Scientific Research Publishing

Advances in Pure Mathematics, 2022, 12, 757-771
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Gaps between Primes

Pal Doroszlai, Horacio Keller

Independent Researcher, Kékkút, Hungary
Email: paldoroszlai@yahoo.com

How to cite this paper: Doroszlai, P. and Keller, H. (2022) The Gaps between Primes. *Advances in Pure Mathematics*, **12**, 757-771.
https://doi.org/10.4236/apm.2022.1212058

Received: November 18, 2022
Accepted: December 27, 2022
Published: December 30, 2022

Abstract

The union of the straight and—of the over a point of reflection—reflected union of the series of the arithmetic progression of primes results the double density of occupation of integer positions by multiples of the primes. The remaining free positions represent diads of equidistant primes to the point of reflection: in case the point of reflection is an even number, they satisfy Goldbach's conjecture. Further, it allows to prove, that the number of twin primes is unlimited. The number of all greater gaps as two between primes has well defined lower limit functions as well: it is evaluated with the local density of diads, multiplied with the total of the density of no-primes of all positions over the distance between the components of the diads (the size of the gaps). The infinity of these lower limit functions proves the infinity of the number of gaps of any size between primes. The connection of the infinite number of diads to the infinity of the number of gaps of any size is the aim of the paper.

Keywords

Twin Primes, Prime Gaps, Goldbach's Conjecture

1. Introduction

Some of the unsolved problems of number theory are related to the equidistance of two primes to a midway point. Such is the infinite set of twin primes equidistantly placed around the series of multiples of six and the infinite sets of any two primes equidistantly placed around an even number composing diads and satisfying Goldbach's conjecture.

The present paper gives the analysis of the number of diads with no primes in between their components, composing gaps: the number of diads composed by consecutive primes. The formula representing the number of diads must be modified correspondingly with the density of no-primes within the gaps. This extension of the number of diads by the density of no-primes within the gap of the components of the diads is the main topic of the present paper. It gives the proof in detail, that the infinity of the number of gaps of any size is based on the infinity of the number of diads of any size, as already stated in reference [1].

First the simplification of the integral of the density of primes resulting the prime-number-formula—as given in detail in reference [2]—is given shorted.

Secondly the same method applied to the symmetric series of the union of the sets of multiples of primes—given in detail in reference [1]—is shortly repeated. Reflecting of the union of the series of multiples of primes around a midway point creates a symmetric assembly. The infinity of the sum of the remaining free integer positions representing primes allows to analyze and solve unsolved problems, such as Goldbach's conjecture, the infinity of the number of twin primes and the presence of primes within the section of the size equal to the square root of the distance to the origin, following any distance, as given in ref. [1].

Third the infinity of the sum of the number of primes symmetrically placed around the central point—composing diads—is used to prove the infinity of the sum of the gaps of any size between primes: by inserting into the formula of the sum of diads and into the low limit of this sum the probability of no-primes between the components of the diads. Because the effect of the probability of no-primes approaches unity for distances rising to infinity, the number of gaps of any size approaches the number of diads, as already stated in ref. [1].

Because the infinity of the sum of prime gaps of any size is an extension of the infinity of twins to larger gaps, the infinity of the twins—given in detail in reference [1]—is shortly repeated in **Annex 1**. The most important results of the analysis of the infinity of the sum of prime gaps of any size are checked and illustrated in **Annex 2** with numeric results.

Similar work by Zhang ref. [3] [4] is non-conclusive following notes from Green ref. [5]. Another remarkable approach to prove the infinity of the number of gaps of any size is given by Vega, ref. [6].

2. Density of Occupation of Subsets of All Integers by the Union of the Series of Multiples of Primes

De la Valée Poussin proved 1899, that ($\pi(c) \approx Li(c)$), the integral of the local logarithmic density ($1/\ln(c)$) of positions free of multiples of all smaller primes, up to the square roots of the distance. Approximating the integral by the sum of the logarithmic density over all integers—see (ref. [1])—is in the following named as sum over all integers:

$$\pi(c) = Li(c) + O(c) \approx \int_2^c \frac{1}{\ln(c)} \mathrm{d}c \; ; \quad \pi_{appr}(c) \approx \sum_{n=2}^{c} \frac{1}{\ln(c)} \tag{1.1}$$

This above sum written as summing up first over all integers within the sections of the length ($\sqrt{c}$) and then summing up over all the ($\sqrt{c}$) sections of the length ($\sqrt{c}$), then taking the average value over each section and summing up over all the sections is a first simplification—in the following used as sum over all sections—gives:

$$\pi_{appr}(c) \approx \sum_{n=2}^{c} \frac{1}{\ln(c)} = \sum_{j=2}^{\sqrt{c}} \left[\sum_{n=(j-1)\cdot\sqrt{c}}^{j\sqrt{c}} \frac{1}{\ln(c)} \right] \approx \sum_{j=1}^{\sqrt{c}} \frac{\sqrt{c}}{\ln\left(j \cdot \sqrt{c}\right)} \tag{1.2}$$

Taking for each of the sections the smallest value of the density—at the upper limit of the sections ($j = \sqrt{c}$)—results the low limit of the number of primes: the well proven prime-number-formula PNF, as second simplification of the above approximation of the integral in (1.1).

$$\sum_{j=1}^{\sqrt{c}} \frac{\sqrt{c}}{\ln\left(j \cdot \sqrt{c}\right)} > \pi_{PNF}(c) = \sum_{j=1}^{\sqrt{c}} \frac{\sqrt{c}}{\ln\left(\sqrt{c} \cdot \sqrt{c}\right)} = \frac{c}{\ln(c)} = S(c) \tag{1.3}$$

Taking for each of the sections the largest value of the density at ($j = 1$) gives the upper limit of the number of primes:

$$\sum_{j=1}^{\sqrt{c}} \frac{\sqrt{c}}{\ln\left(j \cdot \sqrt{c}\right)} < \sum_{j=1}^{\sqrt{c}} \frac{\sqrt{c}}{\ln\left(1 \cdot \sqrt{c}\right)} = \frac{c}{\ln\left(\sqrt{c}\right)} = 2 \cdot S(c) \tag{1.4}$$

Reflecting the series of multiples of any prime over a point at the distance (*c*) from the origin results the double density of occupation by the series of multiples of the prime considered, presumed the prime is not a dividend of the distance (*c*). The union of the series of multiples of all primes up to ($\sqrt{c}$) leaves positions free, representing equidistant primes to the point of the reflection. If the point of reflection is a prime, then the positions left free by the double density of occupation represent primes, which are equidistantly placed around the point of reflection. This, because the straight and the reflected series of multiples of each prime are mutually exclusive. The remaining free positions are composing diads. If a prime is a dividend of (*c*), then the reflection of the series of multiples of this prime does not change the occupation of integer position: Therefore, the double density of occupation by the union of the series of multiples all primes up to ($\sqrt{c}$) leaves the smallest number of positions free.

The local density of free positions left at the distance ($d,\ \ 2 < d < c$) below the point of reflection by the straight series of multiples is ($\frac{1}{\ln(c-d)}$), above the point of reflection by the reflected series is ($\frac{1}{\ln(c+d)}$). The combined local density of free positions is, as elaborated in detail in ref. [1], with the constant ($\delta_2 := 1.32032$) (This constant has the double value of the twin prime constant (C2) defined by G. H. Hardy and John Littlewood):

$$D_{diad}(c,d) = \frac{\delta_2}{\ln(c-d) \cdot \ln(c+d)} > \frac{\delta_2}{\ln(c)^2} \tag{1.5}$$

Similarly to the evaluation of the number of primes as simplification of the integral of the local logarithmic density of primes in (1.2), the best estimate of the total number of diads results as simplification of the integral of this above density by taking the sum over all integers (This same generalization from the primes to the twins, respectively to the k-tuples was made already by G. H. Hardy and John Littlewood):

$$\pi_{diad\_appr\_}(c) \approx \sum_{n=1}^{c} \frac{\delta_2}{\ln\left(n\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c - n\cdot\sqrt{c}\right)} \tag{1.6}$$

This above sum may be written as summing up first over all integers within the sections of the length ($\sqrt{c}$) and then summing up over all the ($\sqrt{c}$) sections of the length ($\sqrt{c}$). Taking the average value over each section and—as a first simplification—summing up over the sections gives:

$$\begin{aligned}\pi_{diad\_appr\_}(c) &= \sum_{j=1}^{\sqrt{c}} \sum_{n=(j-1)\cdot\sqrt{c}}^{j\cdot\sqrt{c}} \frac{\delta_2}{\ln\left(n\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c - n\cdot\sqrt{c}\right)} \\ &= \sum_{j=1}^{\sqrt{c}} \frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c - j\cdot\sqrt{c}\right)}\end{aligned} \tag{1.7}$$

Similarly, to the second simplification in case of the primes in (1.4) the low limit of the best estimate number of diads results with the density taken for all sections of the length ($\sqrt{c}$) with ($j=\sqrt{c}$) at the upper limit of the sections, the diads-number-formula (DNF):

$$\pi_{diad\_appr\_low\_}(c) = \sum_{j=1}^{\sqrt{c}} \frac{\delta_2\cdot\sqrt{c}}{\ln(c)\cdot\ln(2\cdot c - c)} = \frac{\delta_2\cdot\sqrt{c}}{\ln(c)^2}\cdot\left(\sum_{j=1}^{\sqrt{c}} 1\right) = \frac{\delta_2\cdot c}{\ln(c)^2} \tag{1.8}$$

The upper limit of the best estimate number of diads results with the density taken for all sections of the length ($\sqrt{c}$) with ($j=1$) at the lower limit of the distance gives:

$$\pi_{diad\_appr\_high\_}(c) = \sum_{j=1}^{\sqrt{c}} \frac{\delta_2\cdot\sqrt{c}}{\ln\left(\sqrt{c}\right)\cdot\ln\left(2\cdot c - \sqrt{c}\right)} = \frac{2\cdot\delta_2\cdot\sqrt{c}}{\ln(c)\cdot\ln\left(2\cdot c - \sqrt{c}\right)} \tag{1.9}$$

The diads-number-formula DNF allows to prove the infinity of the number of twin primes, see ref. [1]. Twin primes are diads with a gap equal to two between their components. Because their number is growing to infinity with the distance, it offers the possibility to proof the infinity of the number of gaps of any size: the DNF must be modified for growing distances between consecutive primes for this purpose, taking account of the requirement, that the components of the diads are consecutive primes in this case.

The distances between the components of the diads in (1.7) may be subdivided is subsets with identical methods of the evaluation of the sum of the number of diads for each subset. Such subdivision is known to be applied to accelerate finding twins by Eratosthenes sieve. The definition of these subsets is the following:

Definition of subsets of all integers:

With the infinite number of integers ($s \in \mathbb{Z}$, $\mathbb{Z} = 1,2,3,\cdots,\infty$) all integers are part of one of the following infinite subsets:

$$N_{sub_{(a,s)}} = \left\{a + s\cdot P_{(1)}\cdot P_{(2)}; a = 0,1,2,3,4,5\right\} \tag{1.10}$$

All integers in the subsets for ($a = 0,2,3,4$) are divisible either by ($P_{(1)}$), and/or by ($P_{(2)}$). Therefore, all primes ($P_{(m)}, \{m = 3,4,5,\cdots,\infty\}$) are members of one of the infinite subsets ($a = 1$) or ($a = 5$), having the distance ($\frac{N}{2} = 1$) to the members of the set ($a = 0$).

Members of the subset ($a = 0$) with one of their neighboring positions equal to a prime compose the infinite subsets of these subsets:

$$N_{sub_{(5,s)}} = \left\{a + s \cdot P_{(1)} \cdot P_{(2)}; a = 5; s \in \mathbb{Z}\right\}; \quad N_{sub_{(1,s)}} = \left\{a + s \cdot P_{(1)} \cdot P_{(2)}; a = 1; s \in \mathbb{Z}\right\} \quad (1.11)$$

Any two primes with the distance (*N*) between them compose diads. The distance between the primes composing the diads, (*N*) is the distance of the components, an even number. The center point of the distance of the components has the distance (*c*) to the origin and is named in the following the distance of the diads. The only condition is, that the half-distance of the components of any diad must be smaller, then the distance of the diad ($N/2 < c$).

In the table below is listed, which values may take the distances of the components (*N*) of the diads belonging to one of the subsets ($a = 0,1,2,3,4,5$) and ($s = 0,1,2,3,4,5,\cdots$) corresponding to (1.11).

In case the center point of the diads is at the distance ($c(a,s) = a + 6 \cdot s$)—corresponding to the subsets (*a*)—the following distances of the components of the diads (*N*) are possible:

a	$\frac{1}{2} \cdot N(a,s)$	$N(a,s)$	
0	$a + 1 + 6 \cdot s = 1,7,13,19,\cdots$	$2,14,26,38,\cdots$	(1.12)
0	$a - 1 + 6 \cdot s = 5,11,17,23,\cdots$	$10,22,34,46,\cdots$	
1	$a - 1 + 6 \cdot s = 6,12,18,24,\cdots$	$12,24,36,48,\cdots$	(1.13)
2	$a + 1 + 6 \cdot s = 3,9,15,21,\cdots$	$6,18,30,42,\cdots$	(1.14)
3	$a + 1 + 6 \cdot s = 4,10,16,22,\cdots$	$8,20,32,44,\cdots$	(1.15)
3	$a - 1 + 6 \cdot s = 2,8,14,20,\cdots$	$4,16,28,40,\cdots$	
4	$a + 1 + 6 \cdot s = 3,9,15,21,\cdots$	$6,18,30,42,\cdots$	(1.16)
5	$a - 1 + 6 \cdot s = 6,12,18,24,\cdots$	$12,24,36,48,\cdots$	(1.17)

This way the center point ($c(0,s)$) is equal to the smallest positive integers with potential prime positions both sides at the distance ($N/2 = 1$), meaning twins. The set of the distances of the components of the diads (*N*) covers the set of all even integers. The sets of the distances for ($a = 1$) with ($a = 5$) as well as the sets for ($a = 2$) with ($a = 4$) are identical.

If the components of the diads are consecutive primes, then the corresponding distance of the components is equal to a prime gap. The number of diads—at growing distance, with the corresponding even number as distance of their components—is unlimited. With (1.8) in ref. [2] the low limit of the number of diads—including twin primes—is given by the diads-number-formula and grows to infinity.

The best estimate number and the low limit of diads are evaluated in (1.7) and (1.8) as sum over all sections of the length ($\sqrt{c}$). The same formula written for any of the group of distances ($c(a,s)$) gives:

$$\pi_{diad\_appr\_}(c(a,s)) = \sum_{j=1}^{\sqrt{c(a,s)}} \frac{\delta_2 \cdot \sqrt{c(a,s)}}{\ln\left(j \cdot \sqrt{c(a,s)}\right) \cdot \ln\left(2 \cdot c(a,s) - j \cdot \sqrt{c(a,s)}\right)} \quad (1.18)$$

$$\pi_{diad\_appr\_low}(c(a,s)) = \sum_{j=1}^{\sqrt{c(a,s)}} \frac{\delta_2 \cdot \sqrt{c(a,s)}}{\ln\left(\sqrt{c(a,s)}\right)^2} = \frac{\delta_2}{4} \cdot R(c(a,s))^2$$

If between ($c(a,s) - \frac{N}{2}$) and ($c(a,s) + \frac{N}{2}$) there are no primes, then the diads represents a gap between consecutive primes.

As an example, for the diads the twin primes are evaluated as a special case of (1.3), with ($\frac{N}{2} = 1$). In this case there are no primes within the gaps. For ($c(0,s) = 6 \cdot s$) the density of occupation of twins is:

$$D_{Tw\_free}(c,j) = \frac{\delta_2}{\ln\left(j \cdot \sqrt{c} - 1\right) \cdot \ln\left(j \cdot \sqrt{c} + 1\right)} \quad (1.19)$$

The approximating function of the total number of twins up to (c), with density in case of the double occupation by the union of the series of multiples of the primes and with the factor of correction gives the twin-number-formula:

$$TW_{appr\_}(c) = \delta_2 \sqrt{c} \cdot \sum_{j=1}^{\sqrt{c}} D_{tw\_free}(c,j) > TW(c) = \frac{\delta_2 \cdot c}{\ln(c)^2} \quad (1.20)$$

The evaluation and the comparison with the effective number twins are carried out in ref. [2]. Because of the implication of the number of twins to the number of gaps of any size, the evaluation is given in the present paper as well, in **Annex 1**.

The number of diads (1.18) gives the number of potential gaps. The density of primes at any point (c) inside the gap is ($\frac{1}{\ln(c)}$). The density of points being no-primes is ($1 - \frac{1}{\ln\left(j \cdot \sqrt{c}\right)}$). The combined density of two points being no-primes is the product of the density at each point. The combined density of all no-primes within the gaps is the product of the density of all points within the gap: ($\left(1 - \frac{1}{\ln(c)}\right)^{N-1}$). Herewith the density of gaps is the product of the density of diads and the density of no-primes

$$D_{gap}(c,j,N) = \frac{\delta_2}{\ln\left(j \cdot \sqrt{c} - \frac{N}{2}\right) \cdot \ln\left(j \cdot \sqrt{c} + \frac{N}{2}\right)} \cdot \left(1 - \frac{1}{\ln\left(j \cdot \sqrt{c}\right)}\right)^{N-1} \quad (1.21)$$

The approximating number of gaps of a given size (N) is evaluated—analogue to (1.18)—as sum over all sections of the length ($\sqrt{c}$) of the density values, as follows:

$$\pi_{gap\_appr}(c,N)=\sum_{j=1}^{\sqrt{c}}\sqrt{c}\cdot\left(D_{gap}(c,j,N)\right)=\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}\cdot\left(1-\frac{1}{\ln\left(j\cdot\sqrt{c}\right)}\right)^{N-1}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-j\cdot\sqrt{c}\right)} \quad (1.22)$$

The low limit of the number of gaps is evaluated analogue to the well proven prime-number-formula PNF with (1.3) and the diads-number-formula DNF with (1.8) as second simplification. Therefore, it may be considered as proven as well. Taking with (1.8) for each of the sections the smallest value of the density of diads at ($j=\sqrt{c}$), respectively the density of no-primes at ($j=1$) yields the gap-number-formula GNF:

$$\pi_{gap\_appr\_low}(c,N)=\frac{\delta_2\cdot\sqrt{c}\cdot\left(1-\frac{1}{\ln\left(\sqrt{c}\right)}\right)^{N-1}}{\ln(c)^2}\cdot\sum_{j=1}^{\sqrt{c}}1=\frac{\delta_2\cdot c\cdot\left(1-\frac{1}{\ln\left(\sqrt{c}\right)}\right)^{N-1}}{\ln(c)^2} \quad (1.23)$$

Lemma 1.1:

The density of no-gaps between any bounded size of diads approaches unity for distances growing to infinity.

Proof: The limit of the number of no-primes within each gap is (1.22), it is approaching unity:

$$\lim_{c\to\infty}\left(1-\frac{1}{\ln\left(\sqrt{c}\right)}\right)^{N-1}=1 \quad (1.24)$$

as stated in the lemma and concluding the proof.

Therefore, the number of the gaps approaches the number of the diads for distances growing to infinity and the gaps-number-formula as low limit of the number of gaps grows to infinity, like the number of the diads, as proved in ref. [2].

The number of the gaps of the size (N) within the sections of the size ($\sqrt{c}$) is limited to ($\frac{\sqrt{c}}{N}$). This number outgrows the number of the primes within the section ($\frac{\sqrt{c}}{\ln(c)}$) for distances over (c_{lim}):

$$\frac{\sqrt{c}}{N}>\frac{\sqrt{c}}{\ln(c)} \quad \text{for } \left(c>c_{lim}=\mathrm{e}^N\right) \quad (1.25)$$

The effective number of gaps of a given size is evaluated in **Annex 2** on the condition, that any diad representing a gap is composed by consecutive primes. The comparison of the effective number of gaps with its approximation requests the introduction of two constants in Equations (1.22) and (1.23):

$$\pi_{gap\_appr}(c,N)=\sum_{j=1}^{\sqrt{c}}\frac{K_1\cdot\delta_2\cdot\sqrt{c}\cdot\left(1-\frac{1}{\ln\left(j\cdot\sqrt{c}\right)}\right)^{K_2\cdot(N-1)}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-j\cdot\sqrt{c}\right)};$$
$$\pi_{gap\_appr\_low}(c,N)=\frac{K_1\cdot\delta_2\cdot\sqrt{c}\cdot\left(1-\frac{1}{\ln\left(\sqrt{c}\right)}\right)^{K_2\cdot(N-1)}}{\ln(c)^2} \quad (1.26)$$

The introduction of the two constants ($K_1 \ge 2$) and ($K_2 = \frac{5}{4}$) in the formula results from the comparison with the effective number of the gaps in **Annex 2**. These constants do not influence the fact of the infinity of the number of the gaps.

The approximation as best estimate and the effective number of the gaps of the sizes:

($N(a,s) = 2,14,26,38,50,62,74,86,98,110,122,134$)—with the set defined as ($a=1$, $s=1,\cdots,12$)—over the distance is evaluated in **Annex 2** using the routines given in **Figure A2.1**. Because all other sets (1.13) through (1.17) are similar, it is sufficient to conduct the analysis for one set alone.

The number of the gaps for large distances approaches the number of the diads (1.24), as illustrated in the comparison of the effective number of the gaps of the sizes defined above, in the **Figure A2.4**. The effective rise of the number of the gaps of any size with the distance in these figures allows to formulate the following lemma:

Lemma 1.2:

The number of gaps of any size is rising without limit over the distance.

Proof: Within the sections of the size ($\sqrt{c}$) the upper limit of the number of the gaps of any size (N) is:

$$\pi_{gap\_section\_lim}(N,c)=\frac{\sqrt{c}}{N}$$

In case the number of primes within the same section—resulting from the prime-number-formula—is smaller than this number, there may be a gap of the size (N) up to this distance. The distance growing to infinity, the number of the gaps of this size will grow to infinite as well:

$$\left(\frac{\sqrt{c}}{\ln\left(c-c_{lim}\right)}>\frac{\sqrt{c}}{N}>\frac{\sqrt{c}}{\ln(c)}\right) \text{ or } \left(\ln\left(c_{lim}\right)>N>\ln\left(c-c_{lim}\right)\right)$$

This condition corresponds to the condition, that the average distance between primes at (c) is greater than (N):

$$\left(\frac{\sqrt{c}}{\frac{\sqrt{c}}{\ln(c)}}=\ln(c)\right) \text{ or } \left(\ln\left(c_{lim}\right)>N\right)$$

For large distances, the relative difference between the effective number of primes to the value of the prime-number-formula, respectively between its best estimate value to the value of the PNF is approaching zero. The dispersion does not influence this fact. Therefore, in case the distance (*c*) is growing to infinity over the above limit ($N < \ln(c)$, $c \ge c_{lim}$), there are infinite numbers of gaps of the size (*N*), as stated in the lemma and concluding the proof.

The factor (K_1) in Equation (1.22) has a value close to (2) for all gaps being equal to the product of two primes, as evaluated in (A2.2). If the value of the gaps has more prime dividends as two, the factor is greater, as shown in **Figure A2.2**.

The total number of the gaps over the distance up to (*c*) is evaluated in (1.22) by summing up the number of the gaps within the sections of the size ($\sqrt{c}$). The evolution of the low limit of the numbers of the gaps within the sections—relative to their values at the largest distance considered in the analysis—is illustrated in **Figure A2.4**. Obviously, this numbers have a decreasing tendency only for the twins ($N = 2$) and have rising tendency for all greater gaps. This, because the number of twins within the sections, relative to the final value is decreases with the density of the primes. The same relative number for gaps of rising sizes occurs first for sufficiently large sizes of the sections of the size ($\sqrt{c}$) and is growing with the size of the sections. For larger gaps this second effect is dominant.

3. Conclusions

With the infinity of the best estimate of the sum of the number of diads (1.6), proved in ref. [2], the first Hardy—Littlewood conjecture is proved. The connection from the number of diads to the gaps between primes by the introduction of the density of no-primes is new in the present paper. As proved, the low limit of the number of gaps is the number of diads, therefore the number of gaps of any size is growing to infinity with the distance.

The definition and the use of the double density of occupation of the set of integer positions by the union of the straight and reflected series of multiples primes are given in ref. [1], to prove the infinity of the number of gaps of any size, between primes is new and conclusive.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Doroszlai, P. and Keller, H. (2022) The Symmetric Series of Multiples of Primes. *Advances in Pure Mathematics*, **12**, 160-177. https://doi.org/10.4236/apm.2022.123014

[2] Doroszlai, P. and Keller, H. (2022) The Number of Primes. *Advances in Pure Mathematics*, **12**, 81-95. https://doi.org/10.4236/apm.2022.122008

[3] Zhang, Y.T. (2014). Bounded Gaps between Primes. *Annals of Mathematics*, **179**,

1121-1174. https://doi.org/10.4007/annals.2014.179.3.7

[4] Zhang, Y.T. (1019) Closing the Gap between Prime Numbers. Irish Times, Arithmetic, Number Theory.

[5] Green, B. (2014) Bounded Gaps between Primes. arXiv:1402.4849.

[6] Vega, F. (2022) The Smallest Gap between Primes. https://www.academia.edu/90709256/

Annexes

Annex 0. General Data, Vectors, and Functions

In the following annexes all above formula are checked with numeric examples. The set of primes and the listed known formula below are used for the checking. Some vectors of the results of the known formula, which are often used, are evaluated and the results are saved.

The set of primes is read from a file: ($P = \text{READPRN}(\text{"Primes\_large.prn"})$). The number of primes in the set and their numbering are:
($N_P = rows(P) - 1 = 5003713$, $n = 1, 2, \cdots, N_P$).

For the evaluation of the number of free positions up to the distance (c) the routine ($n_{next}(c, n_{last})$) resulting the index (n) of the prime next to any integer is needed ($P_{(n)} \le c < P_{(n+1)}$). The evaluation starts either at the last evaluated index (n_{last}), or at the index resulting from the prime-number-formula. This, to shorten some of the evaluation processes. Further functions are the indexes of the closest primes to any distance, and to the square root of any distance:

```
n_next(c, n_last) := | if c ≠ 0
                     |   | n ← floor(c / ln(c))  if n_last = 1
                     |   | n ← n_last  otherwise
                     |   | while P_(n) ≤ c
                     |   |   n ← n + 1
                     |   | Res ← n − 1
                     | Res ← 0  otherwise
```

$$R_{eff}(c) = n_{next}\left(\sqrt{c}, 1\right)$$
$$S_{eff}(c) = n_{next}(c, 1) \quad \text{(A0.1)}$$

In the following all functions, which are evaluated for illustration, are evaluated at sparse distances, equal to multiples of the square root of the largest distance considered ($c_{sp(k)} = k \cdot \sqrt{P_{(N_P)}}$), resp. at the next smaller prime ($P_{\left(n_{sp(k)}\right)} < c_{sp(k)} < P_{\left(n_{sp(k+1)}\right)}$):

$$\Delta c_{sp} = \sqrt{P_{(N_P)}};\ k_{lim} = \frac{P_{(N_P)}}{\Delta c_{sp}} - 1;\ k = 1, 2, \cdots, k_{lim};$$
$$c_{sp(k)} = k \cdot \Delta c_{sp};\ c_{sp(0)} = 2 \quad \text{(A0.2)}$$

The vector of the indexes of the next smaller primes to these sparse distances are evaluated once and written to a file. They are read from this file:

$$n_{sp(k)} = s_{eff}\left(c_{sp(k)}\right) \quad \text{(A0.3)}$$

They are evaluated once and written to files. They are read from these files:

$$\text{WRITEPRN}(\text{"index\_sparse\_primes.prn"}) = n_{sp};$$

$$P = \text{READPRN}(\text{"index\_sparse\_primes.prn"})$$

Annex 1. The Approximation of Twin Primes and Their Dispersion

The components of twin primes are always adjacent to multiples of (6). Primes with gaps equal to (4) are always at the distance (2) away from multiples of (6) plus (3). These gaps between primes are always free of primes as well. The total number of all these gaps has well defined best estimate functions, with well defined lower limit functions. The infinity of the number of diads—analyzed in reference [1]—proves the infinity of twin primes, as shortly repeated in the following.

The evaluation procedure of the best estimate number of twin primes is given below.

$$\pi_{gap\_appr\_2}(c) = \delta_2 \cdot \sqrt{c} \cdot \sum_{j=1}^{\sqrt{c}} \frac{1}{\ln\left(j \cdot \sqrt{c}\right)^2} \tag{A1.1}$$

The vector of the effective number of twin primes is evaluated elsewhere and the results are written to a file. They are read from this file:

$$\pi_{gap\_eff\_2\_sp} = \text{READPRN}(\text{"k\_TW\_eff\_sp.prn"});$$

$$kk_{lim} = length\left(\pi_{gap\_eff\_2\_sp}\right) - 1 = 9277 \tag{A1.2}$$

$$kk = 2, \cdots, kk_{lim}; \quad kk_{lim} = 9277; \quad c_{sp(kk)} = kk \cdot \Delta c_{sp}: \quad c_{sp(0)} = 2$$

The vector of the best estimating, approximating number of the twin primes is evaluated at sparse distances with (A1.1) and the results are written to a file. They are read from this file:

$$\pi_{gap\_appr\_2\_sp(kk)} = \pi_{gap\_appr\_2\_}\left(c_{sp(kk)}\right) \tag{A1.3}$$

$$\text{WRITEPRN}(\text{"tw\_appr\_sp.prn"}) = \pi_{gap\_appr\_2\_sp};$$

$$\pi_{gap\_appr\_2\_sp} = \text{READPRN}(\text{"tw\_appr\_sp.prn"})$$

The difference between the effective number of twins and their approximating function is:

$$\Delta TW_{sp(kk)} = \pi_{gap\_eff\_2\_sp(kk)} - \pi_{gap\_appr\_2\_sp(kk)};$$

$$length\left(\pi_{gap\_appr\_2\_sp}\right) - 1 = 9277 \tag{A1.4}$$

The relative dispersion around the approximating function ($TW_{appr}(c)$)—starting with ($kk_{start} = 6000$)—are illustrated in the figures below. The start value is chosen, to avoid the larger initial deviation in case of smaller distances.

$$\Delta TW_{rel\_TW} = \frac{\Delta TW(c)}{TW_{appr}(c)}; \quad \Delta TW_{rel\_TW\_sp(kk)} = \frac{\Delta TW_{sp(kk)}}{\pi_{gap\_appr\_2\_sp(kk)}} \tag{A1.5}$$

The standard deviation of the relative dispersion of the number of twin primes around its approximating function and around the approximating number of twins ($TW_{appr}(c)$) are in the range ($kk = kk_{start} + 1, \cdots, kk_{lim-1}$):

$$SD_{\Delta TW\_rel\_TW(kk)} = \sqrt{\frac{1}{kk - kk_{start}} \cdot \sum_{j=kk_{start}}^{kk} \left(\Delta TW_{rel\_TW\_sp(j)}\right)^2} \tag{A1.6}$$

The factor of proportionality for the standard deviation of the twin primes is approaching a constant value:

$$F_{SD\_\Delta TW\_TW} = SD_{\Delta TW\_rel\_TW_{(kk_{lim}-1)}} = 0.0001702 \quad \text{(A1.7)}$$

$$\Delta SD_{\Delta TW\_rel\_F\_SD_{(kk)}} = SD_{\Delta TW\_rel\_TW_{(kk)}} - F_{SD\_\Delta TW\_TW}$$

The standard deviation of the dispersion of the number of twins is therefore about 0.02% of the value of the approximating function (**Figure A1.1**).

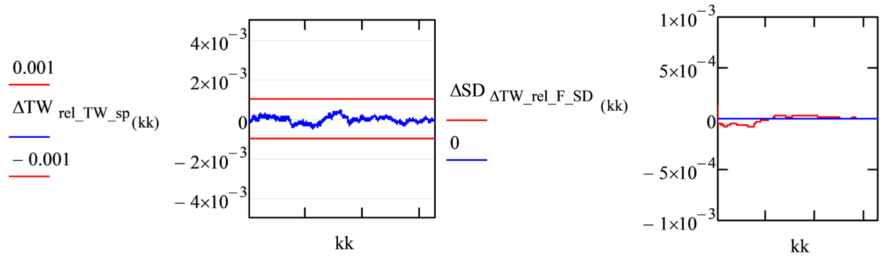

Figure A1.1. Dispersion of the effective number of twins around its approximating function, its limiting boundaries at sparse distances and its standard deviation.

Annex 2. Analysis of the Prime Gaps of a Given Size

The effective number of gaps of the sizes ($N_{(s)}$, $s_{lim} := 14$, $s := 1, \cdots, s_{lim}$) are evaluated for the first set—defined in (1.12)—on the condition, that the difference between consecutive primes is equal to ($N_{(s)} = (s-1) \cdot 12 + 1$). The best estimate approximating number of the gaps of the same sizes is evaluated for the same set with (1.22) with the following formula and the programs in **Figure A2.1**:

```
π_gap_eff_sp := | s ← 1
                | while s ≤ s_lim
                |   | k ← 1
                |   | SS(k,s) ← 0
                |   | while k ≤ k_lim
                |   |   | n ← n_sp(k−1) + 1
                |   |   | n_gap_k ← 0
                |   |   | S ← 0
                |   |   | n_lim ← n_sp(k)
                |   |   | while n ≤ n_lim
                |   |   |   | if P(n+1) = [P(n) + N(s)]
                |   |   |   |   | n_gap_k ← n_gap_k + 1
                |   |   |   |   | S ← n_gap_k
                |   |   |   | n ← n + 1
                |   |   | SS(k,s) ← SS(k−1,s) + S
                |   |   | k ← k + 1
                |   | s ← s + 1
                | SS

π_gap_appr_sp := | s ← 1
                 | while s ≤ s_lim
                 |   | k ← 1
                 |   | while k ≤ k_lim
                 |   |   | j ← 1
                 |   |   | S ← 0
                 |   |   | sqr ← √(c_sp(k))
                 |   |   | j_lim ← floor(sqr)
                 |   |   | while j ≤ j_lim
                 |   |   |   | S ← S + F2(k,j,s) / ln(j·sqr)^2
                 |   |   |   | j ← j + 1
                 |   |   | SS(k,s) ← S·sqr·δ2
                 |   |   | k ← k + 1
                 |   | s ← s + 1
                 | SS
```

Figure A2.1. Routines for the evaluation of the effective number of gaps and of its approximation.

$$F_{section}(k,j,s) := \left[1 - \frac{1}{\ln\left[j \cdot \sqrt{c_{sp_{(k)}}}\right]}\right]^{K_2 \cdot \left[N_{(s)}\right]} \quad \text{(A2.1)}$$

$$\pi_{gap\_appr}(k,s) := \delta_2 \cdot \sqrt{c_{sp_{(k)}}} \cdot \sum_{j=1}^{floor\left[\sqrt{c_{sp(k)}}\right]} \left[\frac{F_{section}(k,j,s)}{\ln\left[j \cdot \sqrt{c_{sp_{(k)}}}\right]^2}\right]$$

The time-consuming evaluation is carried out once and the results are written to files to be read at each evaluation of the present paper:

$$\text{WRITEPRN}(\text{"total\_gap\_eff\_sp.prn"}) := \pi_{gap\_eff\_sp}$$

$$\pi_{gap\_eff\_sp} := \text{READPRN}(\text{"total\_gap\_eff\_sp.prn"})$$

$$\text{WRITEPRN}(\text{"total\_gap\_appr\_sp.prn"}) := \pi_{gap\_appr\_sp}$$

$$\pi_{gap\_appr\_sp} := \text{READPRN}(\text{"total\_gap\_appr\_sp.prn"})$$

The factors of multiplication (K_1) adjusting the best estimate values of the approximation to the effective values of the number of the gaps (A2.1) are evaluated and written to a file. They are read from this file and illustrated in the **Figure A2.2** ($K_1 := \text{READPRN}(\text{"factor\_K1.prn"})$), ($s := 1,\cdots,length(K_1)-1$).

$K_1^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1.64	1.64	1.72	2.2	1.64	2	1.78	2.25	2.9	2.1	1.8

$$s = 1,\cdots,length(K_1); \quad N\_(s) = 12\cdot(s-1)+2$$

$$N_{(s)} = N\_(s); \quad K_{1\_appr_{()s}} = 1.9; \quad K_{1\_appr} = 1.9 \pm 0.25 \quad \text{(A2.2)}$$

The best estimate number of the gaps multiplied with these above factors and the comparison with the effective number are shown below in **Figure A2.3**:

$$\pi_{gap\_appr\_sp\_(k,s)} = K_{1_{(s)}} \cdot \pi_{gap\_appr\_sp_{(k,s)}} \quad \text{(A2.3)}$$

The number of the gaps within the sections of the length ($\sqrt{c}$) and their lo limits are with (A2.1):

$$F_{section\_}(c,j,s) = \frac{\left(1 - \frac{1}{\ln\left(j\cdot\sqrt{c}\right)}\right)^{\frac{5}{4}\cdot\left(N_{(s)}-1\right)}}{\ln\left(j\cdot\sqrt{c}\right)^2}$$

$$> F_{section\_}(c,j,s) = \frac{\left(1 - \frac{1}{\ln\left(\sqrt{c}\right)}\right)^{\frac{5}{4}\cdot\left(N_{(s)}-1\right)}}{\ln(c)^2} \quad \text{(A2.4)}$$

The number of the gaps within the sections, relative to their value at the largest distance ($c_{sp_{(klim)}}$) considered in the present paper, are illustrated in **Figure A2.4** for the first few sets:

$$F_{rel}(k,s)=\frac{F_{section\_low}\left(c_{sp_{(k)}},s\right)}{F_{section\_low}\left(c_{sp_{(k_{lim})}},s\right)} \tag{A2.5}$$

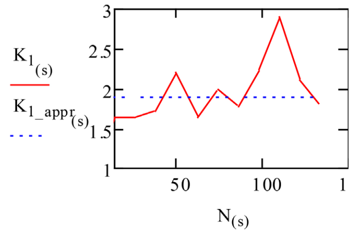

Figure A2.2. The factor of multiplication of the number of gaps, over the size of the gaps.

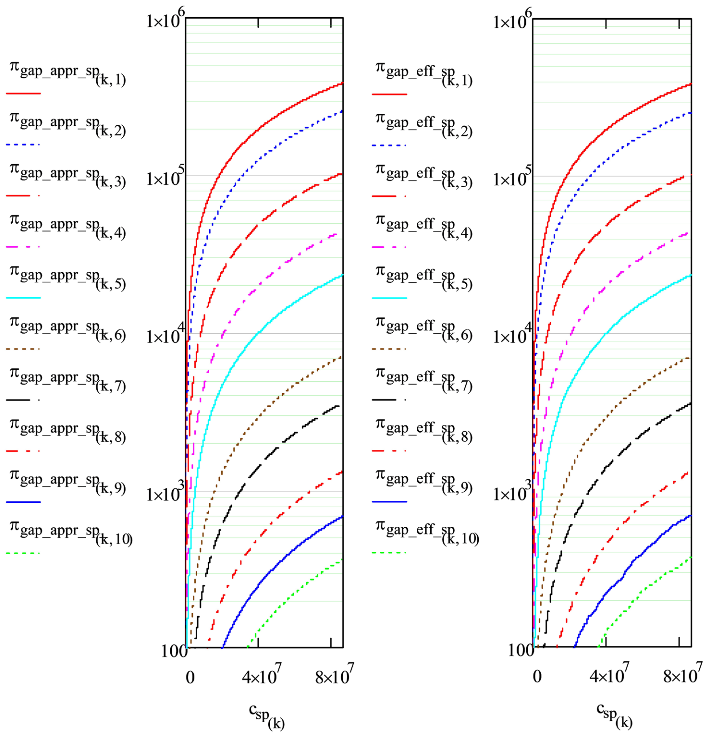

Figure A2.3. The approximating and the effective number of the gaps of varied sizes.

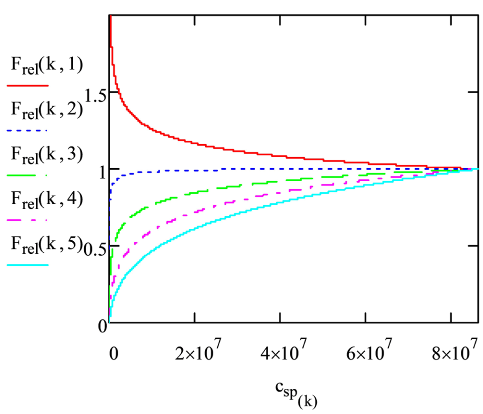

Figure A2.4. The number of gaps within the sections, relative to their final value.