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Mathematical Morphological Distributive Concepts over Unions and Intersections

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Abstract

Mathematical Morphological concepts outline techniques for analysing and processing geometric structures based on set theory. In this paper, we present proofs of our theorems on morphological distributive properties over Unions and Intersections with respect to Dilation and Erosion. These results provide new realizations of Dilation, Erosion and conclude that they are distributive over Unions but non-distributive over Intersections.

Keywords

1. Introduction

Mathematical Morphology is a tool for the extraction of components of images used to describe and represent skeletons, boundaries etc., which involves techniques like morphological thinning, pruning and filtering. Morphological concepts date back to works done by Matheron and Minkowski who used binary mathematical morphology on integral geometry [1], [2]. Matheron and Serra also used the techniques in texture and image analysis [1], [3]. In our recent paper titled "Revised Mathematical Morphological Concepts" [4], we outlined in details some mathematical morphological operators and their algebraic structures when they are linked with unions and intersections. We showed that the partitioning of structural elements before morphological operations is possible. In this paper, we present results on the distributive properties of Dilation and Erosion over unions and intersections. This paper is a continuation of the revised mathematical morphological concept [4] and hence most of the concepts that were developed and discussed in it will be used here without explaining. Therefore, we urge that you read it before going through this paper.

2. Preliminaries

2.1. Definitions

The following definitions are important for our purpose.

Definition 1 (Dilation) *Let the image set X and the structuring element B be subsets of the discrete space* Z^2: $X \subset Z^2$, $B \subset Z^2$. *The dilation of X by B is defined as* $X \oplus B = \{c = x + b \in Z^2 : x \in X; b \in B\}$; *or the Dilation of a binary image A by structure element B, is* $A \oplus B = \{a + b \mid \text{for } a \in A \text{ and } b \in B\}$.

The dilation transform generally causes image objects to grow in size. From the definitions above, dilation is equivalent to a union of translates of the original image with respect to the structure element, that is, $A \oplus B = \bigcup_{b \in B} A_b$.

Definition 2 (Erosion) *Let the image set X and the structuring element B be subsets of the discrete space* Z^2: $X \subset Z^2$, $B \subset Z^2$. *The erosion of X by B is defined as* $X \ominus B = \{c \in Z^2 : \forall b \in B, \exists x \in X \text{ where } c = x - b\}$; *or the Erosion of a binary image A by structure element B, is* $A \ominus B = \{p \mid p + b \in A \; \forall b \in B\}$.

Similarly erosion transform allows image objects to shrink in size, that is, $A \ominus B = \bigcap_{b \in B} A_{-b}$.

2.2. Algebraic Properties of Dilation and Erosion

We note that Dilation is commutative and associative, that is, $A \oplus B = B \oplus A$ and $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, where as Erosion is non-commutative and non-associative, that is, $A \ominus B \neq B \ominus A$ and $(A \ominus B) \ominus C \neq A \ominus (B \ominus C)$ respectively.

Furthermore, Dilation and Erosion are both translation invariant, that is, if x is a vector belonging to A and B ($x \in A$, $x \in B$), then $A \oplus B_x = A_x \oplus B = (A \oplus B)_x$ and $A \ominus B_x = A_x \ominus B = (A \ominus B)_x$. Also both Dilation and Erosion are increasing in A, that is, if an image set A_1 is a subset of A_2 ($A_1 \subset A_2$), then $A_1 \oplus B \subset A_2 \oplus B$ and $A_1 \ominus B \subset A_2 \ominus B$. However, Erosion is decreasing in B, that is, if a structuring element B_1 is a subset of B_2 ($B_1 \subset B_2$), then $A \ominus B_1 \supset A \ominus B_2$. Dilation and Erosion transforms are duals of each other, that is, $(A \oplus B)^c = A^c \ominus \breve{B}$ and $(A \ominus B)^c = A^c \oplus \breve{B}$. Dilation and Erosion are also not the inverse of each other, that is, $(A \oplus B) \ominus B \neq A$ and $(A \ominus B) \oplus B \neq A$. Both the dilation and erosion transforms have an identity set, I, such that $A \oplus I = A$ and $A \ominus I = A$. Dilation transform has an empty set, that is, $\varnothing$ such that $A \oplus \varnothing = \varnothing$.

3. Results

We present results of the distribution of morphological operators over set unions and intersections of two different sets and their extensions. The theorems and their proofs below will facilitate the understanding of the various results.

The Distribution of Morphological Operators over Set Union and Intersection

Theorem 1 (The distribution of Dilation over union with n distinct structural elements)

If $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$ for $n \geq 2$

Then $A \oplus B = (A \oplus B_1) \cup (A \oplus B_2) \cup (A \oplus B_3) \cup \cdots \cup (A \oplus B_n)$

Proof:

If $B = B_1 \cup B_2$

$$(A \oplus B_1) \cup (A \oplus B_2) = \bigcup_{b_1 \in B_1} A_{b_1} \cup \bigcup_{b_2 \in B_2} A_{b_2}$$

Then

$$= \bigcup_{b \in B_1 \cup B_2} \left\{ A_b \cap A_{b_1} \right\} \cup \bigcup_{b \in B_1 \cup B_2} \left\{ A_b \cap A_{b_2} \right\}$$

$$= \bigcup_{b \in B_1 \cup B_2} \left\{ A_b \cap A_{b_1} \cup A_b \cap A_{b_2} \right\}$$

$$= \bigcup_{b \in B_1 \cup B_2} A_b = A \oplus (B_1 \cup B_2) = A \oplus B$$

This implies $A \oplus B = (A \oplus B_1) \cup (A \oplus B_2)$

Let assume that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k$ for $k \geq 2$

Then $A \oplus B = (A \oplus B_1) \cup (A \oplus B_2) \cup (A \oplus B_3) \cup \cdots \cup (A \oplus B_k)$

Now we show that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}$

$$A \oplus B = A \oplus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1})$$

Then

$$= A \oplus \left[(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1} \right]$$

$$= \left[A \oplus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \right] \cup (A \oplus B_{k+1})$$

$$= (A \oplus B_1) \cup (A \oplus B_2) \cup (A \oplus B_3) \cup \cdots \cup (A \oplus B_k) \cup (A \oplus B_{k+1})$$

Theorem 2 (The distribution of Erosion over union with n distinct structural elements)

If $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$ for $n \geq 2$

Then $A \ominus B = (A \ominus B_1) \cup (A \ominus B_2) \cup (A \ominus B_3) \cup \cdots \cup (A \ominus B_n)$

Proof:

If $B = B_1 \cup B_2$

$$(A \ominus B_1) \cup (A \ominus B_2) = \bigcap_{b_1 \in B_1} A_{-b_1} \cup \bigcap_{b_2 \in B_2} A_{-b_2}$$

Then

$$= \bigcap_{b \in B_1 \cup B_2} \left\{ A_{-b} \cap A_{-b_1} \right\} \cup \bigcap_{b \in B_1 \cup B_2} \left\{ A_{-b} \cap A_{-b_2} \right\}$$

$$= \bigcap_{b \in B_1 \cup B_2} \left\{ A_{-b} \cap A_{-b_1} \cup A_{-b} \cap A_{-b_2} \right\}$$

$$= \bigcap_{b \in B_1 \cup B_2} A_{-b} = A \ominus (B_1 \cup B_2) = A \ominus B$$

This implies $A \ominus B = (A \ominus B_1) \cup (A \ominus B_2)$

Let assume that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k$ for $k \geq 2$

Then $A \ominus B = (A \ominus B_1) \cup (A \ominus B_2) \cup (A \ominus B_3) \cup \cdots \cup (A \ominus B_k)$

Now we show that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}$

$$A \ominus B = A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1})$$

Then

$$= A \ominus \left[(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1} \right]$$

$$= \left[A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \right] \cup (A \ominus B_{k+1})$$

$$= (A \ominus B_1) \cup (A \ominus B_2) \cup (A \ominus B_3) \cup \cdots \cup (A \ominus B_k) \cup (A \ominus B_{k+1})$$

The dilation of a set of two different structural elements and taking the union is the

same as taking the union of the structural element and dilating with the set. This shows that morphological dilation distributes over set unions. It also leads to the fact that; if any structural element can be partitioned into n distinct parts then the union of each of the partitions dilation with the set is the same as the set's dilation with the structural element. We note also that provided any structural element can be partitioned into n distinct parts, then the union of each of the partition's erosion with the set is equal to the set's erosion with the structural element.

Theorem 3 (Non-distribution of Erosion over intersection)

If $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$ for $n \geq 2$

Then $A \ominus B = (A \ominus B_1) \cap (A \ominus B_2) \cap (A \ominus B_3) \cap \cdots \cap (A \ominus B_n)$

Proof:

If $B = B_1 \cup B_2$

$$\left[(A \ominus B_1) \cap (A \ominus B_2) \right]^c = (A \ominus B_1)^c \cup (A \ominus B_2)^c$$

Then
$$= (A^c \oplus \breve{B}_1) \cup (A^c \oplus \breve{B}_2)$$

$$= A^c \oplus (\breve{B}_1 \cup \breve{B}_2)$$

$$\Rightarrow (A \ominus B_1) \cap (A \ominus B_2) = \left[A^c \oplus (\breve{B}_1 \cup \breve{B}_2) \right]^c$$

$$= A \ominus (B_1 \cup B_2) = A \ominus B$$

This implies $A \ominus B = (A \ominus B_1) \cap (A \ominus B_2)$

Let assume that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k$ for $k \geq 2$

Then $A \ominus B = (A \ominus B_1) \cap (A \ominus B_2) \cap (A \ominus B_3) \cap \cdots \cap (A \ominus B_k)$

Now we show that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}$

$$A \ominus B = A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1})$$

$$= A \ominus \left[(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1} \right]$$

Then
$$= \left[A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \right] \cap (A \ominus B_{k+1})$$

$$= (A \ominus B_1) \cap (A \ominus B_2) \cap (A \ominus B_3) \cap \cdots \cap (A \ominus B_k) \cap (A \ominus B_{k+1})$$

Theorem 4 (Non-distribution of Dilation over intersection)

If $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$ for $n \geq 2$

Then $A \oplus B = (A \oplus B_1) \cap (A \oplus B_2) \cap (A \oplus B_3) \cap \cdots \cap (A \oplus B_n)$

Proof:

If $B = B_1 \cup B_2$

$$\left[(A \oplus B_1) \cap (A \oplus B_2) \right]^c = (A \oplus B_1)^c \cup (A \oplus B_2)^c$$

Then
$$= (A^c \ominus \breve{B}_1) \cup (A^c \ominus \breve{B}_2)$$

$$= A^c \ominus (\breve{B}_1 \cup \breve{B}_2)$$

$$\Rightarrow (A \oplus B_1) \cap (A \oplus B_2) = \left[A^c \ominus (\breve{B}_1 \cup \breve{B}_2) \right]^c$$

$$= A \oplus (B_1 \cup B_2) = A \oplus B$$

This implies $A \oplus B = (A \oplus B_1) \cap (A \oplus B_2)$

Let assume that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k$ for $k \geq 2$

Then, $A \oplus B = (A \oplus B_1) \cap (A \oplus B_2) \cap (A \oplus B_3) \cap \cdots \cap (A \oplus B_k)$

Now, we show that if $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}$

$$A \oplus B = A \oplus \left(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1} \right)$$

Then,
$$= A \oplus \left[\left(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \right) \cup B_{k+1} \right]$$
$$= \left[A \oplus \left(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \right) \right] \cap \left(A \oplus B_{k+1} \right)$$
$$= \left(A \oplus B_1 \right) \cap \left(A \oplus B_2 \right) \cap \left(A \oplus B_3 \right) \cap \cdots \cap \left(A \oplus B_k \right) \cap \left(A \oplus B_{k+1} \right)$$

The intersection of the erosion of a set with structural elements is equivalent to the union of the structural elements on the erosion of the set. We note that since we are supposed to take the union instead of the intersection, it shows that morphological erosion is non-distributed over set intersection. Similar arguments hold for dilation which leads to the non-distributive property of dilation over intersection.

4. Conclusion

We have presented theorems and their proofs on morphological distribution properties over unions and intersections. Our results show that Dilation and Erosion are distributive over unions but non-distributive over intersections. In addition, our theorems facilitate the partitioning of structural elements in order to implement morphological operations.

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