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The Mass-Critical for the Nonlinear Schrödinger Equation in *d* = 2

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ABSTRACT

This paper studies the global behavior defocusing nonlinear Schrödinger equation in dimension $d = 2$, and we will discuss the case $p=\frac{4}{d}, d=2$. This means that the solutions $u\in C_t\left(\left[0,T\right],L_x^2\right)\cap L_{t,x}^4\left(\left[0,T\right]\times\mathbb{R}^2\right)$, and called critical solution. We show that u scatters forward and backward to a free solution and the solution is globally well posed.

Keywords: NLS; Well Posed

1. Introduction

We consider the Cauchy problem for the nonlinear Schrödinger equation in dimension $d = 2$.

$$\begin{Bmatrix} iu_t+\Delta u=\mu F\left(u\right) \\ u\left(0,x\right)=u_0\left(x\right) \end{Bmatrix} \tag{1.1}$$

where $F\left(u\right)=\left|u\right|^p u$, $\mu=\pm 1$, and $u\left(t,x\right):\mathbb{R}^d\to C$, When $\mu=+1$ (1.1) is called defocusing when $\mu=-1$ (1.1) is called focusing. In this paper we discuss the case when $p=2$ and $\mu=+1$ (defocusing case).

If $u\left(t,x\right)$ is a solution to (1.1) on a time interval $\left[0,T\right]$, then

$$u_\lambda\left(t,x\right)=\lambda u\left(\lambda^2 t,\lambda x\right) \tag{1.2}$$

is a solution to (1.1) on $\left[0,\lambda^{-2}T\right]$ with $u\left(0,t\right)=\lambda u_0\left(\lambda x\right)$. This scaling saves the $L^2\left(\mathbb{R}^2\right)$ norm of u,

$$\left\|\lambda u_0\left(\lambda x\right)\right\|_{L^2\left(\mathbb{R}^2\right)}=u_0\left(x\right)_{L^2\left(\mathbb{R}^2\right)} \tag{1.3}$$

Thus (1.1) under previous hypotheses is called L^2-critical or mass critical.

Proposition 1.1. Suppose that $0<p\le\frac{4}{d}$, and $d\ge 1$ then, for any initial data $u_0\in L^2$, there exist $T>0$ such that there exists a unique solution

$$u\in C_t\left(\left[0,T\right],L_x^2\right)\cap L_{t,x}^{\frac{2(d+2)}{d}}\left(\left[0,T\right]\times\mathbb{R}^d\right)$$

of the nonlinear Schrödinger Equation (1.1). If $p\le\frac{4}{d}$ then $T\ge T\left(\left\|u_0\right\|_{L_x^2}\right)$ for some non-increasing, and if $\left\|u_0\right\|_{L_x^2}$ is sufficiently small u exists globally.

In this paper we will discuss the case, $p=\frac{4}{d}, d=2$. This means that a solution u

$$u\in C_t\left(\left[0,T\right],L_x^2\right)\cap L_{t,x}^4\left(\left[0,T\right]\times\mathbb{R}^2\right),$$

and called critical solution.

Definition 1.1. Let $u:K\times\mathbb{R}^2\to C$, $K\subset\mathbb{R}$ is a solution to (1.1) if for any compact

$$J\subset K, u\in C_t^0L_x^2\left(J\times\mathbb{R}^2\right)\cap L_{t,x}^4\left(J\times\mathbb{R}^2\right)$$

and for all $t,t_0\in K$

$$u\left(t,x\right)=\mathrm{e}^{i(t-t_0)\Delta}u_{t_0}-i\int_t^{t_0}\mathrm{e}^{i(t-r)\Delta}F\left(u\left(r\right)\right)\mathrm{d}r. \tag{1.4}$$

The space $L_{t,x}^4\left(J\times\mathbb{R}^2\right)$ caused from strichartz estimates. This norm is invariant under the scaling (1.2).

Definition 1.2. If there exist $t_0\in K$ a solution u to (1.1) defined on $K\subset R$ blows up forward in time, such that

$$\int_{t_0}^{\sup(K)}\int\left|u\left(t,x\right)\right|^4\mathrm{d}t\mathrm{d}x=\infty \tag{1.5}$$

And u blows up backward in time, such that

$$\int_{\inf(K)}^{t_0}\int\left|u\left(t,x\right)\right|^4\mathrm{d}t\mathrm{d}x=\infty \tag{1.6}$$

Definition 1.3. If there exist $u_+ \in L^2(\mathbb{R}^2)$ we say that a solution u to (1.1) scatter forward in time such that,

$$\lim_{t\to\infty}\left\|e^{it\Delta}u_+ - u(t,x)\right\|_{L^2(\mathbb{R}^2)} = 0 \tag{1.7}$$

A solution is said to scatter backward in time if there exist $u_- \in L^2(\mathbb{R}^2)$

Such that,

$$\lim_{t\to-\infty}\left\|e^{it\Delta}u_- - u(t,x)\right\|_{L^2(\mathbb{R}^2)} = 0 \tag{1.8}$$

We note that the Equation (1.1) has preserved quantities, the mass

$$M(u(t)) = \int |u(t,x)|^2 \,dx = M(u(o)) \tag{1.9}$$

And energy

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\int |\nabla u(t,x)|^2 \,dx + \frac{1}{4}\int |u(t,x)|^4 \,dx \\ &= E(u(0)), \end{aligned} \tag{1.10}$$

For more see [1].

Proposition 1.2. let p be the L_x^2-critical exponent $p = \frac{4}{d}$, then the NLS (1.1) is locally well posed in $L_x^2(\mathbb{R}^d)$ in the critical case. More precisely, given any $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(\mathbb{R}^d) > 0$, such that whenever $u_* \in L_x^2(\mathbb{R}^d)$ has norm at most R, and K is a time interval containing 0 such that

$$\left\|e^{it\Delta/2}u_*\right\|_{L_{t,x}^{\frac{2(d+2)}{d}}(K\times\mathbb{R}^d)} \le \varepsilon_0$$

Then for any u_0 in the ball

$$B = \left\{u_0 \in L_x^2(\mathbb{R}^d) : \|u_0 - u_*\|_{L_x^2(\mathbb{R}^d)} \le \varepsilon_0\right\}$$

there exists a unique strong L_x^2 solution $u \in S^0(K\times R^d)$ to (1.1), and moreover the map $u_0 \to u$, is Lipschitz from B to $S^0(K\times\mathbb{R}^d)$, where $S^0(K\times\mathbb{R}^d)$ defined in Equation (2.5).

Proposition 1.3. let K be a time interval containing t_0 and let $u, u' \in C_{t,x}^2(K\times\mathbb{R}^d \to C)$ be two classical solutions to (1.1) with same initial datum u_0 for some fixed μ and p, assume also that we have the temperate decay hypothesis $u, u' \in L_t^\infty L_x^q(K\times\mathbb{R}^d)$ for $q = 2, \infty$. Then $u = u'$.

Proposition 1.4. Let $t_0 \in R$, given $u_0 \in L^2(\mathbb{R}^d)$ there exists a maximal lifespan solution u to (1.1) define on $k \subset R$, with $u(t_0) = u_0$. Furthermore,

1) k is an open neighborhood of t_0.

2) We say u is a blow up in the contrast direction If $\sup(k)$ or $\inf(k)$ is finite.

3) If we have compact time intervals for bounded sets of initial data, then the map that takes initial data to the corresponding solution is uniformly continuous in these intervals.

4) We say that u scatters forward to a free solution, if $\sup(k) = \infty$ and u does not blow up forward in time. And we say that u scatters backward to a free solution, if $\inf(k) = -\infty$ and u does not blow up backward in time.

To Proof: see [1-3].

2. Strichartz Estimates

In this section we discuss some notation and Strichartz estimates for critical NLS (1.1) and we turn to prove Propositions 1.1 and 1.3.

2.1. Some Notation

If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$, to denote the estimate $X \le cY$ for some c and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$.

We defined the Fourier transform on $\mathbb{R}^2$ by

$$\hat{f}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\xi} f(x)\,dx.$$

We use $L_t^q L_x^r(K\times\mathbb{R}^d)$ to denote the Banach space for any space time slab $K\times\mathbb{R}^d$, of function $K\times\mathbb{R}^d \to C$ with norm is

$$\|u\|_{L_t^q L_x^r(K\times\mathbb{R}^d)} := \left(\int_K \|u(t)\|_{L_x^r}^q \,dt\right)^{\frac{1}{q}} < \infty,$$

With the usual amendments when q or r is equal to infinity. When $q = r$ we cut short $L_t^q L_x^r$ as $L_{t,x}^q$.

Defined the fractional differentiation operators $|\nabla|^s$, $\langle\nabla\rangle^s$ by

$$\widehat{|\nabla|^s f(\xi)} := |\xi|^s \hat{f}(\xi), \quad \widehat{\langle\nabla\rangle^s f(\xi)} := \langle\xi\rangle^s \hat{f}(\xi),$$

where $\langle\xi\rangle := \left(1+|\xi|^2\right)^{\frac{1}{2}}$, specially, we will use ∇ to signify the spatial gradient ∇_x and define the Sobolev norms as

$$\|f\|_{\dot{H}_x^s(\mathbb{R}^2)} := \left\||\nabla|^s\right\|_{L_x^2(\mathbb{R}^2)}$$

$$\|f\|_{H_x^s(\mathbb{R}^2)} := \left\||\nabla|^s\right\|_{L_x^2(\mathbb{R}^2)}$$

Let $e^{it\Delta}$ be the free Schrödinger propagator; in terms of the Fourier transform, this is given by,

$$\widehat{e^{it\Delta} f(\xi)} := e^{-4\pi^2 it|\xi|^2} f(\xi).$$

A Gagliardo-Nirenberg type inequality for Schrödinger equation the generator of the spurious conformal transformation $J \overset{\text{def}}{=} x - 2it\partial$ plays the role of the partial differentiation.

2.2. Strichartz Estimates

Let $e^{it\Delta}$ be the free Schrödinger evolution, from the explicit formula

$$\mathrm{e}^{it\Delta} f(x) = \frac{1}{4\pi it} \int_{\mathbb{R}^2} \mathrm{e}^{\frac{i|x-y|^2}{4t}} f(y)\,\mathrm{d}y \tag{2.1}$$

Specially, as the free propagator saves the L_x^2 -norm,

$$\left\| e^{it\Delta} f(x) \right\|_{L_x^p(\mathbb{R}^2)} \lesssim |t|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L_x^{p'}(\mathbb{R}^2)}$$

For all $t \neq 0$ and $1 \leq p \leq \infty$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Proposition 2.1. There holds that

$$\left\| e^{it\Delta} f \right\|_{L_x^\infty(\mathbb{R}^2)} \leq \frac{1}{4\pi t} \|f\|_{L^1(\mathbb{R}^2)} \tag{2.2}$$

In fact, this follows directly from the formula (2.1).

Definition 2.1. Define an admissible pair to be pair (p,r) with $1 < p \leq \infty$, $2 \leq r \leq \infty$, With $\frac{2}{p}+\frac{d}{r}=\frac{d}{2}$.

Theorem 2.2. If $u(t,x)$ solves the initial value problem

$$iu_t + \Delta u = F(t),$$

$$u(0,x) = u_0,$$

On an interval *K*, then

$$\|u\|_{L_t^p L_x^q(K\times\mathbb{R}^d)} \precsim_{p,q,\tilde{p},\tilde{q}} \|u\|_{L^2(\mathbb{R}^d)} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(K\times\mathbb{R}^d)} \tag{2.3}$$

For all admissible pairs (p,q), $(\tilde{p},\tilde{q})$. $\tilde{p}'$ denotes the Lebesgue dual $\tilde{p}$.

To prove: see [4,5].

Definition 2.2. Define the norm

$$\|u\|_{S^0(K\times\mathbb{R}^2)} := \sup_{(p,q)\text{admissible}} \|u\|_{L_t^p L_x^q(K\times\mathbb{R}^2)}. \tag{2.4}$$

$$S^0(K\times\mathbb{R}^2) = \left\{ u : \|u\|_{S^0(K\times\mathbb{R}^2)} < \infty \right\}. \tag{2.5}$$

We also define the space $N^0(K\times\mathbb{R}^2)$ to be the space dual to $S^0(K\times\mathbb{R}^2)$ with suitable norm. By theorem.2.2,

$$\|u\|_{S^0(K\times\mathbb{R}^2)} := \|u_0\|_{L^2(\mathbb{R}^2)} + \|F\|_{N^0(K\times\mathbb{R}^2)}. \tag{2.6}$$

Theorem 2.3. If $\|u_0\|_{L^2(\mathbb{R}^d)}$ is small, then (1.1) is globally well posed, for more see [6,7].

Proof: by (2.3) and (2.6)

$$\|u\|_{L_{t,x}^4((-\infty,\infty)\times\mathbb{R}^2)} \leq \|u_0\|_{L^2(\mathbb{R}^2)} + \|u\|^3_{L_{t,x}^4((-\infty,\infty)\times\mathbb{R}^2)} \tag{2.7}$$

If $\|u_0\|_{L^2(\mathbb{R}^d)}$ is small enough and by the continuity method, then we have global well-posedness. Furthermore, for any $\varepsilon > 0$ there exist $T(\varepsilon)$ such that

$$\|u\|_{L_{t,x}^4((T,\infty)\times\mathbb{R}^2)} < \varepsilon,$$

Then

$$u(t) = \mathrm{e}^{i(t-T)\Delta} u(T) - i\int_T^t \mathrm{e}^{i(t-r)\Delta} |u(r)|^2 u(r)\,\mathrm{d}r \tag{2.8}$$

So by (2.6), when $t \geq T$,

$$\left\| u(t) - \mathrm{e}^{i(t-T)\Delta} u(T) \right\|_{L_x^2(\mathbb{R}^2)} \leq \varepsilon^3. \tag{2.9}$$

Thus, the limit

$$u_+ = \lim_{k\to\infty} \mathrm{e}^{-iT(2^{-k})\Delta} u\left(T\left(2^{-k}\right)\right) \tag{2.10}$$

Exists, and,

$$\lim_{t\to\infty} \left\| u(t) - \mathrm{e}^{it\Delta} u_+ \right\|_{L_x^2(\mathbb{R}^2)} = 0 \tag{2.11}$$

A conformable argument can be made for $t \to -\infty$. indeed, if $u_{L_{t,x}^4((-\infty,\infty)\times\mathbb{R}^2)} < C$, then $(-\infty,\infty)$ can be division into $\sim C^4$ subintervals *K* with $\|u\|_{L_{t,x}^4((K,\infty)\times\mathbb{R}^2)} < \varepsilon_0$ on each subinterval. Using the Duhamel formula on each interval individually, we obtain global well-posedness and scattering. □

Now we return to prove Proposition 1.1 and Proposition 1.3.

Proof proposition 1.1:

We suppose in what follows that $p = \frac{4}{d}$. Let $L = 2\|u_0\|_{L^2}$ and for some $\delta > 0$ to be chosen, $T > 0$ be such that

$$2\left\| \mathrm{e}^{it\Delta} u(0) \right\|_{L_{t,x}^4([0,T]\times\mathbb{R}^2)} \leq \delta. \tag{2.12}$$

We deem the space

$$S_{L,\delta} = \Big\{ u \in C_t\left([0,T], L_x^2\right) \cap L_{t,x}^4\left([0,T]\times\mathbb{R}^2\right),$$

$$\|u\|_{L_t^\infty([0,T],L_x^2)} \leq L, \|u\|_{L_{t,x}^4} \leq \delta \Big\}$$

And the mapping,

$$\Phi(v)(t) = \mathrm{e}^{it\Delta} u_0 + i\int_0^t \mathrm{e}^{i(t-s)\Delta} \left(|v|^p v\right)(s)\,\mathrm{d}s. \tag{2.13}$$

We want to prove that the δ small adequate, $\Phi : S_{L,\delta} \to S_{L,\delta}$ is contraction. We use first Strichartz estimates, to compute that

$$\left\| \int_0^t \mathrm{e}^{i(t-s)\Delta} \left(|v|^p v\right)(s)\,\mathrm{d}s \right\|_{L_t^\infty L_x^2 \cap L_{t,x}^4}$$

$$\precsim \left\| |v|^{p+1} \right\|_{L_{t,x}^4} \precsim \|u\|_{L_{t,x}^2}^{\theta(p+1)} \|u\|_{L_{t,x}^4}^{(1-\theta)(p+1)}$$

$$\precsim \left(2T^{\frac{1}{2}} L\right)^{\theta(p+1)} \delta^{(1-\theta)(p+1)}$$

where $\theta = \frac{d+2}{2} - \frac{d+4}{2(p+1)}$

Then, distinctly,

$$\left\|\Phi(u)\right\|_{L_t^\infty L_x^2} \le \left\|u(0)\right\|_{L_x^2} + \left(2T^{\frac{1}{2}}L\right)^{\theta(p+1)} \delta^{(1-\theta)(p+1)}$$

$$\left\|\Phi(u)\right\|_{L_{t,x}^4} \le \left\|e^{it\Delta}u(0)\right\|_{L_{t,x}^4} + \left(2T^{\frac{1}{2}}L\right)^{\theta(p+1)} \delta^{(1-\theta)(p+1)}$$

So that for T,δ is small enough, $S_{L,\delta}$ is settled under Φ. In addition,

$$\begin{aligned}
&\left\|\Phi(u)-\Phi(v)\right\|_{L_t^\infty L_x^2 \cap L_{t,x}^4} \\
&\lesssim \left\|\int_0^t e^{i(t-s)\Delta}\left(|u|^p u - |v|^p v\right)(s)\,ds\right\|_{L_t^\infty L_x^2 \cap L_{t,x}^4} \\
&\lesssim \left\||u-v|\left(|u|^p + |v|^p\right)\right\|_{L_{t,x}^4} \\
&\lesssim \|u-v\|_{L_{t,x}^4}\left(\|u\|_{L_{t,x}^4}^p + \|v\|_{L_{t,x}^4}^p\right) \\
&\lesssim 2\|u-v\|_{L_{t,x}^4}\left(2T^{\frac{1}{2}}\right)^0 \delta^2 \\
&= 2\|u-v\|_{L_{t,x}^4}\,\delta^2.
\end{aligned}$$

Again, decreasing may be T,δ, we get a contraction.

If $p = \frac{4}{d}$, then $\theta = 1$, and from Strichartz estimates, we see that if $\|u\|_{L^2}$ is small enough, then (2.12) is satisfied for $T = +\infty$. □

Proof Proposition 1.3: By time translation symmetry we can take $t_0 = 0$. By time reversal symmetry we may assume that K lies in the upper time axis $[0,+\infty]$. Let $u' = u + v$, and then, $v \in C_{t,x}^2\left(K \times \mathbb{R}^d \to C\right)$, $v(0) = 0$ and v obey the variance equation

$$i\partial_t + \Delta v = \left(|u+v|^p (u+v) - |u|^p u\right).$$

Since v and $|u+v|^p (u+v) - |u|^p u$ lies in the $L_t^\infty L_x^2\left(K \times \mathbb{R}^d\right)$ we may calling Duhamel's and conclude

$$v(t) = -i\int_0^t e^{\frac{i(t-s)\Delta}{2}}\left(|u+v|^p (u+v) - |u|^p u\right)(s)\,ds$$

for all. By Minkowski's inequality, and the unitarity of $e^{i(t-s)\Delta}$, conclude that,

$$v(t)_{L_x^2(\mathbb{R}^d)} \le \int_0^t \left(|u+v|^p (u+v) - |u|^p u\right)(s)_{L_x^2(\mathbb{R}^d)}\,ds .$$

Since u and v are in $L_t^\infty L_x^\infty\left(K \times \mathbb{R}^d\right)$, and the function $z \to |z|^p z$ is locally Lipcshitz, we have the bound

$$\begin{aligned}
&\left\|\left(|u+v|^p (u+v) - |u|^p u\right)(s)\right\|_{L_x^2(\mathbb{R}^d)} \\
&\lesssim_p \left(\|u\|_{L_t^\infty L_x^\infty(K\times\mathbb{R}^d)}^{p+1} + \|v\|_{L_t^\infty L_x^\infty(K\times\mathbb{R}^d)}^{p+1}\right)\|v(s)\|_{L_x^2(\mathbb{R}^d)}
\end{aligned}$$

Apply Gronwall's inequality to conclude that $\|v(t)\|_{L_x^2(\mathbb{R}^d)} = 0$ for all $t \in K$ and hence $u = u'$. □

3. Decay Estimates

Consider the defocusing nonlinear Schrödinger Equation (1.1), in $\mathbb{R}^+ \times \mathbb{R}^2$, where $u = u(t,x)$, and $p = 2$, for $d = 1,2$. We suppose that at $t = 0$,

$$u(0,.) = u_0 \in H^1\left(\mathbb{R}^2\right) \tag{3.1}$$

First we have the following result.

Theorem 3.1. Suppose that $p \in (0,+\infty)$, if $d = 1,2,$ and let u be a solution to (1.1), identical to an initial data $u_0 \in H^1\left(\mathbb{R}^2\right)$ such that $\left(1+|\bullet|^2\right)^{\frac{1}{2}} u_0 \in L^2\left(\mathbb{R}^2\right)$. If $d = 2$, let r be such that, $2 \le r < \infty$, then there exists a constant $c > 0$ such that if R is the solution of, $\ddot{R}R = R^{-c_p - 1}$, with $c_p = \min\left(\frac{dp}{2}, 2\right) = (2,2)$, $R(0) = 1$, $\dot{R}(0) = 0$ then

$$\|u(t,.)\|_{L^r(\mathbb{R}^2)} \le cR(t)^{-2\left(\frac{1}{2}-\frac{1}{r}\right)}\ \forall t \ge 0,$$

Furthermore, c depends only on d, p, r and,

$$E_0 := \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{1}{4}\||x|u_0\|_{L^2}^2 + \|u_0\|_{L^4}^4 .$$

The method made up in rescheduling, by the average of a time dependent rescheduling the equation, and to use the energy of the equation, to get by interpolation decay estimates in suitable norms. The asymptotically average, is normally obtained directly by using the pseudo conformal law, the above result was in fact partially proved in [8], under a bit different point of view: look for a time dependent change of coordinates, which maintain the Galilean invariance, and the construction directly a Lyapunov functional by a suitable ansatz. This Lyapunov functional is surely the energy of the rescaled equation. Our aim here is to study with further details the rescaled wave function and its energy. Found to be the method provides rates which are seems completely new in the limiting case of the logarithmic nonlinear Schrödinger equation. Because of the reversibility of the Schrödinger equation and standard results of scattering theory, one cannot foresee the convergence of the rescaled wave function to some a intuition given limiting wave function, but found to be some convexity properties of the energy can be used to state an asymptotically stabilization result. From the general theory of Schrödinger equations, it is well known that the Cauchy problems (1.1)-(3.1) is well posed for any initial data in $H^1(\mathbb{R}^2)$ when $0 < p < \infty$, and that the solution u belongs to

$$C\left(\mathbb{R}^+, H^1\left(R^2\right)\right) \cap L^\infty\left(\mathbb{R}^+, H^1\left(\mathbb{R}^2\right)\right) \cap C^1\left(\mathbb{R}^+, H^{-1}\left(\mathbb{R}^2\right)\right).$$

As usual for Schrödinger equations is critical when $p = \frac{4}{d}$.

Let ϕ be such that

$$u(t,x) = R^{-1}\mathrm{e}^{iS(t)\frac{|x|^2}{2}}\phi\left(\tau(t), \frac{x}{R(t)}\right).$$

where and τ are positive derivable real functions of the time.

It is simple to check that with this change of coordinates, ϕ satisfies the following equation,

$$i\dot{\tau}\phi_r = -\frac{1}{R^2}\Delta\phi + \frac{1}{R^2}|\phi|^2\phi + \frac{R^2}{2}\left(\dot{S} + 2S^2\right)|\xi|^2\phi + i\left(\frac{\dot{R}}{R} - 2S\right)\left(\phi + \xi.\nabla\phi\right),$$

where $\dot{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}$, with the choice $S = \frac{\dot{R}}{2R}$, which means that $\dot{S} = \frac{\ddot{R}}{2R} - 2S^2, \phi$ and u are linked by,

$$\begin{aligned} u(t,x) = R^{-1}\mathrm{e}^{\frac{i}{4}\frac{\dot{R}}{R}|x|^2}\phi\left(\tau, \frac{x}{R}\right) \Leftrightarrow \phi(\tau,\xi) \\ = R\mathrm{e}^{\frac{-i}{4}\dot{R}R|\xi|^2}u(t, R\xi) \end{aligned} \tag{3.2}$$

where $\tau = \tau(t)$ and $\xi = x/R(t)$, and ϕ has to satisfy the following time-dependent defocusing nonlinear Schrödinger equation,

$$i\dot{\tau}\phi_r = -\frac{1}{R^2}\Delta\phi + \frac{1}{R^2}|\phi|^2\phi + \frac{1}{4}\ddot{R}R|\xi|^2\phi \tag{3.3}$$

We note that $|u(t,x)| = R^{-1}\left|\phi\left(\tau, \frac{x}{R}\right)\right|$, so that

$$u(t,\bullet)_{L^2} = \phi\left(\tau(t),\bullet\right)_{L^2} = u_{0L^2}$$

for all $t \geq 0$.

Also we note that if $\dot{R}(0) = \tau(0) = 0$ and $R(0) = 1$, then

$$\phi(0,\bullet) = u_0 \tag{3.4}$$

To extract the controlling impacts as $t \to +\infty$, we fix τ and R such that,

$$\dot{\tau} = \frac{1}{2}\ddot{R}R = R^{-C_p} = R^{-2}$$

where

$$c_p = \min\left(\frac{dp}{2}, 2\right) = 2 \tag{3.5}$$

Because p is critical, this ansatz is actually the only one that sets to 1 at least three of the four coefficients in the equation for ϕ, with $\lim_{t\to+\infty} R(t) = +\infty$. and ϕ solves the equation,

$$i\phi_r = -\Delta\phi + |\phi|^2\phi + \frac{1}{2}|\xi|^2\phi \tag{3.6}$$

With the choice $R(0) = 1$ and $\dot{R}(0) = 0$, integration of (3.5) with respect to t gives $\dot{R}^2 = 2\left(1 - R(t)^{-2}\right)$ and this is possible if, and only if, $R \geq 1$. for all $t \geq 0$ thus the function $t \to R(t)$ is globally defined on $\mathbb{R}^+$ increasing, $\lim_{t\to+\infty} R(t) = 2\sqrt{2}$ and $R(t) \sim t$ as $t \to +\infty$.

Supposing that $\tau(0) = 0$, τ is an increasing positive function such that, $\lim_{t\to+\infty}\tau(t) = \tau_\infty > 0$, where $\tau_\infty < +\infty$ if $p > \frac{2}{d}$.

Consider now the energy functional linked to Equation (3.6)

$$\begin{aligned} E(\tau) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2\,\mathrm{d}\xi + \frac{1}{4}\int_{\mathbb{R}^2}|\xi|^2|\phi|^2\mathrm{d}\xi \\ + \frac{1}{4}\int_{\mathbb{R}^2}|\phi|^4\,\mathrm{d}\xi \end{aligned} \tag{3.7}$$

where R has to be understood as a function of.

Lemma 3.2. Suppose that $p \in (0,+\infty)$, if $d = 1,2$, and let u be a solution to (1.1), identical to an initial data, $u_0 \in H^1(\mathbb{R}^2)$ such that $\left(1+|\bullet|^2\right)^{\frac{1}{2}}u_0 \in L^2(\mathbb{R}^2)$.

With the above notations, E is a decreasing positive functional. Thus $E(\tau)$ is bounded by $E(0) = E_0$, with the notations of Theorem 3.1.

Proof: The proof follows by a direct computation. Because of (3.6), only the coefficients of $\int_{\mathbb{R}^2}|\nabla\phi|^2\mathrm{d}\xi$, and $\int_{\mathbb{R}^2}|\phi|^4\,\mathrm{d}\xi$ contribute to the decay of the energy. For more see [9]. □

Proof of Theorem 3.1: Suppose that p is critical. By Lemma 3.2 and pursuant to the time-dependent rescaling (3.2),

$$\begin{aligned} R^2\int_{\mathbb{R}^2}\left|\nabla u - \frac{i\dot{R}}{2R}ux\right|^2\mathrm{d}x \\ = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2\,\mathrm{d}\xi \leq E(\tau) = E(0) \end{aligned}$$

Thus

$$\int_{\mathbb{R}^2} \left|\nabla |u|\right|^2 \mathrm{d}x = \int_{\mathbb{R}^2} \left|\nabla \left| \mathrm{e}^{\frac{-i\dot{R}}{2R}|x|^2} u \right|\right|^2 \mathrm{d}x$$

$$\le \int_{\mathbb{R}^2} \left|\nabla u - \frac{i\dot{R}}{2R} ux\right|^2 \mathrm{d}x$$

Is bounded by $2E(0)R(t)^{-2}$, the remainder of the proof follows the same lines that in Theorem 7.2.1 of [10], see also [11,12], using maintain the L^2-norm and the Sobolev-Gagliardo-Nirenberg inequality. □

Proposition 3.3. Consider the two-dimensional defocusing cubic NLS (1.1), (is L_x^2-critical). Let $u_0 \in H_x^{0,1}$ then there exists a global L_x^2-well posed solution to (1.1), and moreover the $L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)$ norm of u_0 is finite.

Proof: By time reflection symmetry and adhesion arguments we may heed attention to the time interval $[0,+\infty)$. Since u_0 lies in $H_x^{0,1}$, it lies in L_x^2. Apply the L_x^2 well posedness theory (Proposition 1.2) we can find an L_x^2-well posed solution $u \in S^0([0,T]\times\mathbb{R}^2)$, on some time interval $[0,T]$, with $T>0$ depending on the profile of u_0.

Specially the $L_{t,x}^4([0,T]\times\mathbb{R}^2)$ norm of u is finite. Next we apply the pseudoconformal law to deduce that,

$$E_{pc}\left[u(T),T\right] = E_{pc}\left[u_0,0\right] = \frac{1}{2}\int_{\mathbb{R}^2} |xu_0|^2 \mathrm{d}x < \infty$$

Since $u_0 \in H_x^{0,1}$, we got a solution from $t = 0$ to $t = T$. To go to all the way to $t = +\infty$. We apply the pseudoconformal transformation at time $t = T$, obtaining an initial datum $v\left(\frac{1}{T}\right)$ at time $\frac{1}{T}$ by the formula

$$v\left(\frac{1}{T},x\right) := \frac{1}{i/T}\overline{u(T,Tx)}\,\mathrm{e}^{iT|x|^2/2}.$$

From $E\left[v(t),t\right] = E_{pc}\left[u\left(\frac{1}{t}\right),\frac{1}{t}\right]$ we see that v has finite energy:

$$\frac{1}{2}\int_{\mathbb{R}^2}\left|\nabla v\left(\frac{1}{T},x\right)\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2}\left|v\left(\frac{1}{T},x\right)\right|^4 \mathrm{d}x$$
$$= E_{pc}\left[u(T),T\right] < \infty.$$

And, the pseudoconformal transformation saves mass and hence

$$\int_{\mathbb{R}^2}\left|v\left(\frac{1}{T},x\right)\right|^2 \mathrm{d}x = \int_{\mathbb{R}^2}\left|u(T,x)\right|^2 \mathrm{d}x$$
$$= \int_{\mathbb{R}^2}\left|u_0(x)\right|^2 \mathrm{d}x < \infty$$

So we see that $v(1/T)$ has a finite H_x^1 norm. Thus, we can use the global H_x^1-well posedness theory, backwards in time to obtain an H_x^1-well posed solution $v \in S^1\left(\left[0,\frac{1}{T}\right]\times\mathbb{R}^2\right)$, to the equation $i\partial_t v + \Delta v = |v|^2 v$, particularly, $v \in L_{t,x}^4\left(\left[0,\frac{1}{T}\right]\times\mathbb{R}^2\right)$.

We reverse the pseudoconformal transformation, which defines the original field u on the new slab $\left[\frac{1}{T},\infty\right)\times\mathbb{R}^2$.

We see that the $L_{t,x}^4\left(\left[\frac{1}{T},\infty\right)\times\mathbb{R}^2\right)$, and $C_t^0 L_x^2\left(\left[\frac{1}{T},\infty\right)\times\mathbb{R}^2\right)$ norm of u are finite. This is sufficient to make u an L_x^2-well posed solution to NLS on the time interval $\left[\frac{1}{T},\infty\right)$; for v classical. And for general $u \in S^1\left(\left[0,\frac{1}{T}\right]\times\mathbb{R}^2\right)$, the claim follows by a limiting argument using the L_x^1-well posedness theory. Adhesion together the two intervals $[0,1/T]$ and $(1/T,\infty)$, we have obtained a global $L_{t,x}^4([0,+\infty)\times\mathbb{R}^2)$ solution u to (1.1).

4. Some Lemma

Consider the defocusing case of the NLS (1.1) and if $d=2, p=2$, the energy and mass together will control the H_x^1 norm of the solution:

$$u(t)_{H_x^1}^2 \lesssim E\left[u(t)\right] + M\left[u(t)\right].$$

Conversely, energy and mass are controlled by the H_x^1 norm (the Gagliardo-Nirenberg inequality showed that):

$$E\left[u(t)\right] \lesssim \|u(t)\|_{H_x^1}^2\left(1+\|u(t)\|_{L_x^2}^2\right)$$
$$\lesssim \|u(t)\|_{H_x^1}^2\left(1+\|u(t)\|_{L_x^2}^2\right).$$
$$M\left[u(t)\right] = u(t)_{L_x^2}^2 \lesssim u(t)_{H_x^1}^2.$$

This bound and the energy conservation law and mass conservation law showed that for any H_x^1-well posed solution, the H_x^1 norm of the solution $u(t)$ at time t is bounded by a quantity depending only on the H_x^1 norm of the initial data.

Proposition 4.1. The cubic NLS (1.1) with $\mu = +1$, $d =$ 2 is globally well posed in H_x^1. Actually, for $u_0 \in H_x^1$ and any time interval, K the Cauchy problem (1.1) has a H_x^1 well posed solution

$$u \in S^1\left(K\times\mathbb{R}^2\right) \subseteq C_t^0 H_x^1\left(K\times\mathbb{R}^2\right).$$

Lemma 4.2. If $f(t) \overset{\text{def}}{=} e^{it\Delta}u(t)$ the following holds:

$$\left\|e^{-it\Delta}(xf)\right\|_4^2 \le \left\|e^{-it\Delta}f\right\|_\infty \left\|e^{-it\Delta}\left(x^2f\right)\right\|_2 .$$

Proof: The proof depends on the noticing that;

$$e^{-it\Delta}x = Je^{-it\Delta} .$$

With

$$J = 2ite^{-i\frac{x^2}{4t}}\partial e^{i\frac{x^2}{4t}}$$

Thus

$$\left\|e^{-it\Delta}(xf)\right\|_4^2 = \left\|Je^{-it\Delta}f\right\|_4^2 = 4t^2\left\|e^{-i\frac{x^2}{4t}}\partial e^{i\frac{x^2}{4t}}e^{-it\Delta}f\right\|_4^2$$

By standard Gagliardo-Nirenberg inequality,

$$\begin{aligned}&4t^2\left\|e^{-i\frac{x^2}{4t}}\partial e^{i\frac{x^2}{4t}}e^{-it\Delta}f\right\|_4^2\\ &\lesssim t^2\left\|e^{-it\Delta}f\right\|_\infty\left\|\Delta e^{i\frac{x^2}{4t}}e^{-it\Delta}f\right\|_2\\ &\lesssim \left\|e^{-it\Delta}f\right\|_\infty\left\|J^2e^{-it\Delta}f\right\|_2\\ &\lesssim \left\|e^{-it\Delta}f\right\|_\infty\left\|e^{-it\Delta}\left(x^2f\right)\right\|_2 .\end{aligned}$$

□

Lemma.4.3. Let $d \ge 2$. For any spacetime slab $K\times R^d$, $t_0 \in K$, and for any $\delta > 0$.

$$\begin{aligned}&\|uv\|_{L_t^2L_x^2\left(K\times\mathbb{R}^d\right)}\\ &\le c(\delta)\left(\left\|u(t_0)\right\|_{\dot H^{\frac{-1}{2+\delta}}} + \left\|(i\partial_t+\Delta)u\right\|_{L_t^1H_x^{\frac{-1}{2+\delta}}}\right) \quad (4.1)\\ &\times\left(\left\|v(t_0)\right\|_{\dot H^{\frac{d-1}{2}-\delta}} + \left\|(i\partial_t+\Delta)v\right\|_{L_t^1\dot H^{\frac{d-1}{2}-\delta}}\right)\end{aligned}$$

The estimate (4.1) is very helpful when *u* is high hesitancy and *v* is low hesitancy, as it moves abundance of derivatives onto the low hesitancy term. In particular, this estimate shows that there is little interaction between high and low hesitancy. This estimate is basically the repeated Strichartz estimate of Bourgain in [13]. We make the trivial remark that the $L_{t,x}^2$ norm of *uv* is the same as that of $u\bar v$, $\bar u v$, or $\overline{uv}$, thus the above estimate also applies to expressions of the form $O(uv)$

Proof: We fix δ, and permit our tacit constants to depend on δ. We begin by dealing with homogeneous case, with $u(t) := e^{it\Delta}\zeta$ and $v(t) := e^{it\Delta}\psi$, And consider the more general problem of proving,

$$\|uv\|_{L_{t,x}^2} \lesssim \|\zeta\|_{\dot H^{\alpha_1}}\|\psi\|_{\dot H^{\alpha_2}} \quad (4.2)$$

where $\alpha_1+\alpha_2 = \frac{d}{2}-1$ the scaling invariance of this estimate, first, our objective is to prove this for

$$\alpha_1 = -\frac{1}{2}+\delta \text{ and } \alpha_2 = \frac{d-1}{2}-\delta .$$

May be recast (4.2) using duality and renormalization as

$$\begin{aligned}&\int g\left(\xi_1+\xi_2,|\xi_1|^2+|\xi_2|^2\right)|\xi_1|^{-\alpha_1}\hat\zeta(\xi_1)|\xi_2|^{-\alpha_2}\hat\psi(\xi_2)\,d\xi_1d\xi_2\\ &\lesssim \|g\|_{L^2}\|\zeta\|_{L^2}\|\psi\|_{L^2}\end{aligned} \quad (4.3)$$

Since $\alpha_2 \ge \alpha_1$, we may restrict attention to the interactions with $|\xi_1| \ge |\xi_2|$.

In fact, in the residual case we can multiply by $\left(\frac{|\xi_2|}{|\xi_1|}\right)^{\alpha_2-\alpha_1} \ge 1$ to return to the condition under discussion. In fact, we may further restrict attention to the case where $|\xi_1| > 4|\xi_2|$ since, in the other case, we can move the frequencies between the two factors and reduce the case where $\alpha_2 = \alpha_1$, which can be dealt by $L_{t,x}^4$ Strichartz estimates when $d \ge 2$. Next, we decompose $|\xi_1|$ dyadically and $|\xi_2|$ in dyadic multiples of the size of $|\xi_1|$ by rewriting the quantity to be controlled as (*N*, Λ dyadic):

$$\begin{aligned}&\sum_N\sum_\Lambda\int\int gN\left(\xi_1+\xi_2,|\xi_1|^2+|\xi_2|^2\right)\\ &\cdot|\xi_1|^{-\alpha_1}\widehat{\zeta N}(\xi_1)|\xi_2|^{-\alpha_2}\widehat{\psi\Lambda N}(\xi_2)\,d\xi_1d\xi_2.\end{aligned}$$

Note that subscripts on g,ζ,ψ, have been inserted to invoke the localizations to $|\xi_1+\xi_2| \sim N$, $|\xi_1| \sim N$, $|\xi_2| \sim \Lambda N$, consecutive. In the case $|\xi_1| \ge 4|\xi_2|$, we have that $|\xi_1+\xi_2| \sim |\xi_1|$ and this expound, why *g* may be so localized. By renaming components, we may suppose that $|\xi_1^1| \sim |\xi_1|$ and $|\xi_2^1| \sim |\xi_2|$.

Write $\xi_2 = \left(\xi_2^1,\underline{\xi_2}\right)$. We change variables by writing

$$u = \xi_1+\xi_2, v = |\xi_1|^2+|\xi_2|^2$$

And $du\,dv = J\,d\xi_2^1\,d\xi_1$.

We show that by calculation

$$J = \left|2\left(\xi_1^1 \pm \xi_2^1\right)\right| \sim |\xi_1|.$$

Thus, upon changing variables in the inner two integrals, we encounter

$$\sum_N N^{-\alpha_1}\sum_{\Lambda\le 1}(\Lambda N)^{-\alpha_2}\left(\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^d} gN(u,v)H_{N,\Lambda}\left(u,v,\underline{\xi_2}\right)du\,dv\,d\underline{\xi_2}\right)$$

where

$$H_{N,\Lambda}\left(u,v,\underline{\xi_2}\right) = \frac{\widehat{\zeta N}(\xi_1)\widehat{\psi\Lambda N}(\xi_2)}{J}$$

Apply the Cauchy-Schwarz on the *u*, *v* integration and change back to the original variables to obtain

$$\sum_N N^{-\alpha_1}\|gN\|_{L^2}\sum_{\Lambda\le 1}(\Lambda N)^{-\alpha_2}$$
$$\left(\int_{\mathbb{R}^{d-1}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\frac{\left|\widehat{\zeta N}(\xi_1)\right|^2\left|\widehat{\psi\Lambda N}(\xi_2)\right|^2}{J}\mathrm{d}\xi_1\mathrm{d}\xi_2^1\right]^{\frac{1}{2}}\mathrm{d}\underline{\xi_2}\right)$$

We recall that $J\sim N$ and use Cauchy-Schwarz in the integration, taking into consideration the localization $\underline{\xi_2}\sim\Lambda N$, to get

$$\sum_N N^{-\alpha_1-\frac{1}{2}}\|gN\|_{L^2}\sum_{\Lambda\le 1}(\Lambda N)^{-\alpha_2+\frac{d-1}{2}}\left\|\widehat{\zeta N}\right\|_{L^2}\left\|\widehat{\psi\Lambda N}\right\|_{L^2}$$

Choose $\alpha_1=-\frac{1}{2}+\delta$ and $\alpha_2=\frac{d-1}{2}-\delta$. with $\delta>0$ to obtain

$$\sum_N\|gN\|_{L^2}\left\|\widehat{\zeta N}\right\|_{L^2}\sum_{\Lambda\le 1}\Lambda^{\delta}\left\|\widehat{\psi\Lambda N}\right\|_{L^2}.$$

This summarizes to get the claimed homogeneous estimate. Now we discuss the inhomogeneous estimate (4.1). For simplicity we set, $F:=(i\partial_t+\Delta)u$ and $G:=(i\partial_t+\Delta)v$. Then we use Duhamel's formula to write

$$u=\mathrm{e}^{i(t-t_0)\Delta}u(t_0)-i\int_{t_0}^{t}\mathrm{e}^{i(t-t')\Delta}F(t')\mathrm{d}t'$$

$$v=\mathrm{e}^{i(t-t_0)\Delta}v(t_0)-i\int_{t_0}^{t}\mathrm{e}^{i(t-t')\Delta}G(t')\mathrm{d}t'$$

We obtain

$$\begin{aligned}\|uv\|_{L^2}&\lesssim\left\|\mathrm{e}^{i(t-t_0)\Delta}u(t_0)\mathrm{e}^{i(t-t_0)\Delta}v(t_0)\right\|_{L^2}\\&+\left\|\mathrm{e}^{i(t-t_0)\Delta}u(t_0)\int_{t_0}^{t}\mathrm{e}^{i(t-t')\Delta}G(t')\mathrm{d}t'\right\|_{L^2}\\&+\left\|\mathrm{e}^{i(t-t_0)\Delta}v(t_0)\int_{t_0}^{t}\mathrm{e}^{i(t-t')\Delta}F(t')\mathrm{d}t'\right\|_{L^2}\\&+\left\|\int_{t_0}^{t}\mathrm{e}^{i(t-t')\Delta}G(t')\mathrm{d}t'\int_{t_0}^{t}\mathrm{e}^{i(t-t'')\Delta}G(x,t'')\mathrm{d}t''\right\|_{L^2}\\&:=I_1+I_2+I_3+I_4.\end{aligned}$$

The first term was treated in the first part of the proof. The second and the third are similar and so we consider I_2 only. By the Minkowski inequality,

$$I_2\lesssim\left\|\int_{\mathbb{R}}\mathrm{e}^{i(t-t_0)\Delta}u(t_0)\mathrm{e}^{i(t-t')\Delta}G(t')\right\|_{L^2}\mathrm{d}t'$$

And in this case the lemma follows from the homogeneous estimate proved above. Finally, again by Minkowski's inequality we have

$$I_4\lesssim\int_{\mathbb{R}}\int_{\mathbb{R}}\left\|\mathrm{e}^{i(t-t')\Delta}F(t')\mathrm{e}^{i(t-t'')\Delta}G(t'')\right\|_{L^2}\mathrm{d}t'\mathrm{d}t''$$

And the proof follows by inserting in the integral the homogeneous estimate above. □

Lemma.4.4. Let $u\in C^0_{t,loc}L^2_x(K\times\mathbb{R}^d)$ is nearly periodic modulo *G*. Then there exist functions $x:K\to\mathbb{R}^d$, $\xi:K\to\mathbb{R}^d$ and $N:K\to(0,+\infty)$, and for every $\eta>0$ there exists $0<C(\eta)<\infty$, such that we have the spatial concentration estimate,

$$\int_{|x-x(t)|\ge\frac{C(\eta)}{N(t)}}|u(t,x)|^2\,\mathrm{d}x\le\eta \tag{4.3}$$

And hesitancy concentration estimate,

$$\int_{|\xi-\xi(t)|\ge\frac{C(\eta)}{N(t)}}|\hat{u}(t,\xi)|^2\,\mathrm{d}\xi\le\eta \tag{4.4}$$

For all $t\in K$.

Remark 4.5. Informally, this lemma confirms that the mass $u(t)$ is spatially concentrated in the ball

$$\left\{x:x=x(t)+O\left(\frac{1}{N(t)}\right)\right\},$$

And is hesitancy concentrated in the ball

$$\{\xi:\xi=\xi(t)+O(N(t))\}.$$

Note that we have presently no control about how $x(t)$, $N(t)$, $\xi(t)$ vary in time; (for more see [14-17]).

Proof: By hypothesis, $u(t)$ lay in *GI* for some compact subset *I* in $L^2_x(\mathbb{R}^d)$. For every $\eta>0$, compactness argument shows that there exists $0<C(\eta)<\infty$, (depending on) such that

$$\int_{|x|\ge C(\eta)}|f(x)|^2\,\mathrm{d}x\le\eta$$

And hesitancy concentration estimate

$$\int_{|\xi|\ge C(\eta)}\left|\hat{f}(\xi)\right|\mathrm{d}\xi\le\eta$$

For all $f\in I$. By inspecting what the symmetry group *G* does to the spatial and hesitancy distribution of the mass of a function, then the claim follows. □

Corollary 4.6. Fix μ and *d*, and assume that m_0 is finite. Then there exists a maximal-lifespan solution $u\in C^0_{t,loc}L^2_x(K\times\mathbb{R}^d)$ of mass precisely m_0 which blows up both forward and backward in time, and functions, $x:K\to\mathbb{R}^d$, $\xi:K\to\mathbb{R}^d$ and $N:K\to(0,+\infty)$, with property $0<C(\eta)<\infty$, for every $\eta>0$, (depending on μ, *d*, m_0) such that we have the concentration estimates (4.3), (4.4) For all $t\in K$.

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