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Global Attractors and Their Dimension Estimates for a Class of Generalized Kirchhoff Equations

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Abstract

In this paper, we studied the long-time properties of solutions of generalized Kirchhoff-type equation with strongly damped terms. Firstly, appropriate assumptions are made for the nonlinear source term $g(u)$ and Kirchhoff stress term $M(s)$ in the equation, and the existence and uniqueness of the solution are proved by using uniform prior estimates of time and Galerkin's finite element method. Then, abounded absorption set B_{0k} is obtained by prior estimation, and the Rellich-kondrachov's compact embedding theorem is used to prove that the solution semigroup $S(t)$ generated by the equation has a family of the global attractor A_k in the phase space $E_k = H^{2m+k} \times H^k$. Finally, linearize the equation and verify that the semigroups are Frechet diifferentiable on E_k. Then, the upper boundary estimation of the Hausdorff dimension and Fractal dimension of a family of the global attractor A_k was obtained.

Keywords

Generalized Kirchhoff Equation, The Existence and Uniqueness of Solution, A Family of the Global Attractor, Dimension Estimation

1. Introduction

The objective of this paper is to study the following initial boundary value problem of the generalized Kirchhoff equation

$$u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u) = f(x) \tag{1.1}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1,2,\cdots,2m-1, x \in \partial\Omega \tag{1.2}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega,t>0 \tag{1.3}$$

where $m>1$, $p\ge 2$, $\Omega\subset R^n(n\ge 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $M(s)\in C^2([0,+\infty);R^+)$ is a real function, $\beta(-\Delta)^{2m}u_t(\beta>0)$ denotes strong damping term, $g(u)$ is nonlinear source term, $f(x)$ denotes the external force term. The assumption of $M(s)$ and $g(u)$ will be given later.

In 1883, German physicist G. Kirchhoff [1] first introduced the following model to study the free vibration of elastic strings

$$\rho h\frac{\partial^2 u}{\partial^2 t}+\delta u_t=P_0+\frac{Eh}{2L}\left(\int_0^L|u_x|^2\,\mathrm{d}x\right)+f(x),0<x<L,t>0.$$

where the time variable is *t*, the elastic modulus is *E*, *h* is the cross-sectional area, *L* is the length of the string, ρ is the mass density, P_0 is the initial axial tension, δ is the resistance coefficient, *f* is the external force term, $u=u(x,t)$ is the lateral displacement at the space coordinate *x* and the time *t*.

Since the 1980s, with the progress of science and technology and the continuous development of mathematical physics equations and Kirchhoff equation has been widely used, especially in measuring bridge vibration of engineering physics, so more and more scholars begin to pay close attention to and carries on the thorough study of Kirchhoff equation and a series of theories and research results in recent years, including the existence and uniqueness of the global solutions, global attractor and Hausdorff dimension and fractal dimension, the existence of random attractor, energy decay and blow-up of the solution, exponential attractor and inertial manifolds, etc. Among them, scholars have done a lot of research on the existence of global attractors for Kirchhoff-type equations with strong dissipation terms, the theoretical basis and research results can be found in the references ([2]-[9]).

In reference [10], Guoguang Lin, Yunlong Gao studied the existence and uniqueness of global solutions of a class of generalized Kirchhoff-type equations with nonlinear strong damping and their global attractors

$$u_{tt}+(-\Delta)^m u_t+\left(\alpha+\beta\left(\|D^m u\|^2\right)\right)(-\Delta)^m u+g(u)=f(x),(x,t)\in\Omega\times[0,+\infty)$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,m-1,x\in\partial\Omega,t\in[0,+\infty)$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega$$

By assuming the nonlinear source terms $g(u)$, the author verifies the appropriateness of the solution and proves the existence of the global attractor.

Recently, Lin Guoguang and Guan Liping [11] studied the global attractor of a high-order Kirchhoff-type equation with a strong nonlinear damping term and finite dimensional estimation of its Hausdorff dimension and Fractal dimension

$$u_{tt}+M\left(\|D^m u\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+\Delta g(u)=f(x)$$

where $m>1$, Ω is a bounded domain with smooth homogeneous Dirichlet boundary $\partial\Omega \in R^n$, $f(x)$ denotes the external force, $\Delta g(u)$ denotes second order nonlinear source term, M is a general function, $\beta > 0$, $\beta(-\Delta)^m u_t$ is strong damping term. For more significant research results about the global attractor and its dimension estimation of Kirchhoff equation, please refer to the literature ([12] [13] [14] [15]).

In this paper, on the basis of literature [11], the rigid term $\left\|D^m u\right\|^2$ is extended to $\left\|D^m u\right\|_p^p$, the existence and uniqueness of global solutions of generalized nonlinear Kirchhoff-type equations are proved, and the existence of global attractors and their finite Hausdorff dimension and Fractal dimension of problems (1)-(3) are discussed.

For convenience, define the following spaces and notations

$$H = L^2(\Omega), H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega), H_0^{4m}(\Omega) = H^{4m}(\Omega) \cap H_0^1(\Omega),$$
$$H_0^{2m+k}(\Omega) = H^{2m+k}(\Omega) \cap H_0^1(\Omega), E_0 = H^{2m}(\Omega) \times L^2(\Omega), f(x) \in L^2(\Omega),$$
$$E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega), (k = 1, 2, \cdots, 2m).$$

$(\cdot,\cdot)$ and $\|\cdot\|$ represent the inner product and norms of H respectively, *i.e.*

$$(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x, (u,u) = \|u\|^2, \|\cdot\| = \|u\|_{L^2(\Omega)}, \|\cdot\|_P = \|u\|_{L^P(\Omega)}, \|\cdot\|_\infty = \|u\|_{L^\infty(\Omega)}.$$

Let's call $A_k\ (k = 1, 2, \cdots, 2m)$ is the weak global attractor of E_0 to E_k, B_{ok} is a bounded absorption set in E_k, $C_i > 0\ (i = 0, 1, 2, \cdots)$ is constant.

Assume that the nonlinear source term $g(u)$ in Equation (1.1) satisfies the following conditions

(A1) $g(u) \in C^\infty(R)$;

(A2) $J(u) = \int G(u)\mathrm{d}x$, where $G'(u) = g(u)u_t$;

(A3) $J(u) \ge -\frac{\mu}{4}\left\|\nabla^{2m+k}u\right\|^2 - C$.

The Kirchhoff-type stress term satisfies the following conditions

(A4) $M(s) \in C^2([0, +\infty), R^+)$;

(A5) $\varepsilon + 1 = \mu_0 \le M(s) \le \mu_1$, where $\mu = \begin{cases} \mu_0, \frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^m u\right\|^2 \ge 0 \\ \mu_1, \frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^m u\right\|^2 < 0 \end{cases}$

where μ, μ_0, μ_1 are constant, and

$0 < \varepsilon < \min\left\{\frac{\sqrt{1 + 2\beta\lambda_1^{2m}} - 1}{2}, \frac{\mu_0 + \sqrt{\mu_0^2 - \lambda_1^{-2m}}}{\lambda_1^{-2m}}, \frac{\sqrt{1 + \beta\lambda_1^{2m}} - 1}{2}, \frac{2\mu_0}{\beta + \lambda_1^{-2m}}\right\}$, λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions on Ω.

2. A Priori Estimate of Smooth Solution

Lemma 1. Assume that the nonlinear terms $g(u)$, $M(s)$ satisfies assumptions (A1)-(A5), and $f \in L^2(\Omega)$, $(u_0, v_0) \in E_0 = H^{2m}(\Omega) \times L^2(\Omega)$, then the initial boundary value problem (1.1)-(1.3) has a smooth solution $(u, v) \in E_0$ and $v \in L^2(0, T; H^{2m}(\Omega))$

$$\left\|(u,v)\right\|_{E_0}^2 = \left\|\nabla^{2m}u\right\|^2 + \left\|v\right\|^2 \le \left\|Y(0)\right\| \mathrm{e}^{-\alpha_1 t} + \frac{C_1}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right). \tag{2.1}$$

where $v = u_t + \varepsilon u$, $\alpha_1 = \min\left\{a_1, \frac{a_2}{\mu}, 1\right\}$, $Y(0) = \left\|v_0\right\|^2 + \mu\left\|\nabla^{2m}u_0\right\|^2 + 2J(u_0)$, so there's a non-negative real number $R_0 = \sqrt{\frac{2C_1}{\alpha_1}}$ and $t_1 = \frac{1}{\alpha_1}\ln^{\left(\frac{\alpha_1\|Y(0)\|}{C_1}\right)}$, and $\int_0^T \left\|\nabla^{2m}v\right\|^2 \mathrm{d}t \le C$, such that

$$\left\|(u,v)\right\|_{E_0}^2 = \left\|\nabla^{2m}u\right\|^2 + \left\|v\right\|^2 \le R_0^2, (t > t_1). \tag{2.2}$$

Proof. Set $v = u_t + \varepsilon u$, take the inner product of both sides of Equation (1.1) with v in H, we obtain

$$\left(u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u), v\right) = \left(f(x), v\right). \tag{2.3}$$

$$\left(u_{tt}, v\right) = \left(v_t - \varepsilon v + \varepsilon^2 u, v\right) \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|v\right\|^2 - \frac{2\varepsilon+\varepsilon^2}{2}\left\|v\right\|^2 - \frac{\varepsilon^2}{2}\lambda_1^{-2m}\left\|\nabla^{2m}u\right\|^2. \tag{2.4}$$

$$\begin{aligned}\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u, v\right) &= \left(M\left(\left\|\nabla^m u\right\|_p^p\right)\nabla^{2m}u, \nabla^{2m}\left(u_t + \varepsilon u\right)\right) \\ &= \frac{M\left(\left\|\nabla^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2 + \varepsilon M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^{2m}u\right\|^2 \\ &\ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2 + \varepsilon\mu_0\left\|\nabla^{2m}u\right\|^2.\end{aligned} \tag{2.5}$$

By using the Poincare's inequality, we obtain

$$\begin{aligned}\left(\beta(-\Delta)^{2m} u_t, v\right) &= \beta\left\|\nabla^{2m}v\right\|^2 - \left(\beta\varepsilon(-\Delta)^{2m} u, v\right) \\ &\ge \frac{\beta}{2}\left\|\nabla^{2m}v\right\|^2 + \frac{\beta\lambda_1^{2m}}{2}\left\|v\right\|^2 - \frac{1}{2}\left\|\nabla^{2m}u\right\|^2 - \frac{\beta^2\varepsilon^2}{2}\left\|\nabla^{2m}v\right\|^2.\end{aligned} \tag{2.6}$$

The following estimation can be obtained from hypothesis (A2)

$$\begin{aligned}\left(g(u), v\right) &= \left(g(u), u_t\right) + \varepsilon\left(g(u), u\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int G(u)\mathrm{d}x + \varepsilon\left(g(u), u\right) \\ &\ge \frac{\mathrm{d}}{\mathrm{d}t}\int G(u)\mathrm{d}x + \varepsilon^2\int G(u)\mathrm{d}x \ge \frac{\mathrm{d}}{\mathrm{d}t}J(u) + J(u).\end{aligned} \tag{2.7}$$

By using the weighted Young's inequality, we obtain

$$\left(f(x), v\right) \le \left\|f(x)\right\| \cdot \left\|v\right\| \le \frac{1}{2\varepsilon^2}\left\|f(x)\right\|^2 + \frac{\varepsilon^2}{2}\left\|v\right\|^2. \tag{2.8}$$

Substitute inequality (2.4)-(2.8) into Equation (2.3), therefore

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|v\right\|^2 + \mu\left\|\nabla^{2m}u\right\|^2 + 2J(u)\right] + \left(\beta\lambda_1^{2m} - 2\varepsilon - 2\varepsilon^2\right)\left\|v\right\|^2 + \left(\beta - \beta^2\varepsilon^2\right)\left\|\nabla^{2m}v\right\|^2 \\ &+\left(2\varepsilon\mu_0 - \varepsilon^2\lambda_1^{-2m} - 1\right)\left\|\nabla^{2m}u\right\|^2 + 2J(u) \le \frac{\left\|f(x)\right\|^2}{\varepsilon^2} + C_0.\end{aligned} \tag{2.9}$$

Let $a_1 = \beta\lambda_1^{2m} - 2\varepsilon - 2\varepsilon^2 \ge 0$, $a_2 = 2\varepsilon\mu_0 - \varepsilon^2\lambda_1^{-2m} - 1 \ge 0$, $\beta - \beta^2\varepsilon^2 \ge 0$, and let $\alpha_1 = \min\left\{a_1, \frac{a_2}{\mu}, 1\right\}$, $C_1 = \frac{\|f(x)\|^2}{\varepsilon^2} + C_0$, then Equation (2.9) can be reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) + \alpha_1 Y(t) + \left(\beta - \beta^2 \varepsilon^2\right)\left\|\nabla^{2m} v\right\|^2 \le C_1. \tag{2.10}$$

According to hypothesis (A3)

$$\left(\|v\|^2 + \left\|\nabla^{2m} u\right\|^2\right)\min\left(1, \frac{\mu}{2}\right) \le \|v\|^2 + \frac{\mu}{2}\left\|\nabla^{2m} u\right\|^2 + \left(\frac{\mu}{2}\left\|\nabla^{2m} u\right\|^2 + 2J(u)\right) \le C. \tag{2.11}$$

Then

$$Y(t) = \|v\|^2 + \mu\left\|\nabla^{2m} u\right\|^2 + 2J(u) > 0. \tag{2.12}$$

By using the Gronwall's inequality, we get

$$\|(u,v)\|_{E_0}^2 = \left\|\nabla^{2m} u\right\|^2 + \|v\|^2 \le \|Y(0)\| \mathrm{e}^{-\alpha_1 t} + \frac{C_1}{\alpha_1}\left(1 - \mathrm{e}^{-\alpha_1 t}\right). \tag{2.13}$$

And

$$\overline{\lim_{t\to\infty}} \|(u,v)\|_{E_0}^2 \le \frac{C_1}{\alpha_1}. \tag{2.14}$$

So, there are constants $R_0 = \sqrt{\frac{2C_1}{\alpha_1}}$ and $t_1 = \frac{1}{\alpha_1}\ln^{\left(\frac{\alpha_1\|Y(0)\|}{C_1}\right)} > 0$, we obtain

$$\|(u,v)\|_{E_0}^2 = \left\|\nabla^{2m} u\right\|^2 + \|v\|^2 \le R_0^2, (t > t_1) \tag{2.15}$$

The Lemma 1 is proved.

Lemma 2. Assume that the nonlinear terms $g(u)$, $M(s)$ satisfies assumptions (A1)-(A5), and $f \in H^k(\Omega)$,
$(u_0, v_0) \in E_k = H^{2m+k}(\Omega) \times H^k(\Omega), (k = 1, 2, \cdots, 2m)$, then the initial boundary value problem (1.1)-(1.3) has a smooth solution $u(x,t)$,
$u_t(x,t) + \varepsilon u(x,t) = v(x,t)$ satisfy

$$\|(u,v)\|_{E_k}^2 = \left\|\nabla^{2m+k} u\right\|^2 + \left\|\nabla^k v\right\|^2 \le C_2 \mathrm{e}^{-\alpha_2 t} + \frac{C_2}{\alpha_2}\left(1 - \mathrm{e}^{-\alpha_2 t}\right). \tag{2.16}$$

where $v = u_t + \varepsilon u$, $\alpha_2 = \min\left\{b_1, \frac{b_2}{\mu}\right\}$, $Z(0) = \left\|\nabla^k v(0)\right\|^2 + \mu\left\|\nabla^{2m+k} u(0)\right\|^2$, so there are non-negative real number $R_1 = \sqrt{\frac{2C_2}{\alpha_2}}$ and $t_2 = \frac{1}{\alpha_2}\ln^{\left(\frac{\alpha_2\|Z(0)\|}{C_2}\right)} > 0$, such that

$$\|(u,v)\|_{E_k}^2 = \left\|\nabla^{2m+k} u\right\|^2 + \left\|\nabla^k v\right\|^2 \le R_1^2, (t > t_2). \tag{2.17}$$

Proof. Set $(-\Delta)^k v = (-\Delta)^k u_t + \varepsilon(-\Delta)^k u$, take the inner product of both sides of equation(1.1) with $(-\Delta)^k v$ in H, we obtain

$$\left(u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u), (-\Delta)^k v\right) = \left(f(x), (-\Delta)^k v\right). \tag{2.18}$$

$$\left(u_{tt},(-\Delta)^k v\right)=\left(v_t-\varepsilon v+\varepsilon^2 u,(-\Delta)^k v\right)$$
$$\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^k v\right\|^2-\frac{2\varepsilon+\varepsilon^2}{2}\left\|\nabla^k v\right\|^2-\frac{\varepsilon^2}{2\lambda_1^{2m}}\left\|\nabla^{2m+k}u\right\|^2. \tag{2.19}$$

According to hypothesis (A5), and use a proof method similar to lemma 1, we can get

$$\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u,(-\Delta)^k v\right)$$
$$=M\left(\left\|\nabla^m u\right\|_p^p\right)\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}u\right\|^2+\varepsilon M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^{2m+k}u\right\|^2 \tag{2.20}$$
$$\geq \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}u\right\|^2+\varepsilon\mu_0\left\|\nabla^{2m+k}u\right\|^2.$$

By using Poincare's inequality and Young's inequality, we have

$$\left(\beta(-\Delta)^{2m}u_t,(-\Delta)^k v\right)$$
$$=\beta\left((-\Delta)^{2m}v-\varepsilon(-\Delta)^{2m}u,(-\Delta)^k v\right) \tag{2.21}$$
$$\geq \frac{\beta}{2}\left\|\nabla^{2m+k}v\right\|^2+\frac{\beta\lambda_1^{2m}}{2}\left\|\nabla^k v\right\|^2-\frac{1}{2}\left\|\nabla^{2m+k}u\right\|^2-\frac{\beta^2\varepsilon^2}{2}\left\|\nabla^{2m+k}v\right\|^2.$$

$$\left(g(u),(-\Delta)^k v\right)=\left\|g(u)\right\|_\infty\left\|\nabla^{2k}v\right\|\geq -\frac{\beta}{8}\left\|\nabla^{2m+k}v\right\|^2-C_3. \tag{2.22}$$

where $C_3=\dfrac{2\left\|g(u)\right\|_\infty}{\beta\lambda_1^{2m-k}}$.

$$\left(f(x),(-\Delta)^k v\right)=\left(\nabla^k f(x),\nabla^k v\right)\leq \frac{\left\|\nabla^k f\right\|^2}{2\varepsilon^2}+\frac{\varepsilon^2}{2}\left\|\nabla^k v\right\|^2. \tag{2.23}$$

Substitute inequality (2.19)-(2.23) to (2.18), therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2\right]+\left(\beta\lambda_1^{2m}-2\varepsilon-2\varepsilon^2\right)\left\|\nabla^k v\right\|^2$$
$$+\left(2\varepsilon\mu_0-\frac{\varepsilon^2}{\lambda_1^{2m}}-1\right)\left\|\nabla^{2m+k}u\right\|^2+\left(\frac{3\beta}{4}-\beta^2\varepsilon^2\right)\left\|\nabla^{2m+k}v\right\|^2\leq \frac{\left\|\nabla^k f(x)\right\|^2}{\varepsilon^2}+C_3. \tag{2.24}$$

Obviously, there is a non-negative ε, such that $b_1=\beta\lambda_1^{2m}-2\varepsilon-2\varepsilon^2\geq 0$, $b_2=2\varepsilon\mu_0-\dfrac{\varepsilon^2}{\lambda_1^{2m}}-1\geq 0$, $\dfrac{3\beta}{4}-\beta^2\varepsilon^2\geq 0$, let $\alpha_2=\min\left\{b_1,\dfrac{b_2}{\mu}\right\}$, $C_2=\dfrac{\left\|\nabla^k f(x)\right\|^2}{\varepsilon^2}+C_3$, then (2.24) can reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t)+\alpha_2 Z(t)+\left(\frac{3\beta}{4}-\beta^2\varepsilon^2\right)\left\|\nabla^{2m+k}v\right\|^2\leq C_2. \tag{2.25}$$

where

$$Z(t)=\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2>0. \tag{2.26}$$

By using Gronwall's inequality, we can get

$$\left\|(u,v)\right\|_{E_k}^2=\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2\le Z(0)\mathrm{e}^{-\alpha_2 t}+\frac{C_2}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right). \tag{2.27}$$

and

$$\overline{\lim_{t\to\infty}}\left\|(u,v)\right\|_{E_k}^2\le\frac{C_2}{\alpha_2}. \tag{2.28}$$

So, there are constants $R_1=\sqrt{\frac{2C_2}{\alpha_2}}$ and $t_2=\frac{1}{\alpha_2}\ln^{\left(\frac{\alpha_2\|Z(0)\|}{C_2}\right)}>0$, we obtain

$$\left\|(u,v)\right\|_{E_k}^2=\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k v\right\|^2\le R_1^2,\left(t>t_2\right). \tag{2.29}$$

The Lemma 2 is proved.

3. Existence and Uniqueness of Solutions

Theorem 1 Assume that the nonlinear terms $g(u)$, $M(s)$ satisfies (A1)-(A5), $f\in H$, $(u_0,v_0)\in E_0$, then the initial boundary (1)-(3) exists a unique smooth solution $(u,v)\in L^\infty([0,+\infty);E_0)$. **Proof.** To prove the existence, the application of Calerkin's method is divided into the following three steps.

1) Approximate solution.

Suppose the eigenvector ω_j of $\left((-\Delta)^{2m}\right)\omega_j=\lambda_j^{2m}\omega_j$ generates an orthonormal basis for H^{2m}, where λ_j is the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary on Ω, the k-order approximation $u_k(t)$ is defined as follows: $u_k(t)\in Span[\omega_1,\omega_2,\cdots,\omega_k]$, $u_k(t)=\sum_{j=1}^k g_{jk}(t)\omega_j$.

$$\begin{aligned}&\left(u_k''(t),\omega_j\right)+M\left(\left\|\nabla^m u_k(t)\right\|_p^p\right)\left(\nabla^{2m}u_k(t),\nabla^{2m}\omega_j\right)\\&+\beta\left(\nabla^{2m}u'(t),\nabla^{2m}\omega_j\right)+\left(g(u_k),\omega_j\right)=\left(f(x),\omega_j\right)\end{aligned} \tag{3.1}$$

where $j=1,2,\cdots,k$.

$$u_k(0)=u_{0k},\ u_k'(0)=u_{1k}\ \text{and}\ u_{0k},u_{1k}\in Span[\omega_1,\omega_2,\cdots,\omega_k].$$

In H, $u_{0k}\to u_0, u_{1k}\to v_0$, the system of ordinary differential equations with respect to $g_{jk}(t)$ can be determined on the interval $[0,t_k]$, we need to prove that $t_k=T$.

2) Prior estimate.

According to the conclusion and proof method of lemma 1, $\left(u_k(t),u_k'(t)\right)$ is uniformly bounded on E_0, that is

$$\left\|u_k(t)\right\|\le R \tag{3.2}$$

$$\left\|\nabla^{2m}u_k(t)\right\|\le R \tag{3.3}$$

$$\left\|u_k'(t)\right\|\le R \tag{3.4}$$

where R is a constant independent of k. According to lemma2, we get $u_k'(t)\in L^2\left(0,T;L^2(\Omega)\right)$, therefore $t_k=T$, inequality (3.2)-(3.4) indicate $u_k(t)$ is bounded in $L^\infty\left(0,T;H^{2m}(\Omega)\right)$, $u_k'(t)$ is bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$, in fact, we can get $u_k(t)\in L^\infty\left(0,+\infty;H^{2m}(\Omega)\right)$ and $u_k'(t)\in L^\infty\left(0,+\infty;L^2(\Omega)\right)$.

3) Limit process.

According to Danford-Pttes throrem, space $L^2\left(0,T;H^{2m}(\Omega)\right)$ is conjugated to space $L^2\left(0,T;H^{-2m}(\Omega)\right)$, space $L^\infty\left(0,T;L^2(\Omega)\right)$ is conjugated to space $L'\left(0,T;L^2(\Omega)\right)$, select the subsequence $u_k^*(t)$ from the sequence $u_k(t)$, such that

$u_k^* \to u$ is weak * convergence in $L^\infty\left(0,T;H^{2m}(\Omega)\right)$;
$u_k^{*\prime} \to u'$ is weak * convergence in $L^\infty\left(0,T;L^2(\Omega)\right)$;
$u_k^{*\prime} \to u'$ is weak convergence in $L^2\left(0,T;H^{2m}(\Omega)\right)$.

By the Rellich-Kohdrachov compact embedding theorem, $H^{2m}(\Omega)$ is compact embedded in $L^p(\Omega)$ and $L^2(\Omega)$, then $\left(u_k(t),u_k^{*\prime}\right) \to (u,u')$ converges strongly almost everywhere in $L^2(0,T)$ and $L^2(0,T)$.

$M\left(\|u_k(t)\|_p^p\right) \to M\left(\|u(t)\|_p^p\right)$ converges in R^+;

$M\left(\|u_k(t)\|_p^p\right)\left(\nabla^{2m}u_k(t),\nabla^{2m}\omega_j(t)\right) \to M\left(\|u(t)\|_p^p\right)\left(\nabla^{2m}u,\nabla^{2m}\omega_j\right)$ is weak * convergence in $L^\infty(0,T)$;

$\beta\left(\nabla^{2m}u_k'(t),\nabla^{2m}\omega_j(t)\right) \to \beta\left(\nabla^{2m}u'(t),\nabla^{2m}\omega_j\right)$ is weak * convergence in $L^\infty(0,T)$.

Because of $g(u_k) \in L^\infty\left(H_0^k\right)$, so

$\left(g(u_k),\omega_j\right) \to \left(g(u),\omega_j\right)$ is weak * convergence in $L^\infty(0,T)$.

$\left(u_k''(t),\omega_j\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(u_k'(t),\omega_j\right) \to \left(u'',\omega_j\right)$ is converges in $D'(0,T)$.

From Equation (3.1), the following formula can be derived

$$\begin{aligned}&\left(u'',\omega_j\right)+M\left(\|\nabla^m u\|_p^p\right)\left(\nabla^{2m}u,\nabla^{2m}\omega_j\right)+\beta\left(\nabla^{2m}u'(t),\nabla^{2m}\omega_j\right)\\&+\left(g(u),\omega_j\right)=\left(f(x),\omega_j\right)\end{aligned} \quad \text{this is true for } \forall j.$$

By the density of the base $\omega_1,\omega_2,\cdots,\omega_k,\cdots$, then for $\forall\varphi\in H^{2m}(\Omega)$, the following equation is established

$$\left(u'',\varphi\right)+M\left(\|\nabla^m u\|_p^p\right)\left(\nabla^{2m}u,\nabla^{2m}\varphi\right)+\beta\left(\nabla^{2m}u'(t),\nabla^{2m}\varphi\right)+\left(g(u),\varphi\right)=\left(f(x),\varphi\right)$$

Then $u_k(0)\to u_0$ is weak convergence in $L^2(\Omega)$, and in $H^{2m}(\Omega)$, we establish

$$u_k(0)\to u_0;$$
$$\left(u_k'(0),\omega_j\right)\to\left(u',\omega_j\right)\big|_{t=0}=\left(u'(0),\omega_j\right);$$
$$\left(u_k'(0),\omega_j\right)\to\left(v_0,\omega_j\right)$$

So $\left(u'(0),\omega_j\right)=\left(v_0,\omega_j\right)$ is satisfied for all j, so that existence can be proved.

Then prove the uniqueness of the solution.

Assume u_1 and u_2 are solutions of Equation (1.1), let $w=u_1-u_2$, and substitute u_1 and u_2 into this equation, we can obtain

$$\begin{aligned}&w_{tt}+M\left(\|\nabla^m u_1\|_p^p\right)(-\Delta)^{2m}u_1-M\left(\|\nabla^m u_2\|_p^p\right)(-\Delta)^{2m}u_2\\&+\beta(-\Delta)^{2m}w_t+g(u_1)-g(u_2)=0.\end{aligned} \tag{3.5}$$

Take the inner product of both sides of Equation (3.5) with w_t in *H*, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2+\beta\|\Delta^m w_t\|^2+\left(M\left(\|\nabla^m u_1\|_p^p\right)(-\Delta)^{2m}u_1-M\left(\|\nabla^m u_2\|_p^p\right)(-\nabla)^{2m}u_2,w_t\right)$$
$$+\left(g(u_1)-g(u_2),w_t\right)=0. \tag{3.6}$$

According to Sobolev embedding theorem, $H_0^{2m}(\Omega)\subset L^P(\Omega)$, there exists constant $r>0$, we have

$$\|\nabla^m u_1\|_{L^p(\Omega)}^p\le r\left\|(-\Delta)^m u_1\right\|^{\frac{(2m+n)p-2n}{4m}}\|u_1\|^{\frac{(2m-n)p+2n}{4m}}. \tag{3.7}$$

where

$$\begin{cases}\dfrac{2n}{n+2m}\le p\le\dfrac{2n}{n-2m},n>2m;\\ \dfrac{2n}{n+2m}\le p\le\infty,n\le 2m.\end{cases} \tag{3.8}$$

By lemma 1, lemma 2, differential mean value theorem and Young's inequality, we can obtain

$$\begin{aligned}&\left(M\left(\|\nabla^m u_1\|_p^p\right)(-\Delta)^{2m}u_1-M\left(\|\nabla^m u_2\|_p^p\right)(-\Delta)^{2m}u_2,w_t\right)\\
&=\Big(M\left(\|\nabla^m u_1\|_p^p\right)(-\Delta)^{2m}u_1-M\left(\|\nabla^m u_1\|_p^p\right)(-\Delta)^{2m}u_2\\
&\quad+M\left(\|\nabla^m u_1\|_p^p\right)(-\Delta)^{2m}u_2-M\left(\|\nabla^m u_2\|_p^p\right)(-\Delta)^{2m}u_2,w_t\Big)\\
&\ge\frac{1}{2}M\left(\|\nabla^m u_1\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^m w\|^2-\left(M'(\xi)\left(\|\nabla^m u_1\|_p^p-\|\nabla^m u_2\|_p^p\right)(-\Delta)^{2m}u_2,\omega_t\right)\\
&\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^m w\|^2-\|M'(\xi)\|\cdot\|\nabla^{2m}u_2\|\cdot\|\nabla^{2m}\omega_t\|\cdot\|\nabla^m\omega\|_\infty\cdot C_4\cdot\|\nabla^m u_1\|_{L^{P-1}}^{p-1}\cdot\|\nabla^m u_2\|_{L^{P-1}}^{p-1}\\
&\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^m w\|^2-\frac{\beta}{2}\|\Delta^m w\|^2-\frac{C_4^2}{2\beta}\|\Delta^m w_t\|^2.\end{aligned} \tag{3.9}$$

where $\xi=\theta u_1+(1-\theta)u_2,0<\theta<1$.

According to the hypothesis (A1), we get

$$\begin{aligned}\left(g(u_1)-g(u_2),w_t\right)&\ge-\|g'(\xi)\|_\infty\|w\|\|w_t\|\ge-C_5C_6\|\nabla^m w\|\|w_t\|\\
&\ge-\frac{C_5C_6}{2}\|w_t\|^2-\frac{C_5C_6}{2}\|\nabla^m w\|^2\\
&\ge-\frac{C_5C_6}{2}\|w_t\|^2-\frac{C_5C_6}{2}\lambda_1^{-m}\|\Delta^m w\|^2.\end{aligned} \tag{3.10}$$

Substitute inequality (3.9), (3.10) to (3.6), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\mu\|\Delta^m w\|^2\right)+\left(2\beta-\frac{C_4^2}{\beta}\right)\|\Delta^m w_t\|^2\\
&-\left(\beta+C_5C_6\lambda_1^{-m}\right)\|\Delta^m w\|^2-C_5C_6\|w_t\|^2\le 0\end{aligned} \tag{3.11}$$

By using Gronwall's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\mu\|\Delta^m w\|^2\right)\le C_7\left(\|w_t\|^2+\mu\|\Delta^m w\|^2\right) \tag{3.12}$$

where $C_7 = \max\left\{\dfrac{\beta + C_5 C_6 \lambda_1^{-m}}{\mu}, C_5 C_6\right\}$.

So we can get

$$\left(\left\|w_t(t)\right\|^2 + \mu\left\|\Delta^m w(t)\right\|^2\right) \le \left(\left\|w_t(0)\right\|^2 + \mu\left\|\Delta^m w(0)\right\|^2\right)\mathrm{e}^{C_7 t} = 0. \tag{3.13}$$

Then $w_t(t) = 0, \Delta^m w(t) = 0$, so $w(t) = 0, u_1 = u_2$, then the uniqueness of solutions is proved.

The theorem 1 is proved completely.

4. The Existence of the Family of Global Attractor

Theorem 2 [16] Assume that E is a Banach space, $\{S(t)\}_{t\ge 0}$ is a semigroup operator on E, and $S(t): E \to E$, $S(t+s) = S(t)S(s)(\forall t, s \ge 0)$, $S(0) = I$, where I is unit operator, suppose $S(t)$ satisfies the following conditions:

1) Semigroup $S(t)$ is uniformly bounded in E, that is for all $r > 0$, it exists a constant $C(r)$, such that $\|u\|_E \le r$,

$$\|S(t)u\|_E \le C(r)(\forall t \in [0, \infty));$$

2) It exists a bounded absorbing set $B_0 \subset E$, that is for all $B \subset E$, it exists a constant t_0,

$$S(t)B \subset B_0 (t \ge t_0);$$

3) $S(t)(t \ge 0)$ is completely continuous operator.

Thus there is a compact global attractor A_0 for the semigroup operator $S(t)$.

If the Banach space E is changed to Hilbert space E_k in theorem 2, the existence theorem of the family of the global attractors can be obtained.

Theorem 3 Under the assumption of lemma 1, lemma 2 and theorem 1, problem (1.1)-(1.3) exist a family of the global attractors A_k

$$A_k = \omega(B_{0k}) = \bigcap_{s\ge 0}\overline{\bigcup_{t\ge s} S(t)B_{0k}}.$$

where $B_{0k} = \left\{(u,v) \in E_k, \|(u,v)\|_{E_k}^2 = \|u\|_{H_0^{2m+k}(\Omega)}^2 + \|v\|_{H_0^K(\Omega)}^2 \le C\right\}$, B_{0k} is a bounded absorbing set in E_k, that is exists a compact set $A_k \subset E_k \subset E_0$ satisfies the following conditions:

1) $S(t)A_k = A_k (\forall t > 0)$

2) $\lim\limits_{t\to\infty} dist(S(t)B, A_k) = 0$, ($\forall B \subset E_k(\Omega)$ is bounded set), where

$$\lim_{t\to\infty} dist(S(t)B, A_k) = \sup_{x\in B}\left(\inf_{y\in A_k}\|S(t)x - y\|_{E_k}\right) \to 0, t \to \infty.$$

$S(t)$ is the solution semigroup generated by problem (1.1)-(1.3).

Proof. We need to prove the three conditions of theorem 2, according to theorem 1, lemma 2, we know the problem (1.1)-(1.3) could generate the solution semigroup $S(t)$, $S(t): E_k \to E_k$.

1) According to lemma 1 and lemma 2, we know that for any bounded set $B \subset E_k$, and

$$\left\{\left\|(u,v)\right\|_{E_k}=\|u\|_{H_0^{2m+k}(\Omega)}+\|v\|_{H_0^k(\Omega)}\le R_k\right\},$$

$$\left\|S(t)(u_0,v_0)\right\|_{E_k}^2=\|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2\le C_2'(R_k)+\frac{C_2}{\alpha_2}=C_8.$$

where $t\ge 0$, $(u_0,v_0)\in B$, it indicates $\{S(t)\}(t\ge 0)$ is uniformly bounded in E_k.

2) According to lemma 2, $\forall(u_0,v_0)\in E_k$, for $t\ge\max(t_1,t_2)$

$$\left\|S(t)(u_0,v_0)\right\|_{E_k}^2=\|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2\le R_0^2+R_1^2\le R_k^2.$$

Thus, semigroup $S(t)$ exists bounded absorbing set B_{0k}.

3) $E_k\subset E_0$ is compact embedded, it indicates the bounded set in E^k is the compact set in E_0, so semigroup operator $S(t)$ is completely continuous. Furthermore we can get semigroup operator $S(t)$ exists a compact family of the global attractor $A_k=\omega(B_{0k})=\bigcap_{s\ge 0}\overline{\bigcup_{t\ge s}S(t)B_{0k}},k=1,2,\cdots,2m$.

Theorem 3 is proved.

5. Estimation of the Dimension of the Family of Global Attractors

Let's consider the linearization problem of (1.1)-(1.3)

$$\begin{aligned}&U_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}U\\&+pM'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\int_\Omega\left|\nabla^m u\right|^{p-2}\nabla^m u\cdot\nabla^m U\mathrm{d}x\right)(-\Delta)^{2m}u\\&+\beta(-\Delta)^{2m}U_t+g'(u)U=0.\end{aligned}\tag{5.1}$$

$$U(x,t)=0,\frac{\partial^i U}{\partial v^i},k=1,2,\cdots,2m-1,t>0.\tag{5.2}$$

$$U(x,0)=\xi_1,U_t(x,0)=\xi_2.\tag{5.3}$$

where $(\xi_1,\xi_2)\in E_k$, $(u,u_t)=S(t)(u_0,u_1)$ is the solution of the problem (1.1)-(1.3) which take $(u_0,u_1)\in A_k$ as the initial value. $(u_0,u_1)\in A$, $S(t):E_k\to E_k$, it can be proved that for any $(\xi_1,\xi_2)\in E_k$, linearized initial boundary value problem (5.1)-(5.3) have unique solution $(U(t),U_t(t))\in L^\infty([0,+\infty);E_k)$.

Theorem 4. The Frechet derivative of mapping $S(t):E_k\to E_k$ is the linear operator $T_k:(\xi_1,\xi_2)\to(U(t),U_t(t))$ on $\phi_0=(u_0,u_1)$, thus for any $t>0,R>0$, mapping $S(t):E_k\to E_k$ is Frechet differentiable on E_k, and $(U(t),U_t(t))$ is the solution to Equations (5.1)-(5.3). **Proof.** Let $\phi_0=(u_0,v_0)\in E_k$, $\overline{\phi_0}=(u_0+\xi_1,v_0+\xi_2)\in E_k$, and $\left\|\overline{\phi_0}\right\|_{E_k}\le R$, define $(u,v)=S(t)\phi_0$, $(\overline{u},\overline{v})=S(t)\overline{\phi_0}$, c is a constant. We can get the Lipschitz property of S(t) on the bounded set E_k, that is

$$\left\|S(t)\phi_0-S(t)\overline{\phi_0}\right\|_{E_k}^2\le \mathrm{e}^{ct}\left\|(\xi_1,\xi_2)\right\|_{E_k}^2.\tag{5.4}$$

Let $\sigma=\overline{u}-u-U$ is the solution of problem (1.1)-(1.3), we have

$$\sigma_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}\sigma + \beta(-\Delta)^{2m}\sigma_t = -h, \tag{5.5}$$

$$\sigma(0) = \sigma_t(0) = 0. \tag{5.6}$$

Let $s = \left\|\nabla^m u\right\|_p^p, \bar{s} = \left\|\nabla^m \bar{u}\right\|_p^p$, we obtained

$$\begin{aligned} h &= \left[M(\bar{s}) - M(s)\right](-\Delta)^{2m}\bar{u} - M'(s)s'\nabla^m U(-\Delta)^{2m}u \\ &\quad - \left(g(u) - g(\bar{u}) + g'(u)U\right) \\ &= M'\left(\left\|\xi_1\right\|_p^p\right)\left(\left\|\xi_1\right\|_p^p\right)'\nabla^m(\bar{u} - u)(-\Delta)^{2m}\bar{u} \\ &\quad - M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m(\bar{u} - u)(-\Delta)^{2m}u \\ &\quad + M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m\sigma(-\Delta)^{2m}u - \left(g(u) - g(\bar{u}) + g'(u)U\right) \\ &= h_1 + h_2 + h_3. \end{aligned} \tag{5.7}$$

where $\xi_1 = (1-\alpha)\bar{s} + \alpha s, 0 < s < 1$.

$$h_1 = M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m\sigma(-\Delta)^{2m}u \tag{5.8}$$

$$\begin{aligned} h_2 &= M'\left(\left\|\xi_1\right\|_p^p\right)\left(\left\|\xi_1\right\|_p^p\right)'\nabla^m(\bar{u} - u)(-\Delta)^{2m}\bar{u} \\ &\quad - M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m(\bar{u} - u)(-\Delta)^{2m}u. \end{aligned} \tag{5.9}$$

Let $N'(\xi_1) = M'\left(\left\|\xi_1\right\|_p^p\right)\left(\left\|\xi_1\right\|_p^p\right)'$, then

$$\begin{aligned} h_2 &= \left[\left(N'(\xi_1) - N'(s)\right)(-\Delta)^{2m}\bar{u} + N'(s)(-\Delta)^{2m}(\bar{u} - u)\right]\nabla^m(\bar{u} - u) \\ &= N''(\xi_2)(1-\alpha)\left(\nabla^m(\bar{u} - u)\right)^2(-\Delta)^{2m}\bar{u} + N'(s)(-\Delta)^{2m}(\bar{u} - u)\nabla^m(\bar{u} - u). \end{aligned} \tag{5.10}$$

where $\xi_2 = \theta\xi_1 + (1-\theta)s, 0 < \theta < 1$, then

$$h_3 = -\left(g(u) - g(\bar{u}) + g'(u)U\right). \tag{5.11}$$

Take the inner product of (5.5) with $(-\Delta)^k\sigma_t$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^k\sigma_t\right\|^2 + M(s)\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}\sigma\right\|^2 + \beta\left\|\nabla^{2m+k}\sigma_t\right\|^2 \\ &= \left(h, (-\Delta)^k\sigma_t\right). \end{aligned} \tag{5.12}$$

And we get

$$\begin{aligned} \left|\left(-h, (-\Delta)^k\sigma_t\right)\right| &= \left|\left(-(h_1 + h_2 + h_3), (-\Delta)^k\sigma_t\right)\right| \\ &\le \left|\left(h_1, (-\Delta)^k\sigma_t\right)\right| + \left|\left(h_2, (-\Delta)^k\sigma_t\right)\right| + \left|\left(h_3, (-\Delta)^k\sigma_t\right)\right|. \end{aligned} \tag{5.13}$$

Let $w = u - \bar{u}$, according to lemma 1, lemma 2, differential mean value theorem, interpolation inequality and Poincare's inequality, we obtained

$$\begin{aligned}\left|\left(h_1,(-\Delta)^k\sigma_t\right)\right| &= \left|\left(M'(s)s'(-\Delta)^{2m}u\nabla^k\sigma,(-\Delta)^k\sigma_t\right)\right| \\ &\le C_8\left\|(-\Delta)^{2m}u\right\|_\infty\left\|\nabla^{m+k}\sigma\right\|\left\|\nabla^k\sigma_t\right\| \\ &\le \frac{C_9\lambda_1^{-m}}{2}\left\|\nabla^{2m+k}\sigma\right\|^2+\frac{C_{10}}{2}\left\|\nabla^k\sigma_t\right\|^2.\end{aligned} \tag{5.14}$$

$$\begin{aligned}\left|\left(h_2,(-\Delta)^k\sigma_t\right)\right| &= \Big|\left(N''(\xi_2)(1-\alpha)(-\Delta)^{2m}\bar{u}\left(\nabla^m(\bar{u}-u)\right)^2\right. \\ &\quad \left.+N'(s)(-\Delta)^{2m}(\bar{u}-u)\nabla^m(\bar{u}-u),(-\Delta)^k\sigma_t\right)\Big| \\ &\le \left\|N''(\xi_2)\right\|_\infty\left\|(-\Delta)^{2m}\bar{u}\right\|_\infty\left\|\nabla^m w\right\|^2\left\|\nabla^{2k}\sigma_t\right\| \\ &\quad +\left\|N'(s)\right\|_\infty\left|\int_\Omega(-\Delta)^{2m}(\bar{u}-u)\nabla^m(\bar{u}-u)(-\Delta)^k\sigma_t\mathrm{d}x\right| \\ &\le C_{11}\left\|\nabla^{2k}\sigma_t\right\|\left\|\nabla^{2m+k}w\right\|^2\|w\|^2+C_{12}\left\|\nabla^{2k}\sigma_t\right\|\left\|\nabla^{2m+k}w\right\|^2\|w\|^2 \\ &\le C_{13}\left\|\nabla^{2k}\sigma_t\right\|\left\|\nabla^{2m+k}w\right\|^2+C_{14}\left\|\nabla^{2k}\sigma_t\right\|\left\|\nabla^{2m+k}w\right\|^2 \\ &\le \frac{C_{13}^2+C_{14}^2}{2}\lambda_1^{-k}\left\|\nabla^k\sigma_t\right\|^2+\frac{C_{15}^2+C_{16}^2}{2}\left\|\nabla^{2m+k}w\right\|^4.\end{aligned} \tag{5.15}$$

$$\begin{aligned}\left|\left(h_3,(-\Delta)^k\sigma_t\right)\right| &= \left(-\left(g(u)-g(\bar{u})+g'(u)U\right),(-\Delta)^k\sigma_t\right) \\ &= (-\left(g'(\xi)w,(-\Delta)^k\sigma_t\right)-\left(g'(u)w,(-\Delta)^k\sigma_t\right)-\left(g'(u)\sigma,(-\Delta)^k\sigma_t\right) \\ &\le (\alpha-1)\left(g''(\eta)|w|^2,(-\Delta)^k\sigma_t\right)-\left(g'(u)\sigma,(-\Delta)^k\sigma_t\right) \\ &\le C_{17}\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^k\sigma_t\right\|\right)\left\|\nabla^{2m+k}\sigma_t\right\| \\ &\le C_{18}\left(\left\|\nabla^{2m+k}w\right\|^4+\left\|\nabla^k\sigma_t\right\|^2\right)+\frac{\beta}{2}\left\|\nabla^{2m+k}\sigma_t\right\|^2.\end{aligned} \tag{5.16}$$

Sum up (5.14)-(5.16), we get

$$\left(h,(-\Delta)^k\sigma_t\right)\le C_{19}\left(\left\|\nabla^{2m+k}w\right\|^4+\left\|\nabla^{2m+k}\sigma\right\|^2+\left\|\nabla^k\sigma_t\right\|^2\right)+\beta\left\|\nabla^{2m+k}\sigma_t\right\|^2. \tag{5.17}$$

Then substitute inequality (5.14)-(5.17) to inequality (5.12), and by using Young's inequality, Poincare's inequality can obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\nabla^k\sigma_t\right\|^2+\mu\left\|\nabla^{2m+k}\sigma\right\|^2\right]\le C_{19}\left(\left\|\nabla^k\sigma_t\right\|^2+\mu\left\|\nabla^{2m+k}\sigma\right\|^2+\left\|\nabla^{2m+k}w\right\|^4\right). \tag{5.18}$$

According to Gronwall's inequality, we get

$$\left\|\nabla^k\sigma_t\right\|^2+\mu\left\|\nabla^{2m+k}\sigma\right\|^2\le C_{20}\mathrm{e}^{C_{19}t}\left\|(\xi_1,\xi_2)\right\|_{E_k}^4. \tag{5.19}$$

According to (5.19), we can get $\left\|(\xi_1,\xi_2)^{\mathrm{T}}\right\|_{E_k}^2\to 0$, the following is established

$$\frac{\left\|\bar{\phi}(t)-\phi(t)-U(t)\right\|_{E_k}^2}{\left\|(\xi_1,\xi_2)^{\mathrm{T}}\right\|_{E_k}^2}\le C_{20}\mathrm{e}^{C_{19}t}\left\|(\xi_1,\xi_2)\right\|_{E_k}^2\to 0. \tag{5.20}$$

The theorem 4 is proved.

The following will show that the family of the global attractor $A_k, k=1,2,\cdots,2m$

have finite Hausdorff dimension and Fractal dimension.

Theorem 5 In theorem 3, the family of the global attractor A_k of equation (1.1)-(1.3) have finite Hausdorff dimension and Fractal dimension, and $d_H(A_k) < \frac{1}{6}n, d_F(A_k) < \frac{7}{6}n$.

Proof we can write Equation (1.1) as

$$u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)\nabla^{4m}u + \beta\nabla^{4m}u_t + g(u) = f(x). \tag{5.21}$$

Suppose $\psi = R_\varepsilon\phi = (u,v)^{\mathrm{T}}$, $\phi = (u,u_t)$, $v = u_t + \varepsilon u$, $R_\varepsilon : (u,u_t)^{\mathrm{T}} \to (u,u_t+\varepsilon u)^{\mathrm{T}}$, is a isomorphic mapping, and Equation (5.21) can write as

$$\psi_t + \overline{\Lambda}_\varepsilon\psi = \overline{f}. \tag{5.22}$$

where $\psi = \{u, u_t + \varepsilon u\}^{\mathrm{T}}$, $\overline{f} = \{0, f(x)\}^{\mathrm{T}}$.

$$\overline{\Lambda}_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \left(M\left(\left\|\nabla^m u\right\|_p^p\right) - \beta\varepsilon\right)\nabla^{4m} + \varepsilon^2 I + g(u) & \beta\nabla^{4m} - \varepsilon I \end{pmatrix}$$

$$\psi_t := F'(\psi) = \overline{f} - \overline{\Lambda}_\varepsilon\psi. \tag{5.23}$$

$$P_t = F_t(\psi). \tag{5.24}$$

Assume $F: E_k \to E_k$ is Frechet differential, then linearize Equation (5.23), we get

$$P_t + \Lambda_\varepsilon P = 0. \tag{5.25}$$

where

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \left(M\left(\left\|\nabla^m u\right\|_p^p\right) - \beta\varepsilon\right)\nabla^{4m} + \varepsilon^2 I + D\cdot + g'(u)\cdot U & \beta\nabla^{4m} - \varepsilon I \end{pmatrix}$$

$D\cdot U = pM'\left(\left\|\nabla^m u\right\|_p^p\right)\int_\Omega \left|\nabla^m u\right|^{p-2}\nabla^m u\cdot\nabla^m U\mathrm{d}x\nabla^{4m}u$, and $P = (U, U_t + \varepsilon U)^{\mathrm{T}}$, U is the solution of Equation (5.22).

For a fixed $(u_0, v_0) \in E_k$, assume $\xi_1, \xi_2, \cdots, \xi_n$ are elements of E_k, and suppose $U_1(t), U_2(t), \cdots, U_n(t)$ are solutions of linear Equation (5.1), and corresponding initial values are $U_1(0) = \xi_1, U_2(0) = \xi_2, \cdots, U_n(0) = \xi_n$, so we have

$$\begin{aligned} &\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots U_n(t)\right\|_{\Lambda E_k}^2 \\ &= \left\|\xi_1\Lambda\xi_2\Lambda\cdots\xi_n\right\|_{\Lambda E_k}^2 \exp\left(\int_0^t trF'(\psi(\tau))\cdot Q_n(\tau)\mathrm{d}\tau\right). \end{aligned} \tag{5.26}$$

where Λ denotes the exterior product, tr denotes the trace of the operator, $Q_n(\tau)$ is the orthogonal projection from space E_k to $span\{U_1(t), U_2(t), \cdots, U_n(t)\}$.

For a given time τ, assume $\omega_j(\tau) = (\zeta_j(\tau), \eta_j(\tau))^{\mathrm{T}}, (j = 1, 2, \cdots, n)$ is orthonormal basis of $span\{U_1(t), U_2(t), \cdots, U_n(t)\}$.

Define the inner product of E_k

$$\left(\left(\zeta,\eta\right),\left(\bar{\zeta},\bar{\eta}\right)\right)=\left(\nabla^{2m+k}\zeta,\nabla^{2m+k}\bar{\zeta}\right)+\left(\nabla^{k}\eta,\nabla^{k}\bar{\eta}\right).$$

To sum up, we get

$$\begin{aligned} trF_t\left(\psi\left(\tau\right)\right)Q_n\left(\tau\right) &= \sum_{j=1}^{n}\left(F_t\left(\psi\left(\tau\right)\right)Q_n\left(\tau\right)\omega_j\left(\tau\right),\omega_j\left(\tau\right)\right)_{E_k} \\ &= \sum_{j=1}^{n}\left(F_t\left(\psi\left(\tau\right)\right)\omega_j\left(\tau\right),\omega_j\left(\tau\right)\right)_{E_k}. \end{aligned} \tag{5.27}$$

And

$$\left(F_t\left(\psi\left(\tau\right)\right)\omega_j\left(\tau\right),\omega_j\left(\tau\right)\right)_{E_k}=-\left(\Lambda_\varepsilon\omega_j,\omega_j\right). \tag{5.28}$$

By using Holder inequality, Young's inequality and Poincare inequality

$$\begin{aligned} \left(\Lambda_\varepsilon\omega_j,\omega_j\right) &= \Big(\Big(\varepsilon\zeta_j-\eta_j,\Big(M\left(\left\|\nabla^m u\right\|_p^p\right)-\beta\varepsilon\Big)\nabla^{4m}\zeta_j+\varepsilon^2\zeta_j+D\cdot\zeta_j \\ &\quad +g'\left(u\right)\cdot\zeta_j+\beta\nabla^{4m}\eta_j-\varepsilon\eta_j\Big),\left(\zeta_j,\eta_j\right)\Big) \\ &= \varepsilon\left\|\nabla^{2m+k}\zeta_j\right\|^2+\left(M\left(\left\|\nabla^m u\right\|_p^p\right)-\beta\varepsilon-1\right)\left(\nabla^{2m+k}\zeta_j,\nabla^{2m+k}\eta_j\right) \\ &\quad +\varepsilon^2\left(\nabla^k\zeta_j,\nabla^k\eta_j\right)+\left(\nabla^k\cdot\nabla\zeta_j,\nabla^k\eta_j\right)+\left(\nabla^k\left(g'\left(u\right)\cdot\zeta_j\right),\nabla^k\eta_j\right) \\ &\quad +\beta\left\|\nabla^{2m+k}\eta_j\right\|^2-\varepsilon\left\|\nabla^k\eta_j\right\|^2 \\ &\geq \gamma_1\left\|\nabla^{2m+k}\zeta_j\right\|^2+\gamma_2\left\|\nabla^k\eta_j\right\|^2-C_{23}\left\|\nabla^k\zeta_j\right\|^2. \end{aligned} \tag{5.29}$$

where

$$\gamma_1=\frac{1}{2}\left(3\varepsilon+C_{21}\lambda_j^{-m-k}-\beta\varepsilon-\varepsilon^2\lambda_j^{-2m}-C_{22}\lambda_j^{-2m-k}\right),$$

$$\gamma_2=\frac{1}{2}\left(\left(2\beta+\varepsilon-\beta\varepsilon\right)\lambda_j^{-2m}+C_{21}\lambda_j^{2m}-\frac{C_{23}}{2}-\varepsilon-\varepsilon^2-C_{22}\right).$$

Let $\bar{\delta}=\min\left\{\gamma_1,\gamma_2\right\},\delta=\min\left\{\bar{\delta}\lambda_j^{2m+k},\bar{\delta}\lambda_j^{k}\right\}$, then we get

$$\begin{aligned} &\left(F_t\left(\psi\left(\tau\right)\right)\omega_j\left(\tau\right),\omega_j\left(\tau\right)\right)_{E_k} \\ &= -\left(\Lambda_\varepsilon\omega_j,\omega_j\right)\leq-\bar{\delta}\left(\left\|\nabla^{2m+k}\zeta_j\right\|^2+\left\|\nabla^k\eta_j\right\|^2\right)+C_{23}\left\|\nabla^k\zeta_j\right\|^2 \\ &\leq \delta\left(\left\|\zeta_j\right\|^2+\left\|\eta_j\right\|^2\right)+C_{23}\left\|\nabla^k\zeta_j\right\|^2. \end{aligned} \tag{5.30}$$

Because of $\omega_j\left(\tau\right)=\left(\zeta_j\left(\tau\right),\eta_j\left(\tau\right)\right)^{\mathrm{T}},\left(j=1,2,\cdots,n\right)$ is orthonormal basis, thus

$$\left\|\zeta_j\right\|^2+\left\|\eta_j\right\|^2=1. \tag{5.31}$$

$$\sum_{j=1}^{N}\left(F_t\left(\psi\left(\tau\right)\right)\omega_j\left(\tau\right),\omega_j\left(\tau\right)\right)_{E_k}\leq-N\delta+C_{23}\sum_{j=1}^{N}\left\|\nabla^k\zeta_j\right\|^2. \tag{5.32}$$

There exists $s=\dfrac{k}{2m+k}$, and $0\leq s<1$, we have

$$\sum_{j=1}^{N}\left\|\nabla^k\zeta_j\right\|^2\leq\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{5.33}$$

So

$$trF_t\left(\psi(\tau)\right)\cdot Q_n(\tau)\le -N\delta+C_{23}\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{5.34}$$

Then assume $(u_0,u_1)\in A$, A is bounded absorbing set of E_k, $\psi(t)=\left(u(t),u_t(t)+\varepsilon u(t)\right)\in D(A)$, $D(A)=\left\{u\in H^{2m}, Au\in L^2(\Omega)\right\}$, let

$$q_N(t)=\sup_{\psi_0\in A}\sup_{\substack{\eta_j\in E_k\\ \left\|\nabla^k\eta_j\right\|\le 1}}\left(\frac{1}{t}\int_0^t trF_t\left(S(\tau)\psi_0\right)\cdot Q_n(\tau)\mathrm{d}\tau\right). \tag{5.35}$$

$$q_N=\lim_{t\to\infty}q_N(t). \tag{5.36}$$

By the inequality (5.34)-(5.36), we can get

$$q_N\le -N\delta+C_{23}\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{5.37}$$

Thus, the Lyapunov exponent of B_{0k} is uniformly bounded, that is

$$\sigma_1+\sigma_2+\cdots+\sigma_n\le -N\delta+C_{23}\sum_{j=1}^{N}\lambda_j^{s-1}. \tag{5.38}$$

So, there exists a $s\in[0,1)$, such that

$$\left(q_j\right)_+\le -N\delta+C_{23}\sum_{j=1}^{N}\lambda_j^{s-1}\le C_{23}\sum_{j=1}^{N}\lambda_j^{s-1}\le\frac{N\delta}{7}. \tag{5.39}$$

where λ_j is the eigenvalue of *A*, and $\lambda_1<\lambda_2<\cdots<\lambda_m$, then

$$q_N\le -N\delta\left(1-\frac{C_{23}}{N\delta}\sum_{j=1}^{N}\lambda_j^{s-1}\right)\le -\frac{6}{7}N\delta. \tag{5.40}$$

So

$$\max_{1\le j\le n}\frac{\left(q_j\right)_+}{\left|q_N\right|}\le\frac{1}{6}. \tag{5.41}$$

Thus, we can get the conclusion $d_H\left(A_k\right)<\frac{1}{6}n, d_F\left(A_k\right)<\frac{7}{6}n$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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