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Polynomial Functions Composed of Terms with Non-Integer Powers

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Abstract

Polynomial functions containing terms with non-integer powers are studied to disclose possible approaches for obtaining their roots as well as employing them for curve-fitting purposes. Several special cases representing equations from different categories are investigated for their roots. Curve-fitting applications to physically meaningful data by the use of fractional functions are worked out in detail. Relevance of this rarely worked subject to solutions of fractional differential equations is pointed out and existing potential in related future work is emphasized.

Keywords

Rational, Irrational, Imaginary and Transcendental Powers, Roots of Functions with Variables of Non-Integer Powers, Curve-Fitting by Functions Incorporating Terms with Rational Powers, Fractional Differential Equations

1. Introduction

Euler (1707-1783), arguably the most prolific mathematician of all times who contributed greatly to every branch of applied and pure mathematics, was probably the first to work on functions containing terms with fractional powers. Euler's most praised book *Introductio in Analysin Infinitorum* [1], which is regarded to establish the subject of mathematical analysis, opens with definitions and exercises concerning various functions including *fractional exponents* of variables [1]. Laguerre (1834-1886) gave a survey of the roots of polynomials in usual forms but also considered what may briefly be termed as polynomials of fractional powers [2]. Nevertheless, studies concerning functions with fractional exponents have been quite limited over the years and the relevant literature is scarce. On the other hand, works on differential and integral equations of fractional order are on the rise and some analytical solutions to these types of equations are linked to functions containing terms with fractional powers.

Among numerous works on fractional derivatives and integrals, we mention only a few here for introductory purposes. Diethelm *et al.* [3] presented a selection of algorithms for numerical solution of definite governing equations with derivatives or integrals of fractional order. Daftardar-Gejji and Jafari [4] employed Adomian decomposition method as rectified by Wazwaz [5] for the solution of a multi-order fractional differential equation. Li [6] used cubic B-spline wavelet collocation for the solution of fractional differential equations and obtained excellent agreement with known analytical solutions containing terms of fractional powers. Yüzbaşi [7] solved fractional Riccati type equations by employing Bernstein polynomials as converted to contain fractional powers. Kawala [8] proposed a numerical approach based on Legendre polynomials for the solution of a class of fractional differential equations. An extensive review of fractional calculus in modelling biological phenomena can be found in Ionescu *et al.* [9]. In these and quite many similar works, it is possible to see a definite connection between factional derivatives and fractional functions.

The present study may be viewed as a preliminary work investigating the functions with terms of non-integer powers. First, roots of selected functions are considered; particularly, terms with fractional and transcendental powers are examined and distinct differences between these two categories are pointed out. Then, practical use of such functions is explored for curve-fitting purposes. Satisfactory results are obtained for the fractional functions constructed by employing physically meaningful data. Future applications of these functions are likely to expand into wider domains.

2. Polynomial Functions of Non-Integer Powers

General form of a polynomial of non-integer powers can be written as

$$P_\alpha(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_{n-1} x^{\alpha_{n-1}} + a_n \tag{1}$$

where $a_j, \alpha_j \in \mathbb{R}$ or $\mathbb{C}$ or irrational or transcendental. For covering all cases, the coefficients, a_j, and especially powers, α_j, may briefly be called as non-integer. Due to virtually unlimited possibilities a general treatment of (1) cannot be attempted; therefore, relatively simple particular cases would serve well to embark on the subject. For instance, one may inquire the total number of roots of the following equations.

$$x^{5/2} + 1 = 0, \qquad x^{2/3} + 2x^{1/2} + 1 = 0 \tag{2}$$

Making the cases *apparently* more complicated the determination of all the roots of the below equations may be sought.

$$x^{1.23} + 1 = 0, \qquad x^{5.71} + x^{0.48} + 1 = 0 \tag{3}$$

Still more, we may question the differences in number of roots between the transcendental powers π and e and those of their truncated counterparts such as

$$x^{\pi} + 1 = 0, \qquad x^{3.141} + 1 = 0; \qquad x^{e} + 1 = 0, \qquad x^{2.718} + 1 = 0 \tag{4}$$

Further, roots of the following equations may be questioned.

$$x^{\pi}+x^{ei}+i=0, \qquad x^{3.141}+x^{2.718i}+i=0 \tag{5}$$

In this manner, we could easily go on proposing more and more complicated equations involving more terms and coefficients. However, as we would hardly make any progress by proceeding so, we are going to tackle with problems amenable to treatment at least to some extent while making only some comments on those that can be treated partially, and finally avoid all those which are beyond our capability. Concerning the practical use of these types of functions we present some curve-fitting applications and thereby reveal the challenges lie in extending them to general multi-term forms defined in (1).

3. Roots of Some Particular Functions of Non-Integer Powers

Three different equations containing variables with non-integer powers are considered for demonstrating how the roots can be obtained.

3.1. Example 1

Let us seek the roots of the following fractional polynomial equation.

$$x^{5/2}+1=0 \tag{6}$$

Different approaches to the solution of this problem are possible; two of these are presented here. We first use a very elementary approach as follows.

$$\begin{aligned}
&\ln x^{5/2}=\ln(-1)=\ln\left[e^{i(\pm\pi\pm2\pi k)}\right]\\
&\frac{5}{2}\ln x=i(\pi+2\pi k)\\
&x=\exp\left[i\frac{2}{5}(\pi+2\pi k)\right]=\cos\left[\frac{2}{5}(\pi+2\pi k)\right]+i\sin\left[\frac{2}{5}(\pi+2\pi k)\right]
\end{aligned} \tag{7}$$

in which -1 in the natural logarithm is expressed as $e^{i(\pm\pi\pm2\pi k)}=\cos(\pm\pi\pm2\pi k)+i\sin(\pm\pi\pm2\pi k)$ with $k=0,1,2,\cdots$ and then $\pm$ sign has been replaced by + sign here and hereafter without loss of generality. Note also that k is limited to $k=0,1,2,3,4$ since the quintic roots are evaluated. The above method is of course identical to the usual way of evaluating the power or roots of a complex number. In this particular case the equation is simultaneously squared and its quintic roots are computed; that is, $x_k=\left[\exp i(\pi+2\pi k)\right]^{2/5}$ for $k=0,1,2,3,4$. The corresponding numerical values are

$$\begin{aligned}
&x_0=+0.309017+0.951057i, \quad x_1=-0.809017-0.587785i, \quad x_2=+1+0i\\
&x_3=-0.809017+0.587785i, \quad x_4=+0.309017-0.951057i
\end{aligned} \tag{8}$$

Caution should be observed in rejecting *apparently false* roots such as $x_2=+1+0i$. At first glance this root does not appear to satisfy Equation (6) but this is not the case. Square roots of 1 are +1 and −1 which correspond to $k=0$ and $k=1$ modes respectively in

$1^{1/2} = \left[\cos(0+2\pi k) + i\sin(0+2\pi k)\right]^{1/2} = \cos(0+\pi k) + i\sin(0+\pi k)$. Taking $k=1$ for the second root we have $1^{1/2} = -1$ and consequently $1^{5/2} = (-1)^5 = -1$, which does satisfy $x^{5/2}+1=0$.

Obviously all the roots must be considered when dealing with fractional equations. To clarify this point visually for the present example two graphs of $P_{5/2}(x) = x^{5/2}+1$ are drawn together in Figure 1 by evaluating the square root in $x^{5/2}$ positive $k=0$ and negative $k=1$. Note that for the latter case the function has a zero at $x=1$ as computed.

The second approach introduces a new variable $u = x^{1/2}$ so that (6) becomes

$$u^5 + 1 = 0 \tag{9}$$

which has the roots $u_k = \left[\exp i(\pi + 2\pi k)\right]^{1/5}$ with $k = 0,1,2,3,4$:

$$\begin{aligned} &u_0 = +0.809017 + 0.587785i, \quad u_1 = -0.309017 + 0.951057i, \quad u_2 = -1 + 0i \\ &u_3 = -0.309017 - 0.951057i, \quad u_4 = +0.809017 - 0.587785i \end{aligned} \tag{10}$$

Computing the solutions to (6) by employing $x_k = u_k^2$ naturally renders the results given in (8). Note that according to this approach $u_2 = -1$, which makes it clear that we must select −1 as the appropriate root of +1 in $u_2 = x_2^{1/2} = 1^{1/2}$ to satisfy the equation. Finally, reason for introducing this rather longer approach lies in its use in multi-term equations.

3.2. Example 2

Let us seek the roots of the following multi-term polynomial equation with fractional powers.

$$x^{2/3} + 2x^{1/2} + 1 = 0 \tag{11}$$

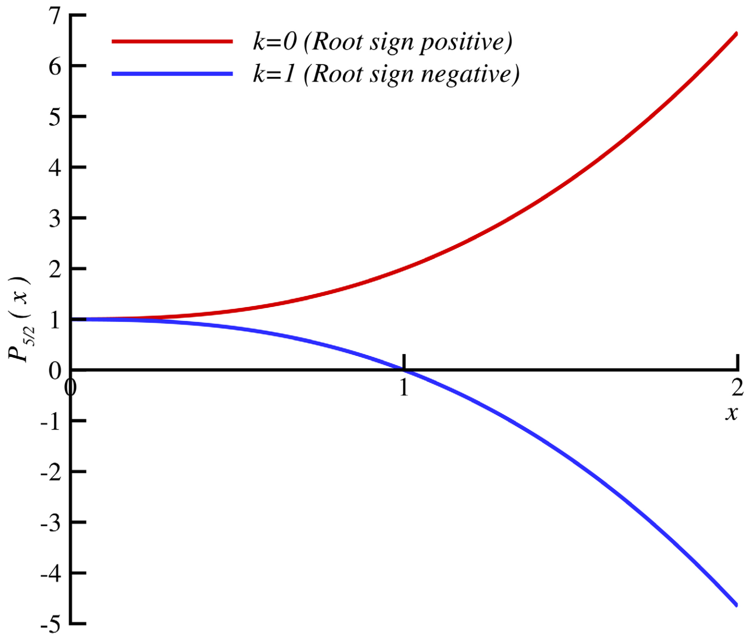

Figure 1. Graphs of $P_{5/2}(x) = x^{5/2}+1$ as computed for the positive $k=0$ (red) and negative $k=1$ (blue) root signs within the domain $0 \le x \le 2$.

As we have more than one unknown term the second approach of Example 1 is needed. Expressing the fractions under a common denominator gives

$$x^{4/6} + 2x^{3/6} + 1 = 0 \tag{12}$$

Introducing a new variable $u = x^{1/6}$ transforms (12) to

$$u^4 + 2u^3 + 1 = 0 \tag{13}$$

which may be shown to have the roots $u_0 = -1$, $u_1 = -1.839286752$, $u_2 = +0.419643375 + 0.60629073i$, and $u_2 = +0.419643375 - 0.60629073i$. From $u = x^{1/6}$ we get $x = u^6$ and obtain the roots of (11) as

$$\begin{aligned} &x_0 = +1 + 0i, \quad x_1 = +38.7165505 + 0i, \\ &x_2 = +0.141724546 - 0.75781919i, \quad x_3 = +0.141724546 + 0.75781919i \end{aligned} \tag{14}$$

As in the first example the roots $x_0 = +1$ and $x_1 = +38.7165505$ give the false impression that they are not true solutions. The key point here hinges on taking the appropriate roots. Numerically, $x_0^{1/2} = 1^{1/2} = -1$ and $x_0^{2/3} = 1^{2/3} = \left(1^{1/3}\right)^2 = 1^2 = 1$ hence $1^{2/3} + 2 \cdot 1^{1/2} + 1 = 1 - 2 + 1 = 0$. In exactly the same manner we can show that for $x_1 = +38.7165505$ we have $38.7165505^{2/3} + 2 \cdot 38.7165505^{1/2} + 1 = +11.44452497 + 2 \cdot \left(-6.22226249\right) + 1 = 0$. Likewise, for the other solutions $x_2 = +0.141724546 - 0.75781919i$ and $x_3 = +0.141724546 + 0.75781919i$, the appropriate roots must be used. This is a somewhat ambiguous point without a strict rule as to which mode, $k = 0$ or $k = 1$, should be selected for the root. At present, we may regard it a peculiarity of fractional equations. In summary, Equation (11) has four solutions altogether and to show that these solutions satisfy the equation the fractional powers must be computed for the appropriate mode, which is $k = 1$ for the square root and $k = 0$ for the cubic root in this case. Also, for a given set of x values $P_{2/3}\left(x\right) = x^{2/3} + 2x^{1/2} + 1$ produces six different sets of values by separate use of three modes $k = 0,1,2$ in evaluating the cubic root of $x^{2/3}$ and two modes $k = 0,1$ for the square root of $x^{1/2}$. Hence, similar to **Figure 1** of $P_{5/2}\left(x\right) = x^{5/2} + 1$, multiple graphs can be drawn for $P_{2/3}\left(x\right)$.

Finally, any kind of equation involving fractional powers of variables can be treated in the above manner. For instance, the equation $x^{5.71} + x^{0.48} + 1 = 0$ in (3) can be expressed in terms of fractions as $x^{571/100} + x^{12/25} + 1 = 0$. In order to get a solvable equation the denominators must be the same hence $x^{571/100} + x^{48/100} + 1 = 0$. Defining the intermediate variable $u = x^{1/100}$ one gets the polynomial form $u^{571} + u^{48} + 1 = 0$, which at least *in principle* can be solved to obtain 571 roots of u and consequently the roots of $x^{5.71} + x^{0.48} + 1 = 0$ by employing $x = u^{100}$.

3.3. Example 3

In this example we essentially tackle with the problem of obtaining and examining roots of transcendental equation

$$x^{\pi} + 1 = 0 \tag{15}$$

which is going to prove to be a very interesting case. Let us first consider

$x^{3.1}+1=0$, which may be viewed as an *approximate* form of (15). Expressing the exponent as fraction $x^{31/10}+1=0$ and introducing $u=x^{1/10}$ result in the polynomial $u^{31}+1=0$ with 31 roots. If we take one more decimal $x^{3.14}+1=0$, which gives rise to $u^{157}+1=0$ with $u=x^{1/50}$ and requires the determination of 157 roots. We may then conclude that increasing the number of digits to define π would increase the number of roots. Continuing in this line of reasoning we may infer that the process would figuratively lead to an equation with infinite number of roots since π is a transcendental number. Correctness of this naive deduction can be demonstrated in a simple way with quite interesting results. We take $x^{\pi}+1=0$ and proceed to solve it as done in the first example.

$$\begin{aligned} \ln x^{\pi} &= \ln\left[\mathrm{e}^{i(\pi+2\pi k)}\right] \\ \pi \ln x &= i(\pi+2\pi k) \\ x_k &= \exp\left[i(1+2k)\right] = \cos(1+2k) + i\sin(1+2k) \end{aligned} \tag{16}$$

where $k=0,1,2,\cdots$. We have thus shown that the πth roots of a complex number can be computed just like any integer roots as customarily done. Moreover, it is observed that there are infinite number of roots for transcendental numbers as conjectured above. The zeroth mode $k=0$ gives $x_0=\mathrm{e}^{i}$, which, upon substituting into the original equation $x^{\pi}+1=0$, reveals no other than the famous Euler formula:

$$\mathrm{e}^{\pi i}+1=0 \tag{17}$$

Also, the general solution renders infinitely many Euler formulas: $\mathrm{e}^{3\pi i}+1=0$, $\mathrm{e}^{5\pi i}+1=0$, $\mathrm{e}^{7\pi i}+1=0$, etc., which correspond to $k=1,2,3,\cdots$. An interesting comparison is possible by making a crude approximation $\pi \simeq 3$ and computing the roots of $x^3+1=0$ as

$$x_0=+0.5000+0.8660i,\quad x_1=-1.0000+0.0000i,\quad x_2=+0.5000-0.8660i \tag{18}$$

On the other hand, from (16) the roots of (15) for the first three modes $k=0,1,2$ are

$$x_0=+0.5403+0.8415i,\quad x_1=-0.9899+0.1411i,\quad x_2=0.2837-0.9589i \tag{19}$$

Closeness of the corresponding roots in (18) and (19) are clearly visible in Figure 2 where the first three roots of $x^{\pi}+1=0$ and all three roots of $x^3+1=0$ are plotted.

Convergence of roots of relevant fractional functions to those of $x^{\pi}+1=0$ can be further demonstrated. Computing the roots of $x^{3.1}+1=0$ and $x^{3.14}+1=0$ give respectively, $x_k=\exp\left[i\frac{10}{31}\pi(1+2k)\right]$ for $k=0,1,2,\cdots,30$ and $x_k=\exp\left[i\frac{50}{157}\pi(1+2k)\right]$ for $k=0,1,2,\cdots,156$ as solutions. Table 1 compares the numerical values of the first ten roots, $k=0,1,2,\cdots,9$, with those of $x^{\pi}+1=0$. As expected the roots of $x^{3.14}+1=0$ are closer to $x^{\pi}+1=0$ than $x^{3.1}+1=0$.

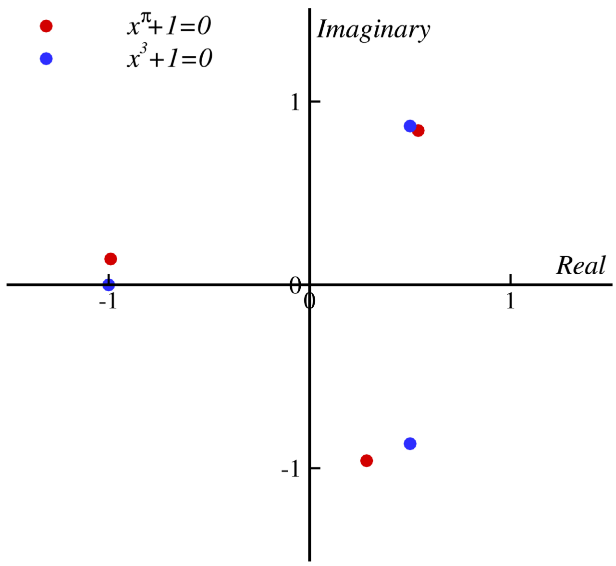

Figure 2. First three roots of $x^{\pi}+1=0$ (red) and all three roots of $x^{3}+1=0$ (blue).

Table 1. First ten roots of $x^{3.1}+1=0$ (second column), $x^{3.14}+1=0$ (third column), and $x^{\pi}+1=0$ (fourth column).

Mode	$x^{3.1}+1=0$	$x^{3.14}+1=0$	$x^{\pi}+1=0$
k	$x_k=\exp\left[i\frac{10}{31}\pi(1+2k)\right]$	$x_k=\exp\left[i\frac{50}{157}\pi(1+2k)\right]$	$x_k=\exp\left[i(1+2k)\right]$
0	$+0.52896+0.84864i$	$+0.53988+0.84175i$	$+0.54030+0.84147i$
1	$-0.99487+0.10117i$	$-0.99021+0.13961i$	$-0.98999+0.14112i$
2	$+0.34731-0.93775i$	$+0.28609-0.95820i$	$+0.28366-0.95892i$
3	$+0.68897+0.72479i$	$+0.75156+0.65966i$	$+0.75390+0.65699i$
4	$-0.95414+0.29936i$	$-0.91300+0.40796i$	$-0.91113+0.41212i$
5	$+0.15143-0.98847i$	$+0.01001-0.99995i$	$+0.00443-0.99999i$
6	$+0.82076+0.57127i$	$+0.90466+0.42614i$	$+0.90745+0.42017i$
7	$-0.87435+0.48530i$	$-0.76461+0.64449i$	$-0.75969+0.65029i$
8	$-0.05065-0.99872i$	$-0.26686-0.96373i$	$-0.27516-0.96140i$
9	$+0.91896+0.39436i$	$+0.98721+0.15940i$	$+0.98870+0.14988i$

We make some remarks concerning equations of the form $x^{\alpha}+1=0$ and their connection to the Euler formula. For instance, the roots of $x^{i}+1=0$ are $x_k=\exp(\pi+2\pi k)$ where $k=0,1,2,\cdots$. Likewise, it is fairly simple to show that $x^{e}+1=0$ has infinitely many solutions of the form $x_k=\exp\left[i(\pi+2\pi k)/e\right]$ with $k=0,1,2,\cdots$ since the ratio π/e is transcendental. All these solutions when substituted into their corresponding equations yield the Euler formula. This should be expected because the solution is initiated by expressing $-1=\exp\left[i(\pi+2\pi k)\right]$.

While we have made good progress in the solution of polynomial equations with terms of fractional powers and the special form $x^\alpha + 1 = 0$ with transcendental and imaginary powers, solution of the multi-term functions with transcendental powers as given in (5) remains a challenging problem.

4. Curve-Fitting Applications of Functions with Fractional Powers

Curve-fitting to a given data is a possible application area for functions containing terms with fractional powers. A general form as in (1) might be considered; however, just as in the case of roots, a corresponding general approach does not seem possible. For this reason our treatment is again confined to few special cases with select data.

A physically meaningful set of points, known as the righting moment lever or *GZ* values, which make up the most essential data representing a ship's stability, is used for demonstrations. To establish a $GZ - \phi$ curve quite complicated computations involving the underwater volume of a ship at definite heel angles ϕ, second moment of water plane area, etc. are necessary. Certain points of the *GZ* curve are particularly important for defining its characteristics. At the zero angle of heel normally *GZ* is zero and the first derivative of the *GZ* curve at zero angle is called the metacentric height *GM*. The maximum value of the righting lever is GZ_m and the vanishing angle of heel ϕ_v is the angle at which *GZ* becomes zero.

All the curve-fitting examples here use the *GZ* data points computed for an actual design given in Table 2. The *GZ* values are normalized by GZ_m so that $GZ_m = 1.0$. The normalized metacentric height is
$GM = \mathrm{d}GZ(0)/\mathrm{d}\phi \simeq \Delta GZ/\Delta\phi|_0 = (0.3182 - 0.0000)/(0.1745 - 0.0000) = 1.8235$,
and the vanishing angle from Table 2 is $\phi_v = 1.2217\ \text{rad} = 70^\circ$.

4.1. Example 1

We begin by selecting a simple fractional function of the form

$$GZ_f(\phi) = a_0\phi^{\alpha_0} + a_1\phi + a_2 \tag{20}$$

where a_0, a_1, a_2, and α_0 are constants to be determined by imposing certain conditions according to the data available in Table 2.

First, we assume $\alpha_0 > 0$ and set $a_2 = 0$ so that $GZ_f(0) = 0$. Next, requiring $\mathrm{d}GZ_f(0)/\mathrm{d}\phi = GM$ and supposing $\alpha - 1 > 0$ give $a_1 = GM$. At this stage it is obvious that α must be a real quantity greater than unity. Naturally, there is no guarantee that the conditions imposed to determine a_0 and α_0 are

Table 2. Dimensionless GZ_i data corresponding to heel angles ϕ_i (radians) computed for an actual design.

i	1	2	3	4	5	6	7	8
ϕ_i	0.0000	0.1745	0.3491	0.5236	0.6981	0.8727	1.0472	1.2217
GZ_i	0.0000	0.3182	0.6834	0.8845	1.0000	0.8571	0.5135	0.0000

going to yield an $\alpha_0 > 1$, as required. The value of α_0 depends on the data points and in some cases it may not be possible to get $\alpha_0 > 1$. In this particular application we impose the conditions $GZ_f(\phi_m) = GZ_m$ and $GZ_f(\phi_v) = 0$. It is emphasized that requiring only $GZ_f(\phi_m) = GZ_m$ does not make a true maximum at ϕ_m; for a true maximum $\mathrm{d}GZ_f(\phi_m)/\mathrm{d}\phi = 0$ must also be imposed as can be seen in the third example below. However, since only two free parameters are available in our present choice of fitting function we cannot impose more conditions. Then, making use of $a_2 = 0$ and $a_1 = GM$, and imposing $GZ_f(\phi_m) = GZ_m$ and $GZ_f(\phi_v) = 0$ to Equation (20) give

$$a_0\phi_m^{\alpha_0} + GM\phi_m = GZ_m, \qquad a_0\phi_v^{\alpha_0} + GM\phi_v = 0 \tag{21}$$

Solving a_0 from the latter as $a_0 = -GM\phi_v^{-\alpha_0}$ and using in the former with some straightforward manipulations give

$$\alpha_0 = \frac{\ln\left[\left(GM\phi_m - GZ_m\right)/GM\phi_v\right]}{\ln\left(\phi_m/\phi_v\right)} \tag{22}$$

which reveals that $GM\phi_m - GZ_m$ must be positive for a real α_0. It is also clear that getting an α_0 greater than unity depends on the numerical values of data.

For comparison purposes we also use a third-degree polynomial

$$GZ_p(\phi) = c_0\phi^3 + c_1\phi^2 + c_2\phi + c_3 \tag{23}$$

Conditions imposed on Equation (20) are now applied to (23) so that $c_2 = GM$, $c_3 = 0$, and

$$\begin{aligned} c_0 &= \left[GM + \frac{GZ_m}{\phi_m\left(\phi_m/\phi_v - 1\right)}\right] \Big/ \left(\phi_m\phi_v\right), \\ c_1 &= -\left[\left(\phi_m + \phi_v\right)GM + \frac{\phi_v GZ_m}{\phi_m\left(\phi_m/\phi_v - 1\right)}\right] \Big/ \left(\phi_m\phi_v\right) \end{aligned} \tag{24}$$

For the data in Table 2 the numerical values of all the constants are computed as $\alpha_0 = 3.75$, $a_0 = -1.05$, $a_1 = 1.82$, $a_2 = 0.00$ and $c_0 = -1.78$, $c_1 = 0.68$, $c_2 = 1.82$, $c_3 = 0.00$ hence the fractional $GZ_f(\phi)$ and polynomial $GZ_p(\phi)$ functions become

$$\begin{aligned} GZ_f(\phi) &= -1.05\phi^{3.75} + 1.82\phi \\ GZ_p(\phi) &= -1.78\phi^3 + 0.68\phi^2 + 1.82\phi \end{aligned} \tag{25}$$

Note that $\alpha_0 = 3.75$ is greater than unity as required for this particular application. Figure 3 depicts the curve-fittings given in (25) against the data $GZ_i(\phi)$ in Table 2. The polynomial function $GZ_p(\phi)$ performs slightly better since the true maximum of the fractional function $GZ_f(\phi)$ is shifted to the right more compared to the polynomial; nevertheless, the general characteristics of both curves are quite similar.

4.2. Example 2

In this application the data and functional forms are kept the same but instead of satisfying the maximum point $GZ(\phi_m) = GZ_m$, the least-squares method is

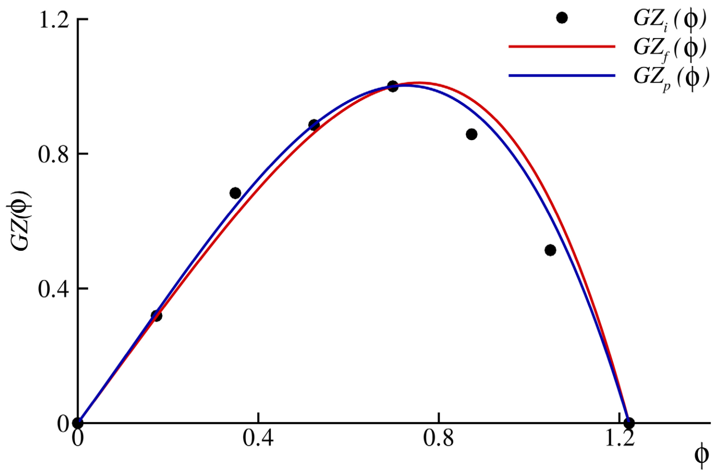

Figure 3. Data $GZ_i(\phi)$ (black dots) against curve-fittings by fractional power $GZ_f(\phi)$ (red) and polynomial $GZ_p(\phi)$ (blue).

employed in determination of constants both for the fractional and polynomial function. In this regard we enounce a *hybrid* approach in which besides satisfying definite points individually the entire data are utilized via the least-squares method. Skipping the parts in common with the first application we consider only the least-squares method as applied to the fractional function $GZ_f(\phi) = a_0\phi^{\alpha_0} + GM\phi$ with the vanishing angle requirement $a_0\phi_v^{\alpha_0} + GM\phi_v = 0$. Solving for a_0 and using it in the function result in

$$GZ_f(\bar{\phi}) = GM\phi_v\left(\bar{\phi} - \bar{\phi}^{\alpha_0}\right) \tag{26}$$

where $\bar{\phi} = \phi/\phi_v$ is defined for the simplicity of notation and α_0 is the only constant to be determined by employing the least-squares method with the data given in **Table 2**. For discreet data points the corresponding total squared error function is

$$E^2(\alpha_0) = \sum_{i=2}^{7}\left[\frac{GZ_i}{GM\phi_v} - \bar{\phi}_i + \bar{\phi}_i^{\alpha_0}\right]^2 \tag{27}$$

where GZ_i is an available data point corresponding to a definite ϕ_i value with $\bar{\phi}_i = \phi_i/\phi_v$. The first and last data points are already satisfied hence they are excluded from the least-squares application so that the running index i is from 2 to 7. Differentiating (27) with respect to α_0 gives the equation to be satisfied for minimizing the total squared error

$$\frac{\partial E^2(\alpha_0)}{\partial \alpha_0} = 2\sum_{i=2}^{7}\left[\frac{GZ_i}{GM\phi_v} - \bar{\phi}_i + \bar{\phi}_i^{\alpha_0}\right]\bar{\phi}_i^{\alpha_0}\ln\left(\bar{\phi}_i\right) = 0 \tag{28}$$

Obviously solving (28) for α_0 is not possible by conventional approaches. Therefore, we proceed by scanning the above function for a range of α_0 values with very small increments: $\alpha_0 = 1.0 + j\Delta\alpha_0$ with $j = 1,2,3,\cdots$ and $\Delta\alpha_0$ in the range of, say, 10^{-2} - 10^{-5} to obtain a functional value as close as possible to zero. At the same time the value of the error function is computed for each α_0.

Figure 4 shows both the error function and its derivative in the range of $1 \le \alpha_0 \le 5$ for the data of Table 2. Note that for approximately $\alpha_0 = 3.39$ the error function $E^2(\alpha_0)$ makes a minimum while its derivative becomes zero $\partial E^2(\alpha_0)/\partial \alpha_0 = 0$. Once α_0 is determined a_0 is computed from $a_0 = -GM\phi_v^{1-\alpha_0}$ while $a_1 = GM$ and $a_2 = 0$ as before. $\alpha_0 = 3.39$ is again greater than unity but nearly 10% less than the one computed in the first example by the use of maximum point GZ_m. The numerical values of the constants both for the fractional function and polynomial are obtained as $a_0 = -1.13$, $a_1 = 1.82$, $a_2 = 0.00$, $\alpha_0 = 3.39$ and $c_0 = -1.62$, $c_1 = 0.49$, $c_2 = 1.82$, $c_3 = 0.00$. The corresponding fractional $GZ_f(\phi)$ and polynomial $GZ_p(\phi)$ functions are

$$\begin{aligned} GZ_f(\phi) &= -1.13\phi^{3.39} + 1.82\phi \\ GZ_p(\phi) &= -1.62\phi^3 + 0.49\phi^2 + 1.82\phi \end{aligned} \tag{29}$$

Figure 5 makes the same comparison as Figure 3 by using the equations in (29). For a consistent comparison the least-squares approach is employed for c_0 of the polynomial function. The polynomial again performs somewhat better while the fractional power function differs only very slightly from the polynomial.

4.3. Example 3

The last application increases the number of terms to satisfy more conditions hence produce a much better curve-fitting function.

$$GZ(\phi) = a_0\phi^{\alpha_0} + a_1\phi^{\alpha_1} + a_2\phi + a_3 \tag{30}$$

The constants are to be determined by applying six different conditions:

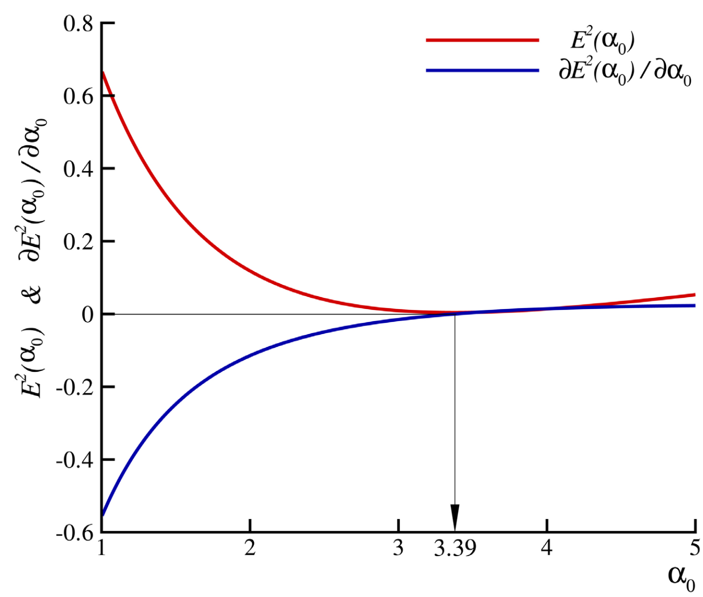

Figure 4. Total squared error $E^2(\alpha_0)$ (red) and its derivative $\partial E^2(\alpha_0)/\partial \alpha_0$ (blue).

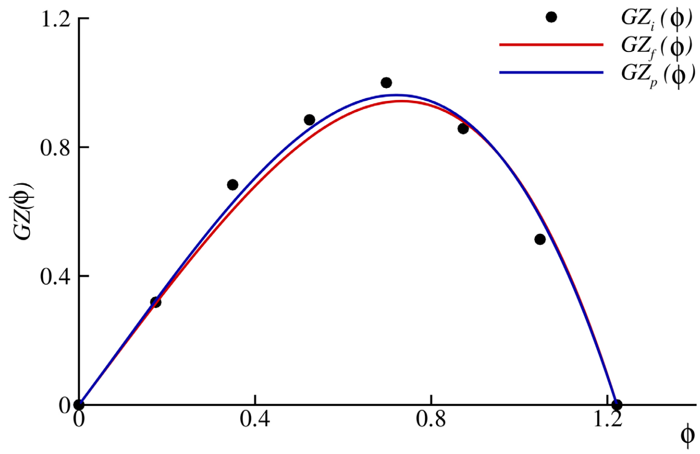

Figure 5. Data $GZ_i(\phi)$ (black dots) against curve-fittings by fractional power $GZ_f(\phi)$ (red) and polynomial $GZ_p(\phi)$ (blue).

$GZ(0)=0$, $\mathrm{d}GZ(0)/\mathrm{d}\phi=GM$, $GZ(\phi_m)=GZ_m$, $\mathrm{d}GZ(\phi_m)/\mathrm{d}\phi=0$, $GZ(\phi_v)=0$, and the total squared error for the GZ_i data $i=2,\cdots,7$ in Table 2 is a minimum. The last condition, used in the preceding application as well, is essential here to avoid perfect symmetry in equations with respect to the terms $a_0\phi^{\alpha_0}$ and $a_1\phi^{\alpha_1}$. Otherwise, the numerical scheme does not converge to a true solution as no preference can be made between these terms; that is, the numerical value of α_0 may well be consigned to that of α_1 and vice versa.

After satisfying the first two conditions listed above the following equations are obtained for $GZ(\phi_m)=GZ_m$ and $\mathrm{d}GZ(\phi_m)/\mathrm{d}\phi=0$:

$$\begin{aligned} a_0\phi_m^{\alpha_0}+a_1\phi_m^{\alpha_1}+GM\phi_m &= GZ_m \\ a_0\alpha_0\phi_m^{\alpha_0-1}+a_1\alpha_1\phi_m^{\alpha_1-1}+GM &= 0 \end{aligned} \tag{31}$$

which ensure that the function has a true maximum at ϕ_m. Solving for a_0 and a_1 gives respectively

$$\begin{aligned} a_0 &= \frac{GM}{\alpha_1-\alpha_0}\left[1+(q-1)\alpha_1\right]\phi_m^{1-\alpha_0} \\ a_1 &= \frac{GM}{\alpha_0-\alpha_1}\left[1+(q-1)\alpha_0\right]\phi_m^{1-\alpha_1} \end{aligned} \tag{32}$$

where $q=GZ_m/(GM\phi_m)$ is defined for notational convenience. a_0 and a_1 given in (32) are substituted into the fifth condition $GZ(\phi_v)=a_0\phi_v^{\alpha_0}+a_1\phi_v^{\alpha_1}+GM\phi_v=0$ and the resulting equation is used to express α_1 as

$$\alpha_1=\frac{\left[1+(q-1)\alpha_0\right]r^{\alpha_1}-r^{\alpha_0}+\alpha_0 r}{(q-1)r^{\alpha_0}+r} \tag{33}$$

where $r=\phi_v/\phi_m$. Similar to the second example we are going to seek an α_0 to make the total squared error minimum. To do so a scan is initiated for a range of α_0 values in the manner $\alpha_0=1.0+j\Delta\alpha_0$ with $j=1,2,3,\cdots$ while $\Delta\alpha_0$ is a small increment on the order of, say, 10^{-2} - 10^{-5}. For a given α_0 Equation (33) is used to obtain the corresponding α_1 value by successive iterations since α_1 appears on both sides of (33). Once α_1 is determined then a_0 and a_1 can be computed from (32). Having thus obtained all the parameters needed the least-squares approach can be initiated by calculating the total squared error for the successive α_0 values

$$E^2(\alpha_0) = \sum_{i=2}^{7} \left[GZ_i - a_0\phi_i^{\alpha_0} - a_1\phi_i^{\alpha_1} - GZ\phi_i \right]^2 \tag{34}$$

To avoid protracted algebra we do not attempt to derive $\mathrm{d}E^2(\alpha_0)/\mathrm{d}\alpha_0$ [1]; instead, numerically seek α_0 value that minimizes $E^2(\alpha_0)$. **Figure 6** shows the variations of $E^2(\alpha_0)$ for small increments of α_0 in the range 1-3. For $\Delta\alpha_0 = 5\times10^{-4}$ the numerically determined minimum is achieved at approximately $\alpha_0 = 2.30$ which in turn gives $\alpha_1 = 2.11$ as determined from (33) by iteration with absolute error tolerance set to 10^{-4} between two successive iterations. The other coefficients corresponding to these values are computed from (32) as $a_0 = -8.75$ and $a_1 = 7.62$; the last coefficient $a_2 = GM = 1.82$. The fractional function satisfying all six conditions is then

$$GZ(\phi) = -8.75\phi^{2.30} + 7.62\phi^{2.11} + 1.82\phi \tag{35}$$

It must be indicated that depending on the tolerance set for the iterative determination of α_1 from Equation (33) the resulting α_1 value may show differences; nevertheless, when all the rest of the parameters are computed according to determined value the resulting curves deviate imperceptibly from one another. In other words, a perceptible difference in a parameter changes the

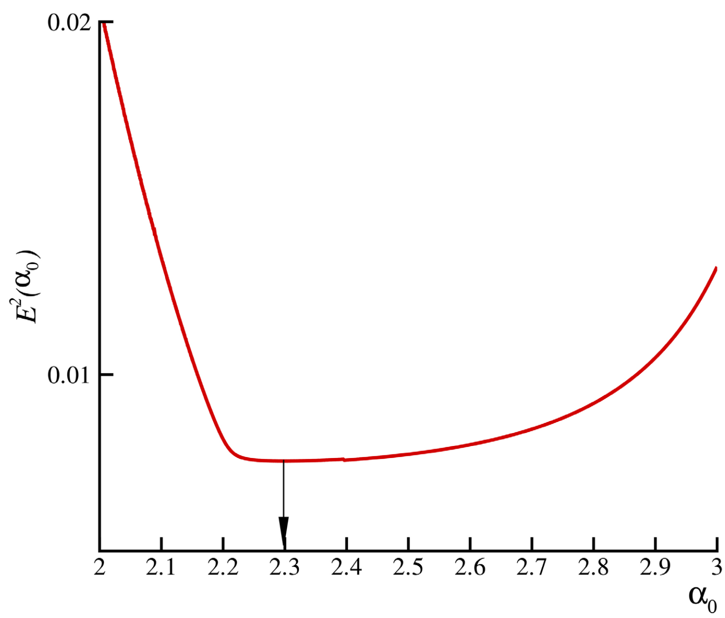

Figure 6. Total squared error $E^2(\alpha_0)$ as a function of α_0 and its numerically determined minimum.

[1]Note that a_0, a_1, and α_1 are all functions of α_0 as given by (32) and (33) hence $\mathrm{d}E^2(\alpha_0)/\mathrm{d}\alpha_0$ becomes an extremely lengthy expression.

other parameters in such a way that the ultimate error is absorbed greatly to become imperceptible. Indeed, from Figure 6, we see not a definite minimum at 2.30 but a nearly constant region within the range 2.25 - 2.35 where the total error remains very low. The point 2.30 is the precise result of computation under set tolerances; however, use of any value between 2.25 and 2.35 produces virtually the same curve, supporting the above argument. This interesting characteristic of the squared error curve is observed for the last curve-fitting shown in Figure 8 too and presumed to be peculiar to fractional functions.

Polynomial representations for this case had to be abandoned due to extremely cumbersome algebra required for dealing with a fifth-order polynomial. Such a serious difficulty should be taken as an indicator of the obvious advantage of fractional functions for satisfying a number of conditions, six in this case.

Figure 7 depicts Equation (35) against the data of Table 2. Agreement with

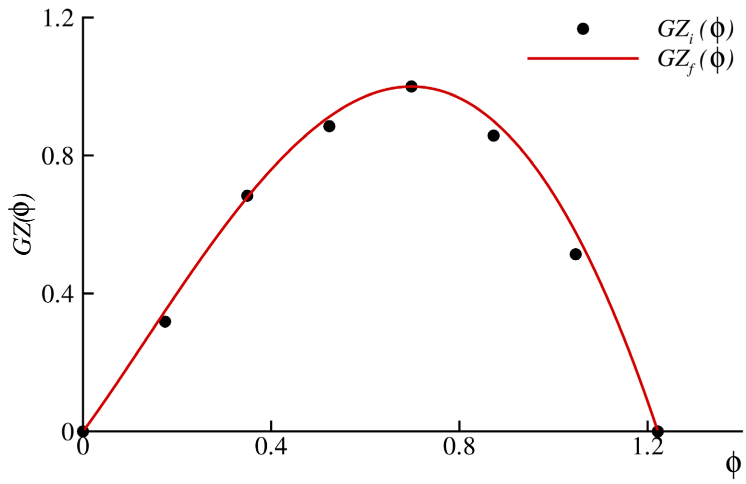

Figure 7. Data $GZ_i(\phi)$ (black dots) against curve-fittings by fractional power $GZ_f(\phi)$ (red).

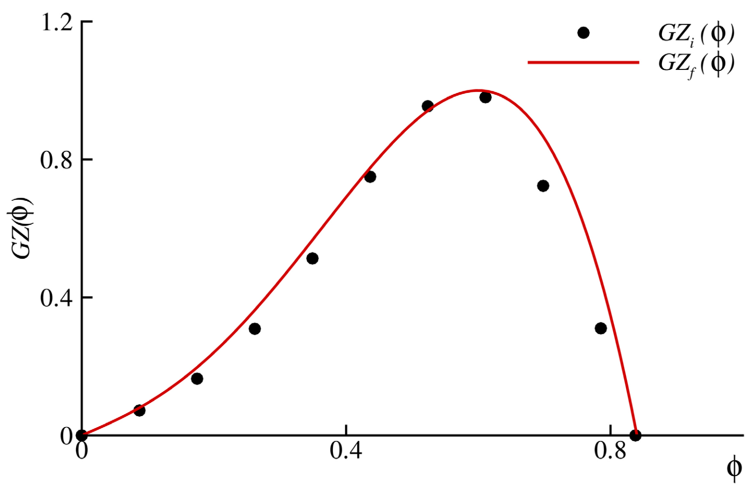

Figure 8. Data $GZ_i(\phi)$ (black dots) against curve-fittings by fractional power $GZ_f(\phi)$ (red).

the actual data is remarkable, especially when compared with **Figure 3** and **Figure 5**; note in particular that the maximum point of the curve coincides exactly with the data maximum.

A more difficult curve-fitting with changing curvatures is done for a different data set ([10], p. 90) by imposing exactly the same conditions. Again, the data are normalized with respect to GZ_m. The resulting function $GZ_f(\phi) = -49.47\phi^{3.45} + 46.16\phi^{3.20} + 0.83\phi$ is plotted against the corresponding data in **Figure 8**. The overall result is quite agreeable especially if allowances are made for the varying characteristics of the data.

5. Concluding Remarks

Polynomial functions composed of terms of non-integer powers are considered for developing methods to obtain their roots. Several representative cases amenable to treatment are examined and some distinct properties of fractional and transcendental powers are revealed. Curve-fitting is recognized as a useful application area of fractional functions and physically meaningful data are employed for computations with satisfactory results. New applications of these functions to diverse fields are likely to emerge in the future.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Euler, L. (1988) Introductio in Analysin Infinitorum (Introduction to Analysis of the Infinite, Book I. Translated by John D. Blanton). Springer-Verlag, New York.

[2] Laguerre, E.N. (1883) Théorie des équations numériques. *Journal de Mathématiques pures et appliquées*, **9**, 1-48.

[3] Diethelm, K., Ford, N.J., Freed, A.D. and Luchko, Y. (2005) Algorithms for the Fractional Calculus: A Selection of Numerical Methods. *Computer Methods in Applied Mechanics and Engineering*, **194**, 743-773. https://doi.org/10.1016/j.cma.2004.06.006

[4] Daftardar-Gejji, V. and Jafari, H. (2007) Solving a Multi-Order Fractional Differential Equation Using Adomian Decomposition. *Applied Mathematics and Computation*, **189**, 541-548. https://doi.org/10.1016/j.amc.2006.11.129

[5] Wazwaz, A.M. (1999) A Reliable Modification of Adomian Decomposition Method. *Applied Mathematics and Computation*, **102**, 77-86. https://doi.org/10.1016/S0096-3003(98)10024-3

[6] Li, X. (2012) A Reliable Modification of Adomian Decomposition Method. *Communications in Nonlinear Science and Numerical Simulation*, **17**, 3934-3946.

[7] YüzbaşI, Ş. (2013) Numerical Solutions of Fractional Riccati Type Differential Equations by Means of the Bernstein Polynomials. *Applied Mathematics and Computation*, **219**, 6328-6343. https://doi.org/10.1016/j.amc.2012.12.006

[8] Kawala, A.M. (2018) Numerical Solution for Initial and Boundary Value Problems of Fractional Order. *Advances in Pure Mathematics*, **8**, 831-884.

https://doi.org/10.4236/apm.2018.812051

[9] Ionescu, C., Lopes, A., Copot, D., Machado, J.A.T. and Bates, J.H.T. (2017) The Role of Fractional Calculus in Modeling Biological Phenomena: A Review. *Communications in Nonlinear Science and Numerical Simulation*, **51**, 141-159. https://doi.org/10.1016/j.cnsns.2017.04.001

[10] DíSouza, V., Detrisac, L., Opdyke, S., Rugnetta, M. and Sultani, K. (2003) Drillship Nereus. International Student Offshore Design Competition, Design Report, University of Michigan's Department of Naval Architecture and Marine Engineering, p. 119.