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Some Inequalities on Polar Derivative of Polynomial Having No Zero in a Disc

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Abstract

Let $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu\le n$, be a polynomial of degree n having no zero in $|z|<k$, $k\ge1$, then Qazi [*Proc. Amer. Math. Soc.*, 115 (1992), 337-343] proved

$$\max_{|z|=1}|p'(z)|\le n\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\max_{|z|=1}|p(z)|.$$

In this paper, we first extend the above inequality to polar derivative of a polynomial. Further, as an application of our result, we extend a result due to Dewan *et al.* [*Southeast Asian Bull. Math.*, 27 (2003), 591-597] to polar derivative.

Keywords

Polynomials, Inequalities, Polar Derivative of a Polynomial, Zeros

1. Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree *n*. Then according to the well-known Bernstein's inequality [1].

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

Equality holds in (1.1) if and only if $p(z)$ has all its zeros at the origin.

If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, then inequality (1.1) can be

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sharpened. It was conjectured by Erdös and later verified by Lax [2] that if $p(z)\neq 0$ in $|z|<1$, then (1.1) can be replaced by

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

Inequality (1.2) is best possible and equality attains for $p(z)=\alpha+\beta z^n$, $|\alpha|=|\beta|$.

Malik [3] extended (1.2) by considering the class of polynomials $p(z)$ of degree n not vanishing in $|z|<k$, $k\geq 1$, and proved

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.3}$$

Qazi [4] considered a more general class of polynomials $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\leq\mu\leq n$, having no zero in $|z|<k$, $k\geq 1$, and obtained the following, which is a generalization as well as an improvement of (1.3).

Theorem A. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\leq\mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then*

$$\max_{|z|=1}|p'(z)|\leq n\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\max_{|z|=1}|p(z)|. \tag{1.4}$$

Inequality (1.4) *is sharp and equality holds for the polynomial* $p(z)=\left(z^\mu+k^\mu\right)^{\frac{n}{\mu}}$ *where n is a multiple of* μ.

By involving $\min_{|z|=k}|p(z)|$, the above theorem was improved by Dewan *et al.* [5] for $1\leq\mu<n$.

Theorem B. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\leq\mu<n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then*

$$\begin{aligned}\max_{|z|=1}|p'(z)|\leq\ & n\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\max_{|z|=1}|p(z)|\\ &-\frac{n}{k^n}\left\{1-\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\min_{|z|=k}|p(z)|.\end{aligned} \tag{1.5}$$

Inequality (1.5) *is best possible for* $p(z)=\left(z^\mu+k^\mu\right)^{\frac{n}{\mu}}$ *where n is a multiple of* μ *with* $1\leq\mu<n$.

Remark 1. Theorem B proved by Dewan *et al.* [5] seems to have a deficiency in the sense that for $\mu=n$ the corresponding result was not specified. In fact, by simple calculation, we find the result to be the equality

$$\max_{|z|=1}|p'(z)|=\frac{n}{1+k^n}\left\{\max_{|z|=1}|p(z)|-\min_{|z|=k}|p(z)|\right\}. \tag{1.6}$$

Let $p(z)$ be a polynomial of degree n and α be any real or complex number, then the polar derivative of $p(z)$, denoted by $D_\alpha p(z)$, is defined as

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z). \tag{1.7}$$

The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ of $p(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p'(z).$$

The polynomial $D_\alpha p(z)$ is called by Laguerre ([6], p. 48) the "émanant" of $p(z)$, by Pólya and Szegö [7] the "derivative of $p(z)$ with respect to the point α" and by Marden ([8], p. 44) simply " the polar derivative of $p(z)$".

Aziz [9] extended (1.3) to the polar derivative of $p(z)$ by showing that if $p(z)$ has no zero in $|z|<k$, $k\ge 1$, then for every real or complex number α with $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\le n\left(\frac{k+|\alpha|}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.8}$$

Inequality (1.8) is best possible and equality holds for $p(z)=(z+k)^n$ with $\alpha\ge 1$ and $k\ge 1$.

Further, by considering a more general class of polynomials $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu\le n$, of degree n having no zero in $|z|<k$, $k\ge 1$, then for every real or complex number α with $|\alpha|\ge 1$, it was Dewan and Singh [10] who proved the following inequality which generalizes inequality (1.8) due to Aziz [9].

$$\max_{|z|=1}|D_\alpha p(z)|\le n\frac{\left(k^\mu+|\alpha|\right)}{1+k^\mu}\max_{|z|=1}|p(z)|. \tag{1.9}$$

In this paper, we first extend Theorem A to polar derivative of a polynomial, which gives an improvement of (1.9). More precisely, we prove.

Theorem 1. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\le n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}|p(z)|. \tag{1.10}$$

Equality in (1.10) *holds for* $\mu=1$ *with* $\alpha\ge 1$, *extremal polynomial being* $p(z)=(z+k)^n$, $k\ge 1$.

Remark 2. To prove that the bound of Theorem 1is better than that of (1.9), it is sufficient to prove that

$$\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\le\frac{\left(k^\mu+|\alpha|\right)}{1+k^\mu},$$

i.e. equivalently,

$$\left(|\alpha|-1\right)\left\{k^\mu+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu+1}\right\}\le\left(|\alpha|-1\right)\left\{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}\right\},$$

i.e.

$$\left(|\alpha|-1\right)(k-1)k^\mu\left\{1-\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu\right\}\ge 0$$

which is true since $|\alpha|\ge 1$, $k\ge 1$, and by (2.5) of Lemma 2.3, *i.e.*, $\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu\le 1$.

Further, if we put $\mu=1$ in Theorem 1, we get the following result which is an improvement of inequality (1.8) due to Aziz [9].

Corollary 1. *If* $p(z)=\sum_{\upsilon=0}^{n}a_\upsilon z^\upsilon$ *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\le n\left\{\frac{(n|a_0|+|a_1|)k^2+|\alpha|(n|a_0|+|a_1|k^2)}{n|a_0|(1+k^2)+2k^2|a_1|}\right\}\max_{|z|=1}|p(z)|. \tag{1.13}$$

Remark 3. Inequality (1.13) is the corresponding polar derivative version of a result proved by Govil *et al.* ([11], Inequality (10)).

Remark 4. As mentioned earlier, inequality (1.13) improves inequality (1.8) and is evident from Remark 2, for the paticularcase $\mu=1$.

It is of interest that as an application of Theorem 1, we have been able to obtain an independent proof of a result proved by Mir and Dar ([12], Theorem 1), which involves $\min_{|z|=k}|p(z)|$ and extends Theorem B to polar derivative which also improves upon Theorem 1 for $1\le\mu\le n$. In fact, we prove

Theorem 2. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge 1$,

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)|\\ &\le\frac{n}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1}+k^{2\mu})}\left[\left\{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)\right\}\max_{|z|=1}|p(z)|\right.\\ &\quad\left.-\left\{\frac{|\alpha|-1}{k^n}\left(k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}\right)\right\}\min_{|z|=k}|p(z)|\right]\quad\text{for } 1\le\mu<n\end{aligned} \tag{1.11}$$

$$\max_{|z|=1}|D_\alpha p(z)|=\frac{n}{k^n+1}\left\{(k^n+|\alpha|)\max_{|z|=1}|p(z)|-(|\alpha|-1)\min_{|z|=k}|p(z)|\right\}\quad\text{for }\mu=n \tag{1.12}$$

Equality occurs in (1.11) *for* $\mu=1$ *with* $\alpha\ge 1$, *extremal polynomial being* $p(z)=(z+k)^n$, $k\ge 1$.

If we divide both sides of the above inequalities (1.11) and (1.12) by $|\alpha|$ and make $|\alpha|\to\infty$, we obtain the inequalities (1.5) and (1.6) respectively.

Remark 5. For $\mu=1$, Theorem 2 gives the following

Corollary 2. *If* $p(z)=\sum_{\upsilon=0}^{n}a_\upsilon z^\upsilon$ *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge 1$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)|&\le\frac{n}{n|a_0|(1+k^2)+2k^2|a_1|}\Big[\left\{(n|a_0|+|a_1|)k^2+|\alpha|(n|a_0|+|a_1|k^2)\right\}\\ &\quad\times\max_{|z|=1}|p(z)|-(|\alpha|-1)\frac{(n|a_0|+|a_1|)k^2}{k^n}\min_{|z|=k}|p(z)|\Big].\end{aligned} \tag{1.14}$$

Inequality (1.14) *is best possible for* $p(z)=(z+k)^n$, *with* $\alpha\ge 1$ *and* $k\ge 1$.

Remark 6. It is obvious that Corollary 2 is an improvement of Corollary 1.

2. Lemmas

The following lemmas are required in the proofs of the theorems.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree n, then on* $|z|=1$,

$$|p'(z)|+|q'(z)| \le n \max_{|z|=1} |p(z)|, \tag{2.1}$$

where

$$q(z)=z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

The above lemma is a special case of a result due to Govil and Rahman [13].

Lemma 2.2. *If* $p(z)$ *is a polynomial of degree n, then for every real or complex number* α, *we have on* $|z|=1$,

$$|D_\alpha p(z)| \le n \max_{|z|=1} |p(z)| + (|\alpha|-1)|p'(z)|. \tag{2.2}$$

Proof of Lemma 2.2. The proof of this lemma is simple and follows as a part ([10], proof of Theorem 1), but for the sake of completeness, we outline it. Let $q(z)=z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. Then it is easy to verify that on $|z|=1$,

$$|q'(z)| = |np(z)-zp'(z)|. \tag{2.3}$$

Now, for every real or complex number α, the polar derivative of $p(z)$ with respect to α is

$$D_\alpha p(z) = np(z)+(\alpha - z)p'(z). \tag{2.4}$$

This implies on $|z|=1$,

$$\begin{aligned}|D_\alpha p(z)| &\le |np(z)-zp'(z)|+|\alpha||p'(z)| \\ &= |q'(z)|+|\alpha||p'(z)| \quad (\text{by } (2.3)) \\ &\le n\max_{|z|=1}|p(z)|+(|\alpha|-1)|p'(z)|, \quad (\text{by Lemma } 2.1)\end{aligned}$$

which completes the proof of Lemma 2.2.

Lemma 2.3. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n} a_\upsilon z^\upsilon$, $1\le \mu \le n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge 1$, *then*

$$|q'(z)| \ge k^{\mu+1} \frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1}+1}{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}} |p'(z)| \text{ on } |z|=1, \tag{2.4}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu \le 1. \tag{2.5}$$

Lemma 2.3 is due to Qazi ([4], Proof and Remark of Lemma 1).

3. Proofs of the Theorems

Proof of Theorem 1. On $|z|=1$, by Lemma 2.1, we have

$$|p'(z)|+|q'(z)| \le n \max_{|z|=1} |p(z)|, \tag{3.1}$$

and by inequality (2.4) of Lemma 2.3, we have

$$\left|p'(z)\right|k^{\mu+1}\frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1}+1}{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}\le\left|q'(z)\right|. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain for $|z|=1$,

$$\left|p'(z)\right|\left\{1+k^{\mu+1}\frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1}+1}{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}\right\}\le n\max_{|z|=1}\left|p(z)\right|,$$

which gives for $|z|=1$,

$$\left|p'(z)\right|\le n\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}\left|p(z)\right|.$$

Now, if $|\alpha|\ge 1$, then multiplying both sides of the above inequality by $\left(|\alpha|-1\right)$, we get

$$\left(|\alpha|-1\right)\left|p'(z)\right|\le n\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\left(|\alpha|-1\right)\max_{|z|=1}\left|p(z)\right|. \tag{3.3}$$

Inequality (3.3) when combined with Lemma 2.2, gives for $|z|=1$ and $|\alpha|\ge 1$,

$$\left|D_\alpha p(z)\right|\le n\max_{|z|=1}\left|p(z)\right|+n\left\{\frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\left(|\alpha|-1\right)\max_{|z|=1}\left|p(z)\right|,$$

which is equivalent to

$$\left|D_\alpha p(z)\right|\le n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}\left|p(z)\right|,$$

from which Theorem1 follows.

Proof of Theorem 2. First, we prove inequality (1.11).

Let $m=\min_{|z|=k}\left|p(z)\right|$. Since $p(z)$ has no zero in $|z|<k$, $k\ge 1$ the polynomial $p(z)+m\beta z^n$ has no zero in $|z|<k$, $k\ge 1$, for every real or complex number β with $|\beta|<\frac{1}{k^n}$. The claim is obvious if $p(z)$ has a zero on $|z|=k$ for then $m=0$ and hence $p(z)+m\beta z^n=p(z)$. If $p(z)$ has no zero on $|z|=k$, then we have $\left|m\beta z^n\right|<\left|p(z)\right|$ on $|z|=k$ and the claim follows from Rouché's theorem. Thus, in any case $p(z)+m\beta z^n$ has no zero in $|z|<k$, $k\ge 1$ and therefore on applying Theorem 1 to the polynomial $p(z)+m\beta z^n$, that is to $a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon+m\beta z^n$, where $1\le\mu<n$, we have for every real or complex number α with $|\alpha|\ge 1$,

$$\max_{|z|=1}\left|D_\alpha\left\{p(z)+m\beta z^n\right\}\right| \le n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}\left|p(z)+m\beta z^n\right|,$$

which implies

$$\max_{|z|=1}\left|D_\alpha p(z)+n\alpha m\beta z^{n-1}\right| \le n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\left\{\max_{|z|=1}|p(z)|+m|\beta|\right\}. \tag{3.4}$$

Let z_0 be a point on the unit circle such that $\max_{|z|=1}|D_\alpha p(z)|=|D_\alpha p(z_0)|$, then (3.4), in particular, gives

$$\left|D_\alpha p(z_0)+n\alpha m\beta z_0^{n-1}\right| \le n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\left\{\max_{|z|=1}|p(z)|+m|\beta|\right\}. \tag{3.5}$$

Now, we choose the argument of β in (3.5) such that

$$\left|D_\alpha p(z_0)+n\alpha m\beta z_0^{n-1}\right| = |D_\alpha p(z_0)|+n|\alpha|m|\beta|.$$

Then (3.5) becomes

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\le n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}|p(z)| \\ &\quad -nm|\beta|\left\{|\alpha|-\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\} \\ &= n\left\{\frac{k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+|\alpha|\left(1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}|p(z)| \\ &\quad -nm|\beta|\left\{\frac{\left(|\alpha|-1\right)\left(k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}\right)}{1+k^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}.\end{aligned}$$

Finally, making $|\beta|\to\frac{1}{k^n}$ in the above inequality, we obtain inequality (1.11).

For $\mu=n$, the polynomial is simply $p(z)=a_0+a_nz^n$ having no zero in $|z|<k$, $k\ge 1$. As $p(z)$ has no

zero in $|z|<k$, therefore $|a_n|k^n \le |a_0|$. Then $\min_{|z|=k}|p(z)| = |a_0| - |a_n|k^n$, $\max_{|z|=1}|p(z)| = |a_0| + |a_n|$, and $\max_{|z|=1}|D_\alpha p(z)| = n(|a_0| + |\alpha||a_n|)$. From these three equations, equality (1.12) follows readily.

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