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Vector-Valued Convex Functions

Mustapha Laayouni

Department of Mathematics, Faculty of Science and Technology, Moulay Ismail University, Errachidia, Morocco
Email: mustaphalaayouni@yahoo.fr

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Abstract

Convex analysis plays a fundamental role in mathematics. In this paper, we extend the concept of convexity to vector-valued functions in Banach lattices. We introduce the notion of “order convexity” (o-convexity) and explore its properties, generalizing several results from real-valued convex analysis. These include a continuity theorem for o-convex functions (Theorem 1.2), an analogue of Bauer’s maximal principle for o-convex functions on compact sets (Theorem 2.2), and a fixed-point theorem for order contraction maps (Theorem 2.3).

Keywords

Order, Riesz Space, Banach Lattice, Convexity and Order Convexity

1. Order Convexity of Vector-Valued Functions

Often in functional analysis one needs local algebraic linearity. Thus, one of the interactions of the algebraic and topological structure of a topological vector is manifested in the important properties of the class of convex functions. So far, we have allowed the convex functions defined on the convex subsets of a vector space to be real valued. We will extend the definition of convexity to the valued functions in a Banach lattice.

Definition 1.1 *Let* E *be a Banach lattice. A function* $f:C\to E$ *on a convex set C in a vector space* X *is*:

1) order convex (denoted by o-convex) if for all $x,y\in C$ and all $0\le\alpha\le1$, $f\left(\alpha x+\left(1-\alpha\right)y\right)\le\alpha f\left(x\right)+\left(1-\alpha\right)f\left(y\right)$.

2) strictly o-convex if for all $x,y\in C$ with $x\ne y$ and all $0<\alpha<1$, $f\left(\alpha x+\left(1-\alpha\right)y\right)<\alpha f\left(x\right)+\left(1-\alpha\right)f\left(y\right)$.

3) o-concave (respectively, strictly o-concave) if $-f$ is an o-convex (respectively, strictly o-concave) function.

It is easy to realize that, f is o-convex if and only if,

$$f\left(\sum_{k=1}^{n}\alpha_k x_k\right)\le\sum_{k=1}^{n}\alpha_k f\left(x_k\right)$$

for every convex combination $\sum_{k=1}^{n}\alpha_k x_k$.

Example 1.1 Here are some familiar examples of o-convex mappings.

- Obviously, any convex real function is o-convex.
- Let E be a Banach lattice. The absolute value $x\mapsto|x|$ is an o-convex mappings from E to E.
- Let A be a commutative unital real Banach algebra. The set of all multiplicative linear functionals on A is denoted by Δ_A. It is well known that Δ_A, endowed with the Gelfand topology, is compact and the Gelfand representation ϕ of A into $C\left(\Delta_A\right)$ is an homomorphism ([1], Theorem 13). Thus ϕ is o-convex.

Proposition 1.1 *A function* $f:C\to E$ *on a convex subset of a vector space into a Banach lattice* E *is o-convex if and only if its epigraph,* $\operatorname{epi}(f)=\left\{(x,\chi)\in C\times E:\chi\ge f(x)\right\}$, *is convex. Similarly,* f *is o-concave if and only if its hypograph is concave.*

Proof. We prove the first part of this proposition. The remaining assertion is identical. Suppose that f is o-convex, then for $\left(x_1,\chi_1\right),\left(x_2,\chi_2\right)\in\operatorname{epi}(f)$ and $\alpha\in[0,1]$ we have

$$\begin{aligned}\alpha_1\chi_1+(1-\alpha)\chi_2&\ge\alpha_1 f\left(x_1\right)+(1-\alpha)f\left(x_2\right)\\&\ge f\left(\alpha x_1+(1-\alpha)x_2\right)\end{aligned}$$

So, $\left(\alpha_1\chi_1+(1-\alpha)\chi_2,\alpha_1x_1+(1-\alpha)x_2\right)\in\operatorname{epi}(f)$. The "only if" part stems from the fact that $\left(x_1,f\left(x_1\right)\right)\in\operatorname{epi}(f)$ and $\left(x_2,f\left(x_2\right)\right)\in\operatorname{epi}(f)$. ■

Proposition 1.2 *The collection of o-convex functions on a fixed convex set* C *into a Banach lattice* E *has the following properties:*

1) Sums and nonnegative scalar multiples of o-convex functions are o-convex.

2) The (finite) pointwise order limit of a net of o-convex functions is o-convex.

3) The (finite) pointwise supremum of a family of o-convex functions is o-convex.

Proof. The first statement is trivial. For the second assertion, consider a net $\left\{f_i\right\}$ of o-convex functions (finite) pointwise order convergent to f, that is, for any finite part F of C, there is a net $\left\{\chi_i\right\}$ (with the same directed set) satisfying $\chi_i\downarrow 0$ and $\left|f_i(z)-f(z)\right|\le\chi_i$ for each i and every $z\in F$. Let $x,y\in C$ and $\alpha\in[0,1]$. For $F=\left\{x,y,\alpha x+(1-\alpha)y\right\}$ we have:

$$\begin{aligned}f\left(\alpha x+(1-\alpha)y\right)&\le\chi_i+f_i\left(\alpha x+(1-\alpha)y\right)\\&\le\chi_i+\alpha f_i(x)+(1-\alpha)f_i(y)\\&\le\chi_i+\alpha\left[f(x)+f_i(x)-f(x)\right]\\&\quad+(1-\alpha)\left[f(y)+f_i(y)-f(y)\right]\\&\le\chi_i+\alpha f(x)+(1-\alpha)f(y)+\chi_i\\&\le\left[2\chi_i+\alpha f(x)+(1-\alpha)f(y)\right]\downarrow\alpha f(x)+(1-\alpha)f(y)\end{aligned}$$

So, f is o-convex.

Now, let $f_1, f_2, \cdots, f_n$ be o-convex functions on a convex set C into a Banach lattice E. For all $x \in C$, we define $f(x) = \bigvee_{1 \le k \le n} f_k(x)$. It is easy to see that:

$$\begin{aligned} f(\alpha x + (1-\alpha) y) &= \bigvee_{1\le k\le n} f_k(\alpha x + (1-\alpha) y) \\ &\le \bigvee_{1\le k\le n} \left[\alpha f_k(x) + (1-\alpha) f_k(y)\right] \\ &\le \bigvee_{1\le k\le n} \left[\alpha f_k(x)\right] + \bigvee_{1\le k\le n} \left[(1-\alpha) f_k(y)\right] \\ &\le \alpha \bigvee_{1\le k\le n} f_k(x) + (1-\alpha) \bigvee_{1\le k\le n} f_k(y) \\ &\le \alpha f(x) + (1-\alpha) f(y). \end{aligned}$$

So, f is o-convex, what completes the proof. ■

Proposition 1.3 *Let* $f : C \to E$ *be an o-convex function,* $x \in C$ *and* $x \mp z \in C$. *Then for all* $\alpha \in [0,1]$,

$$\left|f(x+\alpha z) - f(x)\right| \le \alpha\left(\left[f(x+z) - f(x)\right] \vee \left[f(x-z) - f(x)\right]\right).$$

Proof. $f(x+\alpha z) \le (1-\alpha) f(x) + \alpha f(x+z)$ because the hypothesis and the equality: $x + \alpha z = (1-\alpha) x + \alpha(x+z)$. Rearranging terms yields

$$f(x+\alpha z) - f(x) \le \alpha\left[f(x+z) - f(x)\right] \tag{1.1}$$

$$\le \alpha\left(\left[f(x+z) - f(x)\right] \vee \left[f(x-z) - f(x)\right]\right) \tag{1.2}$$

Replacing z by $-z$ in (1.1) gives

$$f(x-\alpha z) - f(x) \le \alpha\left[f(x-z) - f(x)\right] \tag{1.3}$$

Since $x = \frac{1}{2}(x+\alpha z) + \frac{1}{2}(x-\alpha z)$, we have $f(x) \le \frac{1}{2} f(x+\alpha z) + \frac{1}{2} f(x-\alpha z)$. Multiplying by two and rearranging terms we obtain

$$f(x) - f(x+\alpha z) \le f(x-\alpha z) - f(x) \tag{1.4}$$

(1.3) implies

$$\begin{aligned} f(x) - f(x+\alpha z) &\le f(x-\alpha z) - f(x) \\ &\le \alpha\left[f(x-z) - f(x)\right] \\ &\le \alpha\left(\left[f(x+z) - f(x)\right] \vee \left[f(x-z) - f(x)\right]\right) \end{aligned} \tag{1.5}$$

With definition of the absolute value in mind, (1.2) in conjunction with (1.5) yields the conclusion of the proposition. ■

Recall that a subset A of a Riesz space X is order bounded, from above if there is a vector u (called an upper bound of A) that dominates each element of A, that is, satisfying $a \le u$ for each $a \in A$. Sets order bounded from below are defined similarly. A box or an order interval, is any set of the form

$$[a,b] = \{x \in X : a \le x \le b\}$$

Definition 1.2 *A mapping* $f : X \to E$ *between Riesz spaces is o-bounded above* (*respectively, o-bounded*) *on a subset* V *of* X, *if* $f(V)$ *is order* *bounded from above* (*respectively, if* $f(V) \subset [a,b]$ *for some box* $[a,b]$ *of* E).

We would have liked an o-convex function $f : X \to E$ to be order continuous, but this is not true even in the trivial case when $E = \mathbb{R}$. Indeed, let $X = C[0,1]$, we emphasize: There is no nonzero σ-order continuous linear functional on the Riesz space X (see for example ([2], p. 329)). However, for the topological continuity we have the following, which generalizes a similar result well known for the convex (real) functions.

Theorem 1.1 *If an o-convex function* $f : C \to E$ *is o-bounded above in a neighborhood of an interior point* $x \in C$, *then* f *is continuous at* x.

Proof. We may assume that for some $x \in C$ there exist an open ball V of radius η at 0 and some $\chi \in E$ satisfying $x + V \subset C$ and $f(y) \le f(x) + \chi$ for each $y \in x + V$. Fix $\varepsilon > 0$ and choose some $0 < \alpha < 1$ so that $\alpha\|\chi\| < \varepsilon$. From Proposition1.3, it follows that for each $y \in x + \alpha V$ we have $|f(y) - f(x)| \le \alpha\chi$. Now, the norm of E is lattice, then $\|f(y) - f(x)\| \le \alpha\|\chi\| < \varepsilon$. ■

Remark 1.1

Provided that the interior $Int(E_+)$ of the cone E_+ is non-empty, semicontinuity can be generalized to vector functions as follows (For more details on the impact of a cone's properties on the Riesz space it generates, the reader is referred to [3]).

Definition 1.3 *A mapping* $f : X \to E$ *from a topological space* X *into a Banach lattice* E *is*:

- Lower o-semicontinuous if for each $c \in E$ the set

$$\{x \in X : f(x) - c \in Int(E_+)\}$$

is open.

- Upper o-semicontinuous if for each $c \in E$ the set

$$\{x \in X : c - f(x) \in Int(E_+)\}$$

is open.

Obviously, a mapping f is lower o-semicontinuous if and only $-f$ is upper o-semicontinuous, and vice versa.

The classic example of a lower (resp. upper) o-semicontinuous mapping is given by the lower (resp. upper) semicontinuous real functions. Now assume that E is a Banach lattice with an order unit e. It is well known that the principal ideal E_e generated by e coincides with E which when provided with the norm $\|x\|_\infty = \inf\{\lambda > 0 : |x| \le \lambda e\}$ becomes an AM-space with unit. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous real function. Then the mapping $\tilde{f} : E \to E$ defined by $\tilde{f}(x) = f(\|x\|_\infty)e$ is a lower and upper o-semicontinuous mapping.

The E-valued mapping on a Banach lattice is a useful device, but it needs to be handled with care. For example, the complement of the set $\{x \in X : f(x) \ge c\}$ in X is not at all the set $\{x \in X : f(x) < c\}$. However, the following lemma reduces this difficulty by reducing us to functions with real values.

Proposition 1.4 *If* E_+ (*respectively,* E'_+) *is the positive cone of a Banach lattice* E (*respectively, of* E'), *then* $x \notin E_+$ *if and only if it exists* $\varphi \in E'_+$ *such* *that* $\varphi(x)<0$.

Proof. The definition of the positive cone E'_+ gives a sense of lemma. Conversely, if $x \notin E_+$, since E_+ is closed and convex, it follows from the Hahn-Banach theorem that there is a $\varphi \in E'$ with $\varphi(x)<\varphi(y)$ for all $y \in E_+$. Thus $\varphi(x)<0=f(0)$ and $\varphi(x)<\varphi(ny)$ for all non-negative integer number n. So $\varphi(x)<0 \le \varphi(y)$ for all $y \in E_+$. ■

As a first application of the above definitions, we have the following result.

Theorem 1.2 *For an o-convex mapping* $f: C \to E$ *on an open convex subset* $C \subset X$, *the following are equivalent.*

(a) f is continuous on C.

(b) f is upper o-semicontinuous.

(c) f is o-bounded above on a neighborhood of each point in C.

(d) f is o-bounded above on a neighborhood of some point in C.

(e) f is continuous at some point in C.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c); Assume that f is upper o-semicontinuous. Let $x \in C$ and $a \in Int(E_+)-\{0\}$. Then the set $\{y \in E: f(x)+a-f(y) \in Int(E_+)\}$ is an open neighborhood of x on which f is o-bounded above.

(c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (e) This is Theorem 1.1.

(e) $\Rightarrow$ (a) Suppose f is continuous at the point x, and let y be any other point in C. Since C is open and convex, therefore C does not contain extreme points. This implies that there exist $z \in C$ and $0<\lambda<1$ such that $y=\lambda x+(1-\lambda)z$. Fix $\varepsilon>0$ and choose some circled neighborhood V of zero so that $\|f(x)-f(x+v)\|<\frac{\varepsilon}{\lambda}$ for all $v \in V$. We claim that,

$\|f(y)-f(y+v)\|<\varepsilon$ for all $v \in \lambda V$. Indeed, let $v \in V$, Then

$y+\lambda v=\lambda(x+v)+(1-\lambda)z \in C$ and the o-convexity of f implies

$$\begin{aligned} f(y+\lambda v) &= f(\lambda(x+v)+(1-\lambda)z) \\ &\le \lambda f(x+v)+(1-\lambda)f(z) \end{aligned}$$

and

$$\begin{aligned} f(y) &= f(\lambda x+(1-\lambda)z) \\ &\le \lambda f(x)+(1-\lambda)f(z) \end{aligned}$$

Thus

$$\begin{aligned} f(y+\lambda v)-f(y) &\le \lambda\big(f((x+v)-f(x))\big) \\ &\le \lambda\left|f((x+v)-f(x))\right| \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} f(y)-f(y+\lambda v) &\le \lambda\big(f(x)-f((x+v))\big) \\ &\le \lambda\left|f((x+v)-f(x))\right| \end{aligned} \tag{1.7}$$

This shows that

$$\left|f(y)-f(y+\lambda v)\right|\le\lambda\left|f((x+v)-f(x))\right| \tag{1.8}$$

Then

$$\left\|f(y)-f(y+\lambda v)\right\|\le\lambda\left\|f((x+v)-f(x))\right\|<\varepsilon$$

So, f is continuous at y. ■

2. Order Lipschitzian Vector-Valued Functions

Lipchitzian and contractive real functions have important properties that we want to extend to infinite dimensional analysis. For this purpose, we adopt the following definition.

Definition 2.1 *A mapping f from a subset* B *of a normed space* $(X,\|\|)$ *to a Banach lattice* E *is order Lipschitzian on* B *if there exists* $e\in E_+$ *such that for every* $y,z\in B$

$$\left|f(y)-f(z)\right|\le\|y-z\|e$$

If moreover $\|e\|<1$ then f is called an order contraction.

The following gives examples of order Lipschitzian functions.

Theorem 2.1 *Let* $f:C\to E$ *be a positive o-convex function from a convex subset* $C\subset X$ *into a Banach lattice* E. *If* f *is continuous at some interior point* x *of* C, *then* f *is order Lipschitzian on a neighborhood of* x.

Proof. Since f is continuous at x, it follows from Theorem 1.2 that there exists $e\in E_+$ and $\delta>0$ satisfying $B_{2\delta}(x)\subset C$ and $f(y)\le e$. So, $w,z\in B_{2\delta}(x)$ implies $0\le f(w)\le e$ and $-e\le -f(z)\le 0$. By addition, we achieve $\left|f(w)-f(z)\right|\le e$, for all $w,z\in B_{2\delta}$. Let $y,z\in B_{2\delta}(x)$ and $\alpha=\|y-z\|$. Then $w=y+\frac{\delta}{\alpha}(y+z)$ belongs to $B_{2\delta}$ and we have $y=\frac{\alpha}{\alpha+\delta}w+\frac{\delta}{\alpha+\delta}z$. Therefore

$$f(y)\le\frac{\alpha}{\alpha+\delta}f(w)+\frac{\delta}{\alpha+\delta}f(z)$$

Subtracting $f(z)$ from each side gives

$$\begin{aligned}f(y)-f(z)&\le\frac{\alpha}{\alpha+\delta}\left[f(w)-f(z)\right]\\&\le\frac{\alpha}{\alpha+\delta}e\\&\le\alpha e\end{aligned}$$

Switching the roles of y and z allows us to conclude

$$\left|f(y)-f(z)\right|\le\|y-z\|e$$

■

A net $\{x_\alpha\}$ in a Riesz space E is order convergent to some $x\in E$, written $\{x_\alpha\}\xrightarrow{o}x$, if there is a net $\{q_\alpha\}$ (with the same directed set) satisfying $\{q_\alpha\}\downarrow 0$ and $\left|x_\alpha-x\right|\le q_\alpha$ for each α. A function $f:E\to F$ between two Riesz spaces is order uniformly continuous if $\{y_\alpha - z_\alpha\} \overset{o}{\rightarrow} 0$ in E implies $\{f(y_\alpha) - f(z_\alpha)\} \overset{o}{\rightarrow} 0$ in F.

Proposition 2.1 *If* $f; X \rightarrow E$ *is order Lipschitz continuous and* $\|.\|$ *is an order continuous lattice norm on* X *, then* f *is order uniformly continuous.*

Proof. Obvious. ■

Now we will generalize, to convex order applications, one of the important themes of the analysis, namely the extreme points of a convex functions on a compact convex set. Let C be a convex subset of a vector space X. Recall that an extreme subset of C, is a nonempty subset F of C with the property that if x belongs to F, it cannot be written as a convex combination of points of C outside F. A point x is an extreme point of C if the singleton $\{x\}$ is an extreme set.

Proposition 2.2 *If* $f: C \rightarrow E$ *is o-convex and attains a maximum at some point, then the set of maximizers is an extreme set.*

Proof. Suppose f achieves a maximum on C; that is, f satisfies the identity $\sup\{f(x): x \in C\} = f(e)$ for some $e \in C$. Put $\mathcal{M} = \{x \in C: f(x) = f(e)\}$. Suppose that $x = \alpha y + (1-\alpha) z \in \mathcal{M}$, $0 < \alpha < 1$ and $y, z \in C$. If $y \notin \mathcal{M}$ then $f(y) < f(e)$, so

$$\begin{aligned} f(e) = f(x) &= f(\alpha y + (1-\alpha) z) \\ &\leq \alpha f(y) + (1-\alpha) f(z) \\ &< \alpha f(e) + (1-\alpha) f(e) = f(e) \end{aligned}$$

a contradiction. Hence $y, z \in \mathcal{M}$, so $\mathcal{M}$ is an extreme subset of C. ■

Recall that the order $\leq$ of a Banach lattice E is continuous if $\leq$ is a closed subset of $E \times E$. Let us say that $\leq$ is upper semicontinuous if $\{x \in E: y \leq x\}$ is closed for each y.

Theorem 2.2 *Let* K *be compact and* $f: K \rightarrow E$ *continuous, with* E *a Banach lattice having continuous order. If* $f(K)$ *is closed under suprema* (*condition C*), *then* f *attains a minimum on* K, *and the minimizer set is compact.*

Proof. Let K be a compact of a normed vector space X and let $f: K \rightarrow E$ be a continuous mapping from K to a Banach lattice E. For each $c \in f(K)$, put $F_c = \{x \in K: f(x) \geq c\}$. It follows from the continuity of f and of the order that the nonempty set F_c is closed ($c = f(x)$ implies $x \in F_c$). Moreover, the family $\mathcal{F} = \{F_c: c \in f(K)\}$ has the finite intersection property. In deed, let $F_{c_1}, F_{c_2}, \cdots, F_{c_n}$ be a finite family in $\mathcal{F}$. Since $F(K)$ satisfies condition (C) so, $c_0 = \bigvee_{i=1}^{i=n} c_i \in f(K)$. For all $x \in F_{c_0}$ and $1 \leq i \leq n$ we have

$$0 \leq f(x) - c_0 \leq f(x) - c_i$$

so, $x \in \bigcap_{1 \leq i \leq n} F_{c_i}$ and $F_{c_0} \subset \bigcap_{1 \leq i \leq n} F_{c_i}$. Since K is compact, ([2], Theorem.2.31) implies that the set of minimizers $\bigcap_{c \in f(K)} F_c$ is compact and nonempty. ■

We realize that, in its real context, the assumptions (C) and (C') in Theorem 2.2 are ensured from the fact that the order in $\mathbb{R}$ is total.

A complete lattice is a lattice in which every nonempty subset that is order bounded from above has a supremum. (Equivalently, if every nonempty subset that is bounded from below has an infimum).

Now consider a vector form of the Contraction Mapping Theorem.

Theorem 2.3 *Let* $f : B \to B$ *be an order contraction on a closed subset* B *of a Banach lattice* E, *with contraction modulus* $\|e\| < 1$. *Then f has a unique fixed point* x, *and for any* $x_0 \in B$, *the iterates* $x_{n+1} = f(x_n)$ *converges to* x *with* $\|x_n - x\| \le \|e\|^n \|x_0 - x\|$.

Proof. Let $e \in E_+$ such that $\|e\| < 1$ and $|f(y) - f(z)| \le \|y - z\| e$, for all $y, z \in B$. If $f(x) = x$ and $f(y) = y$ then $|x - y| = |f(x) - f(x)| \le \|x - y\| e$. Since the norm of E is lattice, we have $\|x - y\| \le \|x - y\| \|e\|$ and hence $\|x - y\| = 0$. Thus f can have at most one fixed point.

Now, if x_0 is chosen in B then the formula $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \cdots$ defines inductively the sequence (x_n) which satisfies: $|x_{n+1} - x_n| \le \|x_n - x_{n-1}\| e$, for every $n \ge 1$. The lattice property verified by the norm of E implies that $\|x_{n+1} - x_n\| \le \|x_n - x_{n-1}\| \|e\|$ and by induction, we see that for all $n \geqslant 1$, $\|x_{n+1} - x_n\| \le \|x_1 - x_0\| \|e\|^n$. Hence, for $n > m$ the triangle inequality yields

$$\begin{aligned}\|x_m - x_n\| &\le \sum_{k=m+1}^{n} \|x_k - x_{k-1}\| \\ &\le \|x_1 - x_0\| \sum_{k=m+1}^{n} \|e\|^k \\ &\le \|x_1 - x_0\| \frac{\|e\|^m}{1 - \|e\|}\end{aligned}$$

This implies that (x_n) is a Cauchy sequence. Since B is closed in the complete space E then, $(x_n) \to x \in B$. Obviously, f is continuous, and: $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x)$, so x is the fixed point of f.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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