Advances in Pure Mathematics, 2022, 12, 308-331
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Exponential Function as Split Infinite Product

Pál Doroszlai, Horacio Keller

Independent Researcher, Kékkut, Hungary
Email: paldoroszlai@yahoo.com

How to cite this paper: Doroszlai, P. and Keller, H. (2022) The Exponential Function as Split Infinite Product. *Advances in Pure Mathematics*, **12**, 308-331.
https://doi.org/10.4236/apm.2022.124024

Received: March 8, 2022
Accepted: April 19, 2022
Published: April 22, 2022

Abstract

It is shown that any polynomial written as an infinite product with all positive real roots may be split in two steps into the product of four infinite polynomials: two with all imaginary and two with all real roots. Equations between such infinite products define adjoint infinite polynomials with roots on the adjoint roots (real and imaginary). It is shown that the shifting of the coordinates to a parallel line of one of the adjoint axes does not influence the relative placement of the roots: they are shifted to the parallel line. General relations between original and adjoint polynomials are evaluated. These relations are generalized representations of the relations of Euler and Pythagoras in form of infinite polynomial products. They are inherent properties of split polynomial products. If the shifting of the coordinate system corresponds to the shifting of the imaginary axes to the critical line, then the relations of Euler take the form corresponding to their occurrence in the functional equation of the Riemann zeta function: the roots on the imaginary axes are all shifted to the critical line. Since it is known that the gamma and the zeta functions may be written as composed functions with exponential and trigonometric parts, this opens the possibility to prove the placement of the zeta function on the critical line.

Keywords

Infinite Polynomials, Shifting to the Critical Line, Zeta Function

1. Introduction

One of the open questions of number theory is the placement of the roots of the Riemann's zeta function. To analyze this problem, first, the placement of the roots of other analytical functions must be analyzed. One option to start this analysis offers the infinite polynomial products, because the placement of their roots is well defined. For infinite polynomial products, first, the variable splitting is carried out. The relations between such split products result in the adjoint polynomial functions. Shifting the roots on one of the coordinate axes of one split polynomial product, the roots of the adjoint functions on the other axes are shifted to a parallel line.

Once the laws of the relations between infinite polynomial products are defined, the approximation of basic functions by the corresponding polynomials is established. First the most important function, the exponential function and its adjoint functions, the trigonometric functions are represented by infinite polynomial products. Starting with the binomial coefficients the polynomial representation of the normal distribution and of the exponential function are established. The adjoint functions of the exponential functions give the infinite polynomial product representations of the trigonometric functions.

Shifting the roots on the real axes of the point of symmetry to the critical line, shifts the roots of the adjoint functions to the critical line. Equations between these shifted functions have the identical form to components of the functional equation of the Riemann zeta function. If all components are written in form of infinite polynomial products, then the functional equation defines all complex roots and poles on the critical line.

2. Splitting a Polynomial with all Positive Roots

Is (ζ) a complex number, so any function ($f(\zeta)$) is a complex number again. The sum of such a complex number and of its conjugate, as well as the difference of the complex number and its transpose result in a real number ($f(\zeta)-\overline{f(\zeta)}=f(\zeta)-f(\zeta)=\text{real}$). The sum of such a complex number and of its transpose, as well as the difference of the complex number and its conjugate result in an imaginary number ($f(\zeta)+f(\zeta)^{\mathrm{T}}=f(\zeta)-\overline{f(\zeta)}=\text{imaginary}$). These obvious relations will be applied to polynomial functions.

Definition of the complete polynomial function:

The positive real numbers ($a_{0_{(j)}}>0$, $a_{0_{(j1)}}\neq a_{0_{(j2)}}$, $a_{0_{(j+1)}}>a_{0_{(j)}}$) with ($j,j_1,j_2\in\mathbb{Z}$, $\mathbb{Z}=1,2,\cdots,\infty$) are composing the vector (A_0). The set of the square roots ($a_{1_{(j)}}=\sqrt{a_{0_{(j)}}}$) of these numbers is composing the vector (A_1). The set of the square root of these last numbers ($a_{2_{(j)}}=\sqrt{a_{1_{(j)}}}$) is composing the vector (A_2). The following infinite product ($P(\zeta_0,A_0)$) is the **complete polynomial function** with the complex variables (ζ_0). With all positive real roots at ($a_{0_{(j)}}$) in the normed form it is defined as follows:

$$P(\zeta_0,A_0)=\prod_{j=1}^{\infty}\left(1-\frac{\zeta_0}{a_{0_{(j)}}}\right) \tag{2.1}$$

Definition of the first-degree split polynomials:

Splitting the complex variable (ζ_0) by the substitution ($\zeta_1=\sqrt{\zeta_0}$) the polynomial will be split:

$$P(\zeta_0, A_0) = \prod_{j=1}^{\infty}\left(1-\frac{\zeta_1}{a_{1_{(j)}}}\right)\cdot\prod_{j=1}^{\infty}\left(1+\frac{\zeta_1}{a_{1_{(j)}}}\right)$$
$$= P_p(\zeta_1, A_1)\cdot P_n(\zeta_1, A_1) \quad \left(a_{1_{(j)}} = \sqrt{a_{0_{(j)}}}\right) \tag{2.2}$$

The first of these **first-degree split polynomials**, ($P_p(\zeta_1, A_1)$) has all positive real roots at ($a_{1_{(j)}}$) and is like the complete polynomial ($P(\zeta_0, A_0)$), the other ($P_n(\zeta_1, A_1)$) has all negative real roots at ($-a_{1_{(j)}}$):

$$P_p(\zeta_1, A_1) = \prod_{j=1}^{\infty}\left(1-\frac{\zeta_1}{a_{1_{(j)}}}\right), \quad P_n(\zeta_1, A_1) = \prod_{j=1}^{\infty}\left(1+\frac{\zeta_1}{a_{1_{(j)}}}\right) \tag{2.3}$$

Definition of the second-degree split polynomials:

Continuing the splitting by the variable substitution ($\zeta_2 = \sqrt{\zeta_1}$) of the first-degree split polynomials ($P_p(\zeta_1, A_1)$) and ($P_n(\zeta_1, A_1)$), they will be split into two components resulting in the **second-degree split polynomials**:

$$P(\zeta_0, A_0) = P_p(\zeta_1, A_1)\cdot P_n(\zeta_1, A_1)$$
$$= P_{p\_p}(\zeta_2, A_2)\cdot P_{p\_n}(\zeta_2, A_2)\cdot P_{n\_p}(\zeta_2, i\cdot A_2)\cdot P_{n\_n}(\zeta_2, i\cdot A_2) \tag{2.4}$$

The splitting of the first-degree split polynomial ($P_p(\zeta_1, A_1)$) is like the splitting of the original polynomial ($P(\zeta_0, A_0)$), since the first-degree split polynomials have exclusively positive real roots ($a_{1_{(j)}} > 0$):

$$P_p(\zeta_1, A_1) = \prod_{j=1}^{\infty}\left(1-\frac{\zeta_2}{a_{2_{(j)}}}\right)\cdot\prod_{j=1}^{\infty}\left(1+\frac{\zeta_2}{a_{2_{(j)}}}\right)$$
$$= P_{p\_p}(\zeta_2, A_2)\cdot P_{p\_n}(\zeta_2, A_2) \quad \left(a_{2_{(j)}} = \sqrt{a_{1_{(j)}}}\right) \tag{2.5}$$

The first one of the second-degree split polynomials ($P_{p\_p}(\zeta_2, A_2)$) has all positive real roots at ($a_{2_{(j)}}$) and is similar again to the original polynomial ($P(\zeta_0, A_0)$), the other ($P_{p\_n}(\zeta_2, A_2)$) has all negative real roots at ($-a_{2_{(j)}}$):

$$P_{p\_p}(\zeta_2, A_2) = \prod_{j=1}^{\infty}\left(1-\frac{\zeta_2}{a_{2_{(j)}}}\right), \quad P_{p\_n}(\zeta_2, A_2) = \prod_{j=1}^{\infty}\left(1+\frac{\zeta_2}{a_{2_{(j)}}}\right) \tag{2.6}$$

The splitting of the other first degree split polynomial ($P_n(\zeta_1, A_1)$) with all negative real roots is different: It splits into two second degree split polynomials each of them with all roots on the imaginary axes:

$$P_n(\zeta_1, A_1) = \prod_{j=1}^{\infty}\left(1-\frac{\zeta_2}{i\cdot a_{2_{(j)}}}\right)\cdot\prod_{j=1}^{\infty}\left(1+\frac{\zeta_2}{i\cdot a_{2_{(j)}}}\right)$$
$$= P_{n\_p}(\zeta_2, i\cdot A_2)\cdot P_{n\_n}(\zeta_2, i\cdot A_2) \tag{2.7}$$

The first one of the second-degree split polynomials ($P_{n\_p}(\zeta_2, i\cdot A_2)$) has all positive imaginary roots at ($i\cdot a_{2_{(j)}}$), the other, ($P_{n\_n}(\zeta_2, i\cdot A_2)$) has all negative imaginary roots at ($-i\cdot a_{2_{(j)}}$):

$$P_{n\_p}(\zeta_2, i\cdot A_2) = \prod_{j=1}^{\infty}\left(1-\frac{\zeta_2}{i\cdot a_{2_{(j)}}}\right), \quad P_{n\_n}(\zeta_2, i\cdot A_2) = \prod_{j=1}^{\infty}\left(1+\frac{\zeta_2}{i\cdot a_{2_{(j)}}}\right) \tag{2.8}$$

Definition of the adjoint axes:

This splitting of the negative numbers renders the introduction of the imaginary and of the complex numbers necessary, to increase the degree of liberty. The two independent axes, the real and the imaginary are defined as **adjoint axes**.

From these above relations result per definition the following relations between the first-degree split polynomials:

$$\begin{aligned} P_p(\zeta_1, A_1) &= P_{p\_p}(\zeta_2, A_2) \cdot P_{p\_n}(\zeta_2, A_2) \\ P_n(\zeta_1, A_1) &= P_{n\_p}(\zeta_2, i \cdot A_2) \cdot P_{n\_n}(\zeta_2, i \cdot A_2) \end{aligned} \tag{2.9}$$

The complete polynomial ($P(\zeta_0, A_0)$) after the second-degree split (2.9) has symmetrically placed infinite number of roots on the real and on the imaginary axes. Between the two real roots next to zero there are only imaginary

$$\max\left[-a_{2_{(j)}}\right] < \operatorname{Re}(\zeta_2) < \min\left[a_{2_{(j)}}\right]$$

$$\text{at:} \quad \zeta_2 = i \cdot a_{2_{(j)}}, \quad \zeta_2 = -i \cdot a_{2_{(j)}}, \quad j = 1, 2, \cdots, \infty \tag{2.10}$$

From the definitions follow the symmetry relations: If a polynomial ($P(\zeta_0, A_0)$) may be written in the form (2.1) with all roots positive and real, then it may be split after the substitution ($\zeta_1 = \sqrt{\zeta_0}$) into two first degree split polynomials, one ($P_p(\zeta_1, A_1)$) having only positive, the other ($P_n(\zeta_1, A_1)$) having only negative real roots and each of them is the transpose function of the other:

$$P_p(\zeta_1, A_1) = P_n(\zeta_1, A_1)^{\mathrm{T}} \tag{2.11}$$

Definition of the polynomials bound by their roots:

Similarly, the polynomial ($P_p(\zeta_1, A_1)$) having only positive real roots may be split after the substitution ($\zeta_2 = \sqrt{\zeta_1}$) into two second degree split polynomials one ($P_{p\_p}(\zeta_2, A_2)$) having only positive, the other ($P_{p\_n}(\zeta_2, A_2)$) having only negative real roots of the same set of real numbers and each of them is the transpose function of the other:

$$P_{p\_p}(\zeta_2, A_2) = P_{p\_n}(\zeta_2, A_2)^{\mathrm{T}} \tag{2.12}$$

This pair of second-degree split polynomials has roots on the same axes: they are **bound by their roots** being on the same axes, on the real axes and having the same absolute values.

Taking the polynomial ($P_n(\zeta_1, A_1)$) with all negative real roots, after the substitution ($\zeta_2 = \sqrt{\zeta_1}$) it may be split once more into two second degree split polynomials like (2.8), both having all roots on the imaginary axes. The two second degree split polynomials are symmetrical over the real axes, meaning they are conjugate complex:

$$P_{n\_p}(\zeta_2, i \cdot A_2) = \overline{P_{n\_n}(\zeta_2, i \cdot A_2)} \tag{2.13}$$

This pair of second-degree split polynomials has roots on the same axes: they are **bound by their roots** being on the same axes, on the imaginary axes and having the same absolute values.

Taking the roots of the first-degree split polynomials (2.10) on the imaginary, instead on the real axes, results polynomials formally identical to the polynomials (2.13). Therefore, they are conjugate complex as well:

$$P_p(\zeta_2, i\cdot A_2) = \overline{P_n(\zeta_2, i\cdot A_2)} \tag{2.14}$$

3. Adjoint Polynomials and Equations for Their Roots

Definition of the adjoint polynomials:

The first-degree split polynomials ($R_n(\zeta_2, i\cdot D_2)$), ($Q_n(\zeta_2, i\cdot C_2)$) with roots on the imaginary axes result as sum and difference of the second-degree split polynomials ($P_{p\_p}(\zeta_2, A_2)$), ($P_{p\_n}(\zeta_2, A_2)$) with roots on the adjoint, on the real axes. The second-degree split polynomials ($P_{n\_p}(\zeta_2, A_2)$), ($P_{n\_n}(\zeta_2, A_2)$) with roots on the imaginary axes result as sum and difference of the first-degree split polynomials ($R_p(\zeta_2, D_2)$), ($Q_p(\zeta_2, C_2)$) with roots on the adjoint, on the real, axes. They are mutually **adjoint split polynomials** to the original split polynomials.

Taking the second-degree split polynomials with all real roots (2.6) for any (ζ_2) their sum results in the polynomials ($Q_n(\zeta_2, i\cdot C_1)$) and their difference the polynomials ($R_n(\zeta_2, i\cdot D_1)$), all with imaginary roots:

$$\begin{aligned} &P_{p\_p}(\zeta_2, A_2) + P_{p\_n}(\zeta_2, A_2) \\ &= 2\cdot Q_{n\_p}(\zeta_2, i\cdot C_2)\cdot Q_{n\_n}(\zeta_2, i\cdot C_2) = 2\cdot Q_n(\zeta_2, i\cdot C_1) \end{aligned} \tag{3.1}$$

$$\begin{aligned} &P_{p\_p}(\zeta_2, A_2) - P_{p\_n}(\zeta_2, A_2) \\ &= 2\cdot R_{n\_p}(\zeta_2, i\cdot D_2)\cdot R_{n\_n}(\zeta_2, i\cdot D_2) = 2\cdot R_n(\zeta_2, i\cdot D_1) \end{aligned} \tag{3.2}$$

Taking the second-degree split polynomials with all imaginary roots (2.8), for any (ζ_2) their sum results the ($Q_p(\zeta_2, C_1)$) and their differences the polynomials ($R_p(\zeta_2, D_1)$) all with real roots:

$$\begin{aligned} &P_{n\_p}(\zeta_2, i\cdot A_2) + P_{n\_n}(\zeta_2, i\cdot A_2) \\ &= 2\cdot Q_{p\_p}(\zeta_2, C_2)\cdot Q_{p\_n}(\zeta_2, C_2) = 2\cdot Q_p(\zeta_2, C_1) \end{aligned} \tag{3.3}$$

$$\begin{aligned} &P_{n\_p}(\zeta_2, i\cdot A_2) - P_{n\_n}(\zeta_2, i\cdot A_2) \\ &= 2\cdot R_{p\_p}(\zeta_2, D_2)\cdot R_{p\_n}(\zeta_2, D_2) = 2\cdot i\cdot R_p(\zeta_2, D_1) \end{aligned} \tag{3.4}$$

The above definitions allow to formulate the following lemma:

Lemma 3.1:

Polynomials composed as the sum, or the difference of second-degree split polynomials bound by their roots define their adjoint polynomials with all roots on the adjoint axes.

Proof: Setting the sum or the difference of split polynomials equal to zero result in the equations above for the roots of the adjoint polynomials roots on the adjoint axes.

$$P_{n\_p}(\zeta_2, i\cdot A_2) + P_{n\_n}(\zeta_2, i\cdot A_2) = 2\cdot Q_p(\zeta_2, C_1)$$
$$P_{p\_p}(\zeta_2, A_2) + P_{p\_n}(\zeta_2, A_2) = 2\cdot i\cdot R_P(\zeta_2, i\cdot C_1)$$

$$P_{n\_p}(\zeta_2, i\cdot A_2) - P_{n\_n}(\zeta_2, i\cdot A_2) = 2\cdot i\cdot R_p(\zeta_2, D_1)$$
$$P_{p\_p}(\zeta_2, A_2) - P_{p\_n}(\zeta_2, A_2) = 2\cdot R_n(\zeta_2, i\cdot D_1)$$

These equations may be regarded as the **generalized relations of Euler**. This is because when applied to the split polynomials of the exponential and trigonometric functions, they correspond to the relations of Euler. They define roots on one of the adjoint axes for the adjoint polynomials and are the consequences of the symmetry conditions (2.11) and (2.12), **concluding the proof**.

Therefore, **the relations of Euler are inheriting properties of the split polynomials**.

Multiplying formally all components of the infinite product (2.8) results in the second-degree split polynomial and for its conjugate the following polynomials:

$$\begin{aligned} P_{n\_n}(\zeta_2, i\cdot A_2) &= 1 - k_1\cdot i\cdot\zeta_2 - k_2\cdot\zeta_2^2 + k_3\cdot i\cdot\zeta_2^3 + k_4\cdot\zeta_2^4 \\ &\quad - k_5\cdot i\cdot\zeta_2^5 - k_6\cdot\zeta_2^6 + \cdots \\ P_{n\_p}(\zeta_2, i\cdot A_2) &= 1 + k_1\cdot i\cdot\zeta_2 - k_2\cdot\zeta_2^2 - k_3\cdot i\cdot\zeta_2^3 + k_4\cdot\zeta_2^4 \\ &\quad + k_5\cdot i\cdot\zeta_2^5 - k_6\cdot\zeta_2^6 + \cdots \end{aligned} \tag{3.5}$$

The sum of the above two series is again a polynomial, an even function and the difference is a polynomial, an odd function, with roots on the real axes, in accordance with (3.3) and (3.4):

$$\begin{aligned} &P_{n\_p}(i\cdot\tau, i\cdot A_2) + P_{n\_n}(i\cdot\tau, i, i\cdot A_2) \\ &= 2\cdot\left(1 - k_2\cdot(i\cdot\tau)^2 + k_4\cdot(i\cdot\tau)^4 - k_6\cdot(i\cdot\tau)^6 + \cdots\right) \\ &= 2\cdot\left(1 + k_2\cdot\sigma^2 + k_4\cdot\sigma^4 - k_6\cdot\sigma^6 + \cdots\right) \\ &= 2\cdot\prod_{j=1}^{\infty}\left[1 - \frac{\sigma^2}{c_{1(j)}}\right] = 2\cdot Q_p(\zeta_2, C_1) \end{aligned}$$

$$\begin{aligned} &P_{n\_p}(\zeta_2, i\cdot A_2) - P_{n\_n}(\zeta_2, i\cdot A_2) \\ &= 2\cdot\left(k_1\cdot i\cdot(i\cdot\tau) - k_3\cdot i\cdot(i\cdot\tau)^3 + k_5\cdot i\cdot(i\cdot\tau)^5 - k_7\cdot i\cdot(i\cdot\tau)^7 + \cdots\right) \\ &= 2\cdot i\cdot k_1\cdot\sigma\cdot\left(1 - \frac{k_3}{k_1}\cdot\sigma^2 + \frac{k_5}{k_1}\cdot\sigma^4 - \frac{k_7}{k_1}\cdot\sigma^6 + \cdots\right) \\ &= \frac{2\cdot i\cdot\sigma}{d_{1(0)}}\cdot\prod_{j=1}^{\infty}\left[1 - \frac{\sigma^2}{d_{1(j)}}\right] = 2\cdot i\cdot R_p(\sigma, D_1) \end{aligned} \tag{3.6}$$

Multiplying formally all components of the infinite product (2.6) results in the second-degree split polynomial and for its transpose the following polynomials:

$$P_{p\_p}(\zeta_2, A_2) = 1 - k_1\cdot\zeta_2 + k_2\cdot\zeta_2^2 - k_3\cdot\zeta_2^3 + k_4\cdot\zeta_2^4 - k_5\cdot\zeta_2^5 + k_6\cdot\zeta_2^6 + \cdots \tag{3.7}$$
$$P_{p\_n}(\zeta_2, A_2) = 1 + k_1\cdot\zeta_2 + k_2\cdot\zeta_2^2 + k_3\cdot\zeta_2^3 + k_4\cdot\zeta_2^4 + k_5\cdot\zeta_2^5 + k_6\cdot\zeta_2^6 + \cdots$$

The sum of the above two series is again a polynomial an even function with roots on the imaginary axes, the difference is an odd function with real roots, both in accordance with (3.1) and (3.2):

$$P_{p\_p}\left(\zeta_2, A_2\right) + P_{p\_n}\left(\zeta_2, A_2\right) = 2\cdot\left(1 + k_2\cdot\zeta_2^2 + k_4\cdot\zeta_2^4 + k_6\cdot\zeta_2^6 + \cdots\right)$$
$$= 2\cdot\prod_{j=1}^{\infty}\left[1 - \left(\frac{\zeta_2}{i\cdot c_{1_{(j)}}}\right)^2\right] = 2\cdot Q_n\left(\zeta_2, i\cdot C_1\right)$$
$$P_{p\_p}\left(\zeta_2, A_2\right) - P_{p\_n}\left(\zeta_2, A_2\right) = -2\cdot\left(k_1\cdot\zeta_2 + k_3\cdot\zeta_2^3 + k_5\cdot\zeta_2^5 + k_7\cdot\zeta_2^7 + \cdots\right)$$
$$= -2\cdot k_1\cdot\zeta_2\cdot\left(1 + \frac{k_3}{k_1}\cdot\zeta_2^2 + \frac{k_5}{k_1}\cdot\zeta_2^4 + \frac{k_7}{k_1}\cdot\zeta_2^6 + \cdots\right) \quad (3.8)$$
$$= 2\cdot\frac{-\zeta_2}{d_{1_{(0)}}}\cdot\prod_{j=1}^{\infty}\left[1 - \left(\frac{\zeta_2}{i\cdot d_{1_{(j)}}}\right)^2\right] = 2\cdot R_n\left(\zeta_2, i\cdot D_1\right)$$

All polynomials ($Q_n\left(\zeta_2, i\cdot C_1\right)$), ($Q_p\left(\zeta_2, C_1\right)$), ($R_n\left(\zeta_2, i\cdot D_1\right)$) resp. ($R_p\left(\zeta_2, D_1\right)$) have an infinite number of roots (C_1), respectively (D_1), symmetrically distributed around zero on the adjoint axes. The polynomials ($R_p\left(\zeta_2, D_1\right)$) and ($R_n\left(\zeta_2, i\cdot D_1\right)$) have additional roots at zero.

Formally all four functions may be taken at any complex value (ζ): the functions ($Q_p\left(\zeta_2, C_1\right)$) and ($Q_n\left(\zeta_2, i\cdot C_1\right)$) remain in this case symmetrical over the imaginary resp. over the real axes and the functions ($R_p\left(\zeta_2, D_1\right)$) and ($R_n\left(\zeta_2, i\cdot D_1\right)$) skew symmetrical over the origin:

$$-2\cdot R_p\left(\tau, i\cdot D_1\right) = P_{n\_p}\left(\zeta, i\cdot A_2\right) - P_{n\_n}\left(\zeta, i\cdot A_2\right)$$
$$= P_{n\_n}\left(-\zeta, i\cdot A_2\right) - P_{n\_p}\left(-\zeta, i\cdot A_2\right) = 2\cdot R_p\left(-\tau, i\cdot D_1\right)$$
$$2\cdot R_n\left(\sigma, i\cdot D_1\right) = P_{p\_p}\left(\zeta, A_2\right) - P_{p\_n}\left(\zeta, A_2\right)$$
$$= P_{p\_n}\left(-\zeta, A_2\right) - P_{p\_p}\left(-\zeta, A_2\right) = -2\cdot R_n\left(-\sigma, D_1\right) \quad (3.9)$$
$$2\cdot Q_n\left(\tau, i\cdot C_1\right) = P_{p\_p}\left(\zeta, A_2\right) + P_{p\_n}\left(\zeta, A_2\right)$$
$$= P_{p\_p}\left(-\zeta, A_2\right) + P_{p\_n}\left(-\zeta, A_2\right) = 2\cdot Q_n\left(-\tau, i\cdot C_1\right)$$
$$2\cdot Q_p\left(\sigma, C_1\right) = P_{n\_p}\left(\zeta, i\cdot A_2\right) + P_{n\_n}\left(\zeta, i\cdot A_2\right)$$
$$= P_{n\_p}\left(-\zeta, i\cdot A_2\right) + P_{n\_n}\left(-\zeta, i\cdot A_2\right) = 2\cdot Q_p\left(-\sigma, C_1\right)$$

Addition and subtraction of the adjoint polynomials ($Q_p\left(\sigma, C_1\right)$) and ($R_n\left(\tau, i\cdot D_1\right)$) allows to write the following equations expressing the original split polynomials by their adjoint polynomials on the adjoint axes:

$$P_{p\_p}\left(\zeta_2, A_2\right) = \prod_{j=1}^{\infty}\left[1 + \frac{\tau^2}{c_{1_{(j)}}}\right] + \frac{\tau}{i\cdot d_{2_{(0)}}}\prod_{j=1}^{\infty}\left[1 - \frac{\tau^2}{d_{1_{(j)}}}\right] = Q_n\left(\tau, i\cdot C_1\right) + R_n\left(\tau, i\cdot D_1\right) \quad (3.10)$$
$$P_{p\_n}\left(\zeta_2, A_2\right) = \prod_{j=1}^{\infty}\left[1 + \frac{\tau^2}{c_{1_{(j)}}}\right] - \frac{\tau}{i\cdot d_{2_{(0)}}}\prod_{j=1}^{\infty}\left[1 - \frac{\tau^2}{d_{1_{(j)}}}\right] = Q_n\left(\tau, i\cdot C_1\right) - R_n\left(\tau, i\cdot D_1\right)$$

The sum and the difference of the adjoint polynomials ($Q_p\left(\sigma, C_1\right)$) and ($R_p\left(\sigma, D_1\right)$) allows to write the following equations expressing the original split polynomials by their adjoint polynomials on the adjoint axes:

$$P_{n\_p}(\zeta_2, i\cdot A_2) = \prod_{j=1}^{\infty}\left[1-\frac{\sigma^2}{c_{1_{(j)}}}\right] - \frac{i\cdot\sigma}{d_{1_{(0)}}}\prod_{j=1}^{\infty}\left[1-\frac{\sigma^2}{d_{1_{(j)}}}\right] = Q_p(\sigma, C_1) + i\cdot R_p(\sigma, D_1) \quad (3.11)$$

$$P_{n\_n}(\zeta_2, i\cdot A_2) = \prod_{j=1}^{\infty}\left[1-\frac{\sigma^2}{c_{1_{(j)}}}\right] + \frac{i\cdot\sigma}{d_{1_{(0)}}}\prod_{j=1}^{\infty}\left[1-\frac{\sigma^2}{d_{1_{(j)}}}\right] = Q_p(\sigma, C_1) - i\cdot R_p(\sigma, D_1)$$

With this the following lemma may be formulated:

Lemma 3.2:

The sum and the difference of the adjoint polynomials define the roots of the original split polynomials on the corresponding adjoint axes.

Proof: The equations (3.10) and (3.11) define roots of the original split polynomials on the adjoint axes, as stated in the lemma and **concluding the proof**.

The resulting polynomials (3.1) of the adjoint polynomials may be written in the product form similarly to the first-degree split polynomials (2.3) and therefore may be formally split into the following second-degree split polynomials of their own: ($Q_{p\_p}(\zeta_2, C_2)$), ($Q_{p\_n}(\zeta_2, C_2)$), $R_{p\_p}(\zeta_2, D_2)$ and $R_{p\_n}(\zeta_2, D_2)$.

$$Q_p(\zeta_1, C_1) = Q_{p\_p}(\zeta_2, C_2)\cdot Q_{p\_n}(\zeta_2, C_2)$$
$$= \prod_{j=1}^{\infty}\left[1-\frac{\zeta_2}{c_{2_{(j)}}}\right]\cdot\prod_{j=1}^{\infty}\left[1+\frac{\zeta_2}{c_{2_{(j)}}}\right] = \prod_{j=1}^{\infty}\left[1-\frac{\zeta_1}{c_{1_{(j)}}}\right] \quad (3.12)$$

$$R_p(\zeta_1, D_1) = R_{p\_p}(\zeta_2, D_2)\cdot R_{p\_n}(\zeta_2, D_2)$$
$$= \prod_{j=1}^{\infty}\left[1-\frac{\zeta_2}{d_{2_{(j)}}}\right]\cdot\prod_{j=1}^{\infty}\left[1+\frac{\zeta_2}{d_{2_{(j)}}}\right] = \prod_{j=1}^{\infty}\left[1-\frac{\zeta_1}{d_{1_{(j)}}}\right]$$

$$Q_{p\_p}(\zeta_2, C_2) = \prod_{j=1}^{\infty}\left[1-\frac{\zeta_2}{c_{2_{(j)}}}\right];\quad Q_{p\_n}(\zeta_2, C_2) = \prod_{j=1}^{\infty}\left[1+\frac{\zeta_2}{c_{2_{(j)}}}\right]$$

$$R_{p\_p}(\zeta_2, D_2) = \prod_{j=1}^{\infty}\left[1-\frac{\zeta_2}{d_{2_{(j)}}}\right];\quad R_{p\_n}(\zeta_2, D_2) = \prod_{j=1}^{\infty}\left[1+\frac{\zeta_2}{d_{2_{(j)}}}\right]$$

These split components of the adjoint polynomials are formally identical to the split components (2.6) and (2.8), only the roots (A_2) are replaced by (C_2) respectively by (D_2).

Similarly, to (3.1), (3.2), (3.3) and (3.4) these polynomials may be written with roots on the imaginary axes as well. Their sum and difference define adjoint polynomials on the adjoint axes of their own:

They are the adjoint polynomials of the adjoint polynomials: **The second degree adjoint polynomials**:

($QQ_{p\_n}(\zeta_2, CC_2)$, $QR_{p\_n}(\zeta_2, CD_2)$), $RQ_{p\_n}(\zeta_2, RC_2)$ and $RR_{p\_n}(\zeta_2, RD_2)$

The sum and the difference of these second degree adjoint polynomials define similarly to (3.10) and (3.11) the corresponding first degree adjoint polynomials on the adjoint axes.

Definition of the generalized relations of Pythagoras:

Multiplying these above equations pair wise with each other gives relations, which may be regarded, by the same reasoning as at lemma 1, as the **generalized relations of Pythagoras**:

$$P_p(\sigma, A_1) = P_{p\_p}(\sigma, A_2)\cdot P_{p\_n}(\sigma, A_2) = Q_p(\sigma, C_1)^2 + R_p(\sigma, D_1)^2 \quad (3.12)$$

$$P_n(\sigma, A_1) = P_{n\_p}(\sigma, i\cdot A_2)\cdot P_{n\_n}(\sigma, i\cdot A_2) = Q_n(\sigma, C_1)^2 + R_n(\sigma, D_1)^2$$

4. Shifting the Infinite Polynomials on One of the Adjoint Axes

What is the representation of a polynomial having all its roots on one of the adjoint axes, if the coordinate system is modified that way, that the same axes are shifted parallel by a constant value? Taking the polynomials ($P_{n\_p}(\sigma, i\cdot A_2)$) and ($P_{n\_n}(\sigma, i\cdot A_2)$) defined in (3.8), with all roots on the imaginary axis ($i\cdot a_{2_{(j)}}$) and ($-i\cdot a_{2_{(j)}}$) and shifting the roots to the line ($s_2 = \zeta_2 - \frac{b}{2}$), then they will be the complex numbers:

$$(a_{2\_tr_{(j)}} = \frac{b}{2} + i\cdot a_{2_{(j)}}) \text{ and } (a_{2\_tr_{(j)}} = \frac{b}{2} - i\cdot a_{2_{(j)}}) \quad (4.1)$$

The variable (ζ_2) will be replaced by the variable ($s_2 = \zeta_2 - \frac{b}{2}$), and the imaginary axes will be replaced by a line starting from ($\sigma = \frac{b}{2}$), instead from the origin. For ($b = 1$) this corresponds to the critical line of the Riemann Zeta function. For each of the components of the infinite product it may be written:

$$1 - \frac{\zeta_2}{i\cdot a_{2_{(j)}}} = 1 - \frac{s_2 + \frac{b}{2}}{a_{2\_tr_{(j)}} - \frac{b}{2}} = \frac{a_{2\_tr_{(j)}}}{a_{2\_tr_{(j)}} - \frac{b}{2}} - \frac{s_2}{a_{2\_tr_{(j)}} - \frac{b}{2}} = \frac{a_{2\_tr_{(j)}}}{a_{2\_tr_{(j)}} - \frac{b}{2}}\cdot\left[1 - \frac{s_2}{a_{2\_tr_{(j)}}}\right] \quad (4.2)$$

$$1 + \frac{\zeta_2}{i\cdot a_{2_{(j)}}} = 1 - \frac{s_2 + \frac{b}{2}}{a_{2\_tr_{(j)}} - \frac{b}{2}} = \frac{a_{2\_tr_{(j)}}}{a_{2\_tr_{(j)}} - \frac{b}{2}} + \frac{s_2}{a_{2\_tr_{(j)}} - \frac{b}{2}} = \frac{a_{2\_tr_{(j)}}}{a_{2\_tr_{(j)}} - \frac{b}{2}}\cdot\left[1 + \frac{s_2}{a_{2\_tr_{(j)}}}\right]$$

The polynomials (2.8) will have the form:

$$\prod_{j=1}^{\infty}\frac{a_{2\_tr_{(j)}}}{a_{2\_tr_{(j)}} - \frac{b}{2}}\cdot\prod_{j=1}^{\infty}\left[1 - \frac{s_2}{a_{2\_tr_{(j)}}}\right] \text{ and } \prod_{j=1}^{\infty}\frac{a_{2\_tr_{(j)}}}{a_{2\_tr_{(j)}} - \frac{b}{2}}\cdot\prod_{j=1}^{\infty}\left[1 + \frac{s_2}{a_{2\_tr_{(j)}}}\right] \quad (4.3)$$

In this representation the point of central symmetry is changed from the origin to the point ($\sigma = \frac{b}{2}$). Correspondingly the roots of the second-degree split functions with all roots on the real axes will be shifted by the same value ($s_2 = \zeta_2 - \frac{b}{2}$) and their roots will be ($a_{2\_tr_{(j)}} = \frac{b}{2} - a_{2_{(j)}}$) and ($a_{2\_tr_{(j)}} = \frac{b}{2} + a_{2_{(j)}}$).

DOI: 10.4236/apm.2022.124024

Again, for each of the components of the infinite product it may be written:

$$1-\frac{\zeta_2}{a_{2(j)}}=\prod_{j=1}^{\infty}\frac{a_{2\_tr(j)}}{a_{2\_tr(j)}-\frac{b}{2}}\cdot\prod_{j=1}^{\infty}\left[1-\frac{s_2}{a_{2\_tr(j)}}\right]$$
$$1+\frac{\zeta_2}{a_{2(j)}}=\prod_{j=1}^{\infty}\frac{a_{2\_tr(j)}}{a_{2\_tr(j)}-\frac{b}{2}}\cdot\prod_{j=1}^{\infty}\left[1+\frac{s_2}{a_{2\_tr(j)}}\right] \tag{4.4}$$

The corresponding polynomials (2.8) will have the same form as (4.3).

The infinite products as first components in (4.3) and (4.4) are approaching a constant value, especially in case of the exponential function and ($b=1$) gives:

$$\lim_{x\to\infty}\prod_{j=1}^{\infty}\left[\frac{a_{2\_tr(j)}}{a_{2\_tr(j)}-\frac{b}{2}}\right]=\mathrm{e}^{\frac{b}{2}}=\sqrt{\mathrm{e}}=1.649 \tag{4.5}$$

without taking account of this limit, the equation defining the adjoint functions on the adjoint axes by setting equal the two infinite polynomial products, the above limits cancel out, resulting in the complex or real roots ($a_{2\_tr(j)}$):

$$\prod_{j=1}^{\infty}\left[1-\frac{s_2}{a_{2\_tr(j)}}\right]=\prod_{j=1}^{\infty}\left[1+\frac{s_2}{a_{2\_tr(j)}}\right] \tag{4.6}$$

Shifting the roots of the second-degree split polynomials from the imaginary axes to the critical line ($\frac{b}{2}+i\cdot\tau$), respectively shifting the roots on the real axes by ($\frac{b}{2}$) to ($\sigma+\frac{b}{2}$) yields the complex roots on the shifted second-degree split polynomials, with the imaginary part of the roots remaining unchanged:

$$(a_{2\_i\_tr(j)}=\frac{b}{2}+i\cdot a_{2(j)}),\ (a_{2\_i\_tr(j)}=\frac{b}{2}-i\cdot a_{2(j)}),$$
$$(a_{2\_tr(j)}=\frac{b}{2}+a_{2(j)}) \text{ and } (a_{2\_tr(j)}=\frac{b}{2}-a_{2(j)}) \tag{4.7}$$

with this shifting the point of central symmetry is moved from the origin to the point ($\sigma=\frac{b}{2}$).

With the definitions (3.1), (3.2), (3.3) and (3.4) the equations defining the roots of the symmetric and of the askew symmetric adjoint functions on the adjoint axes will be:

$$P_{p\_p\_tr}\left(\zeta_2,A_{2\_tr}\right)+P_{p\_n\_tr}\left(\zeta_2,A_{2\_tr}\right)=2\cdot Q_{n\_tr}\left(\zeta_2,C_{1\_i\_tr}\right)$$
$$c_{1\_i\_tr(j)}=\frac{b}{2}+i\cdot c_{1(j)} \tag{4.8}$$

$$P_{p\_p\_tr}\left(\zeta_2,A_{2\_tr}\right)-P_{p\_n\_tr}\left(\zeta_2,A_{2\_tr}\right)=2\cdot R_{n\_tr}\left(\zeta_2,D_{1\_i\_tr}\right)$$
$$d_{1\_i\_tr(j)}=\frac{b}{2}+i\cdot d_{1(j)} \tag{4.9}$$

$$P_{n\_p\_tr}\left(s_2, A_{2\_i\_tr}\right) + P_{n\_n\_tr}\left(s_2, A_{2\_i\_tr}\right) = 2 \cdot Q_{p\_tr}\left(s_2, C_{1\_tr}\right)$$
$$c_{1\_tr(j)} = \frac{b}{2} + c_{1(j)} \tag{4.10}$$

$$P_{n\_p\_tr}\left(s_2, A_{2\_i\_tr}\right) - P_{n\_n\_tr}\left(s_2, A_{2\_i\_tr}\right) = 2 \cdot i \cdot R_{p\_tr}\left(s_2, D_{1\_tr}\right)$$
$$d_{1\_tr(j)} = \frac{b}{2} + d_{1(j)} \tag{4.11}$$

Similarly, to (3.10) and (3.11) addition and subtraction of the adjoint polynomial functions result in the original split polynomial functions in the infinite product form:

$$Q_{n\_tr}\left(s_2, C_{1\_i\_tr}\right) + R_{n\_tr}\left(s_2, D_{1\_i\_tr}\right) = P_{p\_p\_tr}\left(s_2, A_{2\_tr}\right) \tag{4.12}$$

$$Q_{n\_tr}\left(s_2, C_{1\_i\_tr}\right) - R_{n\_tr}\left(s_2, D_{1\_i\_tr}\right) = P_{p\_n\_tr}\left(s_2, A_{2\_tr}\right) \tag{4.13}$$

$$Q_{p\_tr}\left(s_2, C_{1\_tr}\right) + i \cdot R_{p\_tr}\left(s_2, D_{1\_tr}\right) = P_{n\_p\_tr}\left(s_2, A_{2\_i\_tr}\right) \tag{4.14}$$

$$Q_{p\_tr}\left(s_2, C_{1\_tr}\right) - i \cdot R_{p\_tr}\left(s_2, D_{1\_tr}\right) = P_{n\_n\_tr}\left(s_2, A_{2\_i\_tr}\right) \tag{4.15}$$

Thus, any infinite polynomial having polynomial quotient equations of similar form, will have all the roots either on the critical line ($\frac{b}{2} + i \cdot \tau$) or on the real axes ($\sigma - \frac{b}{2}$) within the shifted coordinate system, corresponding to roots either on the imaginary axes or on the real axes within the original coordinate system.

Lemma 4.1:

The complex roots of polynomials in a shifted coordinate system, with shift of the imaginary axes parallel by a constant value ($\frac{b}{2}$ **) will be exclusively on the critical line (** $\frac{b}{2} + i \cdot \tau$ **), if the roots of the same polynomials in the original coordinate system are all on the imaginary axes.**

Proof: The roots of a polynomial being exclusively on the imaginary axes are moved by the shifting from the imaginary axes to the critical line at the distance ($\frac{b}{2}$), because with (4.8) and (4.9) the imaginary components of the complex roots remain unchanged, as stated in the lemma and **concluding the proof**.

5. The Product Representation of the Exponential Function

The exponential function may be written as a polynomial in the form of an infinite product with the aid of the binomial coefficients (Ref. [1]). The binomial coefficients are for any positive integer ($n = 1, 2, \cdots, \infty$) defined as:

$$B(j, n) = \frac{n!}{j! \cdot (n-j)!} = \frac{\Gamma(n+1)}{\Gamma(j+1) \cdot \Gamma(n-j+1)}, \quad j = 1, 2, \cdots, n \tag{5.1}$$

The maximum is: ($j = \frac{n}{2}$):

$$B_{\max}(n)=\frac{n!}{\left(\left(\frac{n}{2}\right)!\right)^2}=\prod_{j=1}^{\frac{n}{2}}\frac{j+\frac{n}{2}}{j}=\prod_{j=1}^{\frac{n}{2}}\left(1+\frac{n}{2\cdot j}\right) \tag{5.2}$$

Normed with this maximum the binomial coefficients will be:

$$B_{norm}(j,n)=\frac{B(j,n)}{B_{\max}(n)}=\frac{\left[\left(\frac{n}{2}\right)!\right]^2}{j!\cdot(n-j)!}=\frac{\Gamma\left(\frac{n}{2}+1\right)^2}{\Gamma(j+1)\cdot\Gamma(n-j+1)} \tag{5.3}$$

Using the following linear shifting of the coordinates on the abscise and neglecting unity versus ($\frac{n}{2}$), the normed binomial coefficients will be centered around the origin:

$$\zeta_1(j,n)=\left(j-\frac{n}{2}\right)^2\cdot\frac{n}{2};\ \ \zeta_2(j,n)=\left(j-\frac{n}{2}\right)\cdot\sqrt{\frac{2}{n}};\ \ j=\zeta_2(n)\cdot\sqrt{\frac{n}{2}}+\frac{n}{2} \tag{5.4}$$

It can be proved, that this function is equal to the normal distribution of the variable ($\zeta_1(n)$). The complete proof is not given in the present paper, only the formal identity is demonstrated in **Annex A1**:

$$\lim_{n\to\infty}\mathrm{e}^{-\zeta_1(n)}=\lim_{n\to\infty}\mathrm{e}^{-(\zeta_2(n))^2}=\lim_{n\to\infty}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+\zeta_2(n)\cdot\sqrt{\frac{n}{2}}\right)}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}-\zeta_2(n)\cdot\sqrt{\frac{n}{2}}\right)}\right) \tag{5.5}$$

Using the definition of the gamma function from Gauss (Ref. [2]):

$$\Gamma(x)=\lim_{m\to\infty}\left[m^x\frac{(m-1)!}{x\cdot(x+1)\cdot(x+2)\cdots(x+m-1)}\right]=\lim_{m\to\infty}\left(\frac{m^x}{m}\cdot\prod_{k=0}^{m-1}\frac{k+1}{k+x}\right) \tag{5.6}$$

This infinite polynomial product written for ($\frac{n}{2}$), ($\frac{n}{2}+\zeta_2\sqrt{\frac{n}{2}}$) and for ($\frac{n}{2}-\zeta_2\sqrt{\frac{n}{2}}$), shortened gives the following quotients, composed of an exponential part and of a trigonometric part:

$$\begin{gathered}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+\zeta_2(n)\right)}=\Gamma_{exp}(\zeta_2(n))\cdot\Gamma_{tri}(\zeta_2(n))\\ \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}-\zeta_2(n)\right)}=\Gamma_{exp}(-\zeta_2(n))\cdot\Gamma_{tri}(-\zeta_2(n))\end{gathered} \tag{5.7}$$

$$\Gamma_{exp}(\zeta_2(n))=\lim_{n\to\infty}\left(n^{-(\zeta_2(n))\cdot\sqrt{\frac{n}{2}}}\right)$$

$$\Gamma_{tri}\left(\zeta_2(n)\right)=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\zeta_2(n)}{\left(k+\frac{n}{2}\right)\cdot\sqrt{\frac{2}{n}}}\right)\right] \tag{5.8}$$

Inserted into the normal distribution (4.5) eliminates the exponential components. Using the definitions of the following positive real numbers, leaking out to infinity, but on the real axes:

$$a_1(k,n)=\frac{(2\cdot k+n)^2}{2\cdot n}\,;\;\lim_{n\to\infty}a_1(k,n)=\infty\,;\;a_2(k,n)=\frac{2}{3}\cdot a_1(k,n)=\frac{(2\cdot k+n)^2}{3\cdot n} \tag{5.9}$$

$$\mathrm{e}^{-\zeta_1}(n)=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\zeta_1(n)}{a_1(k,n)}\right)\right]$$

Herewith the normal distribution function with roots ($\zeta_1(n)=a_1(k,n)$) on the real axes but leaking out to infinity results in the exponential function with real roots ($\zeta_2(n)=a_2(k,n)$) and ($\zeta_2(n)=-a_2(k,n)$) on the real axes but leaking out to infinity as well. They are transposed to each other:

$$\begin{aligned}P_{p\_p\_e}(\zeta_2,A_2)&=\mathrm{e}^{-\zeta_2}=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\zeta_2}{a_2(k,n)}\right)\right]\\P_{p\_n\_e}(\zeta_2,A_2)&=\mathrm{e}^{\zeta_2}=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\zeta_2}{a_2(k,n)}\right)\right]\end{aligned} \tag{5.10}$$

The exponential functions with roots ($\zeta_2(n)=i\cdot a_2(k,n)$) and ($\zeta_2(n)=-i\cdot a_2(k,n)$) on the imaginary axes, but leaking out to infinity are conjugate complex to each other:

$$\begin{aligned}P_{n\_p\_e}(\zeta_2,A_2)&=\mathrm{e}^{-i\cdot\zeta_2}=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\zeta_2}{i\cdot a_2(k,n)}\right)\right]\\P_{n\_n\_e}(\zeta_2,A_2)&=\mathrm{e}^{i\cdot\zeta_2}=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\zeta_2}{i\cdot a_2(k,n)}\right)\right]\end{aligned} \tag{5.11}$$

6. The Product Representation of the Trigonometric Functions

With lemma 2.1, the relations of Euler, the sum and the difference of the second-degree split polynomials of the normal distribution function, the exponential functions (4.10) and (4.11) written with roots on one of the adjoint axes are defining roots for the adjoint functions on the other adjoint axes. Taking the roots of the exponential functions on the real axes ($\zeta_2=i\cdot\tau$), define the roots of the adjoint functions, the trigonometric functions, with roots on the imaginary axes ($\zeta_1=\zeta_2^2=\sigma$):

$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\sigma}{a_2(k,n)}\right)\right]+\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\sigma}{a_2(k,n)}\right)\right]=2\cdot i\cdot\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\tau^2}{c_1(k)}\right)\right] \tag{6.1}$$

$$P_{p\_p\_e}(\zeta_2,A_2)+P_{p\_n\_e}(\zeta_2,A_2)=2\cdot Q_{n\_e}(\tau,i\cdot C_1)\;;\;\mathrm{e}^{-\sigma}+\mathrm{e}^{\sigma}=2\cdot\cosh(\sigma)=2\cdot\cos(i\cdot\tau)$$

$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\sigma}{a_2(k,n)}\right)\right]-\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\sigma}{a_2(k,n)}\right)\right]$$
$$=-2\cdot i\cdot\lim_{n\to\infty}\left[\frac{\tau}{d_1(0)}\cdot\prod_{k=0}^{n}\left(1+\frac{\tau^2}{d_1(k)}\right)\right]$$
$$P_{p\_p\_e}(\zeta_2,A_2)-P_{p\_n\_e}(\zeta_2,A_2)=2\cdot R_{n\_e}(\tau,i\cdot D_1)\ ;\ \mathrm{e}^{-\sigma}-\mathrm{e}^{\sigma}=2\cdot i\cdot\sinh(\sigma)=2\cdot\sin(i\cdot\tau)$$

Taking the roots of the exponential functions (1.11) on the imaginary axes ($\zeta_2=i\cdot\tau$), define the adjoint functions, the trigonometric functions, on the real axes ($\zeta_1=\zeta_2^2=\sigma^2$):

$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{i\cdot\tau}{a_2(k,n)}\right)\right]+\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{i\cdot\tau}{a_2(k,n)}\right)\right]=2\cdot\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\tau^2}{c_1(k)}\right)\right]\quad(6.2)$$
$$P_{n\_p\_e}(\zeta_2,A_2)+P_{n\_n\_e}(\zeta_2,A_2)=2\cdot Q_{p\_e}(\sigma,C_1);$$
$$\mathrm{e}^{-i\cdot\tau}+\mathrm{e}^{i\cdot\tau}=2\cdot\cosh(i\cdot\tau)=2\cdot\cos(\sigma)$$

$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{i\cdot\tau}{a_2(k,n)}\right)\right]-\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{i\cdot\tau}{a_2(k,n)}\right)\right]$$
$$=-2\cdot i\cdot\lim_{n\to\infty}\left[\frac{\sigma}{d_1(0)}\cdot\prod_{k=0}^{n}\left(1-\frac{\sigma^2}{d_1(k)}\right)\right]$$
$$P_{n\_p\_e}(\zeta_2,A_2)-P_{n\_n\_e}(\zeta_2,A_2)=2\cdot i\cdot R_{p\_e}(\sigma,D_1);$$
$$\mathrm{e}^{-i\cdot\tau}-\mathrm{e}^{i\cdot\tau}=-2\cdot i\cdot\sinh(i\cdot\tau)=2\cdot i\cdot\sin(\sigma)$$

Addition and subtraction of the above equations give with the imaginary roots of the trigonometric functions the real roots of the exponential function:

$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\tau^2}{c_1(k)}\right)\right]-\lim_{n\to\infty}\left[\frac{i\cdot\tau}{d_1(0)}\cdot\prod_{k=0}^{n}\left(1-\frac{\tau^2}{d_1(k)}\right)\right]=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\sigma}{a_2(k,n)}\right)\right]\quad(6.3)$$
$$Q_{n\_e}(\tau,i\cdot C_1)+R_{n\_e}(\tau,i\cdot D_1)=P_{p\_p\_e}(\sigma,A_2);\quad\cos(i\cdot\tau)+i\cdot\sin(i\cdot\tau)=\mathrm{e}^{-\sigma}$$
$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\tau^2}{c_1(k)}\right)\right]+\lim_{n\to\infty}\left[\frac{i\cdot\tau}{d_1(0)}\cdot\prod_{k=0}^{n}\left(1-\frac{\tau^2}{d_1(k)}\right)\right]=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\sigma}{a_2(k,n)}\right)\right]$$
$$Q_{n\_e}(\tau,i\cdot C_1)-R_{n\_e}(\tau,i\cdot D_1)=P_{p\_n\_e}(\sigma,A_2);\quad\cos(i\cdot\tau)-i\cdot\sin(i\cdot\tau)=\mathrm{e}^{\sigma}$$

Addition and subtraction of the above equations give for real roots of the trigonometric functions the following equations, especially for ($\sigma=\pi$) the identity of Euler:

$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\sigma^2}{c_1(k)}\right)\right]-\lim_{n\to\infty}\left[\frac{\sigma}{d_1(0)}\cdot\prod_{k=0}^{n}\left(1-\frac{\sigma^2}{d_1(k)}\right)\right]=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{i\cdot\tau}{a_2(k,n)}\right)\right]\quad(6.4)$$
$$Q_{p\_e}(\sigma,C_1)+i\cdot R_{p\_e}(\sigma,D_1)=P_{n\_p\_e}(\tau,i\cdot A_2);\quad\cos(\sigma)-i\cdot\sin(\sigma)=\mathrm{e}^{-i\cdot\sigma}$$
$$\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\sigma^2}{c_1(k)}\right)\right]+\lim_{n\to\infty}\left[\frac{\sigma}{d_1(0)}\cdot\prod_{k=0}^{n}\left(1-\frac{\sigma^2}{d_1(k)}\right)\right]=\lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{i\cdot\tau}{a_2(k,n)}\right)\right]$$
$$Q_{p\_e}(\sigma,C_1)-i\cdot R_{p\_e}(\sigma,D_1)=P_{n\_n\_e}(\tau,i\cdot A_2);\quad\cos(\sigma)+i\cdot\sin(\sigma)=\mathrm{e}^{i\cdot\sigma}$$

In (5.4) the sum of two complex functions results in a periodic function. From where does the periodicity originate? This will be clear if the products of the sum are evaluated as infinite series:

$$e^{i\cdot x} = \prod_{k=0}^{n=\infty}\left(1+\frac{i\cdot x}{a_2(k,n)}\right) \tag{6.5}$$

$$= 1 + i\cdot h_1\cdot x - h_2\cdot x^2 - i\cdot h_3\cdot x^3 + h_4 x^4 + i\cdot h_5\cdot x^5 - h_6 x^6 + \cdots$$

$$e^{-i\cdot x} = \prod_{k=0}^{n=\infty}\left(1-\frac{i\cdot x}{a_2(k,n)}\right)$$

$$= 1 - i\cdot h_1\cdot x - h_2\cdot x^2 + i\cdot h_3\cdot x^3 + h_4 x^4 - i\cdot h_5\cdot x^5 - h_6 x^6 + \cdots$$

The sum and the difference of these series result in the known series for the trigonometric functions:

$$\cos(x) = \cosh(i\cdot x) = 1 - h_2\cdot x^2 + h_4\cdot x^4 - h_6\cdot x^6 + \cdots,$$

$$\text{with } h_2 = \frac{1}{2!}, h_4 = \frac{1}{4!}, h_6 = \frac{1}{6!}, \cdots \tag{6.6}$$

$$\sin(x) = \sinh(i\cdot x) = h_1\cdot x - h_3\cdot x^3 + h_5\cdot x^5 + \cdots, \text{ with } h_1 = \frac{1}{1!}, h_3 = \frac{1}{3!}, h_5 = \frac{1}{5!}, \cdots$$

In these series the value of the last term always overtakes the sum of all previous terms.

Since the sign of the last term is varying, the function becomes periodic, the root values converging to odd respectively to even multiples of ($\frac{\pi}{2}$). In fact, the value ($\frac{\pi}{2}$) is defined by this convergence, as already recognized by Euler.

Odd multiples of ($\frac{\pi}{2}$) give the zeros for the split cosine and for the cosine functions at:

$$\left(c_2(k) = (2\cdot k-1)\cdot\frac{\pi}{2},\quad c_1(k) = (2\cdot k-1)^2\cdot\left(\frac{\pi}{2}\right)^2\right)$$

defining the vectors (C_2) and (C_1). (6.7)

Even multiples of ($\frac{\pi}{2}$) give the zeros for the split sine and for the sine functions at:

($d_2(k) = 2\cdot k\cdot\frac{\pi}{2},\quad d_1(k) = (2\cdot k)^2\cdot\left(\frac{\pi}{2}\right)^2$) defining the vectors (D_2) and (D_1).

Equations (6.4) may be written for even and for odd powers of (x) separated as follows:

$$e^{i\cdot x} = \sum_{k=0}^{n=\infty}\frac{(-1)^k\cdot x^{2\cdot k}}{(2\cdot k)!} - i\cdot\sum_{k=0}^{n=\infty}\frac{(-1)^k\cdot x^{2\cdot k+1}}{(2\cdot k+1)!} = \cos(x) + i\cdot\sin(x) \tag{6.8}$$

$$e^{-i\cdot x} = \sum_{k=0}^{n=\infty}\frac{(-1)^k\cdot x^{2\cdot k}}{(2\cdot k)!} + i\cdot\sum_{k=0}^{n=\infty}\frac{(-1)^k\cdot x^{2\cdot k+1}}{(2\cdot k+1)!} = \cos(x) - i\cdot\sin(x)$$

The relations of Pythagoras (2.13) give the well-known formula:

$$Q_{p\_e}(\sigma,C_1)^2+R_{p\_e}(\sigma,D_1)^2=\cos(\sigma)^2+\sin(\sigma)^2=P_{p\_e}=1 \tag{6.9}$$

$$Q_{n\_e}(\tau,C_1)^2+R_{n\_e}(\tau,D_1)^2=\cos(i\cdot\tau)^2+\sin(i\cdot\tau)^2=P_{n\_e}=1$$

From (6.1) and (6.2) follows, that the zeros of the trigonometric functions on the adjoint axes are defined with ($k=1,2,3,\cdots,\infty$) by the following equations:

$$P_{p\_p\_e}(\sigma,A_2)=-P_{p\_n\_e}(\sigma,A_2);\ \ \mathrm{e}^{-\sigma}=-\mathrm{e}^{\sigma};$$

$$\cosh(\sigma)\ \text{ defines roots for }\ \cos(i\cdot\tau) \tag{6.10}$$

$$P_{p\_p\_e}(\sigma,A_2)=P_{p\_n\_e}(\sigma,A_2);\ \ \mathrm{e}^{-\sigma}=\mathrm{e}^{\sigma};\ \ \sinh(\sigma)\ \text{ defines roots for }\ \sin(i\cdot\tau)$$

$$P_{n\_n\_e}(\tau,A_2)=-P_{n\_p\_e}(\tau,A_2);\ \ \mathrm{e}^{-i\cdot\tau}=\mathrm{e}^{i\cdot\tau};\ \ \cosh(i\cdot\tau)\ \text{ defines roots for }\ \cos(\sigma)$$

$$P_{n\_n\_e}(\tau,A_2)=P_{n\_p\_e}(\tau,A_2);\ \ \mathrm{e}^{-i\cdot\tau}=-\mathrm{e}^{i\cdot\tau};\ \ \sinh(i\cdot\tau)\ \text{ defines roots for }\ \sin(\sigma)$$

From (6.3) and (6.4) follows, that the zeros of the exponential functions on the adjoint axes (leaking out to infinity) are defined by the following equations:

$$Q_{n\_e}(\tau,i\cdot C_1)=-R_{n\_e}(\tau,i\cdot D_1);$$

$$\cos(i\cdot\tau)=-i\cdot\sin(i\cdot\tau)\ \text{ defines roots for }\ \mathrm{e}^{-\sigma} \tag{6.11}$$

$$Q_{n\_e}(\tau,i\cdot C_1)=R_{n\_e}(\tau,i\cdot D_1);\ \ \cos(i\cdot\tau)=i\cdot\sin(i\cdot\tau)\ \text{ defines roots for }\ \mathrm{e}^{\sigma}$$

$$Q_{p\_e}(\sigma,C_1)=-i\cdot R_{p\_e}(\sigma,D_1);\ \ \cos(\sigma)=-i\cdot\sin(\sigma)\ \text{ defines roots for }\ \mathrm{e}^{i\cdot\tau}$$

$$Q_{p\_e}(\sigma,C_1)=i\cdot R_{p\_e}(\sigma,D_1);\ \ \cos(\sigma)=i\cdot\sin(\sigma)\ \text{ defines roots for }\ \mathrm{e}^{-i\cdot\tau}$$

These equations are in accordance with the relations of Euler:

$$\cos(\sigma)+i\cdot\sin(\sigma)=\mathrm{e}^{-i\cdot\tau};\ \ \cos(\sigma)-i\cdot\sin(\sigma)=\mathrm{e}^{i\cdot\tau} \tag{6.12}$$

$$\cos(i\cdot\tau)+i\cdot\sin(i\cdot\tau)=\mathrm{e}^{-\sigma};\ \ \cos(i\cdot\tau)-i\cdot\sin(i\cdot\tau)=\mathrm{e}^{\sigma}$$

$$\cosh(i\cdot\tau)=\cos(\sigma);\ \ \sinh(\sigma)=-i\cdot\sin(i\cdot\tau)$$

The identity of the exponential and of the trigonometric functions with their representation as infinite polynomial products is demonstrated in **Annex 2**.

With (3.12) the first-degree split infinite polynomial product ($Q_{p\_e}(\sigma,n)$) may be split resulting in the split trigonometric functions, demonstrated in **Annex 3**:

$$Q_{p\_p\_e}(\sigma,C_2)=\prod_{j=1}^{\infty}\left(1-\frac{\sigma}{c_{2(j)}}\right)=\prod_{j=1}^{\infty}\left(1-\frac{\sigma}{2\cdot j-1}\right) \tag{6.13}$$

$$Q_{p\_n\_e}(\sigma,C_2)=\prod_{j=1}^{\infty}\left(1+\frac{\sigma}{c_{2(j)}}\right)=\prod_{j=1}^{\infty}\left(1+\frac{\sigma}{2\cdot j-1}\right)$$

The sum and the difference of these second-degree split infinite polynomial products are:

$$QQ_{p\_p\_e}(i\cdot\tau,C_2)=\frac{1}{2}\left(Q_{p\_p\_e}(i\cdot\tau,C_2)+Q_{p\_n\_e}(i\cdot\tau,C_2)\right) \tag{6.14}$$

$$RQ_{p\_p\_e}(i\cdot\tau,C_2)=\frac{1}{2}\left(Q_{p\_p\_e}(i\cdot\tau,C_2)-Q_{p\_n\_e}(i\cdot\tau,C_2)\right)$$

Formally these equations are like the Equations (6.4). Therefore, their sum and difference are like the relations of Euler defining the cosine hyperbolic and sine hyperbolic functions:

$$\cosh(i\cdot\tau)=\frac{1}{2}\cdot\left(e^{j\cdot\tau}+e^{-j\cdot\tau}\right)=\cos(\sigma);\ \ \sinh(i\cdot\tau)=\frac{1}{2\cdot i}\cdot\left(e^{i\cdot\tau}-e^{-i\cdot\tau}\right)=\sin(\sigma)\quad(6.15)$$

This similarity of these functions (6.14) and (6.15) results in the interesting fact: Multiplying with the constant value (6) the arguments result in equality, as demonstrated in **Annex 4**. More important is, that the components on the right from (6.14) are mutually transposed, therefore define roots for the adjoint functions on the imaginary axes.

7. Shifting the Roots from the Imaginary Axes to the Critical Line

Equations (6.3) define roots on the imaginary axes of the trigonometric function sine and cosine. This results from setting the symmetric respectively central symmetric functions cosine hyperbolic respectively sine hyperbolic functions equal to zero. Applying the linear shift, corresponding to lemma 3.1, of the hyperbolic functions from the imaginary axes to the critical line by replacing (σ) by ($\sigma-\frac{1}{2}$) and ($0+i\cdot\tau$) by ($\frac{1}{2}+i\cdot\tau$) gives:

$$\frac{1}{2}\cdot\left(e^{\sigma}+e^{-\sigma}\right)=0;\ \ \frac{1}{2}\cdot\left(e^{\sigma-\frac{1}{2}}+e^{-\left(\sigma-\frac{1}{2}\right)}\right)=0;$$

$$\frac{1}{2}\cdot\left(e^{\sigma-\frac{1}{2}}+e^{-\left(\sigma-\frac{1}{2}\right)}\right)\cdot\sqrt{e}=\frac{1}{2}\cdot\left(e^{\sigma}+e^{1-\sigma}\right)=0;\ \ e^{\sigma}=-e^{1-\sigma}$$

$$\frac{1}{2}\left(e^{\sigma}-e^{-\sigma}\right)=0;\ \ \frac{1}{2}\cdot\left(e^{\sigma-\frac{1}{2}}-e^{-\left(\sigma-\frac{1}{2}\right)}\right)\cdot\sqrt{e}=\frac{1}{2}\cdot\left(e^{\sigma}-e^{1-\sigma}\right)=0;\ \ e^{\sigma}=e^{1-\sigma}\quad(7.1)$$

The imaginary components of the roots for ($\cos(i\cdot\tau)$) and ($\sin(i\cdot\tau)$) for both are independent from the shifting on the real axes: ($\cos\left(\frac{1}{2}+i\cdot\tau\right)$) and ($\sin\left(\frac{1}{2}+i\cdot\tau\right)$).

$$\cosh\left(\frac{1}{2}+i\cdot\tau\right)=\frac{1}{2}\cdot\left(e^{\frac{1}{2}+i\cdot\tau}+e^{\frac{1}{2}-i\cdot\tau}\right)=0;$$

$$\frac{1}{2}\cdot\left(e^{\frac{1}{2}+i\cdot\tau}+e^{\frac{1}{2}-i\cdot\tau}\right)\cdot\frac{1}{\sqrt{e}}=\frac{1}{2}\cdot\left(e^{i\cdot\tau}+e^{-i\cdot\tau}\right)=0;\ \ e^{i\cdot\tau}=-e^{-i\cdot\tau}\quad(7.2)$$

$$\cosh\left(\frac{1}{2}+i\cdot\tau\right)=\frac{1}{2}\cdot\left(e^{\frac{1}{2}+i\cdot\tau}+e^{\frac{1}{2}-i\cdot\tau}\right)=0;$$

$$\frac{1}{2}\cdot\left(e^{\frac{1}{2}+i\cdot\tau}+e^{\frac{1}{2}-i\cdot\tau}\right)\cdot\frac{1}{\sqrt{e}}=\frac{1}{2}\cdot\left(e^{i\cdot\tau}+e^{-i\cdot\tau}\right)=0\,;\quad e^{i\cdot\tau}=-e^{-i\cdot\tau}$$

The real roots for ($\cos(\sigma)$) and for ($\sin(\sigma)$) both are shifted with reference to the critical line: ($\cos\left(\sigma-\frac{1}{2}\right)$) and ($\sin\left(\sigma-\frac{1}{2}\right)$).

These above equations may be written in the following form:

$$\cosh\left(\frac{1}{2}+i\cdot\tau\right)=\frac{1}{2}\cdot\left(e^{\frac{1}{2}+i\cdot\tau}+e^{\frac{1}{2}-i\cdot\tau}\right)=\frac{1}{2}\cdot\left(e^{i\cdot\tau}+e^{-i\cdot\tau}\right)=0\quad\text{defining}\quad\cos\left(\sigma-\frac{1}{2}\right)\tag{7.3}$$

$$\sinh\left(\frac{1}{2}+i\cdot\tau\right)=\frac{1}{2\cdot j}\cdot\left(e^{\frac{1}{2}+i\cdot\tau}-e^{\frac{1}{2}-i\cdot\tau}\right)=\frac{1}{2}\cdot\left(e^{i\cdot\tau}-e^{-i\cdot\tau}\right)=0\quad\text{defining}\quad\sin\left(\sigma-\frac{1}{2}\right)$$

With (5.11) the roots of the exponential function in the central form being on the imaginary axes, shifted to the critical line results the following form:

$$\lim_{n\to\infty}\prod_{k=0}^{n}\left(1-\frac{s}{\frac{1}{2}+i\cdot a_2(k,n)}\right)=\lim_{n\to\infty}\prod_{k=0}^{n}\left(1+\frac{1-s}{\frac{1}{2}+i\cdot a_2(k,n)}\right)\tag{7.4}$$

This equation defines the complex roots for ($\sin\left(\frac{1}{2}+i\cdot\tau\right)$) on the critical line.

Shifting the ($\cos(\sigma)$) respectively the ($\sin(\sigma)$) functions from the symmetric respectively central symmetric positions with respect to the imaginary axes to the critical line, they will become symmetric respectively central symmetric with respect to the critical line:

$$\cos(\sigma)+i\cdot\sin(\sigma)=0\,;\quad\cos\left(\sigma-\frac{1}{2}\right)+i\cdot\sin\left(\sigma-\frac{1}{2}\right)=0\tag{7.5}$$

$$\cos(\sigma)-i\cdot\sin(\sigma)=0\,;\quad\cos\left(\sigma-\frac{1}{2}\right)-i\cdot\sin\left(\sigma-\frac{1}{2}\right)=0$$

The imaginary components of the complex roots ($e^{\frac{1}{2}-i\cdot\tau}$) and for ($e^{\frac{1}{2}+i\cdot\tau}$) are indifferent to the shifting on the real axes: ($e^{-i\cdot\tau}$) and ($e^{i\cdot\tau}$):

$$\cos(i\cdot\tau)+i\cdot\sin(i\cdot\tau)=0\quad\text{and}\quad\cos\left(\frac{1}{2}+i\cdot\tau\right)+i\cdot\sin\left(\frac{1}{2}+i\cdot\tau\right)=0\tag{7.6}$$

$$\cos(i\cdot\tau)-i\cdot\sin(i\cdot\tau)=0\quad\text{and}\quad\cos\left(\frac{1}{2}+i\cdot\tau\right)-i\cdot\sin\left(\frac{1}{2}+i\cdot\tau\right)=0$$

Thus, the real roots for (e^{σ}) and ($e^{-\sigma}$) are shifted on the real axes: ($e^{\sigma-\frac{1}{2}}$) and ($e^{-\left(\sigma-\frac{1}{2}\right)}$).

Addition and subtraction of these equations result in the hyperbolic functions:

$$\cos\left(\sigma-\frac{1}{2}\right)=0\quad\text{defines roots for}\quad\cosh\left(\frac{1}{2}+i\cdot\tau\right),$$

the imaginary part equal to the roots of $\cosh(i\cdot\tau)$

$\sin\left(\sigma-\frac{1}{2}\right)=0$ defines roots for $\sinh\left(\frac{1}{2}+i\cdot\tau\right)$,

the imaginary part equal to the roots of $\sinh(i\cdot\tau)$

$$\cos\left(\frac{1}{2}+i\cdot\tau\right)=0 \text{ defines the shifted roots of } \cosh\left(\sigma-\frac{1}{2}\right) \quad (7.7)$$

$$\sin\left(\frac{1}{2}+i\cdot\tau\right)=0 \text{ defines the shifted roots of } \sinh\left(\sigma-\frac{1}{2}\right)$$

with (6.14) similar relations are valid for the split cosine functions:

$$Q_{p\_p\_e}(i\cdot\tau,n)+Q_{p\_n\_e}(i\cdot\tau,n)=0 \text{ and}$$

$$Q_{p\_p\_e}\left(\frac{1}{2}+i\cdot\tau,n\right)+Q_{p\_n\_e}\left(\frac{1}{2}+i\cdot\tau,n\right)=0$$

define roots for the imaginary components

$$QQ_{p\_p\_e}(\sigma) \text{ and } QQ_{p\_p\_e}\left(\sigma-\frac{1}{2}\right) \quad (7.8)$$

$$Q_{p\_p\_e}(i\cdot\tau,n)-Q_{p\_n\_e}(i\cdot\tau,n)=0 \text{ and}$$

$$Q_{p\_p\_e}\left(\frac{1}{2}+i\cdot\tau,n\right)-Q_{p\_n\_e}\left(\frac{1}{2}+i\cdot\tau,n\right)=0$$

define roots on the real axes $RQ_{p\_p\_e}(\sigma)$ and $RQ_{p\_p\_e}\left(\sigma-\frac{1}{2}\right)$.

The symmetric positions with respect to the imaginary axes of the coordinate ($\sigma=0$) define the same imaginary values for the roots, then the symmetric positions with respect to the critical line of the coordinate ($\sigma=\frac{1}{2}$).

The invariance of the imaginary roots of the trigonometric functions with respect to the shifting of the hyperbolic functions on the real axes is demonstrated in **Annex 5**.

8. Conclusions

The functional equation of the Riemann zeta-function Ref. [3] may be written as follows:

$$\xi(s)=\Gamma\left(\frac{s}{2}\right)\cdot\pi^{-\frac{s}{2}}\cdot\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\cdot\pi^{-\frac{1-s}{2}}\cdot\zeta(1-s)=\xi(1-s).$$

The second term of this equation may be written as:

$$\pi^{-\frac{s}{2}}=\pi^{-\frac{1-s}{2}};\quad e^{-s\cdot\ln\left(\sqrt{\pi}\right)}=e^{-(1-s)\cdot\ln\left(\sqrt{\pi}\right)}$$

with (7.1) this equation defines roots on the critical line for the adjoint function.

It is known that the gamma function may be written as composed of an exponential and a trigonometric component, which is the split cosine function. For the trigonometric component similar relations are valid as (7.8).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Fowler, D. (1996) The Binomial Coefficient Function. *The American Mathematical Monthly*, **103**, 1-17. https://doi.org/10.2307/2975209

[2] Davis, P.J. (1959) Leonhard Euler's Integral: A Historical Profile of the Gamma Function: In Memoriam: Milton Abramowitz. *American Mathematical Monthly*, **66**, 849-869. https://doi.org/10.2307/2309786

[3] Titmarsh, E.C. and Heath-Brown, D.R. (1994) The Theory of the Riemann Zeta-Function. 2nd Edition, Oxford University Press, Oxford.

Annex 1. The Normal Distribution and the Exponential Function

The normal distribution and the exponential functions are shown in the figure below for the ranges ($n = 400$), for ($j = 1, 2, \cdots, n$) and for ($\sigma = -4, -3.9, -3.8, \cdots, 3.9, 4$).

The complex variables are defined in (5.4):

$$\zeta_1(j,n) = \left(j - \frac{n}{2}\right)^2 \cdot \frac{2}{n}; \quad \zeta_2(j,n) = \left(j - \frac{n}{2}\right) \cdot \sqrt{\frac{2}{n}} \tag{A1.1}$$

The infinite polynomial products for the normal distribution and for the exponential functions are defined in (5.9) and (5.10) as well as their roots:

$$a_2(k,n) = \frac{(2 \cdot k + n)^2}{3 \cdot n}; \quad \lim_{n \to \infty} a_2(k,n) = \infty \tag{A1.2}$$

$$F_1(j,n) = \prod_{k=0}^{n} \left(1 - \frac{\zeta_1(j,n)}{a_2(k,n)}\right); \quad F_2(\sigma) = \prod_{k=0}^{n} \left(1 - \frac{\sigma^2}{a_2(k,n)}\right)$$

$$P_{p\_n\_e}(\sigma) = \prod_{k=0}^{n} \left(1 + \frac{\sigma}{a_2(k,n)}\right); \quad P_{p\_p\_e}(\sigma) = \prod_{k=0}^{n} \left(1 - \frac{\sigma}{a_2(k,n)}\right)$$

The first-degree split polynomial product of the exponential functions is equal to unity: (**Figure A1.1**)

$$P_{p\_e}(\sigma) = P_{p\_p\_e}(\sigma) \cdot P_{p\_n\_e}(\sigma) = e^{\sigma} \cdot e^{-\sigma} = 1 \tag{A1.3}$$

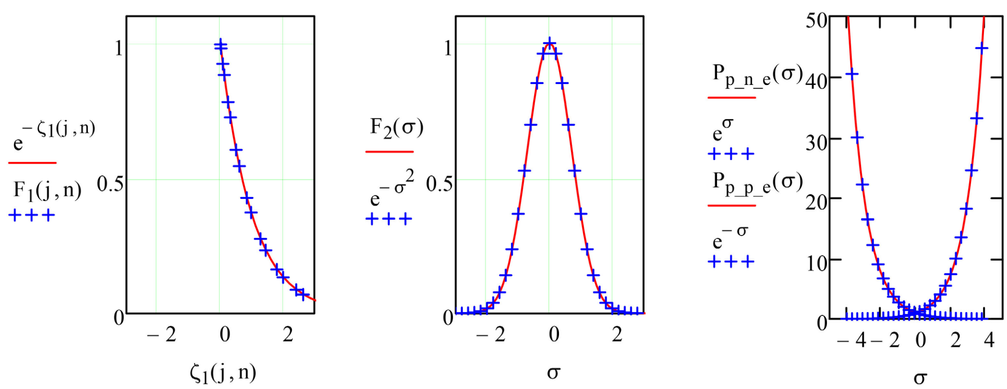

Figure A1.1. Normal distribution and exponential function, all compared with their infinite product representation.

Annex 2. The Infinite Products of Sine and Cosine Functions

The infinite product polynomials of the ($\cos(\sigma)$) and of the ($\sin(\sigma)$) from (6.2) are compared with the functions themself.

Odd multiples of ($\frac{\pi}{2}$) give the zeros of the cosine function at ($c_1(k) = (2 \cdot k - 1)^2 \cdot \left(\frac{\pi}{2}\right)$), even multiples of ($\frac{\pi}{2}$) give the zeros of the sine function at ($d_1(k) = k^2 \cdot \pi^2$) defining the vectors (C_1) and (D_1). For the comparison it is sufficient to take the first ($n = 100$) terms of the infinite product (**Figure A2.1**).

The range of the comparison is ($\sigma = -4, -3.9, -3.8, \cdots, 3.9, 4$).

$$Q_{p\_e}(\sigma, n) = \prod_{k=1}^{n}\left(1 - \frac{\sigma^2}{c_1(k)}\right); \quad R_{p\_e}(\sigma, n) = \sigma \prod_{k=1}^{n}\left(1 - \frac{\sigma^2}{d_1(k)}\right) \tag{A2.1}$$

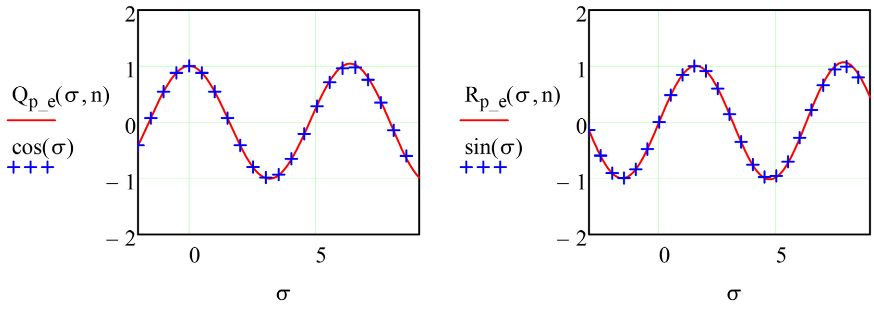

Figure A2.1. The cosine and the sine functions compared with their infinite polynomial products.

Annex 3. The Split Trigonometric Components of the Gamma Function

The second-degree split polynomials ($Q_{p\_p\_e(\zeta_2, CC_2)}$), ($Q_{p\_n\_e(\zeta_2, CD_2)}$) and their product, the cosine function are with (6.13):

$$Q_{p\_n\_e}(\sigma, n) = \prod_{j=1}^{n}\left(1 + \frac{\sigma}{2 \cdot j - 1}\right); \quad Q_{p\_p\_e}(\sigma, n) = \prod_{j=1}^{n}\left(1 - \frac{\sigma}{2 \cdot j - 1}\right)$$

$$Q_{p\_e}(\sigma, n) = Q_{p\_n\_e}(\sigma, n) \cdot Q_{p\_p\_e}(\sigma, n) = \cos\left(\frac{\pi}{2} \cdot \sigma\right) \tag{A3.1}$$

The split components of the cosines function are shown in **Figure A3.1** below with the parameter ($n = 50000$), for the range ($\sigma = -10, -9.8, -9.7, \cdots, 9.8, 10$), as well as their product. The split component ($Q_{p\_p\_e}(\sigma)$) of the cosine function has only positive real roots at odd multiples of ($\pi/2$) and its absolute value is decreases, the other split component with only negative real roots. The split components are the transpose to each other. Both components are equal to unity at zero: ($Q_{p\_p\_e}(0, n) = 1$, $Q_{p\_n\_e}(0, n) = 1$).

$$Q_{p\_p\_e}\left(5 \cdot \frac{\pi}{2}, n\right) = 2.037280934236391 \times 10^{-18}$$

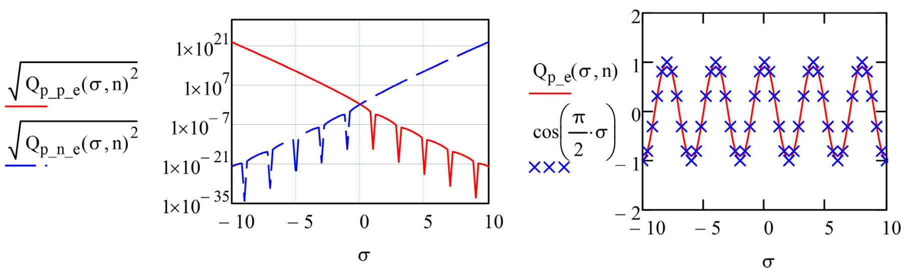

Figure A3.1. The split cosine functions and their product, the cosine function.

Annex 4. The Split Trigonometric Components of the Gamma Function

With (6.14) and (6.15) the sum and the difference of these functions define the first-degree split functions, which formally correspond to the cosine hyperbolic and sine hyperbolic functions:

$$QQ_{p\_p\_e}(\sigma,n)=\frac{1}{2}\cdot\left(Q_{p\_p\_e}(\sigma,n)+Q_{p\_n\_e}(\sigma,n)\right)$$

$$RQ_{p\_p\_e}(\sigma,n)=\frac{1}{2}\cdot\left(Q_{p\_p\_e}(\sigma,n)-Q_{p\_n\_e}(\sigma,n)\right)$$

with the parameter ($n=50000$) and for the range ($\sigma=-1,-0.9,-0.8,\cdots,0.9,1$) they are shown for real arguments in the figure below. In fact, they are proportional to the cosine hyperbolic and sine hyperbolic functions, with the factor of proportionality of the arguments equal to (6) (**Figure A4.1**):

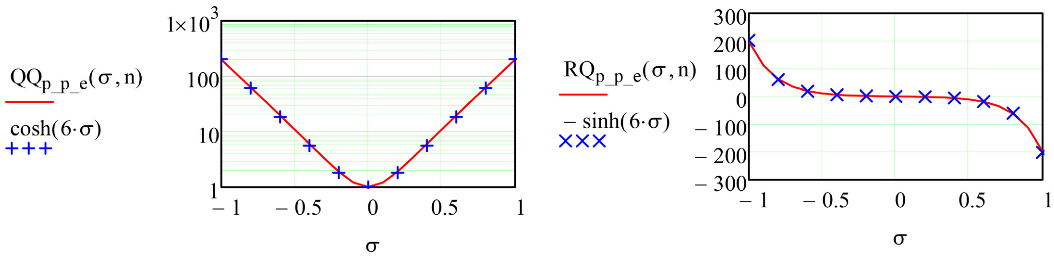

Figure A4.1. The sum and difference of the split cosine function for real arguments.

Annex 5. Shifting to the Critical line the Hyperbolic and Trigonometric Functions

In the range ($\sigma=-3,-2.8,-2.6,\cdots,2.8,3$) the sine hyperbolic and the cosine hyperbolic functions are shown in central and in shifted positions (**Figure A5.1** and **Figure A5.2**):

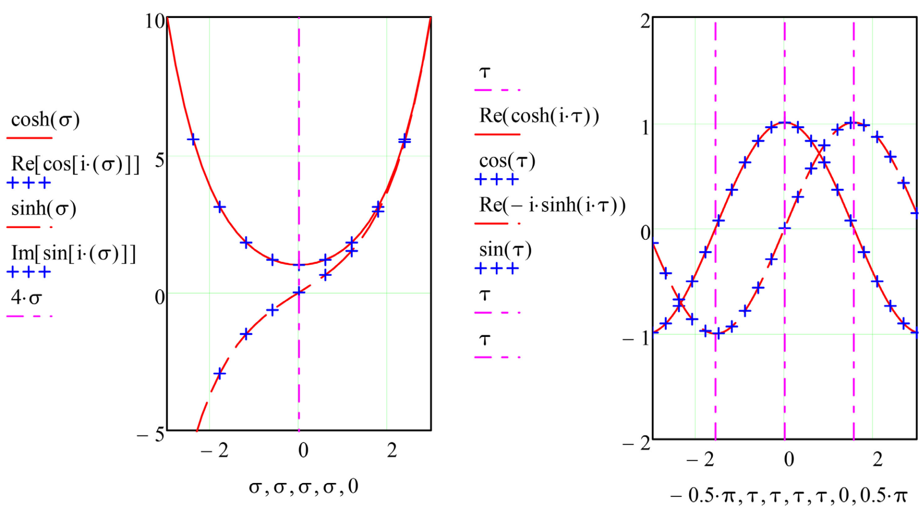

Figure A5.1. The cosine and the sine hyperbolic functions and the roots of their adjoint functions in central position on the imaginary axes.

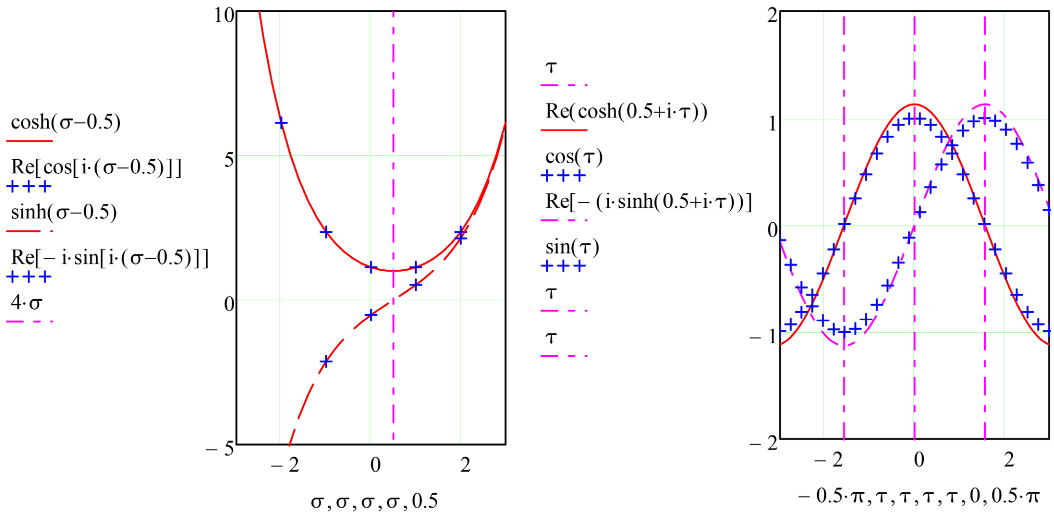

Figure A5.2. The cosine and the sine hyperbolic functions shifted to the critical line and the imaginary components of the complex roots of their adjoint functions on the critical line.