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Caristi's Fixed Point Theorem in *G*-Cone Metric Spaces and Application

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Abstract

In this work, we introduce a few versions of Caristi's fixed point theorems in *G*-cone metric spaces which extend Caristi's fixed point theorems in metric spaces. Analogues of such fixed point theorems are proved in this space. Our work extends a good number of results in this area of research.

Keywords

Caristi's Fixed Point, *G*-Cone Metric Spaces, Unique Fixed Point, Cauchy Sequence

1. Introduction

The study of metric fixed-point theory has been researched extensively in the past decades, since fixed point theory plays a vital role in mathematics and applied sciences. Different mathematicians tried to generalize the usual notion of metric space (X, d) to extend the known metric space theorems in a more general setting [1]-[15]. But different authors have proved that these attempts are invalid [1] [7] [8] [10] [11] [12]. In 2004, Mustafa and Sims introduced a generalized metric space, the generalization of the usual metric space (X, d) [13].

In 2010, Beg, Abbas and Nazir introduced a concept of *G*-cone metric space by replacing the set of real numbers with ordered Banach space. They also introduced new fixed point theories in this new structure [15].

In the last decades, Caristi's fixed point theorem has been generalized and extended in several directions and the related references therein. The following are basic definitions and theorems.

Definition 1.1. [1] Let X be a non-empty set. Suppose that $d : X \times X \to [0, \infty)$ satisfies:

$$0 \le d(x,y) \quad \forall x,y \in X \quad \text{and} \quad d(x,y)=0 \quad \text{if and only if} \quad x=y,$$

$$d(x,y)=d(y,x) \quad \forall x,y \in X$$

$$d(x,y) \le d(x,z)+d(z,y) \quad \forall x,y,z \in X$$

Then d is called a metric on X, and (X, d) is called a metric space.

Definition 1.2. [5] Let X be a non-empty set and $G: X \times X \times X \to [0,\infty)$ be a function satisfying the following properties:

$$G(x,y,z)=0 \quad \text{if and only if} \quad x=y=z,$$

$$G(x,x,y)>0 \quad \forall x,y \in X, \text{ with } x \ne y$$

$$G(x,x,y)<G(x,y,z) \quad \forall x,y,z, \in X, \text{ with } z \ne y$$

$$G(x,y,z)=G(\rho(x,y,z)) \quad \text{(symmetry)}$$

where ρ denotes the permutation function.

$$G(x,y,z) \le G(x,a,a)+G(a,y,z) \quad \forall a,x,y,z \in X \quad \text{(rectangle inequality).}$$

Then the function G is called a G-metric on X.

Definition 1.3. Let X be a non-empty set. Suppose that $d: X \times X \to E$ satisfies:

$$0 \le d(x,y) \quad \forall x,y \in X \quad \text{and} \quad d(x,y)=0 \quad \text{if and only if} \quad x=y,$$

$$d(x,y)=d(y,x) \quad \forall x,y \in X$$

$$d(x,y) \le d(x,z)+d(z,y) \quad \forall x,y,z \in X.$$

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.4. Let X be a non-empty set. Suppose $G: X \times X \times X \to E$ satisfies:

(G_1) $G(x,y,z)=0$ if $x=y=z$;

(G_2) $0<G(x,x,y)$; whenever $x \ne y$ $\forall x,y \in X$;

(G_3) $G(x,x,y) \le G(x,y,z)$; whenever $y \ne z$;

(G_4) $G(x,y,z)=G(x,z,y)=G(x,y,z)=\cdots$ (symmetric in all the three variables);

(G_5) $G(x,y,z) \le G(x,a,a)+G(a,y,z)$ $\forall x,y,z,a \in X$.

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or G-cone metric space.

Definition 1.5. A G-cone metric space X is symmetric if $G(x,y,y)=G(y,x,x)$ $\forall x,y \in X$.

Proposition 1.6. Let X be a G-cone metric space define $d_G: X \times X \to E$ by $d_G(x,y)=G(x,y,y)+G(y,x,x)$. Then (X, d_G) is a cone metric space. It can be noted that $G(x,y,y) \le 2/3\, d_G(x,y)$. If X is a symmetric G-cone metric space, then $d_G(x,y)=2G(x,y,y)$ $\forall x,y, \in X$.

Definition 1.7. Let X be a G-cone metric space and x_n be a sequence in X. We say that x_n is:

1) Cauchy sequence if for every $c \in E$ there is N such that $\forall m,n>N$

$G(x_n, x_m, x_l) \ll c$.

2) Convergent sequence if for every $c \in E$ with $0 \ll c$, there is N such that $\forall m, n > N$, $G(x_n, x_m, x) \ll c$ for some fixed $x \in X$. Here x is called the limit of the sequence x_n and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

3) A G-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Proposition 1.8. Let X be a G-cone metric space then the following are equivalent.

1) x_n is convergent to x;

2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

3) $G(x_n, x, x) \to 0$ as $n \to \infty$;

4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 1.9. Let X be a G cone metric space. x_m, y_n and z_l be sequences in X such that $x_m \to x$, $y_n \to y$ and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.

Lemma 1.10. Let x_n be a sequence in G-cone metric space X and $x \in X$. If x_n converges to x and x_n converges to y, then $x = y$.

Lemma 1.11. Let x_n be a sequence in G-cone metric space X and if x converges to x $\forall x \in X$, then $G(x_n, x_m, x) \to 0$ as $m, n \to \infty$.

Lemma 1.12. Let x_n be a sequence in G-cone metric space X and $x \in X$, if x_n converges to $x \in X$, then x_n is a Cauchy sequence.

Lemma 1.13. Let x_n be a sequence in a G-cone metric space X and if x_n is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to 0$, as $m, n, l \to \infty$.

Theorem 1.14. Let (*X*, *d*) be a complete metric space and let $T : X \to X$ be a mapping such that:

$$d(x, Tx) \le \phi(x) - \phi(Tx)$$

for all $x \in X$, where $\phi : X \to [0, \infty)$ is a lower semi continuous mapping. Then T has at least a fixed point.

Theorem 1.15. Let (*X*, *d*) be a complete metric space and let $T : X \to CB(X)$ be a mapping such that:

$$H(Tx, Ty) \le \eta(d(x, y))d(x, y)$$

for all $x, y \in X$, where $\eta : (0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{r \to x^i} \eta(r) < 1$, for all $r \in [0, \infty)$. Then T has a fixed point.

Theorem 1.16. Let (*X*, *d*) be a complete metric space, and let $T : X \to X$ be a mapping such that:

$$d(x, y) \le \phi(x, y) - \phi(Tx, Ty)$$

for all $x, y \in X$, where $\phi : X \to [0, \infty)$ is lower semicontinuous with respect to the first variable. Then T has a unique fixed point.

Theorem 1.17. Let (*X*, *d*) be a complete metric space and let $T : X \to X$ be a mapping such that for some $a \in [0, 1)$:

$$d(Tx, Ty) \le ad(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Problem 1.18. Let (*X*, *G*) be a complete *G*-cone metric space and let $T: X \to CB(X)$ be a multivalued mapping such that:

$$H(Tx, Ty, Tz) \le \mu G(x, y, z)$$

for all $x, y, z \in X$ where $\mu: R^+ \to R^+$ is continuous and increasing map such that $\mu(t) < t$, for all $t > 0$. Does T have a fixed point?

Problem 1.19. Let (*X*, *G*) be a complete G-cone metric space and let $T: X \to CB(X)$ be a mapping such that:

$$H(Tx, Ty, Tz) \le \eta(G(x, y, z)) G(x, y, z)$$

for all $x, y, z \in X$, where $\eta: [0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{r \to t^+} \eta(r) < 1$, for all $r \in (0, +\infty)$. Does T have a fixed point?

Theorem 1.20. Let (*X*, *d*) be a complete metric space and let $T: X \to X$ be a mapping such that:

$$d(Tx, Ty) \le \eta(d(x, y))$$

where $\eta: [0, \infty) \to [0, \infty)$ is a lower semi continuous mapping such that $\eta(t) < t$, for each $t > 0$, and $\frac{\eta(t)}{t}$ is a non decreasing map. Then T has a unique fixed point.

2. Main Result

Theorem 2.1

Let (*X*, *G*) be a complete *G*-cone metric space, and let $T: X \to X$ be a mapping such that:

$$G(x, x, y) \le \phi(x, y, y) - \phi(Tx, Tx, Ty)$$

for all $x, y \in X$, where $\phi: X \to [0, \infty)$ is lower semi continuous with respect to the first variable. Then T has a unique fixed point.

Proof:

For each $x \in X$, let $y = Tx$ and $\psi(x) = \phi(x, Tx, Tx)$. Then for each $x \in X$:

$$G(x, x, Tx) \le \psi(x) - \psi(Tx)$$

and ψ is a lower semi continuous mapping. Thus, applying Theorem 1.3 leads us to conclude the desired result.

To see the uniqueness of the fixed point suppose u and v are two distinct fixed points for *T*. Then:

$$d(u, v) \le \phi(u, v) - \phi(Tu, Tv) = \phi(u, v) - \phi(u, v) = 0$$

Thus, $u = v$.

Theorem 2.2

(Banach contraction principle in *G*-cone metric spaces) Let (*X*, *G*) be a complete *G*-cone metric space and let $T: X \to X$ be a mapping such that for some $a \in [0, 1)$:

$$G(Tx, Ty, Tz) \le aG(x, y, z) \qquad \text{(i)}$$

for all $x,y,z\in X$. Then T has a unique fixed point.

Proof

Define:

$$\phi(x,y,z)\le G(x,y,z)/(1-a)$$

(i) shows that:

$$(1-a)G(x,y,z)\le G(x,y,z)-G(Tx,Ty,Tz).$$

That is:

$$G(x,y,z)\le\frac{G(x,y,z)}{1-a}-\frac{G(Tx,Ty,Tz)}{1-a}$$

and so:

$$G(x,y,z)\le\phi(x,y,z)-\phi(Tx,Ty,Tz)$$

applying Theorem 2.1, one can conclude that T has a unique fixed point.

Theorem 2.3

Let (*X*, *G*) be a complete *G*-cone metric space and let $T:X\to X$ be a mapping such that:

$$G(Tx,Ty,Tz)\le\eta\left(G(x,y,z)\right)\tag{ii}$$

where $\eta:[0,\infty)\to[0,\infty)$ is a lower semi continuous mapping such that $\eta(t)<t$, for each $t>0$, and $\frac{\eta(t)}{t}$ is a non decreasing map. Then T has a unique fixed point.

Proof:

Define $\phi(x,y,z)=\dfrac{G(x,y,z)}{1-\dfrac{\eta\left(G(x,y,z)\right)}{G(x,y,z)}}$, if $x\ne y$ and otherwise $\phi(x,x,x)=0$.

Then (ii) shows that:

$$\left(1-\frac{\eta\left(G(x,y,z)\right)}{G(x,y,z)}\right)G(x,y,z)\le G(x,y,z)-G(Tx,Ty,Tz)$$

It means that:

$$G(x,y,z)\le\frac{G(x,y,z)}{1-\dfrac{\eta\left(G(x,y,z)\right)}{G(x,y,z)}}-\frac{G(Tx,Ty,Tz)}{1-\dfrac{\eta\left(G(x,y,z)\right)}{G(x,y,z)}}$$

Since $\frac{\eta(t)}{t}$ *is* non decreasing and $G(Tx,Ty,Tz)<G(x,y,z)$,

$$G(x,y,z)\le\frac{G(x,y,z)}{1-\dfrac{\eta\left(G(x,y,z)\right)}{G(x,y,z)}}-\frac{G(Tx,Ty,Tz)}{1-\dfrac{\eta\left(G(x,y,z)\right)}{G(x,y,z)}}=\phi(x,y,z)-\phi(Tx,Ty,Tz)$$

and so by applying Theorem 2.1, one can conclude that T has a unique fixed point.

The following results are the main results of this paper and play a crucial role to find the partial answers for Problem1.7 and Problem 1.8. Compare to the work of Farshid Khojasteh1, Erdal Karapinar and Hassan Khandani dealing with distance in a straight line, our work consider distance round a triangle.

Theorem 2.3

Let (X, G) be a complete G-cone metric space, and let $T : X \to CB(X)$ be a nonexpansive mapping such that, for each $x \in X$, and for all $y \in Tx$, there exists $z \in Ty$ such that:

$$G(x,y,y) \le \phi(x,y,y) - \phi(y,z,z) \tag{iii}$$

where $\phi : X \times X \to [0,\infty)$ is lower semicontinuous with respect to the first variable. Then T has a fixed point.

Proof:

Let $x_0 \in X$ and let $x_1 \in Tx_0$. If $x_0 = x_1$ then x_0 is a fixed point and we are through. Otherwise, let $x_0 \ne x_1$. By assumption there exists $x_2 \in Tx_1$ such that:

$$G(x_0,x_1,x_1) \le \phi(x_0,x_1,x_1) - \phi(x_1,x_2,x_2)$$

Alternatively, one can choose $x_n \in Tx_{n-1}$, such that $x_n \ne x_{n-1}$ and find $x_{n+1} \in Tx_n$ such that:

$$0 < G(x_{n-1},x_n,x_n) \le \phi(x_{n-1},x_n,x_n) - \phi(x_n,x_{n+1},x_{n+1}) \tag{iv}$$

which means that $[\phi(x_{n-1},x_n,x_n)]_n$ is a non-increasing sequence, bounded below, so it converges to some $r \ge 0$. By taking the limit on both sides of (iv) we have $\lim_{n\to\infty} G(x_{n-1},x_n,x_n) = 0$. Also, for all $m,n \in N$ with $m > n$,

$$\begin{aligned} G(x_n,x_m,x_m) &\le \sum_{i=n+1}^{m} G(x_{i-1},x_i,x_i) \\ &\le \sum_{i=n+1}^{m} \left(\phi(x_{i-1},x_i,x_i) - \phi(x_i,x_{i+1},x_{i+1})\right) \\ &\le \phi(x_n,x_{n-1},x_{n-1}) - \phi(x_m,x_{m+1},x_{m+1}) \end{aligned} \tag{v}$$

Therefore, by taking the limsup on both sides of (v) we have:

$$\limsup_{n\to\infty}\left(G(x_n,x_m,x_m)\right) = 0 .$$

It means that (x_n) is a Cauchy sequence and so it converges to $v \in X$. Now we show that v is a fixed point of T. We have:

$$\begin{aligned} G(v,Tv,Tv) &\le G(v,x_n,x_n) + G(x_n,Tv,Tv) \\ &= G(v,x_n,x_n) + H(Tx_n,Tv,Tv) \\ &\le G(v,x_n,x_n) + G(x_n,v,v) \end{aligned} \tag{vi}$$

By taking the limit on both sides of (vi), we get $G(x,x,Tx) = 0$ and this means that $x \in Tx$.

Theorem 2.4

Let (X, G) be a complete G-cone metric space, and let $T : X \to CB(X)$ be a multivalued function such that:

$$H(Tx,Ty,Ty) \le \eta G(x,y,y)$$

for all $x, y \in X$ where $\eta : [0,\infty) \to [0,\infty)$ is a lower semi continuous map such that $\eta(t)<t$, for all $t\in(0,+\infty)$, and $\frac{\eta(t)}{t}$ is nondecreasing. Then T has a fixed point.

Proof:

Let $x\in X$ and $y\in Tx$ then T has a fixed point and the proof is complete, so we suppose that $x\neq y$.

Define $\theta(t)=\frac{\eta(t)+t}{2}$ for all $t\in(0,\infty)$. We have:

$$H(Tx,Ty,Ty)\leq\eta(G(x,y,y))<\theta(G(x,y,y))<G(x,y,y).$$

Thus there exists $\epsilon_0>0$ such that $\theta(G(x,y,y))=H(Tx,Ty,Ty)+\epsilon_0$. So there exists $z\in Ty$ such that:

$$G(y,z,z)<H(Tx,Ty,Ty)+\epsilon_0=\theta(G(x,y,y))<G(x,y,y).$$

We again suppose that $z\neq y$; therefore

$G(x,y,y)-\theta(d(x,y,y))\leq G(x,y,y)-G(y,z,z)$ or equivalently:

$$G(x,y,y)<\frac{G(x,y,y)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}}+\frac{G(y,z,z)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}}$$

Since $\frac{\eta(t)}{t}$ is also a nondecreasing function and $G(y,z,z)<G(x,y,y)$ we get:

$$G(x,y,y)<\frac{G(x,y,y)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}}+\frac{G(y,z,z)}{1-\frac{\theta(G(y,z,z))}{G(y,z,z)}}$$

Define $\phi(x,y,y)=\frac{G(x,y,y)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}}$ if $x\neq y$, otherwise 0 for all $x,y\in X$. It means that:

$$G(x,y,y)=\phi(x,y,y)-\phi(y,z,z).$$

Therefore, T satisfies (iii) of Theorem 2.3 and so we conclude that T has a unique fixedpoint u and the proof is completed.

Existence of bounded solutions of functional equations [4]

Mathematical optimization is one of the fields in which the methods of fixed point theory are widely used. It is well known that dynamic programming provides useful tools for mathematical optimization and computer programming. In this setting, the problem of dynamic programming related to a multistage process reduces to solving the functional equation:

$$p(x)=\sup_{y\in\tau}\left\{f(x,y)+\vartheta\big(x,y,p(\eta(x,y))\big)\right\},x\in Z \tag{a}$$

where $\eta:Z\times\tau\to Z$, $f:Z\times\tau\to R$ and $\vartheta:Z\times\tau\times R\to R$. We assume that M and N are Banach spaces, $Z\subset M$ is a state space, and $T\subset N$ is a decision space.

The studied process consists of a state space, which is the set of the initial state, actions, and a transition model of the process, and *a decision space*, which is the set of possible actions that are allowed for the process.

Here, we study the existence of the bounded solution of the functional equation. Let *B*(*Z*) denote the set of all bounded real-valued functions on W and, for an arbitrary $h \in B(Z)$, define $\|h\| = \sup_{x\in Z}\left[h(x)\right]$. Clearly, $\left(B(W), \|.\|\right)$ endowed with the metric d defined by:

$$d(h,k) = \sup_{x\in Z}\left|h(x) - k(x)\right| \tag{b}$$

for all $h, k \in B(Z)$, is a Banach space. Indeed, the convergence in the space *B*(*Z*) with respect to $\|.\|$ is uniform. Thus, if we consider a Cauchy sequence $\{h_n\}$ in *B*(*Z*), then $\{h_n\}$ converges uniformly to a function, say h_*, that is bounded and so $h \in B(Z)$. We also define $S : B(Z) \to B(Z)$ by:

$$Sh(x) = \sup_{y\in\tau}\left\{f(x,y) + \vartheta\left(x, y, h\left(\eta(x,y)\right)\right)\right\} \tag{c}$$

for all $h \in B(Z)$ and $x \in Z$.

Remark: We can extend this to cone and G-cone metric spaces.

$$G(h,k,k) = \sup_{x\in Z}\left|h(x) - k(x)\right| \text{ for all } h,k \in B(Z).$$

We will prove the following theorems.

Theorem 2.5

Let $S : B(Z) \to B(Z)$ be an upper semi continuous operator defined by (c) and assume that the following conditions are satisfied:

1) $f : Z \times T \to R$ and $\vartheta : Z \times T \times R \to R$ are continuous and bounded;

2) for all $h, k \in B(Z)$, if:

$$0 < d(h,k) < 1 \text{ implies } \left|\vartheta\left(x,y,h(x)\right) - \vartheta\left(x,y,k(x)\right)\right| \le \frac{1}{2}d^2(h,k),$$

$$d(h,k) \ge 1 \text{ implies } \left|\vartheta\left(x,y,h(x)\right) - \vartheta\left(x,y,k(x)\right)\right| \le \frac{2}{3}d(h,k) \tag{d}$$

where $x \in Z$ and $y \in \tau$. Then the functional Equation (a) has a bounded solution.

Proof:

Note that (*B*(*Z*), *d*) is a complete cone metric space, where d is the cone metric given by (b).

Let μ be an arbitrary positive number, $x \in Z$, and $h_1, h_2 \in B(Z)$, then there exist $y_1, y_2 \in \tau$ such that:

$$S(h_1)(x) < f(x,y_1) + \vartheta\left(x, y_1, h_1\left(\eta(x,y_1)\right)\right) + \mu \tag{e}$$

$$S(h_2)(x) < f(x,y_2) + \vartheta\left(x, y_2, h_2\left(\eta(x,y_2)\right)\right) + \mu \tag{f}$$

$$S(h_1)(x) \ge f(x,y_1) + \vartheta\left(x, y_1, h_1\left(\eta(x,y_1)\right)\right) \tag{g}$$

$$S(h_2)(x) \ge f(x,y_2) + \vartheta\left(x, y_2, h_2\left(\eta(x,y_2)\right)\right) \tag{h}$$

Let $\partial:[0,\infty)\to[0,\infty)$ be defined by:

$$\partial(t)=\begin{cases}\frac{1}{2}t^2, & 0<t<1\\ \frac{1}{2}t, & t\geq 1\end{cases}$$

Then we can say that (d) is equivalent to:

$$\left|\vartheta(x,y,h(x))-\vartheta(x,y,k(x))\right|\leq\partial(d(h,k)), \tag{i}$$

for all $h,k\in B(Z)$. It is easy to see that $\partial(t)<t$, for all $t>0$ and $\frac{\partial(t)}{t}$ is a non decreasing function.

Therefore, by using (e), (h) and (i), it follows that:

$$\begin{aligned}S(h_1)(x)-S(h_2)(x)&<\vartheta(x,y_1,h_1(\eta(x,y_1)))-\vartheta(x,y_2,h_2(\eta(x,y_2)))+\mu\\&\leq\left|\vartheta(x,y_1,h_1(\eta(x,y_1)))-\vartheta(x,y_2,h_2(\eta(x,y_2)))\right|+\mu\\&\leq\partial(d(h_1,h_2))+\mu\end{aligned}$$

Then we get:

$$S(h_1)(x)-S(h_2)(x)<\partial(d(h_1,h_2))+\mu \tag{j}$$

Analogously, by using (f) and (g), we have:

$$S(h_2)(x)-S(h_1)(x)<\partial(d(h_1,h_2))+\mu \tag{k}$$

Hence, from (j) and (k) we obtain:

$$\left|S(h_1)(x)-S(h_2)(x)\right|<\partial(d(h_1,h_2))+\mu$$

that is,

$$d(S(h_1),S(h_2))<\partial(d(h_1,h_2))+\mu$$

Since the above inequality does not depend on $x\in Z$, $\mu>0$ is taken arbitrary, we conclude immediately that:

$$d(S(h_1),S(h_2))\leq\partial(d(h_1,h_2))$$

so we deduce that the operator S is a ∂-contraction. Thus, due to the continuity of S, Theorem 2.4 applies to the operator S, which has a fixed point $h^*\in B(Z)$, that is, h^* is a bounded solution of the functional Equation (a).

Theorem 2.6

Let $S:B(Z)\to B(Z)$ be an upper semi continuous operator defined by (c) and assume that the following conditions are satisfied:

1) $f:Z\times T\to R$ and $\vartheta:Z\times T\times R\to R$ are continuous and bounded;

2) for all $h,k\in B(Z)$, if

$$0<G(h,k,k)<1 \text{ implies } \left|\vartheta(x,y,h(x))-\vartheta(x,y,k(x))\right|\leq\frac{1}{2}G^2(h,k,k),$$

$$G(h,k,k)\geq 1 \text{ implies } \left|\vartheta(x,y,h(x))-\vartheta(x,y,k(x))\right|\leq\frac{2}{3}G(h,k,k) \tag{d}$$

where $x \in Z$ and $y \in \tau$. Then the functional Equation (a) has a bounded solution.

Proof:

Note that $(B(Z), d)$ is a complete G-cone metric space, where G is the G-cone metric define by $G(h,k,k) = \sup_{x \in Z} |h(x) - k(x)|$.

Let μ be an arbitrary positive number, $x \in Z$, and $h_1, h_2 \in B(Z)$, then there exist $y_1, y_2 \in \tau$ such that:

$$S(h_1)(x) < f(x,y_1) + \vartheta(x,y_1,h_1(\eta(x,y_1))) + \mu \tag{e}$$

$$S(h_2)(x) < f(x,y_2) + \vartheta(x,y_2,h_2(\eta(x,y_2))) + \mu \tag{f}$$

$$S(h_1)(x) \geq f(x,y_1) + \vartheta(x,y_1,h_1(\eta(x,y_1))) \tag{g}$$

$$S(h_2)(x) \geq f(x,y_2) + \vartheta(x,y_2,h_2(\eta(x,y_2))) \tag{h}$$

Let $\partial : [0,\infty) \to [0,\infty)$ be defined by:

$$\partial(t) = \begin{cases} \frac{1}{2}t^2, & 0 < t < 1 \\ \frac{1}{2}t, & t \geq 1 \end{cases}$$

Then we can say that (d) is equivalent to:

$$|\vartheta(x,y,h(x)) - \vartheta(x,y,k(x))| \leq \partial(G(h,k,k)), \tag{i}$$

for all $h,k \in B(Z)$. It is easy to see that $\partial(t) < t$, for all $t > 0$ and $\frac{\partial(t)}{t}$ is a nondecreasing function.

Therefore, by using (e), (h) and (i), it follows that:

$$\begin{aligned} S(h_1)(x) - S(h_2)(x) &< \vartheta(x,y_1,h_1(\eta(x,y_1))) - \vartheta(x,y_2,h_2(\eta(x,y_2))) + \mu \\ &\leq |\vartheta(x,y_1,h_1(\eta(x,y_1))) - \vartheta(x,y_2,h_2(\eta(x,y_2)))| + \mu \\ &\leq \partial(G(h_1,h_2,h_2)) + \mu \end{aligned}$$

Then we get:

$$S(h_1)(x) - S(h_2)(x) < \partial(G(h_1,h_2,h_2)) + \mu \tag{j}$$

Analogously, by using (f) and (g), we have:

$$S(h_2)(x) - S(h_1)(x) < \partial(G(h_1,h_2,h_2)) + \mu \tag{k}$$

Hence, from (j) and (k) we obtain:

$$|S(h_1)(x) - S(h_2)(x)| < \partial(G(h_1,h_2,h_2)) + \mu$$

that is,

$$d(S(h_1),S(h_2)) < \partial(G(h_1,h_2,h_2)) + \mu .$$

Since the above inequality does not depend on $x \in Z$, $\mu > 0$ is taken arbitrary, we conclude immediately that:

$$d\left(S\left(h_1\right),S\left(h_2\right)\right)\le\partial\left(G\left(h_1,h_2,h_2\right)\right)$$

so we deduce that the operator S is a ∂-contraction. Thus, due to the continuity of S, Theorem 2.4 applies to the operator S, which has a fixed point $h^*\in B\left(Z\right)$, that is, h^* is a bounded solution of the functional Equation (a).

Authors' Contributions

All the authors have made equal contributions.

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Conflicts of Interest

The authors declare that they have no competing interests.

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