *can be used as bounds of* $L_n/2$ *for all* $n \geq k$*, where bounded integer* $k > 20541$ *means value of k is uncertain but there exists upper bound* $N > 20542$ *for k.*

6. Conclusion

In this paper, we presented there are connections between Goldbach conjecture and prime number theorem and these links seem to arise from existence of largest strong Goldbach numbers and Goldbach steps. We can expect distribution of Goldbach steps is asymptotically expressed as $Q(n) \sim n/\log n$ same as the prime number theorem. So, by introducing like-prime and like-prime gap, it is expected that distribution of twin like-primes can be asymptotically expressed as $Q_2(n) \sim 2C_2 n/(\log n)^2$, distribution of triplet like-primes can be asymptotically expressed as $Q_3(n) \sim C_3 n/(\log n)^3$, distribution of quadruplet like-primes can be asymptotically expressed as $Q_4(n) \sim C_4 n/(\log n)^4$ and these asymptotic expressions are obviously akin to prime number theorem and correspond to some special cases of the first Hardy-Littlewood conjecture. It means that Goldbach steps have like-prime nature which not only shows in distribution of like-primes but also in distribution of like-prime gaps. Based on gap between like-primes, general like-prime gap conjecture is made, which states that there are infinitely many like-primes $n1$ such that $n1 + k$ is also like-prime for every natural number k. The weakest form is that there are infinitely many like-prime gaps whose length k is uncertain (value of k is unknown) but bounded by a finite integer $N > 1$. We proved that many such conjectures will imply Goldbach conjecture. Our study on bounds of L_n and bounds of $L_n/2$ seem to be supported by numerical evidence such that every $L_n/B_{\text{low}}(n) > 1$ and every $(L_n/2)/A_{\text{low}}(n) > 1$ for $20542 \leq n \leq 4000000000$. There is a general trend such that the relative error between L_n and $2P_n$ is smaller and smaller with growth of n and one can expect that the relative error between L_n and $2P_n$ approaches 0 as n grows without bound, thus, the general trend will lead the relative error between lower bound of L_n and lower bound of $2P_n$ and also the relative error between lower bound of $L_n/2$ and lower bound of P_n to be smaller and smaller with growth of n so that we can expect every $L_n/B_{\text{low}}(n) > 1$ or every $(L_n/2)/A_{\text{low}}(n) > 1$ for $n > 4000000000$. It is obvious that if it can be proven that every $L_n/B_{\text{low}}(n) > 1$ or every $(L_n/2)/A_{\text{low}}(n) > 1$ for $n \geq 20542$ then Goldbach conjecture is true. A weak form of the statement is that if there is a bounded integer $k > 20541$ such that bounds of prime can be used as bounds of $L_n/2$ for all $n \geq k$ then Goldbach conjecture is true, where bounded integer $k > 20541$ means value of k is uncertain (value of k is unknown) but there exists upper bound $N > 20542$ for k.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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