Scientific Research Publishing

Advances in Pure Mathematics, 2025, 15(4), 291-302
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Order Compactness in Riesz Spaces

Mustapha Laayouni

Department of Mathematics, Faculty of Science and Technology, U. M. I., Errachidia, Morocco
Email: mustaphalaayouni@yahoo.fr

How to cite this paper: Laayouni, M. (2025) Order Compactness in Riesz Spaces. *Advances in Pure Mathematics*, **15**, 291-302.
https://doi.org/10.4236/apm.2025.154014

Received: January 9, 2025
Accepted: April 19, 2025
Published: April 22, 2025

Abstract

Compact sets have important properties, and their studies have contributed to the development of functional analysis, particularly in the field of compact operators. In this paper, we introduce the concept of order compactness in Riesz spaces as an analog to topological compactness in the absence of a topology. We define order compact sets based on order convergence of nets and subnets, explore properties of these sets (e.g., closure, boundedness, preservation under order continuous maps), and prove results analogous to those in topological spaces, including an order version of the Banach-Stone theorem (Theorem 4.4) and a fixed point theorem (Theorem 4.4). When we introduce a continuous Banach lattice norm of order, we will show that the compactness of order coincides with the compactness of norm.

Keywords

Banach Lattices, Order, Compact, Order Convergence, Order Continuity, Riesz Spaces

1. Introduction

The concept of order compactness extends the notion of compactness to spaces without a defined topology. This allows for the generalization of key results related to compactness in Riesz spaces, enriching the theory and potentially offering new tools for analysis in these spaces.

If Γ is a subset of the partially ordered set Ω, we say that Γ is up-directed if and only if every finite subset of Γ has an upper bound in Γ. "Directed" will be used to denote "up-directed". A net in a Riesz space E is an arbitrary function from a non-empty directed set Γ to the space E. Nets will be denoted by $\{x_\alpha\}_{\alpha\in\Gamma}$, where x_α is the point of E assigned to the element α of the directed set Γ. Several authors have given various meanings to the statement "Net $\{x_\alpha\}_{\alpha\in\Gamma}$ order converges to the element x". In the literature on Riesz spaces theory, nets order convergence is defined in one of three ways.

Definition 1.1 ([1]) *A net* $\{x_\alpha\}_{\alpha\in\Gamma}$ *is order convergent to* x, *if there exists a net* $\{y_\beta\}_{\beta\in\Delta}$ *such that*:

1) $y_\beta \downarrow 0$, and
2) $|x_\alpha - x| \le y_\beta$ for all $\alpha\in\Gamma$.

Definition 1.2 ([2]) *A net* $\{x_\alpha\}_{\alpha\in\Gamma}$ *is order convergent to* x, *written* $x_\alpha \xrightarrow{o} x$, *if there exists a net* $\{y_\beta\}_{\beta\in\Delta}$ *such that*:

1) $y_\beta \downarrow 0$, and
2) for each $\beta\in\Delta$ there exists some $\alpha_0\in\Gamma$ satisfying $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_0$.

It should be noticed that definition 1.1 does not satisfy our understanding of the word "convergence". A converging net must remain converging even if we attach additional terms at the "beginning of the net. On the other hand, we hope that the definition adopted will not conflict with that used when the trellis is provided with a topology: If E is any topological (Riesz) space with a family $\mathbf{T}$ of "open" sets, the convergence of a net $\{x_\alpha\}_{\alpha\in\Gamma}$ of points of E is defined by the rule: $x_\alpha \xrightarrow{\mathbf{T}} x$ means that for every open set U containing x, β_U exists such that $x_\alpha \in U$ for all $\alpha \ge \beta_U$.

We now give the Aliprantis-Border definition of order convergence of a net.

Definition 1.3 ([2], p. 322) *A net* $\{x_\alpha\}$ *in a Riesz space E is order convergent to some* $x\in E$, *written* $x_\alpha \xrightarrow{o} x$, *if there is a net* $\{y_\alpha\}$ (*with the same directed set*) *satisfying*:

1) $y_\alpha \xrightarrow{o} 0$ and
2) $|x_\alpha - x| \le y_\alpha$ for each α

In order to ensure compatibility with the results when we introduce a topology in the Riesz space, we will adopt this last definition.

Finally, we will need the following definition.

Definition 1.4 ([2], p. 31) *A net* $\{y_\lambda\}_{\lambda\in\Lambda}$ *is a subnet of a net* $\{x_\alpha\}_{\alpha\in\Gamma}$ *if there is a function* $\varphi:\Lambda\to\Gamma$ *satisfying*

1) $y_\lambda = x_{\varphi_\lambda}$ for each $\lambda\in\Lambda$, where φ_λ stands for $\varphi(\lambda)$; and
2) for each $\alpha_0\in\Gamma$ there exists some $\lambda_0\in\Lambda$ such that $\lambda\ge\lambda_0$ implies $\varphi_\lambda \ge \alpha_0$.

2. Order Compactness in Riesz Spaces

The famous topological properties lacking in the general context of Riesz spaces, we take for definition of order compactness an equivalent for the order of its important topological property, namely the Bolzano-Weierstrass characterization. For a complete account of Riesz spaces, the reader is referred to [3]-[9].

Definition 2.1 *A subset* $\mathcal{K}$ *of a Riesz space E is*:

- *order compact if every net of* $\mathcal{K}$ *has an order convergent subnet with limit point belonging to* $\mathcal{K}$.
- σ*-order compact if every sequence of* $\mathcal{K}$ *has an order convergent subsequence with limit point belonging to* $\mathcal{K}$.

Note that if $\mathcal{K}$ is order compact then $\mathcal{K}$ is σ-order compact.

Proposition 2.1 *Let E be a Riesz space. The following assertions are fulfilled*:

i) *Finite unions of order compact subsets are order compact.*

ii) *Intersections of order compact parts are order compact.*

iii) *Order closed subsets of order compact subsets are order compact.*

Proof. i) Let $\{x_\alpha\}$ be a net in $\mathcal{K} = \bigcup_{1\le i\le p} \mathcal{K}_i$, where $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_p$ are order compact subsets. It is easy to see that at least one $\mathcal{K}_i$ contains an infinity of terms x_α. The order compactness of $\mathcal{K}_i$ ensures the existence of a convergent subnet of $\{x_\alpha\}$ in $\mathcal{K}_i \subset \mathcal{K}$, and the proof of assertion (i) is complete.

ii) is trivial.

iii) derives easily from the fact that a subset $\mathcal{K}$ of a Riesz space is order closed if $\{x_\alpha\} \subset \mathcal{K}$ and $x_\alpha \xrightarrow{o} x$ imply $x \in \mathcal{K}$. ■

With the same terms as for the whole space, the definition of the order completion for a part can be formulated as follows.

Definition 2.2 *A subset* $\mathcal{K}$ *of a Riesz space* E *is order complete*, *or Dedekind complete if both of the following conditions are fulfilled*:

1) *all its non-empty parts that are order bounded from above have a supremum in* $\mathcal{K}$.

2) *Every nonempty subset of* $\mathcal{K}$ *that is bounded from below has an infimum in* $\mathcal{K}$.

Theorem 2.1 *Let* $\mathcal{K}$ *be an order compact subset of a Dedekind complete Riesz space E such that* $x \vee y \in \mathcal{K}$ *and* $x \wedge y \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$. *Then* $\mathcal{K}$ *is order complete.*

Proof. Let *A* be an order bounded from the above subset of $\mathcal{K}$. The order ≥ on the set $S \subset \mathcal{K}$ of suprema of finite subsets of *A* is a direction: for each pair $x, y \in S$, we have $x \le x \vee y$, $y \le x \vee y$, and $x \vee y \in S$. Furthermore, *S* has the same upper bounds as *A*. Let a be upper bound of *A* in *E*. Then a is upper bound of *S* in *E*. So, there exists $\{x_\alpha\}_{\alpha \in L}$, $L \subset S$ such that $x_\alpha \uparrow a$. Since $\mathcal{K}$ is order compact, it follows that there exists some subnet of $\{x_\alpha\}$ convergent in $\mathcal{K}$. Thus $a \in \mathcal{K}$.

Now if *A* is an order bounded from below subset of $\mathcal{K}$. Condition (2) of the above definition is derived from the same method applied to the set $I \subset \mathcal{K}$ of infimum of finite subsets of *A*. The order ≤ on *I* is a direction: for each pair $x, y \in I$, we have $x \ge x \wedge y$, $y \ge x \wedge y$, and $x \wedge y \in I \subset \mathcal{K}$. ■

We need to manipulate functions that are defined only on parts of a Riesz space, a fact that allows the following definition.

Definition 2.3 *A function* $f : X \to F$ *from a subset X of a Riesz space E into a Riesz space F is order continuous on X if every net* $\{x_\alpha\} \subset X$ *satisfies*: $x_\alpha \xrightarrow{0} x$ *in E* (*with* $x \in X$) *implies* $f(x_\alpha) \xrightarrow{0} f(x)$ *in F*.

A subset $\mathcal{S}$ of a Riesz space is order closed if $\{x_\alpha\} \subset \mathcal{S}$ and $x_\alpha \xrightarrow{o} x$ imply $x \in \mathcal{S}$.

Let *E* and *F* be Riesz spaces. The Cartesian product $E \times F$ is a Riesz space under the usual ordering where $a_1 = (x_1, y_1) \ge a_2 = (x_2, y_2)$ whenever $x_1 \ge x_2$ and

$y_1 \geq y_2$. The infimum and supremum of two vectors a_1 and a_2 are given by $a_1 \vee a_2 = (x_1 \vee x_2, y_1 \vee y_2)$ and $a_1 \wedge a_2 = (x_1 \wedge x_2, y_1 \wedge y_2)$. Consider a function $f : \mathcal{S} \to \mathcal{K}$ between two subsets of E and F, respectively.

Remember that the graph of f is $G_f = \{(x, f(x)) : x \in \mathcal{S}\}$. If f is order continuous then the graph of f is order closed. Indeed, let $\{(x_\alpha, f(x_\alpha))\} \subset G_f$ such that $(x_\alpha, f(x_\alpha)) \xrightarrow{o} (x, y)$, which in terms of the lattice product, reads as $x_\alpha \xrightarrow{o} x$ and $f(x_\alpha) \xrightarrow{o} y$. It follows from the order continuity of f and the uniqueness of the order limit, that $f(x_\alpha) \xrightarrow{o} f(x) = y$. Thus $(x, y) \in G_f$ and G_f is order closed.

Proposition 2.2 *If a subset* $\mathcal{K}$ *of a Riesz space E is order compact then it is order closed.*

Proof. Let $\{x_\alpha\}$ a net in $\mathcal{K}$ such that $x_\alpha \xrightarrow{o} x \in E$. From the order compactness of $\mathcal{K}$ we infer that there exists a subnet $\{x_{\varphi(\alpha)}\}$ of $\{x_\alpha\}$ such that $\{x_{\varphi(\alpha)}\} \xrightarrow{o} a \in \mathcal{K}$. The uniqueness of the order limit of $\{x_{\varphi(\alpha)}\}$ implies that $x = a \in \mathcal{K}$. ■

Proposition 2.3 *Let* $\mathcal{K}_1, \mathcal{K}_3, \cdots, \mathcal{K}_n$ *be order compact subsets of Riesz spaces* $E_1, E_2, \cdots, E_n$, *respectively. Then* $\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$ *is order compact in the Riesz space* $E_1 \times E_2 \times \cdots \times E_n$

Proof. Let $\{(x_{\alpha 1}, x_{\alpha 2}, \cdots, x_{\alpha n})\}_\alpha$ a net in $\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$. The order compactness of $\mathcal{K}_1$ implies that there exists a subnet $\{x_{\varphi_1(\alpha)1}\}_{\varphi_1(\alpha)}$ which is order convergent to a_1 in $\mathcal{K}_1$. The net $\{x_{\varphi_1(\alpha)2}\}_{\varphi_1(\alpha)}$ contains a subnet $\{x_{\varphi_2 \circ \varphi_1(\alpha)2}\}_{\varphi_2 \circ \varphi_1(\alpha)}$ which is order convergent to a_2 in $\mathcal{K}_2$. A finite induction gives that $\{(x_{\alpha 1}, x_{\alpha 2}, \cdots, x_{\alpha n})\}_\alpha$ admits a subnet

$$\left\{\left(x_{\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1(\alpha)1}, x_{\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1(\alpha)2}, \cdots, x_{\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1(\alpha)n}\right)\right\}_{\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1(\alpha)}$$

which is order convergent to $(a_1, a_2, \cdots, a_n)$ in $\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$. ■

Proposition 2.4 *Let* $\mathcal{K}$ *be an order compact in an order complete Riesz space E such that* $x \vee y \in \mathcal{K}$ *and* $x \wedge y \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$. *Then* $\mathcal{K}$ *is order bounded.*

Proof. It follows from Proposition 1.3 and Theorem 1.4 that $\mathcal{K} \times \mathcal{K}$ is order compact and the order continuous mappings $(x, y) \to x \vee y$, $(x, y) \to x \wedge y$ from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$ achieve their maximum and minimum respectively. Which means that $\mathcal{K}$ is order bounded. ■

We will show that continuous order mappings preserve order compactness.

Theorem 2.2 *Let* $f : \mathcal{K} \to F$ *be an order continuous function from an order compact* $\mathcal{K}$ *of Riesz E into a Riesz space F. Then* $f(\mathcal{K})$ *is an order compact subset of F.*

Proof. Let $\{y_\alpha\}$ be a net in $f(\mathcal{K})$. There is $\{x_\alpha\}$ in $\mathcal{K}$ such that $f(x_\alpha) = y_\alpha$ for all α. Since $\mathcal{K}$ is order compact, there exists a subnet $\{x_{\varphi(\alpha)}\}$ which converges in order to $x \in \mathcal{K}$. It follows from the order continuity of f that $\{y_{\varphi(\alpha)}\} = \{f(x_{\varphi(\alpha)})\} \xrightarrow{o} y = f(x) \in f(\mathcal{K})$. Thus $f(\mathcal{K})$ is an order compact in F. ■

A similar result is obtained for functions with values in a Riesz space when a stability condition is required (which is obviously verified in the case of the real line)

Theorem 2.3 *Let* $\mathcal{K}$ *be an order compact subset of a Riesz space E. Let* $f:\mathcal{K}\to F$ *be an order continuous function defined on* $\mathcal{K}$ *into an order complete Riesz space F such that* $x\vee y\in f(\mathcal{K})$ *and* $x\wedge y\in f(\mathcal{K})$ *for all* $x,y\in f(\mathcal{K})$. *Then* f *achieves its supremum and infimum values.*

Proof. According to Theorem 1.1 and Theorem 1.2, $f(\mathcal{K})$ is order compact and order complete. So, to conclude the proof, it suffices to apply the definition 1.2. ■

Theorem 2.4 *Let* $\mathcal{K}$ *be an order compact subset of an order complete Riesz space E. Denote by* $f_{\mathcal{K}}$ *the mapping* $x\to\inf\{|x-a|:a\in\mathcal{K}\}$ *from E to* E_+. *Then* $f_{\mathcal{K}}$ *is well defined and satisfies:*

(i) $\forall x\in E$, $x\in\mathcal{K}\Leftrightarrow f_{\mathcal{K}}(x)=0$

(ii) $\forall x,y\in E$, $|f_{\mathcal{K}}(x)-f_{\mathcal{K}}(y)|\le|x-y|$.

Proof. (i) Pick any x in *E*, in view of ([2] (a) Corollary 8.7, p. 318) we can conclude that the mapping g_x: $a\to|a-x|$ from $\mathcal{K}$ to E_+ satisfies

$$|g_x(a)-g_x(b)|=\big||x-a|-|x-b|\big|\le|a-b|$$

for all $a,b\in\mathcal{K}$.

Now take $\{a_\alpha\}_\alpha$ a net in $\mathcal{K}$ with $a_\alpha\xrightarrow{o}a$. and $|a_\alpha-a|\le u_\alpha\downarrow 0$. Then $|g_x(a_\alpha)-g_x(a)|\le|a_\alpha-a|\le u_\alpha\downarrow 0$ and g_x is order continuous. It follows from Theorem.1.4 that g_x achieves its infimums values in $\mathcal{K}$. Then the function $f_{\mathcal{K}}$ is well defined and the proof of (i) is obviously obtained.

(ii) Let $x,y\in E$, for every $a\in\mathcal{K}$ we have we have:

$$\begin{aligned}|x-a|&\le|x-a-y+a|+|y-a|\\&\le|x-y|+|y-a|\end{aligned}$$

Taking the infimum on *a*, we get

$$f_{\mathcal{K}}(x)\le|x-y|+f_{\mathcal{K}}(y) \tag{1}$$

Interchanging x and y in the identity (1), we obtain (ii) ■

One of the theorems that is applied in analysis is that of the fixed point. Here is a version of the order.

Theorem 2.5 *Let* $\mathcal{K}$ *be a nonempty order compact subset of an Archimedean Riesz space E and let* f *be a function from* $\mathcal{K}$ *to* $\mathcal{K}$. *If there is some* $c\in]0,1[$ *such that* $|f(x)-f(y)|\le c|x-y|$, *for all* $x,y\in\mathcal{K}$, *then* f *has a unique fixed point.*

Proof. Pick any point $x_0\in\mathcal{K}$, and then define a sequence $\{x_n\}$ inductively by the formula $x_{n+1}=f(x_n)$, $n=0,1,\cdots$. For $n\ge1$ we have

$$|x_{n+1}-x_n|=|f(x_n)-f(x_{n-1})|\le c|x_n-x_{n-1}|$$

and by induction, we see that $|x_{n+1}-x_n|\le c^n|x_1-x_0|$. Hence, for $m>n$ the triangle inequality yields

$$
\begin{aligned}
|x_m - x_n| &\le \sum_{k=n+1}^{n} |x_k - x_{k-1}| \\
&\le \sum_{k=n+1}^{n} c^{k-1} |x_1 - x_0| \\
&\le \left[\sum_{k=n+1}^{+\infty} c^{k-1} \right] |x_1 - x_0| \\
&\le \frac{c^n}{1-c} |x_1 - x_0|.
\end{aligned} \tag{2}
$$

Now, since $\mathcal{K}$ is order compact, there exists a subsequence $\{x_{n_k}\}$ which converges in order to some $a \in \mathcal{K}$. According to (2), for every $n_k > n$ we have $|x_{n_k} - x_n| \le \frac{c^n}{1-c}|x_1 - x_0|$. By letting $n_k \to \infty$, it follows from ([2] Lemma 8.15) that $|a - x_n| \le \frac{c^n}{1-c}|x_1 - x_0|$. From this, the hypothesis on c, and the fact that E is Archimedean, we conclude that $\{x_n\} \xrightarrow{o} a$ and consequently its subsequence $\{x_{n+1}\} \xrightarrow{o} a$. On the other hand, the order continuity of f implies that $\{x_{n+1}\} = \{f(x_n)\} \xrightarrow{o} f(a)$. However, a sequence can have at most one order limit, then $a = f(a)$. That is, a is also a fixed point of f.

Now if $b = f(b)$, then clearly $|a-b| = |f(a) - f(b)| \le c|a-b|$ and $(c-1)|a-b| \in -E_+ \cap E_+ = \{0\}$, so $b = a$. Hence, a is the only fixed point of f. ■

3. Topological Cases

Recall that a linear topology τ on a Riesz space E is locally solid (and (E,τ) is called a locally solid Riesz space) if τ has a base at zero consisting of solid neighborhoods. A function $f: E \to F$ between two topological vector spaces is uniformly continuous if for each neighborhood W of zero in F there is a neighborhood V of zero in E such that $x - y \in V$ implies $f(x) - f(y) \in W$. It is well known that (E,τ) is a locally solid Riesz space if and only if the lattice operations are uniformly continuous with respect to τ. A locally solid linear topology τ on a Riesz space E which is moreover locally convex is called locally convex-solid topology. In other words, τ has a base $\mathcal{B}$ at zero consisting of neighborhoods that are simultaneously closed, solid, and convex.

Let (E,τ) be a locally convex-solid Riesz space. The gauge p_V of $V \in \mathcal{B}$ $p_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$ is a lattice seminorm. With the notation above, we have:

Definition 3.1 *A locally convex-solid topology* τ *is:*

- *order continuous if* $x_\alpha \downarrow 0$ *implies* $p_V(x_\alpha) \downarrow 0$ *for all* $V \in \mathcal{B}$.
- σ *-order continuous if* $x_n \downarrow 0$ *implies* $p_V(x_n) \downarrow 0$ *for all* $V \in \mathcal{B}$.

Theorem 3.1 *Let* $\mathcal{K}$ *be an order compact subset of a locally convex-solid Riesz space* (E,τ). *If* τ *is order continuous then* $\mathcal{K}$ *is compact* (*in the topology sense*).

Proof. According to ([2] Theorem 2.31 p. 39), it is enough to show that every net of $\mathcal{K}$ has a τ-convergent subnet. To this end, consider a net $\{x_\alpha\}_\alpha$ in $\mathcal{K}$. It follows from the order compactness of $\mathcal{K}$, that there is a subnet $\{x_{\varphi(\alpha)}\}_{\varphi(\alpha)} \xrightarrow{o} a \in \mathcal{K}$. So, there is a net $\{u_{\varphi(\alpha)}\}_{\varphi(\alpha)}$ in E_+ satisfying: $u_{\varphi(\alpha)} \downarrow 0$ and $|x_{\varphi(\alpha)} - a| \leq u_{\varphi(\alpha)}$. Since the locally convex-solid topology τ is order continuous, we get $p_V(x_{\varphi(\alpha)} - a) \leq p_V(u_{\varphi(\alpha)})$ and $p_V(u_{\varphi(\alpha)}) \downarrow 0$ for all $V \in \mathcal{B}$. This shows that $\{x_{\varphi(\alpha)}\}_{\varphi(\alpha)} \xrightarrow{\tau} a$. This proves that $\mathcal{K}$ is topologically compact. ■

An important special case of topological Riesz spaces is that of Banach lattices (which means that the Banach norm $\|\|$ is compatible with order: *i.e.*, $|x| \leq |y|$ in E implies $\|x\| \leq \|y\|$). The norm is order continuous if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$.

A simple consequence of Theorem 3.1 is the following.

Theorem 3.2 *Let E be a Banach lattice with order continuous norm. Every order compact of E is compact* (*with respect to this norm*).

In the Banach lattice literature, disjoint sequences are important. The following result combines them with order compactness.

Theorem 3.3 *Let* $\mathcal{K}$ *be an order compact in a Banach lattice with order continuous norm such that* $x \vee y \in \mathcal{K}$ *and* $x \wedge y \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$. *If* $0_E \notin \mathcal{K}$ *then there is now disjoint sequence in* $\mathcal{K}$.

Proof. If $\{x_n\}_n$ is a disjoint sequence in $\mathcal{K}$. From ([2] (3) Theorem 9.22, p. 356) follows that E is order complete. Looking at Proposition1.4 and ([2] (7) Theorem 9.22), we end up with $\{x_n\}_n \xrightarrow{\|\|} 0_E$ and $0_E \in \mathcal{K}$. ■

The following gives the converse of Theorem 3.1 in the case of a Banach lattice with the order continuous norm. In this situation, the two notions of compactness are identical.

Theorem 3.4 *Let* $\mathcal{K}$ *be a compact subset of a Banach lattice* $(E, \|\|)$ *with order continuous norm. Then* $\mathcal{K}$ *is order compact.*

Proof. Suppose that $\mathcal{K}$ is a compact of $(E, \|\|)$ and consider a net $\{x_\alpha\}_\alpha$ in $\mathcal{K}$. Then there is a subnet $\{x_{\varphi(\alpha)}\}_{\varphi(\alpha)}$ which satisfies $(x_{\varphi(\alpha)} - a) \xrightarrow{\|\|} 0$. Therefore, the fact that Banach lattices are local-solid-convex Hausdorff topological Riesz spaces ([2] Theorem 8.41, p. 334) implies that $(|x_{\varphi(\alpha)} - a|) \xrightarrow{\|\|} 0$. We can assume without loss of generality that $\|(|x_{\varphi(\alpha)} - a|)\| \downarrow 0$. According to ([2] (2) Theorem 8.43), we can conclude that $|x_{\varphi(\alpha)} - a| \downarrow 0$. Thus $x_{\varphi(\alpha)} \xrightarrow{o} a$, as required. ■

If the norm of a Banach lattice E is not order continuous then the two compactnesses are different in E. Indeed, there exists a net $\{x_\alpha\} \subset E$ satisfying $x_\alpha \downarrow 0$ but, $\|x_\alpha\| \nrightarrow 0$. Obviously, $\mathcal{K} = \{x_\alpha\} \cup \{0\}$ is order compact without being norm compact.

Theorem 3.5 *Let* $T: E \to F$ *be a lattice homomorphism from a Banach lattice with order continuous norm into an Archimedean Riesz space. If* $\mathcal{K}$ *is a com-*

pact in E then $T(\mathcal{K})$ *is order compact in F.*

Proof. Let $\mathcal{K}$ be a compact of *E*. It follows from Theorem 3.4 that $\mathcal{K}$ is order compact. Then the desired conclusion follows from Theorem 2.2. ■

4. Some Functional Results

In functional analysis, the space $C(X)$ (*X* being a topological space) has nice properties and allows to achieve important results. In this section, we will establish order version of the Banach-Stone Theorem. $\mathcal{C}_n(\mathcal{K})$ means the Riesz space, under pointwise algebraic operations and order, of order continuous functions from $\mathcal{K}$ into $\mathbb{R}$. Let $\mathcal{K}$ and $\mathcal{L}$ be order compacts in the Riesz spaces *E* and *F*, respectively. Now, let $\phi:\mathcal{C}_n(\mathcal{K})\to\mathcal{C}_n(\mathcal{L})$ be a Riesz isomorphism satisfying the condition

(C): $0\notin f(\mathcal{K})$ if and only if $0\notin\phi(f)(\mathcal{L})$ for each $f\in\mathcal{C}_n(\mathcal{K})$.

We will show that $\mathcal{K}$ is Lattice homeomorphic to $\mathcal{L}$. The proof will be based upon the following order version of the Urysohn's lemma.

4.1. Order Urysohn's lemma

Recall that a function $f:X\to F$ from a subset *X* of a Riesz space *E* into a Riesz space *F* is order continuous on *X* if every net $\{x_\alpha\}\subset X$ satisfies: $x_\alpha\xrightarrow{0}x$ in *E* (with $x\in X$) implies $f(x_\alpha)\xrightarrow{0}f(x)$ in *F*.

Suppose that $\mathcal{K}$ is an order compact in a Riesz space *E*. By $\mathcal{C}_n(\mathcal{K})$ we mean the Riesz space, under pointwise algebraic operations and order, of order continuous functions from $\mathcal{K}$ into $\mathbb{R}$. As a straightforward application of Theorem 2.3, we get the following.

Theorem 4.1 $\mathcal{C}_n(\mathcal{K})$ *is a Banach lattice under the norm defined by*:

$$\|f\|_c=\sup_{x\in\mathcal{K}}|f(x)|$$

Proof. The stability of $\mathcal{C}_n(\mathcal{K})$ under supremum and infimum follows from ([2], (4) Lemma 8.15, p. 322). So, $\mathcal{C}_n(\mathcal{K})$ is a Riesz space. Moreover, if $f\in\mathcal{C}_n(\mathcal{K})$, combining Theorems 2.2 and 3.4, we deduce then that $f(\mathcal{K})$ is a compact subset of $\mathbb{R}$. It follows from Theorem 2.3 that $\sup_{x\in\mathcal{K}}|f(x)|\in\mathbb{R}_+$ for all $f\in\mathcal{C}_n(\mathcal{K})$. It is routine to verify that the correspondence $f\to\|f\|_c$ becomes a norm which satisfies $0\le f\le g$ implies $\|f\|_c\le\|g\|_c$.

Next, we shall establish the completeness of $\left(\mathcal{C}_n(\mathcal{K}),\|\,\|_c\right)$. To this end, let $\{f_k\}$ be a Cauchy sequence in $\mathcal{C}_n(\mathcal{K})$. Then, given $\epsilon>0$ there exists k_0 such that

$$\|f_k-f_l\|_c<\epsilon\ \left(\text{for all } k,l\ge k_0\right). \tag{3}$$

In particular, note that for each $x\in\mathcal{K}$, the inequality $|f_k(x)-f_l(x)|\le\|f_k-f_l\|_c$ implies that $\{f_k(x)\}$ is a Cauchy sequence of real numbers. Thus, $\{f_k(x)\}$ converges in $\mathbb{R}$ for each $x\in\mathcal{K}$, let $f(x)=\lim f_k(x)$. Now we show that $f\in\mathcal{C}_n(\mathcal{K})$ and $f_k\xrightarrow{\|\,\|_c}f$. Suppose $x_\alpha\xrightarrow{o}a$ in $\mathcal{K}$. We now show that $f(x_\alpha)\xrightarrow{o}f(a)$ in $\mathbb{R}$, that is $|f(x_\alpha)-f(a)|\to 0$. Since for all $k\ge k_0$ $\|f_k-f_{k_0}\|_c<\epsilon$ and f_{k_0} is order continuous, we derive that there exists some α_0

such that $\left|f_{k_0}(x_\alpha)-f_{k_0}(a)\right|<\varepsilon$ for all $\alpha\ge\alpha_0$ and

$$\begin{aligned}\left|f(x_\alpha)-f(a)\right| &\le \left|f(x_\alpha)-f_{k_0}(x_\alpha)\right|+\left|f_{k_0}(x_\alpha)-f_{k_0}(a)\right|+\left|f_{k_0}(a)-f(a)\right| \\ &\le \varepsilon+\varepsilon+\varepsilon=3\varepsilon\end{aligned}$$

Thus, $f(x_\alpha)\overset{o}{\rightarrow}f(a)$, so f is an order continuous function. On the other hand, it easily follows from the inequality (3) that $\left|f_k(x)-f(x)\right|<\varepsilon$ for all $k\ge k_0$ and all $x\in\mathcal{K}$. This last inequality in turn implies that $\left\|f_k-f\right\|_c\le\varepsilon$ for all $k\ge k_0$. Hence, $\lim_{k\to\infty}f_k=f$, and so $\mathcal{C}_k(\mathcal{K})$ is complete. ■

A subset X of a Riesz space is order closed if $\{x_\alpha\}\subset X$ and $x_\alpha\overset{o}{\rightarrow}x$ imply $x\in X$. In topological spaces, the normal parts have famous properties. In order to profit from the non-topological case, we consider the following.

Definition 4.1 *A subset X of a Riesz space E is order normal if every pair A and B of disjoint order closed subsets of X can be separated by an order continuous function, which means that, there exists an order continuous function* $f:X\to[0,1]$ *such that* $f(a)=0$ *for each* $a\in A$ *and* $f(b)=1$ *for each* $b\in B$.

A linear functional $\varphi:E\to\mathbb{R}$ on a Riesz space E is strictly positive if $x>0_E$ implies $\varphi(x)>0$. Obviously, if $E=\mathbb{R}$ then any non-trivial positive linear form is strictly positive. Note that there are Riesz spaces that have no strictly positive linear functional (See [2], Example 8.21, p. 326).

Lemma 4.1 (*Order-Urysohn's lemma*) *Let E be an order complete Riesz space having a strictly positive linear functional which is order continuous. Then every order compact* $\mathcal{K}$ *of E is order normal.*

Proof. Let $\mathcal{K}$ be an order compact of E and φ be a strictly positive linear functional which is order continuous. Let A and B be disjoint order closed subsets of $\mathcal{K}$. It follows from Proposition 2.1 (iii) that A and B are order compacts. The Theorem 2.4 implies that the mappings: $x\to g_A(x)$ and $x\to g_B(x)$ defined from $\mathcal{K}$ to E_+ by $g_A(x)=\inf\{|x-a|:a\in A\}$ and $g_B(x)=\inf\{|x-b|:b\in B\}$ satisfy:

(i) $x\in A$ (respectively, $x\in B$) if and only if $g_A(x)=0$ (respectively, $g_B(x)=0$), for all $x\in\mathcal{K}$.

(ii) $\forall x,y\in\mathcal{K}$, $\left|g_A(x)-g_A(y)\right|\le|x-y|$ (respectively, $\left|g_B(x)-g_B(y)\right|\le|x-y|$)

On the other hand, since $\varphi\circ g_A$ is order continuous the theorems 2.3 and 3.2 tell us that $\varphi\circ g_A(\mathcal{K})$ is a compact in $\mathbb{R}_+$. Then there exists some $\lambda\in\mathbb{R}$ satisfying $0\le\varphi\circ g_A(x)<\lambda$ for each $x\in\mathcal{K}$. Put $h=\frac{1}{\lambda}\varphi$ and consider the mapping $f:\mathcal{K}\to[0,1]$ given by

$$f(x)=\frac{h\circ g_A(x)\left[1-h\circ g_B(x)\right]}{h\circ g_A(x)+h\circ g_B(x)} \tag{4}$$

The strict positivity of h added to the assertion (i) above and the disjointness of the order compacts A and B show that $h\circ g_A(x)+h\circ g_B(x)>0$ for all $x\in\mathcal{K}$. The assertion (ii) above and (4) imply that f is an order continuous mapping satisfying $f(a)=1$ and $f(b)=0$ for all $a\in A$, $b\in B$. Which concludes the proof of the claim. ■

This lemma is based on the existence of a strictly positive linear function of *E*. However, as it is well-known, there are Riesz spaces that have no strictly positive linear function (for instance, the Riesz space $\mathbb{R}^{\mathbb{N}}$ of all real sequences). We may now raise the following question.

Question: Does the order-Urysohn's lemma remain true even if E lacks a strictly positive linear form?

4.2. Order Banach-Stone Theorem

Theorem 4.2 *Let* $\mathcal{K}$ *be an order compact of an order complete Riesz space E such that* $x\vee y\in\mathcal{K}$ *for all* $x,y\in\mathcal{K}$. *For every* $x\in\mathcal{K}$ *define* $\phi_x:\mathcal{C}_n(\mathcal{K})\to\mathbb{R}$ *by* $\phi_x(f)=f(x)$, $\forall f\in\mathcal{C}_n(\mathcal{K})$. *Then the following assertions are fulfilled*:

(*i*) ϕ_x *is a Riesz homomorphism and* $\phi_x(1_{\mathcal{K}})=1$, *where* $1_{\mathcal{K}}$ *means the constant function equals* 1.

(*ii*) *For every Riesz homomorphism* $\phi:\mathcal{C}_n(\mathcal{K})\to\mathbb{R}$ *with* $\phi(1_{\mathcal{K}})=1$ *there exists a unique* $x\in\mathcal{K}$ *such that* $\phi=\phi_x$.

Proof. (i) is obvious.

(ii) Let $\phi:\mathcal{C}_n(\mathcal{K})\to\mathbb{R}$ be a Riesz homomorphism with $\phi(1_{\mathcal{K}})=1$. Assume to the contrary that for every $x\in\mathcal{K}$ there exists $f_x\in\mathcal{C}_n(\mathcal{K})$ for which $\phi(f_x)\neq\phi_x(f_x)$. For each $x\in\mathcal{K}$, the function $h_x=\left|f_x-\phi(f_x)1_{\mathcal{K}}\right|$ is so that

$$h_x(x)=\left|f_x(x)-\phi(f_x)\right|=\left|\phi_x(f_x)-\phi(f_x)\right|>0 \tag{5}$$

while

$$\phi(h_x)=\left|\phi(f_x)-\phi(f_x)\phi(1_{\mathcal{K}})\right|=0 \tag{6}$$

Now the order $\geq$ on the set $\mathcal{F}$ of suprema of finite subsets of $\mathcal{K}$ is a direction: for each pair $\alpha,\beta\in\mathcal{F}$, we have $\alpha\leq\alpha\vee\beta$, $\beta\leq\alpha\vee\beta$, and $\alpha\vee\beta\in\mathcal{F}$. Furthermore, $\mathcal{F}\subset\mathcal{K}$ and $\mathcal{F}$ has the same upper bounds as $\mathcal{K}$.

Let $\alpha=x_1\vee x_2\vee\cdots\vee x_n\in\mathcal{F}$, consider the element g_α of $\mathcal{C}_n(\mathcal{K})$ given by $g_\alpha=h_{x_1}\vee h_{x_2}\vee\cdots\vee h_{x_n}$ and consider the element x_α of $\mathcal{K}$ such that $x_\alpha=x_1\vee x_2\vee\cdots\vee x_n=\alpha$. Obviously, the two nets $\{x_\alpha\}$ and $\{g_\alpha\}$ are increasing in $\mathcal{K}$ and $\mathcal{C}_n(\mathcal{K})_+$, respectively. Moreover, by (6) it is enough to achieve that

$$\phi(g_\alpha)=\phi\left(h_{x_1}\vee h_{x_2}\vee\cdots\vee h_{x_n}\right)=\bigvee_{i=1}^{i=n}\phi\left(h_{x_i}\right)=0 \tag{7}$$

On the other hand, invoking the fact that $\mathcal{K}$ is order compact, we see that there exists a subnet $\left\{x_{\varphi(\alpha)}\right\}_{\varphi(\alpha)}$ which is order convergent to a in $\mathcal{K}$. More precisely, we have $x_\alpha\uparrow a$ and a is the upper bound of $\mathcal{K}$ and so, for every $x\in\mathcal{K}$ we have

$$g_a(x)\geq g_x(x)\geq h_x(x)>0 \tag{8}$$

Keeping in mind Theorems 2.2 and 3.4, the order compact $g_a(\mathcal{K})$ is a compact in $\mathbb{R}_+^*$ which implies that there exists some $\varepsilon>0$ such that for each $x\in\mathcal{K}$ we have $g_a(x)\geq\varepsilon$. It follows that $\phi(g_a)\geq\phi(\varepsilon 1_{\mathcal{K}})=\varepsilon$, contrary to (7). Therefore, there exists $x \in \mathcal{K}$ such that $\phi = \phi_x$.

Now, assume that x, y are elements in $\mathcal{K}$ such that $x \neq y$. It follows from Order-Urysohn's lemma applied to the order compacts $\{x\}$ and $\{y\}$, that there exists $f \in \mathcal{C}_n(\mathcal{K})$ such that $f(x) = 0$ and $f(y) = 1$, *i.e.*, $\phi_x(f) = 0$ and $\phi_y(f) = 1$. This concludes the proof of assertion (ii). ■

Theorem 4.3 *Let* $\mathcal{K}$ *and* $\mathcal{L}$ *be order compacts in two Riesz spaces* E, F *respectively, such that E is order complete and having an order continuous strictly positive linear functional and* $x \vee y$, $x \wedge y \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$. *If* $\psi : \mathcal{C}_n(\mathcal{K}) \to \mathcal{C}_n(\mathcal{L})$ *is a Riesz homomorphism satisfying* $\psi(1_{\mathcal{K}}) = 1_{\mathcal{L}}$ *then there is a unique order continuous map* $\sigma : \mathcal{L} \to \mathcal{K}$ *such that* $\psi(f) = f \circ \sigma$ *for all* $f \in \mathcal{C}_n(\mathcal{K})$.

Proof. Each $y \in \mathcal{L}$ defines the map $f \to \phi(f) = \psi(f)(y)$ from $\mathcal{C}_n(\mathcal{K})$ to $\mathbb{R}$, which is a Riesz homomorphism satisfying $\phi(1_{\mathcal{K}}) = 1$. By Theorem 4.2, we see that there exists a unique element $\sigma(y)$ of $\mathcal{K}$ such that $\psi(f)(y) = f(\sigma(y))$ for every $f \in \mathcal{C}_n(\mathcal{K})$. Thus we obtain the map $\sigma : \mathcal{L} \to \mathcal{K}$ who satisfies $\psi(f) = f \circ \sigma$, for all $f \in \mathcal{C}_n(\mathcal{K})$. To complete the proof, it must be shown that σ is order continuous. To this end we take $y_\alpha \xrightarrow{o} y$ in $\mathcal{L}$, and we proceed to show that $\sigma(y_\alpha) \xrightarrow{o} \sigma(y)$ in $\mathcal{K}$. Now, since $\psi(f) \in \mathcal{C}_n(\mathcal{L})$ for all $f \in \mathcal{C}_n(\mathcal{K})$, we conclude that $f \circ \sigma(y_\alpha) = \psi(f)(y_\alpha) \xrightarrow{o} \psi(f)(y) = f \circ \sigma(y)$, for every $f \in \mathcal{C}_n(\mathcal{K})$. From ([2], Theorem 8.16, p. 323) we infer that

$$\liminf_\alpha f(\sigma(y_\alpha)) = \limsup_\alpha f(\sigma(y_\alpha)) \text{ for all } f \in \mathcal{C}_n(\mathcal{K}) \tag{9}$$

We claim that, $\liminf_\alpha \sigma(y_\alpha) = \limsup_\alpha \sigma(y_\alpha)$. Suppose on the contrary that is false. By order-Urysohn's lemma 1.1 there exists an $f \in \mathcal{C}_n(\mathcal{K})$ such that $f(\liminf_\alpha \sigma(y_\alpha)) = 0$ and $f(\limsup_\alpha \sigma(y_\alpha)) = 1$. Since f is order continuous, then $\liminf_\alpha f(\sigma(y_\alpha)) = 0$ and $\limsup_\alpha f(\sigma(y_\alpha)) = 1$. Thus, $\liminf_\alpha \psi \circ f(y_\alpha) = 0$ and $\limsup_\alpha \psi \circ f(y_\alpha) = 1$, which contradicts (9), and therefore σ is order continuous. ■

Invoking the above theorem, we get the following.

Theorem 4.4 (***Order-Banach-Stone Theorem***) *Let* $\mathcal{K}$ *and* $\mathcal{L}$ *be order compacts in two order complete Riesz spaces E, F respectively which both admit strictly positive and order continuous form such that* $\mathcal{K}$ *and* $\mathcal{L}$ *stable under* $\wedge$ *and* $\vee$. *If* $\mathcal{C}_n(\mathcal{K})$ *and* $\mathcal{C}_n(\mathcal{L})$ *are Riesz isomorphic, then* $\mathcal{K}$ *and* $\mathcal{L}$ *are Riesz homeomorphic.*

Proof. Let ϕ be a Riesz isomorpism of $\mathcal{C}_n(\mathcal{K})$ onto $\mathcal{C}_n(\mathcal{L})$. Let $g_0 = \phi(1_{\mathcal{K}})$ and fix $f_0 \in \mathcal{C}_n(\mathcal{K})$ such that $\phi(f_0) = 1_{\mathcal{L}}$. From Theorem 2.3, which assures us that f_0 achieves its supremum on $\mathcal{K}$, follows that $f_0 \le c1_{\mathcal{K}}$ for some $c > 0$. Then $1_{\mathcal{L}} = \phi(f_0) \le c\phi(1_{\mathcal{K}}) = cg_0$, so $g_0(y) > 0$ for every $y \in \mathcal{L}$. and the mapping $\psi : \mathcal{C}_n(\mathcal{K}) \to \mathcal{C}_n(\mathcal{L})$ defined by

$$\psi(f)(y) = \frac{\phi(f)(y)}{g_0(y)}$$

for all $f \in \mathcal{C}_n(\mathcal{K})$, $y \in \mathcal{L}$, becomes a Riesz isomorphism from $\mathcal{C}_n(\mathcal{K})$ onto $\mathcal{C}_n(\mathcal{L})$ such that $\psi(1_{\mathcal{K}}) = 1_{\mathcal{L}}$. By Theorem 4.3, there exists an order continuous map $\sigma:\mathcal{L}\to\mathcal{K}$ and $\varrho:\mathcal{K}\to\mathcal{L}$ such that $\psi(f)=f\circ\sigma$ and $\psi^{-1}(g)=g\circ\varrho$ for all $f\in\mathcal{C}_n(\mathcal{K})$, $g\in\mathcal{C}_n(\mathcal{L})$. On the other hand, if $x\in\mathcal{K}$ then

$$f(x)=\psi^{-1}\circ\psi(f)(x)=(f\circ\sigma)\circ\varrho(x)=f(\sigma\circ\varrho(x))$$

for all $f\in\mathcal{C}_n(\mathcal{K})$. As a consequence of the order Urysohn's lemma 4.1, we have $\sigma\circ\varrho(x)=x$. Similarly, $\varrho\circ\sigma(y)=y$ for every $y\in\mathcal{L}$. Thus $\varrho=\sigma^{-1}$ and $\mathcal{K}$ and $\mathcal{L}$ are Riesz homeomorphic. ■

Acknowledgements

The author thanks the referees for their helpful recommendations.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Aliprantis, C.D. and Burkinshaw, O. (1998) Principles of Real Analysis. 3rd Edition, Academic Press.

[2] Aliprantis, C.D. and Border, K.C. (2006) Infinite Dimensional Analysis. 3rd Edition, Springer. https://doi.org/10.1007/978-3-662-03961-8

[3] Meyer-Nieberg, P. (1991) Banach Lattices. Springer-Verlag. https://doi.org/10.1007/978-3-642-76724-1

[4] Luxemburg, W.A.J. and Zaanen, A.C. (1971) Riesz Spaces I. North Holland.

[5] Vulikh, B.Z. (1967) Introduction to the Theory of Partially Ordered Spaces. Wolters Noordhoff Scientific Publications, Ltd.

[6] Schaefer, H.H. (1974) Banach Lattices and Positive Operators. Spinger-Verlag. https://doi.org/10.1007/978-3-642-65970-6

[7] Wickstead, A.W. (1977) Representation and Duality of Multiplication Operators on Archimedean Riesz Spaces. *Compositio Mathematica*, **35**, 225-238.

[8] Zaanen, A.C. (1983) Riesz Spaces II. North Holland.

[9] Zaanen, A.C. (1997) Introduction to Operator Theory in Riesz Spaces. Springer. https://doi.org/10.1007/978-3-642-60637-3