

A Kind of Doubly Periodic Riemann Boundary Value Problem on Two Parallel Curves

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Abstract

We proposed a kind of doubly periodic Riemann boundary value problem on two parallel curves. By using the method of complex functions, we investigated the method for solving this kind of doubly periodic Riemann boundary value problem of normal type and gave the general solutions and the solvable conditions for it.

Keywords

Normal Type, Doubly Periodic, Riemann Boundary Problem

1. Introduction

Various kinds of Riemann boundary value problems (BVPs) for analytic functions on closed curves or on open arcs, doubly periodic Riemann BVPs, doubly periodic or quasi-periodic Riemann BVPs and Dirichlet Problems, and BVPs for polyanalytic functions have been widely investigated in papers [1]-[8]. The main approach is to use the decomposition of polyanalytic functions and their generalization to transform the boundary value problems to their corresponding boundary value problems for analytic functions. Recently, inverse Riemann BVPs for generalized analytic functions or bianalytic functions have been investigated in papers [9]-[13].

In this paper, we consider a kind of doubly periodic Riemann boundary value problem on two parallel curves. By using the method of complex functions, we investigate the method for solving kind of doubly periodic Riemann boundary value problem of normal type and give the general solutions and the solvable conditions for it.

2. A Kind of Doubly Periodic Riemann Boundary Value Problem on Two Parallel Curves

Suppose that ω_1, ω_2 are complex constants with $\text{Im}(\omega_1/\omega_2) \neq 0$, and \mathbf{P} denotes the fundamental period pa-

rallelogram with vertices $\pm\omega_1 \pm \omega_2$. The function

$$\zeta(z) = 1/z + \sum'_{m,n} \left[1/(z - \Omega_{mn}) + 1/\Omega_{mn} + z/\Omega_{mn}^2 \right]$$

is called the Weierstrass ζ -function, where $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$, and $\sum'_{m,n}$ denotes the sum for all $m, n = 0, \pm 1, \pm 2, \dots$, except for $m = n = 0$.

Let $L_0 = \sum_{j=1}^2 L_{0j}$ be the set of two parallel curves, lying entirely in the fundamental period parallelogram \mathbf{P} ,

not passing the origin O , with endpoints being periodic congruent and having the same tangent lines at the periodic congruent points. Let D_1, D_2, D_3 denote the domains entirely in the fundamental period parallelogram \mathbf{P} , cut by L_{01} and L_{02} , respectively. Without loss of generality, we suppose that $O \in D_2$, see **Figure 1**. Let L_{01}^*, L_{02}^* be the curves periodically extended for L_{01} and L_{02} with period $2\omega_1$, respectively. And L_{nj}^* ($j = 1, 2; n = 0, \pm 1, \dots$) be the curves periodically extended for L_{0j}^* with $2n\omega_2$.

Our objective is to find sectionally holomorphic doubly periodic functions $F(z)$ and $\Omega(z)$, satisfying the following boundary conditions

$$\begin{cases} F^+(\tau) = D_1(\tau)\Omega^-(\tau) + g_1(\tau), & \tau \in L_{01}, \\ \Omega^+(\tau) = D_2(\tau)F^-(\tau) + g_2(\tau), & \tau \in L_{02}, \end{cases} \tag{1}$$

where $D_j(\tau), g_j(\tau) \in H$, and be doubly periodic with $2\omega_1, 2\omega_2$. $F^\pm(\tau)$ are the boundary values of the function $F(z)$, which is analytic in D_1 and D_3 , belonging to the class $h(a_j)$ on L_{0j} , satisfying the boundary conditions (1), and $\Omega^\pm(\tau)$ are the boundary values of the function $\Omega(z)$, which is analytic in D_2 , belonging to the class $h(a_j)$ on L_{0j} , satisfying the boundary conditions (1).

Since a_j plays the same roles as other points on L_{0j} ($j = 1, 2$), it is natural to require that the unknown functions are bounded at $z = a_j$, that is, the unknown functions $F(z)$ and $\Omega(z)$ are both bounded on L_{01}^* and L_{02}^* .

Problem (1) is called the normal type if $D_j(\tau) \neq 0$ ($j = 1, 2$), otherwise the non-normal type. And if we allow the solution $\Omega(z)$ has poles of order m at $z = 0$, it is actually to solve problem (1) in DR_m .

3. Preliminary Notes

Since $D_j(\tau) \in H$ with $D_j(\tau) \neq 0$ ($j = 1, 2$), by taking logarithm of $\log D_j(\tau)$ for some branch on L_{0j} , we may obtain a continuous single-valued function such as

$$\begin{aligned} -\frac{1}{2\pi i} \log D_j(a_j) &= \alpha_{a_j} + i\beta_{a_j}, & j = 1, 2, \\ \frac{1}{2\pi i} \log D_j(a_j + 2\omega_j) &= -\alpha_{a_j} - i\beta_{a_j}, & j = 1, 2. \end{aligned}$$

with $0 \leq \alpha_{a_j} < 1$. Now we call the integer $\kappa = \kappa_1 + \kappa_2$ the index of problem (1), where κ_j is the integer satisfying

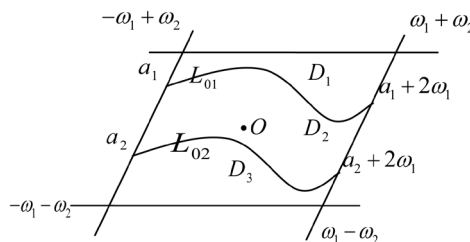


Figure 1. Parallel curves in the fundamental period parallelogram \mathbf{P} .

$$0 \leq -\alpha_{a_j} - \kappa_j < 1, \quad j = 1, 2.$$

Since κ_j can only be 0 and -1 , the index κ can only take $0, -1, -2$.

Set

$$D_* = D_{1*} + D_{2*} = \frac{1}{2\pi i} \int_{L_{01}} \log D_1(\tau) + \frac{1}{2\pi i} \int_{L_{02}} \log D_2(\tau) d\tau \tag{2}$$

$$\gamma_j(z) = \frac{1}{2\pi i} \int_{L_{0j}} \log D_j(\tau) \xi(\tau - z) d\tau, \quad z \notin L_{0j}, \quad j = 1, 2 \tag{3}$$

We can easily see that $1/e^{\gamma_j(z)}$ will have singularities at most less than one order near the endpoints a_j and $a_j + 2\omega_1$ ($j = 1, 2$). Let

$$e^{\gamma(z)} = e^{\gamma_1(z)} e^{\gamma_2(z)} \tag{4}$$

then we have

$$e^{\gamma(z+2\omega_j)} = e^{-2\eta_j D_*} e^{\gamma(z)}, \quad j = 1, 2,$$

where $\eta_j = \zeta(\omega_j)$ ($j = 1, 2$) and $2\omega_2 \eta_1 - 2\omega_1 \eta_2 = \pi i$. Thus $e^{\gamma(z)}$ is not doubly periodic generally. In fact, $e^{\gamma(z)}$ is doubly periodic if and only if

$$\eta_j D_* = k_j \pi i, \quad k_j \text{ is positive integer for } j = 1, 2. \tag{5}$$

Lemma 1. Formula (5) is valid if and only if

$$\eta_1 / \eta_2 = k_1 / k_2, \quad D_* = 2k_1 \omega_2 - 2k_2 \omega_1.$$

And if both $D_* = 2l_1 \omega_1 + 2l_2 \omega_2$ and $\eta_j D_* = k_j \pi i$ are true, then we have $l_1 = -k_2$ and $l_2 = k_1$, where l_j, k_j are all integers.

4. Solution for Problem (1) of Normal Type

Problem (1) can be transferred by using (3) as

$$\begin{cases} \frac{F^+(\tau)}{e^{\gamma_1^+(\tau)}} = \frac{\Omega^-(\tau)}{e^{\gamma_1^-(\tau)}} + \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)}}, & \tau \in L_{01}, \\ \frac{\Omega^+(\tau)}{e^{\gamma_2^+(\tau)}} = \frac{F^-(\tau)}{e^{\gamma_2^-(\tau)}} + \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)}} & \tau \in L_{02}. \end{cases} \tag{6}$$

Multiplying $\frac{1}{e^{\gamma_2^+(\tau)}}$ to the two sides of the first identity in equations (6), and multiplying $\frac{1}{e^{\gamma_1^-(\tau)}}$ to the two sides of the second identity in Equations (6), gives

$$\begin{cases} \frac{F^+(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^+(\tau)}} = \frac{\Omega^-(\tau)}{e^{\gamma_1^-(\tau)} e^{\gamma_2^+(\tau)}} + \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^+(\tau)}}, & \tau \in L_{01}, \\ \frac{\Omega^+(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}} = \frac{F^-(\tau)}{e^{\gamma_2^-(\tau)} e^{\gamma_1^-(\tau)}} + \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}} & \tau \in L_{02}. \end{cases} \tag{7}$$

The function $1/e^{\gamma_j(z)}$ always has singularities at most less than one order near the endpoints a_j and $a_j + 2\omega_1$ ($j = 1, 2$) whatever $\kappa = 0, -1, -2$. And then, $\frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^+(\tau)}}$, $\frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}}$ must belong to class H or class H^* on L_{01} and L_{02} , respectively.

Case 1. If formula (5) holds, that is, $e^{\gamma(z)}$ is doubly periodic, then by Lemma 1 we have

$$D_* \equiv 0 \pmod{2\omega_1, 2\omega_2}. \tag{8}$$

Let

$$\Psi_1(z) = \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)}} [\zeta(\tau - z) + \zeta(z)] d\tau, \quad z \notin L_{01} \tag{9}$$

$$\Psi_2(z) = \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}} [\zeta(\tau - z) + \zeta(z)] d\tau, \quad z \notin L_{02} \tag{10}$$

Then by formulas (9) and (10), we may rewrite (7) as

$$\begin{cases} \frac{F^+(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)}} - \Psi_1^+(\tau) - \Psi_2^+(\tau) = \frac{\Omega^-(\tau)}{e^{\gamma_1^-(\tau)} e^{\gamma_2^-(\tau)}} - \Psi_1^-(\tau) - \Psi_2^+(\tau), & \tau \in L_{01} \\ \frac{\Omega^+(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}} - \Psi_1^-(\tau) - \Psi_2^+(\tau) = \frac{F^-(\tau)}{e^{\gamma_2^-(\tau)} e^{\gamma_1^-(\tau)}} - \Psi_1^-(\tau) - \Psi_2^-(\tau), & \tau \in L_{02} \end{cases} \tag{11}$$

Now we introduce the function

$$\Phi_1(z) = \begin{cases} \frac{F^+(z)}{e^{\gamma(z)}} - \Psi_1^+(z) - \Psi_2^+(z), & z \in D_1, \\ \frac{\Omega^-(z)}{e^{\gamma(z)}} - \Psi_1^-(z) - \Psi_2^+(z), & z \in D_2, \\ \frac{F^-(z)}{e^{\gamma(z)}} - \Psi_1^-(z) - \Psi_2^-(z), & z \in D_3, \end{cases}$$

then $\Phi_1(z)$ has n -order at $z = 0$, and has singularities at most less than one order near the endpoints a_j and $a_j + 2\omega_1$ ($j = 1, 2$). Thus we can get the following results.

1° When $m > 0$, problem (1) is solvable without any restrictive conditions and the general solution is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)} [c_0 + c_1 \zeta'(z) + \dots + c_{m-1} \zeta^{m-1}(z) + \Psi_1^+(z) + \Psi_2^+(z)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)} [c_0 + c_1 \zeta'(z) + \dots + c_{m-1} \zeta^{m-1}(z) + \Psi_1^-(z) + \Psi_2^+(z)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)} [c_0 + c_1 \zeta'(z) + \dots + c_{m-1} \zeta^{m-1}(z) + \Psi_1^-(z) + \Psi_2^-(z)], & z \in D_3, \end{cases} \tag{12}$$

where c_0, c_1, \dots, c_{m-1} are arbitrary constants.

2° When $m = 0$, problem (1) is solvable if and only if the restrictive conditions

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)}} d\tau = 0, \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}} d\tau = 0, \end{cases} \tag{13}$$

are satisfied, and now the solution is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)} [c + \Psi_1^+(z) + \Psi_2^+(z)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)} [c + \Psi_1^-(z) + \Psi_2^+(z)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)} [c + \Psi_1^-(z) + \Psi_2^-(z)], & z \in D_3, \end{cases} \tag{14}$$

where c is arbitrary constant.

3° When $m < 0$, if and only if the restrictive conditions (13) and

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)}} \zeta^k(\tau) d\tau = 0 \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)}} \zeta^k(\tau) d\tau = 0 \end{cases} \quad k = 1, 2, \dots, -m-1, \tag{15}$$

(when $m = -1$, the condition (15) is unnecessary) are necessary, problem (1) is solvable and the solution can still be given by (14) but with

$$c = -\frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)\zeta^k(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}} d\tau - \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)\zeta^k(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}} d\tau,$$

Case 2. If formula (5) fails to hold, then by Lemma 1 we see that $D_* \neq 0$. Let

$$h_*(z) = \sigma(z)/\sigma(z - D_*),$$

then the function $e^{\gamma(z)}h_*(z)$ become doubly periodic, and function $\frac{1}{e^{\gamma(z)}h_*(z)}$ has singularities at most less than one order near the endpoints a_j and $a_j + 2\omega_1$ ($j = 1, 2$). Thus now, we can transform (6) to

$$\begin{cases} \frac{F^+(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)} = \frac{\Omega^-(\tau)}{e^{\gamma_1^-(\tau)}e^{\gamma_2^+(\tau)}h_*(\tau)} + \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}, & \tau \in L_{01}, \\ \frac{\Omega^+(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)} = \frac{F^-(\tau)}{e^{\gamma_2^-(\tau)}e^{\gamma_1^+(\tau)}h_*(\tau)} + \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}, & \tau \in L_{02}. \end{cases} \tag{16}$$

where $\frac{g_1(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}$, $\frac{g_2(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}$ belong to class H or class H^* on L_{01} and L_{02} , respectively. Write

$$\Psi_{*1}(z) = \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)} [\zeta(\tau - z) + \zeta(z)] d\tau, \quad z \notin L_{01} \tag{17}$$

$$\Psi_{*2}(z) = \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)} [\zeta(\tau - z) + \zeta(z)] d\tau, \quad z \notin L_{02} \tag{18}$$

By (17) and (18), we can rewrite (16) as

$$\begin{cases} \frac{F^+(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)} - \Psi_{*1}^+(\tau) - \Psi_{*2}^+(\tau) = \frac{\Omega^-(\tau)}{e^{\gamma_1^-(\tau)}e^{\gamma_2^+(\tau)}h_*(\tau)} - \Psi_{*1}^-(\tau) - \Psi_{*2}^-(\tau), & \tau \in L_{01}, \\ \frac{\Omega^+(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)} - \Psi_{*1}^-(\tau) - \Psi_{*2}^+(\tau) = \frac{F^-(\tau)}{e^{\gamma_2^-(\tau)}e^{\gamma_1^+(\tau)}h_*(\tau)} - \Psi_{*1}^-(\tau) - \Psi_{*2}^-(\tau), & \tau \in L_{02}. \end{cases} \tag{19}$$

Now we will meet two kinds of situations in solving problem (1) in DR_m .

(a) When $D_* \equiv 0 \pmod{(2\omega_1, 2\omega_2)}$, the function $h_*(z)$ is an entire function. And we can write it without counting nonzero constant as

$$h_*(z) = \exp\{2(l_1\eta_1 + l_2\eta_2)\},$$

where l_1, l_2 are determined by the identity $D_* = 2l_1\omega_1 + 2l_2\omega_2$.

1° When $m > 0$, problem (1) is solvable without any restrictive conditions and the general solution is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)}h_*(z) [c_0 + c_1\zeta'(z) + \dots + c_{m-1}\zeta^{m-1}(z) + \Psi_{*1}^+(z) + \Psi_{*2}^+(z)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)}h_*(z) [c_0 + c_1\zeta'(z) + \dots + c_{m-1}\zeta^{m-1}(z) + \Psi_{*1}^-(z) + \Psi_{*2}^-(z)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)}h_*(z) [c_0 + c_1\zeta'(z) + \dots + c_{m-1}\zeta^{m-1}(z) + \Psi_{*1}^-(z) + \Psi_{*2}^-(z)], & z \in D_3, \end{cases} \tag{20}$$

where c_0, c_1, \dots, c_{m-1} are arbitrary constants.

2° When $m = 0$, problem (1) is solvable if and only if the restrictive conditions

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)} h_*(\tau)} d\tau = 0, \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)} h_*(\tau)} d\tau = 0, \end{cases} \tag{21}$$

are satisfied, and the general solution for (1) is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)} h_*(z) [c + \Psi_{*1}^+(z) + \Psi_{*2}^+(z)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)} h_*(z) [c + \Psi_{*1}^-(z) + \Psi_{*2}^+(z)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)} h_*(z) [c + \Psi_{*1}^-(z) + \Psi_{*2}^-(z)], & z \in D_3, \end{cases} \tag{22}$$

where c is arbitrary constant.

3° When $m < 0$, if and only if the restrictive conditions (21) and

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{02}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)} h_*(\tau)} \zeta^{(k)}(\tau) d\tau = 0, \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)} h_*(\tau)} \zeta^{(k)}(\tau) d\tau = 0, \end{cases} \quad k = 1, 2, \dots, m-1. \tag{23}$$

(when $m = -1$, the condition (23) is unnecessary) are both necessary, problem (1) is solvable and the solution can still be given by (22) but with

$$c = -\frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)} h_*(\tau)} d\tau - \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)} h_*(\tau)} d\tau.$$

(b) When $D_* \equiv 0 \pmod{2\omega_1, 2\omega_2}$ fails to hold, the function $\frac{1}{e^{\gamma(z)} h_*(z)}$ has singularity of one order at $z = 0$,

has singularities at most less than one order near the endpoints a_j and $a_j + 2\omega_1$ ($j = 1, 2$), and has a zero of order one at $z = D_*$. Write

$$\Phi_1(z) = \begin{cases} \frac{F^+(z)}{e^{\gamma(z)} h_*(z)} - \Psi_{*1}^+(z) - \Psi_{*2}^+(z), & z \in D_1, \\ \frac{\Omega(z)}{e^{\gamma(z)} h_*(z)} - \Psi_{*1}^-(z) - \Psi_{*2}^+(z), & z \in D_2, \\ \frac{F^-(z)}{e^{\gamma(z)} h_*(z)} - \Psi_{*1}^-(z) - \Psi_{*2}^-(z), & z \in D_3, \end{cases} \tag{24}$$

then $\Phi_1(z)$ must be at most $m + 1$ ordered at $z = 0$, and has singularities less than one order at $z = a_j$ ($j = 1, 2$).

1° When $m \geq 0$, problem (1) is solvable without any restrictive conditions and the general solution is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)} h_*(z) [c_0 + c_1 \zeta'(z) + \dots + c_m \zeta^m(z) + \Psi_{*1}^+(z) + \Psi_{*2}^+(z)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)} h_*(z) [c_0 + c_1 \zeta'(z) + \dots + c_m \zeta^m(z) + \Psi_{*1}^-(z) + \Psi_{*2}^+(z)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)} h_*(z) [c_0 + c_1 \zeta'(z) + \dots + c_m \zeta^m(z) + \Psi_{*1}^-(z) + \Psi_{*2}^-(z)], & z \in D_3, \end{cases} \tag{25}$$

with the restrictive condition that

$$c_0 = -c_1 \zeta'(D_*) - \dots - c_m \zeta^m(D_*) - \Psi_{*1}(D_*) - \Psi_{*2}(D_*) = 0,$$

or

$$c_0 = -c_1 \zeta'(D_*) - \dots - c_{m-1} \zeta^{(m)}(D_*) - \Psi_{*1}(D_*) - \Psi_{*2}(D_*),$$

where c_1, c_2, \dots, c_m are arbitrary constants, which is to ensure that $\Phi_1(D_*) = 0$, that is, to ensure $F^\pm(D_*)$ and $\Omega(D_*)$ be bounded.

2° When $m = -1$, problem (1) is solvable if and only if the restrictive conditions

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)} h_*(\tau)} d\tau = 0, \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)} h_*(\tau)} d\tau = 0, \end{cases} \tag{26}$$

are satisfied, and now the solution is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)} h_*(z) [\Psi_{*1}^+(z) + \Psi_{*2}^+(z) - \Psi_{*1}(D_*) - \Psi_{*2}(D_*)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)} h_*(z) [\Psi_{*1}^-(z) + \Psi_{*2}^-(z) - \Psi_{*1}(D_*) - \Psi_{*2}(D_*)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)} h_*(z) [\Psi_{*1}^-(z) + \Psi_{*2}^-(z) - \Psi_{*1}(D_*) - \Psi_{*2}(D_*)], & z \in D_3, \end{cases} \tag{27}$$

which is finite at $z = D_*$ owing to its structure.

3° When $m < -1$, problem (1) is solvable if and only if both conditions (26) and the following conditions

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau) [\zeta(\tau) - \zeta(\tau - D_*)]}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)} h_*(\tau)} d\tau = 0, \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau) [\zeta(\tau) - \zeta(\tau - D_*)]}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)} h_*(\tau)} d\tau = 0, \end{cases} \tag{28}$$

$$\begin{cases} \frac{1}{2\pi i} \int_{L_{01}} \frac{g_1(\tau)}{e^{\gamma_1^+(\tau)} e^{\gamma_2^-(\tau)} h_*(\tau)} \zeta^{(k)}(\tau) d\tau = 0, \\ \frac{1}{2\pi i} \int_{L_{02}} \frac{g_2(\tau)}{e^{\gamma_2^+(\tau)} e^{\gamma_1^-(\tau)} h_*(\tau)} \zeta^{(k)}(\tau) d\tau = 0, \end{cases} \quad k = 1, 2, \dots, -m-1 \tag{29}$$

(when $m = -2$, (28) is unnecessary) are necessary, and the solution is given by

$$\begin{cases} F^+(z) = e^{\gamma(z)} h_*(z) [\Psi_{*1}^+(z) + \Psi_{*2}^+(z) - \Psi_{*1}(0) - \Psi_{*2}(0)], & z \in D_1, \\ \Omega(z) = e^{\gamma(z)} h_*(z) [\Psi_{*1}^-(z) + \Psi_{*2}^-(z) - \Psi_{*1}(0) - \Psi_{*2}(0)], & z \in D_2, \\ F^-(z) = e^{\gamma(z)} h_*(z) [\Psi_{*1}^-(z) + \Psi_{*2}^-(z) - \Psi_{*1}(0) - \Psi_{*2}(0)], & z \in D_3, \end{cases} \tag{30}$$

which is finite at $z = D_*$ owing to its structure.

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References

[1] Balk, M.B. (1991) *Polyanalytic Functions*. Akademie Verlag, Berlin.
 [2] Begehr, H. and Kumar, A. (2005) Boundary Value Problems for the Inhomogeneous Polyanalytic Equation I. *Analysis: International Mathematical Journal of Analysis and its Application*, **25**, 55-71.
 [3] Du, J.Y. and Wang, Y.F. (2003) On Boundary Value Problems of Polyanalytic Functions on the Real Axis. *Complex Variables*, **48**, 527-542. <http://dx.doi.org/10.1080/0278107031000103412>

- [4] Fatulaev, B.F. (2001) The Main Haseman Type Boundary Value Problem for Metaanalytic Function in the Case of Circular Domains. *Mathematical Modelling and Analysis*, **6**, 68-76.
- [5] Lu, J.K. (1993) Boundary Value Problems for Analytic Functions. World Scientific, Singapore.
- [6] Mshimba, A.S. (2002) A Mixed Boundary Value Problem for Polyanalytic Function of Order n in the Sobolev Space $W_n, p(D)$. *Complex Variables*, **47**, 278-1077.
- [7] Muskhelishvili, N.I. (1993) Singular Integral Equations. World Scientific, Singapore.
- [8] Wanf, Y.F. and Du, J.Y. (2006) Hilbert Boundary Value Problems of Polyanalytic Functions on the Unit Circumference. *Complex Variables and Elliptic Equations*, **51**, 923-943. <http://dx.doi.org/10.1080/17476930600667692>
- [9] Xing, L. (1995) A Class of Periodic Riemann Boundary Value Inverse Problems. *Proceedings of the Second Asian Mathematical Conference*, Nakhon Ratchasima, October 1995, 397-400.
- [10] Wang, M.H. (2006) Inverse Riemann Boundary Value Problems for Generalized Analytic Functions. *Journal of Ningxia University of Natural Resources and Life Sciences Education*, **27**, 18-24.
- [11] Wen, X.Q. and Li, M.Z. (2004) A Class of Inverse Riemann Boundary Value Problems for Generalized Holomorphic Functions. *Journal of Mathematical*, **24**, 457-464.
- [12] Cao, L.X., Li, P.-R. and Sun, P. (2012) The Hilbert Boundary Value Problem With Parametric Unknown Function on Upper Half-Plane. *Mathematics in Practice and Theory*, **42**, 189-194.
- [13] Cao, L.X. (2013) Riemann Boundary Value Problem of Non-Normal Type on the Infinite Straight Line. *Applied Mathematics*, **4**, 1126-1230.

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