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Bifurcations and Traveling Wave Solutions of a Generalized b-Family of Novikov Equation

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Abstract

The Novikov equation is an important shallow water wave model with broad applications in fields, such as fluid mechanics and physics. In this paper, a generalized b-family of Novikov equation is studied by the bifurcation theory method of dynamical system. Firstly, this model is transformed into a planar Hamiltonian system through the traveling wave transformation. Then, the phase portraits of the planar dynamical system under different parameters are then generated using Maple. And two new types of implicit traveling wave solutions are obtained.

Keywords

A Generalized b-Family of Novikov Equation, Traveling Wave Solution, Bifurcation Theory

1. Introduction

In this paper, we study the following generalized b-family of Novikov equation

$$u_t - u_{xxt} + (b+1)u^{2n}u_x = buu_xu_{xx} + u^2u_{xxx}, \tag{1}$$

where $b \in R$, $n \in N^+$. When $n = 1$, $b = 3$, the Equation (1) becomes the classical Novikov equation [1]. The Equation (1) has been extensively investigated, which has led to significant findings in this field (see, e.g., [2]-[6]). When $n = 1$, $b = -2m$, $m \in Z^+$, Meng and He [7] studied the equation by the bifurcation theory. Moreover, when $n = 1$ and arbitrary real number b, Li and Wen [8] obtained the bifurcations and many exact traveling wave solutions for the modified Novikov equation.

The investigation of exact solutions of nonlinear evolution equations plays an important role in nonlinear mathematical physics. Some new and important methods for obtaining exact solutions of nonlinear evolution equations have been presented. Especially, Li introduced a new powerful method based on the bifurcation theory method of dynamical systems (see, e.g., [9]). This method has been used to study of travelling wave solutions of many classes of wave equations (see, e.g. [10]-[14]). The main goal of this paper is to show that Equation (1) has some traveling wave solutions by using the bifurcation theory of planar dynamical systems in [9].

The remainder of this paper is organized as follows. In Section 2, we discuss bifurcations of phase portraits. In Section 3, the parametric expressions of traveling wave solutions are obtained. A conclusion is given in Section 4.

2. Bifurcations of Phase Portraits

Firstly, make the following traveling wave transformation

$$u(x,t)=\varphi(\xi)=\varphi(ax-ct),\ a\neq 0,\ c>0. \tag{2}$$

Substitute (2) into (1), then (1) is transformed into the following ODE:

$$-c\varphi'+a^2c\varphi'''+a(b+1)\varphi^{2n}\varphi'=a^3b\varphi\varphi'\varphi''+a^3\varphi^2\varphi'''. \tag{3}$$

where "′" $=\frac{\mathrm{d}}{\mathrm{d}\xi}$. Integrating the above Equation (3) yields:

$$\left(a^3\varphi^2-a^2c\right)\varphi''=-c\varphi+\frac{a(b+1)}{2n+1}\varphi^{2n+1}-\frac{1}{2}a^3(b-2)\left(\varphi(\varphi')^2-\int(\varphi')^2\,\mathrm{d}\varphi\right). \tag{4}$$

Setting $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=g$, then $\frac{\mathrm{d}y^2}{\mathrm{d}\varphi}=2\varphi''$. Thus, the equivalent form of Equation (4) below is achievable:

$$\frac{1}{2}\left(a^3\varphi^2-a^2c\right)\frac{\mathrm{d}y^2}{\mathrm{d}\varphi}=-c\varphi+\frac{a(b+1)}{2n+1}\varphi^{2n+1}-\frac{1}{2}a^3(b-2)\left(\varphi y^2-\int y^2\mathrm{d}\varphi\right). \tag{5}$$

Taking the derivative of the two sides of the above Equation (5) yields:

$$\frac{1}{2}\left(a^3\varphi^2-a^2c\right)\frac{\mathrm{d}^2y^2}{\mathrm{d}\varphi^2}+\frac{1}{2}a^3b\varphi\frac{\mathrm{d}y^2}{\mathrm{d}\varphi}=-c+a(b+1)\varphi^{2n}. \tag{6}$$

Let $g=\frac{\mathrm{d}y^2}{\mathrm{d}\varphi}$, then the equation is transformed into a first-order ordinary differential equation:

$$\frac{\mathrm{d}g}{\mathrm{d}\varphi}=-\frac{b\varphi}{\varphi^2-\frac{c}{a}}g+\frac{-2c+2a(b+1)\varphi^{2n}}{a^3\left(\varphi^2-\frac{c}{a}\right)}. \tag{7}$$

Using the method of constant variation and $b=2$, it infers

$$g=\left(-\frac{2c}{a^3}\varphi+\frac{6}{a^2(2n+1)}\varphi^{2n+1}\right)\left(\varphi^2-\frac{c}{a}\right)^{-1}. \tag{8}$$

According to (8) and $\frac{\mathrm{d}y^2}{\mathrm{d}\varphi}=g$, it has the first integral of (1):

$$H(\varphi,y)=y^2+\frac{c}{a}\ln\left|\varphi^2-\frac{c}{a}\right| -\frac{3}{(2n+1)a^2}\left[\sum_{k=1}^{n}\frac{1}{k}C_n^k\left(\frac{c}{a}\right)^{n-k}\left(\varphi^2-\frac{c}{a}\right)^k+\left(\frac{c}{a}\right)^n\ln\left|\varphi^2-\frac{c}{a}\right|\right] =h. \tag{9}$$

The following singular systems can be obtained from (9):

$$\begin{cases}\dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\dfrac{\partial H}{\partial y}=2y,\\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi}=-\dfrac{\partial H}{\partial \varphi}=\left(\dfrac{6}{(2n+1)a^2}\varphi^{2n+1}-\dfrac{2c}{a}\varphi\right)\left(\varphi^2-\dfrac{c}{a}\right)^{-1}.\end{cases} \tag{10}$$

Let $M(\varphi_i,y_i)$ be the coefficient matrix of the linearized system of (10). At this point, the determinant of $M(\varphi_i,y_i)$ has the form

$$J(\varphi_i,y_i)=-4\frac{\left(\frac{3}{a^2}\varphi_i^{2n}-\frac{c}{a}\right)\left(\varphi_i^2-\frac{c}{a}\right)-2\varphi_i\left(\frac{3}{(2n+1)a^2}\varphi_i^{2n+1}-\frac{c}{a}\varphi_i\right)}{\left(\varphi_i^2-\frac{c}{a}\right)^2}. \tag{11}$$

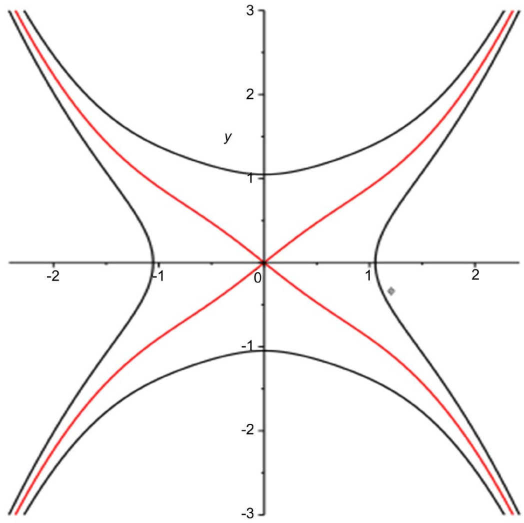
By the theory of planar dynamical systems(see, e.g., [9] [15]), we know that for an equilibrium point of a planar integrable system, if $J<0$, then the equilibrium point is a saddle point; if $J>0$ and $\mathrm{Trace}(M(\varphi_i,y_i))=0$, then it is a center point; if $J>0$ and $(\mathrm{Trace}(M(\varphi_i,y_i)))^2-4J(\varphi_i,y_i)>0$, then it is a node point; if $J=0$ and the index of the equilibrium point is 0, then it is a cusp, otherwise, it is a higher order equilibrium point. Based on the theoretical analysis above, we have obtained the following proposition.

Proposition 1. 1) When $n=2,a<0,c>0$, the origin $E_0(0,0)$ is the only equilibrium point of the system (10) which is a saddle point. The phase portraits of system (10) with $n=2,a=-1,c=1$ is shown in **Figure 1(a)** by mathematical software Maple 18.

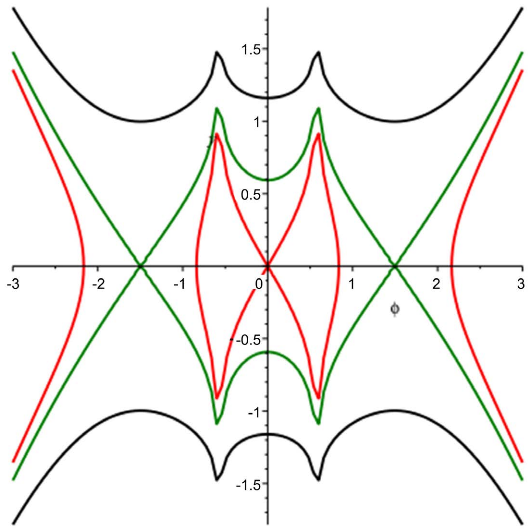
2) When $n=2,a>0,c>0$, the system (10) has three equilibrium points $E_0(0,0)$, $E_1\left(-\sqrt[4]{\frac{5}{3}ac},0\right)$, $E_2\left(\sqrt[4]{\frac{5}{3}ac},0\right)$. The phase portraits of system (10) with $n=2$, $a=3$, $c=1$ is shown in **Figure 1(b)** by mathematical software Maple 18.

Proposition 2. 1) When $n=3,a<0,c>0$, the origin $E_0(0,0)$ is the only equilibrium point of the system (10) which is a saddle point. The phase portraits of system (10) with $n=3,a=-1,c=1$ is shown in **Figure 1(c)** by mathematical software Maple 18.

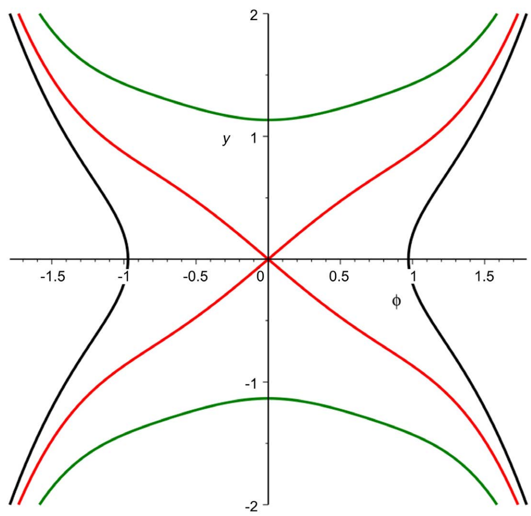
2) When $n=3,a>0,c>0$, the system (10) has three equilibrium points $E_0(0,0)$, $E_3\left(-\sqrt[6]{\frac{7}{3}ac},0\right)$, $E_4\left(\sqrt[6]{\frac{7}{3}ac},0\right)$. The phase portraits of system (10) with $n=3$, $a=3$, $c=1$ is shown in **Figure 1(d)** by mathematical software Maple 18.



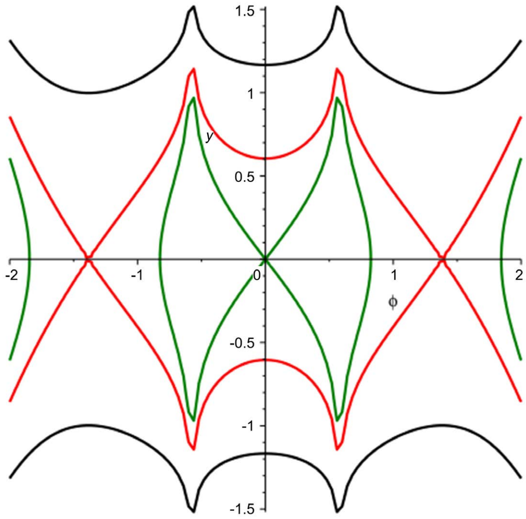
(a) $n = 2, a = -1, c = 1$



(b) $n = 2, a = 3, c = 1$



(c) $n = 3, a = -1, c = 1$



(d) $n=3, a=3, c=1$

Figure 1. The phase portraits of system (10) under different parameter conditions.

3. Parametric Expressions of Traveling Wave Solutions of Equation (1)

In this section, by using the bifurcations of phase portraits in **Figure 1** and the direct integration method, two new types of implicit traveling wave solutions of the Equation (1) are obtained.

Case1. When $n=2$, we deduce from (9) that

$$y^2 = h - \frac{c}{a}\ln\left|\varphi^2 - \frac{c}{a}\right| + \frac{3}{5a^2}\left[\frac{1}{2}\left(\varphi^2 - \frac{c}{a}\right)^2 + \frac{2c}{a}\left(\varphi^2 - \frac{c}{a}\right) + \frac{c^2}{a^2}\ln\left|\varphi^2 - \frac{c}{a}\right|\right]. \quad (12)$$

In view of (12) and the first Equation of (10), it obtains

$$\int_{-\infty}^{\phi} \frac{d\phi}{\sqrt{h - \frac{c}{a}\ln\left|\varphi^2 - \frac{c}{a}\right| + \frac{3}{5a^2}\left[\frac{1}{2}\left(\varphi^2 - \frac{c}{a}\right)^2 + \frac{2c}{a}\left(\varphi^2 - \frac{c}{a}\right) + \frac{c^2}{a^2}\ln\left|\varphi^2 - \frac{c}{a}\right|\right]}} = 2|\xi|.$$

Case2. When $n=3$, from (9) we find

$$y^2 = h - \frac{c}{a}\ln\left|\varphi^2 - \frac{c}{a}\right| + \frac{3}{7a^2}\left[\frac{1}{3}\left(\varphi^2 - \frac{c}{a}\right)^3 + \frac{3c}{2a}\left(\varphi^2 - \frac{c}{a}\right)^2 + \frac{3c^2}{a^2}\left(\varphi^2 - \frac{c}{a}\right) + \frac{c^3}{a^3}\ln\left|\varphi^2 - \frac{c}{a}\right|\right].$$

Then substituting it into the first Equation of (10), we can obtain the following parametric expressions of traveling wave solutions of (1)

$$\int_{-\infty}^{\phi} \frac{d\phi}{\sqrt{h + \left(\frac{3c^3}{7a^5} - \frac{c}{a}\right)\ln\left|\varphi^2 - \frac{c}{a}\right| + \frac{3}{7a^2}\left[\frac{1}{3}\left(\varphi^2 - \frac{c}{a}\right)^3 + \frac{3c}{2a}\left(\varphi^2 - \frac{c}{a}\right)^2 + \frac{3c^2}{a^2}\left(\varphi^2 - \frac{c}{a}\right)\right]}}$$
$$= 2|\xi|.$$

4. Conclusion

In this paper, a generalized b-family of Novikov equation is studied by the bifurcation theory method of dynamical systems (see, e.g., [9]). By applying the traveling wave transformation combined with sophisticated computations, the first integral and its associated planar dynamical system are obtained. When $n=2$ and $n=3$, the phase portraits of system (10) are obtained by mathematical software Maple 18. Then by using **Figure 1** and the direct integration method, two new types of implicit traveling wave solutions of the Equation (1) are obtained which enriched the results of this equation.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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