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Listing Prime Numbers Periodically

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Abstract

Over millennia, people have considered prime numbers as unpredictable and hard to list. This study confirms that all positive integers without factors of 2, 3, 5 and 7 can be regarded as the offspring of 48 roots in the interval [11, 220]. These roots are to generate a Factor-Pair Table composed of 48 columns, each of which forms a basic binary matrix to indicate the locations of composite numbers. For a given interval, these composites are removed via mapping effect. After primality checking of the rest of the numbers, we identify all primes within the interval. We then form a Formula of Primes for prime prediction. Moreover, the Periodic Table of Primes (PTP) [1] is reconstructed based on the proposed algorithm to illustrate the distribution of primes and composites.

Keywords

Prime Listing, Factor-Pair Table, Mapping Effect

1. Introduction

The problem of listing all primes up to a given limit has always been a challenging task. The prime number theorem [2] states that the prime-counting function $\pi(n)$ is approximated by $n/\log n$ when n is large, but listing all primes in a given interval has been considered as a much more difficult problem.

Except for trial division, which is too time-consuming, most of the existing algorithms are designed for finding all primes from the start, namely, from 2 to the given limit. The ancient Sieve of Eratosthenes iteratively removes composites as the multiples of primes already found [3]. In one of his notes [4], Dijkstra proposed a prime number algorithm that is more space-saving than the Sieve of Eratosthenes and has lower time complexity than trial division. Among other variants of the Sieve of Eratosthenes, Atkin's [5] and Pritchard's [6] improve the algorithm with preprocessing efforts, but require a trade-off between time and space complexity.

In contrast, little has been done on the prime listing within a given interval that starts from a large number. Helfgott [7] introduced an improved version of the Sieve of Eratosthenes that can construct all primes in an interval $[n-\Delta, n+\Delta]$ in space $O(\Delta)$ and time $O(\Delta \log n)$.

In this paper, we propose an efficient algorithm to list primes in a given interval that starts from any number. The algorithm is based on the Periodic Table of Primes (PTP) developed by Li *et al.* [1]. Although similar roots were proposed in 1982 as Wheel factorization [8], the PTP [1] is an infinite table generated from the Cyclic Table of Composites through rigorous mathematical proofs. Based on the algorithm presented in this paper, the PTP [1] is further reconstructed.

Our algorithm first restricts the numbers to be tested to a smaller subset, specifically those in the given interval and the PTP. Then, the mapping effect determines that, in expectation, more than 54.75% of them are composite numbers in constant time. Combined with existing efficient primality test algorithms, including the AKS primality test [9] and Miller-Rabin primality test [10] [11], it reveals the primality of the rest of the numbers and lists all primes in that given interval.

Notations

- $N_0 = \{0, 1, 2, \cdots\}$.
- $N_+ = \{1, 2, 3, \cdots\}$.
- $\Lambda = \{n \in N_+ \mid n > 1 \text{ and } n \text{ has no factors of 2, 3, 5 and 7}\}$.
- $S = \{r_i \mid r_i \text{ is an integer}, 11 \le r_i \le 220, r_i \in \Lambda\}$
  $= \{r_1 = 11, r_2 = 13, r_3 = 17, \cdots, r_{48} = 211\}$.
- $Q = \{q_j \mid q_j = r_j \in S\} = \{q_1 = 11, q_2 = 13, q_3 = 17, \cdots, q_{48} = 211\}$.
- $[a,b]_N = \{n \in N \mid a \le n \le b\}$.
- $\lceil x \rceil$ = ceiling function of the smallest integer greater than or equal to *x*.
- $\lfloor x \rfloor$ = floor function of the largest integer less than *x*.

2. Basic Theory

This study proposes a novel way, called Periodic-Table-of-Primes (PTP) method, to list all primes within a pre-specified integer interval. The deduction of this method is based on the following observations:

2.1. Roots of Integers

By choosing the first four primes, 2, 3, 5 and 7, we can group every $2 \times 3 \times 5 \times 7 = 210$ numbers in one period starting from 11, *i.e.*, $[11, 220]_N, [221, 430]_N, [431, 640]_N, \cdots$. Within each of these periods, there are 48 numbers without factors of 2, 3, 5 and 7. In particular, in $[11, 220]_N$, there exist 48 numbers called roots, and we denote

$$S = \{r_1 = 11, r_2 = 13, r_3 = 17, r_4 = 19, r_5 = 23, r_6 = 29, r_7 = 31, r_8 = 37, r_9 = 41, r_{10} = 43, r_{11} = 47, r_{12} = 53, r_{13} = 59, r_{14} = 61, r_{15} = 67, r_{16} = 71, r_{17} = 73, r_{18} = 79, r_{19} = 83, r_{20} = 89, r_{21} = 97, r_{22} = 101, r_{23} = 103, r_{24} = 107, r_{25} = 109, r_{26} = 113, r_{27} = 121, r_{28} = 127, r_{29} = 131, r_{30} = 137, r_{31} = 139, r_{32} = 143, r_{33} = 149, r_{34} = 151, r_{35} = 157, r_{36} = 163, r_{37} = 167, r_{38} = 169, r_{39} = 173, r_{40} = 179, r_{41} = 181, r_{42} = 187, r_{43} = 191, r_{44} = 193, r_{45} = 197, r_{46} = 199, r_{47} = 209, r_{48} = 211\}.$$

Theorem 1. For any number $\alpha \in \Lambda$, there exists a unique $r_i \in S$ and $k \in N_0$ such that

$$\alpha = r_i + 210k. \tag{1}$$

From Theorem 1, we know that each number $\alpha \in \Lambda$ has a root and a period. In other words, α is the k^{th} descendant of r_i.

From notations indicating that $Q = S$ and $q_j = r_j$, $j = 1, 2, \cdots, 48$, we can also know that any factor of α can be expressed as $q_j + 210\theta$, for some $q_j \in Q$ and $\theta \in N_0$. The following observation further analyzes the structure of α's factors.

2.2. Factor-Pair Table with Generation Information

Given $q_j \in Q$ and $r_i \in S$, denote $q_{\hat{j}|i}$ as the unique element in Q such that $q_j \times q_{\hat{j}|i} - r_i$ is a multiplier of 210, where $i, j \in \{1, 2, \cdots, 48\}$. Here $\left(q_j, q_{\hat{j}|i}\right)$ is called a factor-pair with respect to r_i. We also observe there exists an integer

$$t_{j|i} = \frac{q_j q_{\hat{j}|i} - r_i}{210} \tag{2}$$

to specify the generation of r_i in which the descendant of q_j and $q_{\hat{j}|i}$, *i.e.*, $q_j \times q_{\hat{j}|i}$ lies.

Collecting all $\left(q_j, q_{\hat{j}|i}\right)$ and $t_{j|i}$ for all $i, j \in \{1, 2, \cdots, 48\}$ together, we then form a Factor-Pair Table with generation information. It is a 48×48 table made of element $\left(q_j, q_{\hat{j}|i}\right), t_{j|i}$.

We denote each column of the table as V_j, where

$$V_j = \left(\left(q_j, q_{\hat{j}|1}\right), t_{j|1}; \cdots; \left(q_j, q_{\hat{j}|i}\right), t_{j|i}; \cdots; \left(q_j, q_{\hat{j}|48}\right), t_{j|48}\right)^{\mathrm{T}} \quad \text{for} \quad j = 1, 2, 3, \cdots, 48.$$

Take $j = 1$ for instance,

$$V_1 = \left((11, 211), 11; (11, 173), 9; (11, 97), 5; (11, 59), 3; \cdots; (11, 191), 9\right)^{\mathrm{T}},$$

where $\left(q_1, q_{\hat{1}|1}, t_{1|1}\right) = \left(q_1, q_{48}, t_{1|1}\right) = (11, 211, 11)$, as shown in Table 1.

The Factor-Pair Table with generation information has the following properties:

1) In each row of the table, there are 48 factor-pairs.

2) If $q_j = q_{\hat{j}|i}$, then $\left(q_j, q_{\hat{j}|i}\right)$ is called a co-factor-pair. Co-factor-pairs only exist in 6 rows of the table. In each row corresponding to $r_{18} = 79$, $r_{25} = 109$, $r_{27} = 121$, $r_{34} = 151$, $r_{38} = 169$ and $r_{48} = 211$, there are 8 co-factor-pairs among the 48 factor-pairs.

Table 1. Factor-pair table with generation information.

i \ j		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	r_i \ q_j	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
1	11	(11, 211), 11	(13, 17), 1	(17, 13), 1	(19, 89), 8	(23, 37), 4	(29, 109), 15	(31, 41), 6	(37, 23), 4	(41, 31), 6	(43, 137), 28	(47, 103), 23	(53, 127), 32	(59, 139), 39	(61, 131), 38	(67, 113), 36	(71, 151), 51
2	13	(11, 173), 9	(13, 211), 13	(17, 149), 12	(19, 67), 6	(23, 101), 11	(29, 167), 23	(31, 163), 24	(37, 199), 35	(41, 113), 22	(43, 181), 37	(47, 179), 40	(53, 131), 33	(59, 107), 30	(61, 193), 56	(67, 19), 6	(71, 83), 28
3	17	(11, 97), 5	(13, 179), 11	(17, 211), 17	(19, 23), 2	(23, 19), 2	(29, 73), 10	(31, 197), 29	(37, 131), 23	(41, 67), 13	(43, 59), 12	(47, 121), 27	(53, 139), 35	(59, 43), 12	(61, 107), 31	(67, 41), 13	(71, 157), 53
4	19	(11, 59), 3	(13, 163), 10	(17, 137), 11	(19, 211), 19	(23, 83), 9	(29, 131), 18	(31, 109), 16	(37, 97), 17	(41, 149), 29	(43, 103), 21	(47, 197), 44	(53, 143), 36	(59, 11), 3	(61, 169), 49	(67, 157), 50	(71, 89), 30
5	23	(11, 193), 10	(13, 131), 8	(17, 199), 16	(19, 167), 15	(23, 211), 23	(29, 37), 5	(31, 143), 21	(37, 29), 5	(41, 103), 20	(43, 191), 39	(47, 139), 31	(53, 151), 38	(59, 157), 44	(61, 83), 24	(67, 179), 57	(71, 163), 55
6	29	(11, 79), 4	(13, 83), 5	(17, 187), 15	(19, 101), 9	(23, 193), 21	(29, 211), 29	(31, 89), 13	(37, 137), 24	(41, 139), 27	(43, 113), 23	(47, 157), 35	(53, 163), 41	(59, 61), 17	(61, 59), 17	(67, 107), 34	(71, 169), 57
7	31	(11, 41), 2	(13, 67), 4	(17, 113), 9	(19, 79), 7	(23, 47), 5	(29, 59), 8	(31, 211), 31	(37, 103), 18	(41, 11), 2	(43, 157), 32	(47, 23), 5	(53, 167), 42	(59, 29), 8	(61, 121), 35	(67, 13), 4	(71, 101), 34
8	37	(11, 137), 7	(13, 19), 1	(17, 101), 8	(19, 13), 1	(23, 29), 3	(29, 23), 3	(31, 157), 23	(37, 211), 37	(41, 47), 9	(43, 79), 16	(47, 41), 9	(53, 179), 45	(59, 143), 40	(61, 97), 28	(67, 151), 48	(71, 107), 36
9	41	(11, 61), 3	(13, 197), 12	(17, 163), 13	(19, 179), 16	(23, 157), 17	(29, 139), 19	(31, 191), 28	(37, 143), 25	(41, 211), 41	(43, 167), 34	(47, 193), 43	(53, 187), 47	(59, 79), 22	(61, 11), 3	(67, 173), 55	(71, 181), 61
10	43	(11, 23), 1	(13, 181), 11	(17, 89), 7	(19, 157), 14	(23, 11), 1	(29, 197), 27	(31, 103), 15	(37, 109), 19	(41, 83), 16	(43, 211), 43	(47, 59), 13	(53, 191), 48	(59, 47), 13	(61, 73), 21	(67, 79), 25	(71, 113), 38
11	47	(11, 157), 8	(13, 149), 9	(17, 151), 12	(19, 113), 10	(23, 139), 15	(29, 103), 14	(31, 137), 20	(37, 41), 7	(41, 37), 7	(43, 89), 18	(47, 211), 47	(53, 199), 50	(59, 193), 54	(61, 197), 57	(67, 101), 32	(71, 187), 63
12	53	(11, 43), 2	(13, 101), 6	(17, 139), 11	(19, 47), 4	(23, 121), 13	(29, 67), 9	(31, 83), 12	(37, 149), 26	(41, 73), 14	(43, 11), 2	(47, 19), 4	(53, 211), 53	(59, 97), 27	(61, 173), 50	(67, 29), 9	(71, 193), 65
13	59	(11, 139), 7	(13, 53), 3	(17, 127), 10	(19, 191), 17	(23, 103), 11	(29, 31), 4	(31, 29), 4	(37, 47), 8	(41, 109), 21	(43, 143), 29	(47, 37), 8	(53, 13), 3	(59, 211), 59	(61, 149), 43	(67, 167), 53	(71, 199), 67
14	61	(11, 101), 5	(13, 37), 2	(17, 53), 4	(19, 169), 15	(23, 167), 18	(29, 89), 12	(31, 151), 22	(37, 13), 2	(41, 191), 37	(43, 187), 38	(47, 113), 25	(53, 17), 4	(59, 179), 50	(61, 211), 61	(67, 73), 23	(71, 131), 44
15	67	(11, 197), 10	(13, 199), 12	(17, 41), 3	(19, 103), 9	(23, 149), 16	(29, 53), 7	(31, 97), 14	(37, 121), 21	(41, 17), 3	(43, 109), 22	(47, 131), 29	(53, 29), 7	(59, 83), 23	(61, 187), 54	(67, 211), 67	(71, 137), 46
16	71	(11, 121), 6	(13, 167), 10	(17, 103), 8	(19, 59), 5	(23, 67), 7	(29, 169), 23	(31, 131), 19	(37, 53), 9	(41, 181), 35	(43, 197), 40	(47, 73), 16	(53, 37), 9	(59, 19), 5	(61, 101), 29	(67, 23), 7	(71, 211), 71
17	73	(11, 83), 4	(13, 151), 9	(17, 29), 2	(19, 37), 3	(23, 131), 14	(29, 17), 2	(31, 43), 6	(37, 19), 3	(41, 53), 10	(43, 31), 6	(47, 149), 33	(53, 41), 10	(59, 197), 55	(61, 163), 47	(67, 139), 44	(71, 143), 48
18	79	(11, 179), 9	(13, 103), 6	(17, 17), 1	(19, 181), 16	(23, 113), 12	(29, 191), 26	(31, 199), 29	(37, 127), 22	(41, 89), 17	(43, 163), 33	(47, 167), 37	(53, 53), 13	(59, 101), 28	(61, 139), 40	(67, 67), 21	(71, 149), 50
19	83	(11, 103), 5	(13, 71), 4	(17, 79), 6	(19, 137), 12	(23, 31), 3	(29, 97), 13	(31, 23), 3	(37, 59), 10	(41, 43), 8	(43, 41), 8	(47, 109), 24	(53, 61), 15	(59, 37), 10	(61, 53), 15	(67, 89), 28	(71, 13), 4
20	89	(11, 199), 10	(13, 23), 1	(17, 67), 5	(19, 71), 6	(23, 13), 1	(29, 61), 8	(31, 179), 26	(37, 167), 29	(41, 79), 15	(43, 173), 35	(47, 127), 28	(53, 73), 18	(59, 151), 42	(61, 29), 8	(67, 17), 5	(71, 19), 6
21	97	(11, 47), 2	(13, 169), 10	(17, 191), 15	(19, 193), 17	(23, 59), 6	(29, 83), 11	(31, 37), 5	(37, 31), 5	(41, 197), 38	(43, 139), 28	(47, 11), 2	(53, 89), 22	(59, 23), 6	(61, 67), 19	(67, 61), 19	(71, 167), 56
22	101	(11, 181), 9	(13, 137), 8	(17, 43), 3	(19, 149), 13	(23, 187), 20	(29, 199), 27	(31, 71), 10	(37, 173), 30	(41, 151), 29	(43, 17), 3	(47, 163), 36	(53, 97), 24	(59, 169), 47	(61, 191), 55	(67, 83), 26	(71, 31), 10
23	103	(11, 143), 7	(13, 121), 7	(17, 179), 14	(19, 127), 11	(23, 41), 4	(29, 47), 6	(31, 193), 28	(37, 139), 24	(41, 23), 4	(43, 61), 12	(47, 29), 6	(53, 101), 25	(59, 137), 38	(61, 43), 12	(67, 199), 63	(71, 173), 58
24	107	(11, 67), 3	(13, 89), 5	(17, 31), 2	(19, 83), 7	(23, 169), 18	(29, 163), 22	(31, 17), 2	(37, 71), 12	(41, 187), 36	(43, 149), 30	(47, 181), 40	(53, 109), 27	(59, 73), 20	(61, 167), 48	(67, 11), 3	(71, 37), 12
25	109	(11, 29), 1	(13, 73), 4	(17, 167), 13	(19, 61), 5	(23, 23), 2	(29, 11), 1	(31, 139), 20	(37, 37), 6	(41, 59), 11	(43, 193), 39	(47, 47), 10	(53, 113), 28	(59, 41), 11	(61, 19), 5	(67, 127), 40	(71, 179), 60
26	113	(11, 163), 8	(13, 41), 2	(17, 19), 1	(19, 17), 1	(23, 151), 16	(29, 127), 17	(31, 173), 25	(37, 179), 31	(41, 13), 2	(43, 71), 14	(47, 199), 44	(53, 121), 30	(59, 187), 52	(61, 143), 41	(67, 149), 47	(71, 43), 14
27	121	(11, 11), 0	(13, 187), 11	(17, 143), 11	(19, 139), 12	(23, 197), 21	(29, 149), 20	(31, 31), 4	(37, 43), 7	(41, 131), 25	(43, 37), 7	(47, 83), 18	(53, 137), 34	(59, 59), 16	(61, 181), 52	(67, 193), 61	(71, 191), 64
28	127	(11, 107), 5	(13, 139), 8	(17, 131), 10	(19, 73), 6	(23, 179), 19	(29, 113), 15	(31, 187), 27	(37, 151), 26	(41, 167), 32	(43, 169), 34	(47, 101), 22	(53, 149), 37	(59, 173), 48	(61, 157), 45	(67, 121), 38	(71, 197), 66
29	131	(11, 31), 1	(13, 107), 6	(17, 193), 15	(19, 29), 2	(23, 97), 10	(29, 19), 2	(31, 11), 1	(37, 83), 14	(41, 121), 23	(43, 47), 9	(47, 43), 9	(53, 157), 39	(59, 109), 30	(61, 71), 20	(67, 143), 45	(71, 61), 20
30	137	(11, 127), 6	(13, 59), 3	(17, 181), 14	(19, 173), 15	(23, 79), 8	(29, 193), 26	(31, 167), 24	(37, 191), 33	(41, 157), 30	(43, 179), 36	(47, 61), 13	(53, 169), 42	(59, 13), 3	(61, 47), 13	(67, 71), 22	(71, 67), 22
31	139	(11, 89), 4	(13, 43), 2	(17, 107), 8	(19, 151), 13	(23, 143), 15	(29, 41), 5	(31, 79), 11	(37, 157), 27	(41, 29), 5	(43, 13), 2	(47, 137), 30	(53, 173), 43	(59, 191), 53	(61, 109), 31	(67, 187), 59	(71, 209), 70
32	143	(11, 13), 0	(13, 11), 0	(17, 169), 13	(19, 107), 9	(23, 61), 6	(29, 157), 21	(31, 113), 16	(37, 89), 15	(41, 193), 37	(43, 101), 20	(47, 79), 17	(53, 181), 45	(59, 127), 35	(61, 23), 6	(67, 209), 66	(71, 73), 24
33	149	(11, 109), 5	(13, 173), 10	(17, 157), 12	(19, 41), 3	(23, 43), 4	(29, 121), 16	(31, 59), 8	(37, 197), 34	(41, 19), 3	(43, 23), 4	(47, 97), 21	(53, 193), 48	(59, 31), 8	(61, 209), 60	(67, 137), 43	(71, 79), 26
34	151	(11, 71), 3	(13, 157), 9	(17, 83), 6	(19, 19), 1	(23, 107), 11	(29, 179), 24	(31, 181), 26	(37, 163), 28	(41, 101), 19	(43, 67), 13	(47, 173), 38	(53, 197), 49	(59, 209), 58	(61, 61), 17	(67, 43), 13	(71, 11), 3
35	157	(11, 167), 8	(13, 109), 6	(17, 71), 5	(19, 163), 14	(23, 89), 9	(29, 143), 19	(31, 127), 18	(37, 61), 10	(41, 137), 26	(43, 199), 40	(47, 191), 42	(53, 209), 52	(59, 113), 31	(61, 37), 10	(67, 181), 57	(71, 17), 5
36	163	(11, 53), 2	(13, 61), 3	(17, 59), 4	(19, 97), 8	(23, 71), 7	(29, 107), 14	(31, 73), 10	(37, 169), 29	(41, 173), 33	(43, 121), 24	(47, 209), 46	(53, 11), 2	(59, 17), 4	(61, 13), 3	(67, 109), 34	(71, 23), 7
37	167	(11, 187), 9	(13, 29), 1	(17, 121), 9	(19, 53), 4	(23, 199), 21	(29, 13), 1	(31, 107), 15	(37, 101), 17	(41, 127), 24	(43, 209), 42	(47, 151), 33	(53, 19), 4	(59, 163), 45	(61, 137), 39	(67, 131), 41	(71, 97), 32
38	169	(11, 149), 7	(13, 13), 0	(17, 47), 3	(19, 31), 2	(23, 53), 5	(29, 71), 9	(31, 19), 2	(37, 67), 11	(41, 209), 40	(43, 43), 8	(47, 17), 3	(53, 23), 5	(59, 131), 36	(61, 199), 57	(67, 37), 11	(71, 29), 9
39	173	(11, 73), 3	(13, 191), 11	(17, 109), 8	(19, 197), 17	(23, 181), 19	(29, 187), 25	(31, 53), 7	(37, 209), 36	(41, 163), 31	(43, 131), 26	(47, 169), 37	(53, 31), 7	(59, 67), 18	(61, 113), 32	(67, 59), 18	(71, 103), 34
40	179	(11, 169), 8	(13, 143), 8	(17, 97), 7	(19, 131), 11	(23, 163), 17	(29, 151), 20	(31, 209), 30	(37, 107), 18	(41, 199), 38	(43, 53), 10	(47, 187), 41	(53, 43), 10	(59, 181), 50	(61, 89), 25	(67, 197), 62	(71, 109), 36
41	181	(11, 131), 6	(13, 127), 7	(17, 23), 1	(19, 109), 9	(23, 17), 1	(29, 209), 28	(31, 121), 17	(37, 73), 12	(41, 71), 13	(43, 97), 19	(47, 53), 11	(53, 47), 11	(59, 149), 41	(61, 151), 43	(67, 103), 32	(71, 41), 13
42	187	(11, 17), 0	(13, 79), 4	(17, 11), 0	(19, 43), 3	(23, 209), 22	(29, 173), 23	(31, 67), 9	(37, 181), 31	(41, 107), 20	(43, 19), 3	(47, 71), 15	(53, 59), 14	(59, 53), 14	(61, 127), 36	(67, 31), 9	(71, 47), 15
43	191	(11, 151), 7	(13, 47), 2	(17, 73), 5	(19, 209), 18	(23, 127), 13	(29, 79), 10	(31, 101), 14	(37, 113), 19	(41, 61), 11	(43, 107), 21	(47, 13), 2	(53, 67), 16	(59, 199), 55	(61, 41), 11	(67, 53), 16	(71, 121), 40
44	193	(11, 113), 5	(13, 31), 1	(17, 209), 16	(19, 187), 16	(23, 191), 20	(29, 137), 18	(31, 13), 1	(37, 79), 13	(41, 143), 27	(43, 151), 30	(47, 89), 19	(53, 71), 17	(59, 167), 46	(61, 103), 29	(67, 169), 53	(71, 53), 17
45	197	(11, 37), 1	(13, 209), 12	(17, 61), 4	(19, 143), 12	(23, 109), 11	(29, 43), 5	(31, 47), 6	(37, 11), 1	(41, 97), 18	(43, 29), 5	(47, 31), 6	(53, 79), 19	(59, 103), 28	(61, 17), 4	(67, 191), 60	(71, 127), 42
46	199	(11, 209), 10	(13, 193), 11	(17, 197), 15	(19, 121), 10	(23, 173), 18	(29, 101), 13	(31, 169), 24	(37, 187), 32	(41, 179), 34	(43, 73), 14	(47, 107), 23	(53, 83), 20	(59, 71), 19	(61, 79), 22	(67, 97), 30	(71, 59), 19
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15	67	(149, 23), 16	(151, 127), 91	(157, 181), 135	(163, 79), 61	(167, 101), 80	(169, 193), 155	(173, 89), 73	(179, 113), 96	(181, 157), 135	(187, 61), 54	(191, 107), 97	(193, 169), 155	(197, 11), 10	(199, 13), 12	(209, 143), 142	(211, 67), 67
16	71	(149, 109), 77	(151, 191), 137	(157, 173), 129	(163, 137), 106	(167, 13), 10	(169, 29), 23	(173, 157), 129	(179, 79), 67	(181, 41), 35	(187, 143), 127	(191, 151), 137	(193, 107), 98	(197, 43), 40	(199, 89), 84	(209, 139), 138	(211, 71), 71
17	73	(149, 47), 33	(151, 13), 9	(157, 169), 126	(163, 61), 47	(167, 179), 142	(169, 157), 126	(173, 191), 157	(179, 167), 142	(181, 193), 166	(187, 79), 70	(191, 173), 157	(193, 181), 166	(197, 59), 55	(199, 127), 120	(209, 137), 136	(211, 73), 73
18	79	(149, 71), 50	(151, 109), 78	(157, 157), 117	(163, 43), 33	(167, 47), 37	(169, 121), 97	(173, 83), 68	(179, 11), 9	(181, 19), 16	(187, 97), 86	(191, 29), 26	(193, 193), 177	(197, 107), 100	(199, 31), 29	(209, 131), 130	(211, 79), 79
19	83	(149, 157), 111	(151, 173), 124	(157, 149), 111	(163, 101), 78	(167, 169), 134	(169, 167), 134	(173, 151), 124	(179, 187), 159	(181, 113), 97	(187, 179), 159	(191, 73), 66	(193, 131), 120	(197, 139), 130	(199, 107), 101	(209, 127), 126	(211, 83), 83
20	89	(149, 181), 128	(151, 59), 42	(157, 137), 102	(163, 83), 64	(167, 37), 29	(169, 131), 105	(173, 43), 35	(179, 31), 26	(181, 149), 128	(187, 197), 175	(191, 139), 126	(193, 143), 131	(197, 187), 175	(199, 11), 10	(209, 121), 120	(211, 89), 89
21	97	(149, 143), 101	(151, 187), 134	(157, 121), 90	(163, 199), 154	(167, 71), 56	(169, 13), 10	(173, 179), 147	(179, 173), 147	(181, 127), 109	(187, 151), 134	(191, 17), 15	(193, 19), 17	(197, 41), 38	(199, 163), 154	(209, 113), 112	(211, 97), 97
22	101	(149, 19), 13	(151, 41), 29	(157, 113), 84	(163, 47), 36	(167, 193), 153	(169, 59), 47	(173, 37), 30	(179, 139), 118	(181, 11), 9	(187, 23), 20	(191, 61), 55	(193, 167), 153	(197, 73), 68	(199, 29), 27	(209, 109), 108	(211, 101), 101
23	103	(149, 167), 118	(151, 73), 52	(157, 109), 81	(163, 181), 140	(167, 149), 118	(169, 187), 150	(173, 71), 58	(179, 17), 14	(181, 163), 140	(187, 169), 150	(191, 83), 75	(193, 31), 28	(197, 89), 83	(199, 67), 63	(209, 107), 106	(211, 103), 103
24	107	(149, 43), 30	(151, 137), 98	(157, 101), 75	(163, 29), 22	(167, 61), 48	(169, 23), 18	(173, 139), 114	(179, 193), 164	(181, 47), 40	(187, 41), 36	(191, 127), 115	(193, 179), 164	(197, 121), 113	(199, 143), 135	(209, 103), 102	(211, 107), 107
25	109	(149, 191), 135	(151, 169), 121	(157, 97), 72	(163, 163), 126	(167, 17), 13	(169, 151), 121	(173, 173), 142	(179, 71), 60	(181, 199), 171	(187, 187), 166	(191, 149), 135	(193, 43), 39	(197, 137), 128	(199, 181), 171	(209, 101), 100	(211, 109), 109
26	113	(149, 67), 47	(151, 23), 16	(157, 89), 66	(163, 11), 8	(167, 139), 110	(169, 197), 158	(173, 31), 25	(179, 37), 31	(181, 83), 71	(187, 59), 52	(191, 193), 175	(193, 191), 175	(197, 169), 158	(199, 47), 44	(209, 97), 96	(211, 113), 113
27	121	(149, 29), 20	(151, 151), 108	(157, 73), 54	(163, 127), 98	(167, 173), 137	(169, 79), 63	(173, 167), 137	(179, 179), 152	(181, 61), 52	(187, 13), 11	(191, 71), 64	(193, 67), 61	(197, 23), 21	(199, 199), 188	(209, 89), 88	(211, 121), 121
28	127	(149, 53), 37	(151, 37), 26	(157, 61), 45	(163, 109), 84	(167, 41), 32	(169, 43), 34	(173, 59), 48	(179, 23), 19	(181, 97), 83	(187, 31), 27	(191, 137), 124	(193, 79), 72	(197, 71), 66	(199, 103), 97	(209, 83), 82	(211, 127), 127
29	131	(149, 139), 98	(151, 101), 72	(157, 53), 39	(163, 167), 129	(167, 163), 129	(169, 89), 71	(173, 127), 104	(179, 199), 169	(181, 191), 164	(187, 113), 100	(191, 181), 164	(193, 17), 15	(197, 103), 96	(199, 179), 169	(209, 79), 78	(211, 131), 131
30	137	(149, 163), 115	(151, 197), 141	(157, 41), 30	(163, 149), 115	(167, 31), 24	(169, 53), 42	(173, 19), 15	(179, 43), 36	(181, 17), 14	(187, 131), 116	(191, 37), 33	(193, 29), 26	(197, 151), 141	(199, 83), 78	(209, 73), 72	(211, 137), 137
31	139	(149, 101), 71	(151, 19), 13	(157, 37), 27	(163, 73), 56	(167, 197), 156	(169, 181), 145	(173, 53), 43	(179, 131), 111	(181, 169), 145	(187, 67), 59	(191, 59), 53	(193, 103), 94	(197, 167), 156	(199, 121), 114	(209, 71), 70	(211, 139), 139
32	143	(149, 187), 132	(151, 83), 59	(157, 29), 21	(163, 131), 101	(167, 109), 86	(169, 17), 13	(173, 121), 99	(179, 97), 82	(181, 53), 45	(187, 149), 132	(191, 103), 93	(193, 41), 37	(197, 199), 186	(199, 197), 186	(209, 67), 66	(211, 143), 143
33	149	(149, 211), 149	(151, 179), 128	(157, 17), 12	(163, 113), 87	(167, 187), 148	(169, 191), 153	(173, 13), 10	(179, 151), 128	(181, 89), 76	(187, 167), 148	(191, 169), 153	(193, 53), 48	(197, 37), 34	(199, 101), 95	(209, 61), 60	(211, 149), 149
34	151	(149, 149), 105	(151, 211), 151	(157, 13), 9	(163, 37), 28	(167, 143), 113	(169, 109), 87	(173, 47), 38	(179, 29), 24	(181, 31), 26	(187, 103), 91	(191, 191), 173	(193, 127), 116	(197, 53), 49	(199, 139), 131	(209, 59), 58	(211, 151), 151
35	157	(149, 173), 122	(151, 97), 69	(157, 211), 157	(163, 19), 14	(167, 11), 8	(169, 73), 58	(173, 149), 122	(179, 83), 70	(181, 67), 57	(187, 121), 107	(191, 47), 42	(193, 139), 127	(197, 101), 94	(199, 43), 40	(209, 53), 52	(211, 157), 157
36	163	(149, 197), 139	(151, 193), 138	(157, 199), 148	(163, 211), 163	(167, 89), 70	(169, 37), 29	(173, 41), 33	(179, 137), 116	(181, 103), 88	(187, 139), 123	(191, 113), 102	(193, 151), 138	(197, 149), 139	(199, 157), 148	(209, 47), 46	(211, 163), 163
37	167	(149, 73), 51	(151, 47), 33	(157, 191), 142	(163, 59), 45	(167, 211), 167	(169, 83), 66	(173, 109), 89	(179, 103), 87	(181, 197), 169	(187, 11), 9	(191, 157), 142	(193, 89), 81	(197, 181), 169	(199, 23), 21	(209, 43), 42	(211, 167), 167
38	169	(149, 11), 7	(151, 79), 56	(157, 187), 139	(163, 193), 149	(167, 167), 132	(169, 211), 169	(173, 143), 117	(179, 191), 162	(181, 139), 119	(187, 157), 139	(191, 179), 162	(193, 163), 149	(197, 197), 184	(199, 61), 57	(209, 41), 40	(211, 169), 169
39	173	(149, 97), 68	(151, 143), 102	(157, 179), 133	(163, 41), 31	(167, 79), 62	(169, 47), 37	(173, 211), 173	(179, 157), 133	(181, 23), 19	(187, 29), 25	(191, 13), 11	(193, 101), 92	(197, 19), 17	(199, 137), 129	(209, 37), 36	(211, 173), 173
40	179	(149, 121), 85	(151, 29), 20	(157, 167), 124	(163, 23), 17	(167, 157), 124	(169, 11), 8	(173, 103), 84	(179, 211), 179	(181, 59), 50	(187, 47), 41	(191, 79), 71	(193, 113), 103	(197, 67), 62	(199, 41), 38	(209, 31), 30	(211, 179), 179
41	181	(149, 59), 41	(151, 61), 43	(157, 163), 121	(163, 157), 121	(167, 113), 89	(169, 139), 111	(173, 137), 112	(179, 89), 75	(181, 211), 181	(187, 193), 171	(191, 101), 91	(193, 187), 171	(197, 83), 77	(199, 79), 74	(209, 29), 28	(211, 181), 181
42	187	(149, 83), 58	(151, 157), 112	(157, 151), 112	(163, 139), 107	(167, 191), 151	(169, 103), 82	(173, 29), 23	(179, 143), 121	(181, 37), 31	(187, 211), 187	(191, 167), 151	(193, 199), 182	(197, 131), 122	(199, 193), 182	(209, 23), 22	(211, 187), 187
43	191	(149, 169), 119	(151, 11), 7	(157, 143), 106	(163, 197), 152	(167, 103), 81	(169, 149), 119	(173, 97), 79	(179, 109), 92	(181, 131), 112	(187, 83), 73	(191, 211), 191	(193, 137), 125	(197, 163), 152	(199, 59), 55	(209, 19), 18	(211, 191), 191
44	193	(149, 107), 75	(151, 43), 30	(157, 139), 103	(163, 121), 93	(167, 59), 46	(169, 67), 53	(173, 131), 107	(179, 197), 167	(181, 73), 62	(187, 19), 16	(191, 23), 20	(193, 211), 193	(197, 179), 167	(199, 97), 91	(209, 17), 16	(211, 193), 193
45	197	(149, 193), 136	(151, 107), 76	(157, 131), 97	(163, 179), 138	(167, 181), 143	(169, 113), 90	(173, 199), 163	(179, 163), 138	(181, 167), 143	(187, 101), 89	(191, 67), 60	(193, 149), 136	(197, 211), 197	(199, 173), 163	(209, 13), 12	(211, 197), 197
46	199	(149, 131), 92	(151, 139), 99	(157, 127), 94	(163, 103), 79	(167, 137), 108	(169, 31), 24	(173, 23), 18	(179, 41), 34	(181, 109), 93	(187, 37), 32	(191, 89), 80	(193, 13), 11	(197, 17), 15	(199, 211), 199	(209, 11), 10	(211, 199), 199
47	209	(149, 31), 21	(151, 89), 63	(157, 107), 79	(163, 143), 110	(167, 127), 100	(169, 41), 32	(173, 193), 158	(179, 61), 51	(181, 29), 24	(187, 137), 121	(191, 199), 180	(193, 173), 158	(197, 97), 90	(199, 191), 180	(209, 211), 209	(211, 209), 209
48	211	(149, 179), 126	(151, 121), 86	(157, 103), 76	(163, 67), 51	(167, 83), 65	(169, 169), 135	(173, 17), 13	(179, 149), 126	(181, 181), 155	(187, 73), 64	(191, 11), 9	(193, 37), 33	(197, 113), 105	(199, 19), 17	(209, 209), 207	(211, 211), 211

3) If $\left(q_j, q_{\hat{j}|i}\right)$ is not a co-factor-pair, then $\left(q_{\hat{j}|i}, q_j\right)$ is a factor-pair with the same generation $t_{j|i}$ as $\left(q_j, q_{\hat{j}|i}\right)$.

4) For any j and i, we have $t_{j|i} \le r_j$.

We found that the factors of a composite number α are closely related to factor-pairs in the Factor-Pair Table with generation information:

Theorem 2. If $q_j + 210\theta$ ($\theta \in N_0$) is a factor of $\alpha = r_i + 210k$, then $\frac{\alpha}{q_j + 210\theta}$ is a descendant of $q_{\hat{j}|i}$, *i.e.*,

$$\frac{\alpha}{q_j + 210\theta} = q_{\hat{j}|i} + 210\hat{\theta}. \tag{3}$$

Theorems 1 and 2 confirm that any composite number α without factors of 2, 3, 5 and 7 can be written as

$$\alpha = r_i + 210k = \left(q_j + 210\theta\right)\left(q_{\hat{j}|i} + 210\hat{\theta}\right), \tag{4}$$

where $\theta, \hat{\theta} \in N_0$ and $j \in \{1, 2, \cdots, 48\}$.

From Equation (4), we have

$$r_i + 210k = q_j q_{\hat{j}|i} + 210\theta q_{\hat{j}|i} + 210\hat{\theta} q_j + 210^2 \theta\hat{\theta}. \tag{5}$$

Combining Equations (2) and (5), we get

$$t_{j|i} = k - \theta q_{\hat{j}|i} - \hat{\theta} q_j - 210\theta\hat{\theta}. \tag{6}$$

When j is fixed, Equation (6) is a quadratic equation with only two unknowns θ and $\hat{\theta}$ in N_0.

Theorem 3. To check whether a number $\alpha = r_i + 210k$ is a composite number, it suffices to check if Equation (6) has a solution $\left(\theta, \hat{\theta}\right) \in N_0 \times N_0$ for some $j \in \{1, 2, \cdots, 48\}$.

We then propose an algorithm to solve the equation.

Without loss of generality, we assume $q_j + 210\theta \le q_{\hat{j}|i} + 210\hat{\theta}$, since we can replace q_j and θ with $q_{\hat{j}|i}$ and $\hat{\theta}$ correspondingly if $q_j + 210\theta > q_{\hat{j}|i} + 210\hat{\theta}$. Given $\alpha = \left(q_j + 210\theta\right)\left(q_{\hat{j}|i} + 210\hat{\theta}\right)$, we see

$$q_j + 210\theta \le \sqrt{\alpha}. \tag{7}$$

Therefore,

$$\theta \le \frac{\sqrt{\alpha} - q_j}{210} \le \frac{\sqrt{\alpha}}{210}. \tag{8}$$

We can derive a direct corollary from Theorem 3:

Corollary 1. Checking whether a number $\alpha = r_i + 210k \in \Lambda$ is a composite number is equivalent to checking if

$$\hat{\theta} = \frac{k - t_{j|i} - \theta q_{\hat{j}|i}}{q_j + 210\theta} \tag{9}$$

is an integer for some $\theta \in \left\{0, 1, \cdots, \left\lceil \frac{\sqrt{\alpha}}{210} \right\rceil\right\}$.

Based on this Corollary, we can develop a new algorithm for primality test:

Algorithm 1:

Step 1: Input a natural number α. If α is divisible by 2, 3, 5 or 7, then α is not a prime, and the algorithm terminates.

Step 2: Set $r_i = \alpha - \left\lfloor \frac{\alpha - 10}{210} \right\rfloor \times 210$.

Step 3: For each $j \in \{1, 2, \cdots, 48\}$, examine the Factor-Pair Table with generation information for the factor pair $(q_j, q_{\hat{j}|i})$ and generation $t_{j|i}$. For each $\theta \in \left\{0, 1, \cdots, \left\lfloor \frac{\sqrt{\alpha}}{210} \right\rfloor\right\}$, compute Equation (9). According to Corollary 1, if $\hat{\theta} \in N_0$, then α is not a prime, and the algorithm terminates.

Step 4: If the loop completes (having traversed all j) without finding a solution, then α is a prime.

The following theorem gives the complexity of Algorithm 1:

Theorem 4. Algorithm 1 involves at most 48 linear searches over $\left\lfloor \frac{\sqrt{\alpha}}{210} \right\rfloor + 1$ integer values. In other words, the worst case is searching through about $\sqrt{\alpha}/4$ integer values.

However, to list all primes within a pre-specified interval, the above algorithm for primality test costs too much time. Therefore, we need to find a more efficient way to locate primes.

3. Method

Notice that in a given interval, primes only appear with the form of $r_i + 210k$, where $r_i \in S$ and $k \in N_0$, since the rest is divisible by 2, 3, 5 or 7. So we only need to consider the primality of those numbers.

Furthermore, in our PTP method to list primes in the following observations, the cyclic behavior of the Factor-Pair Table with generation information is observed and utilized to ease computation burden.

3.1. Basic Binary Matrices

According to the 4th property of the Factor-Pair Table with generation information, we know

$$t_{j|i} + r_j \le 2r_j \,. \tag{10}$$

To represent the generation information of each factor pair more clearly, we convert each column of the Factor-Pair Table with generation information, V_j, into a basic binary matrix of composites $B_j^o = \left[b_j^0(i,k)\right]_{48\times 2r_j}$ for $i = 1, 2, \cdots, 48$, $k = 0, 1, 2, \cdots, 2r_j - 1$, where $b_j^0(i,k) \in \{0,1\}$. In this matrix, $b_j^0(i,k) = 1$ indicates that the k^{th} descendant of r_i is a composite with a factor being q_j.

Specifically, we specify $b_j^0(i,k)$ following the rule below:

The entry $b_j^0(i,k)$ is set as 1 only under two cases:

Case 1 ($i = j$): if $k = r_i$ or $k = 0$, $b_j^0(i,k) = 1$.

Case 2 ($i \neq j$): if $k = t_{j|i}$ or $k = t_{j|i} + r_j$, $b_j^0(i,k) = 1$.

Otherwise, $b_j^0(i,k) = 0$.

Inequality (10) guarantees that the rule is properly defined.

$V_1(q_1, q_{\hat{1}|i}, t_{1|i})$ ⟶ B_1^0

		$j = 1, r_j = 11$		
i	r_i	q_1	$q_{\hat{1}\mid i}$	$t_{1\mid i}$
1	11	11	211	11
2	13	11	173	9
3	17	11	97	5
4	19	11	59	3
5	23	11	193	10
6	29	11	79	4
7	31	11	41	2
8	37	11	137	7
9	41	11	61	3
10	43	11	23	1
11	47	11	157	8
12	53	11	43	2
13	59	11	139	7
14	61	11	101	5
15	67	11	197	10
16	71	11	121	6
17	73	11	93	4
18	79	11	179	9
19	83	11	103	5
20	89	11	199	10
21	97	11	47	2
22	101	11	181	9
23	103	11	143	7
24	107	11	67	3
25	109	11	29	1
26	113	11	163	8
27	121	11	11	0
28	127	11	107	5
29	131	11	31	1
30	137	11	127	6
31	139	11	89	4
32	143	11	13	0
33	149	11	109	5
34	151	11	171	3
35	157	11	167	8
36	163	11	53	2
37	157	11	187	9
38	169	11	149	7
39	173	11	73	3
40	179	11	169	8
41	181	11	131	6
42	187	11	17	0
43	191	11	151	7
44	193	11	113	5
45	197	11	137	1
46	199	11	209	10
47	209	11	19	0
48	211	11	191	9

i \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1											1										
2										1											1	
3						1											1					
4				1											1							
5											1											1
6					1											1						
7			1											1								
8								1											1			
9				1											1							
10		1											1									
11									1											1		
12			1											1								
13								1											1			
14						1											1					
15											1											1
16							1											1				
17					1											1						
18										1											1	
19						1											1					
20											1											1
21			1											1								
22										1											1	
23								1											1			
24				1											1							
25		1											1									
26									1											1		
27	1											1										
28						1											1					
29		1											1									
30							1											1				
31					1											1						
32	1											1										
33						1											1					
34				1											1							
35									1											1		
36			1											1								
37										1											1	
38								1											1			
39				1											1							
40									1											1		
41							1											1				
42	1											1										
43								1											1			
44						1											1					
45		1											1									
46											1											1
47	1											1										
48										1											1	

Figure 1. Converting V_1 to B_1^0.

We observe the following properties in B_j^0:

1) There are exactly two non-zero entries for each i, which means either Case 1 or Case 2 always sets two entries in B_j^0 as 1.

2) Every matrix B_j^0 consists of two identical submatrices of the size $48\times r_j$ placed side by side.

3) Each non-zero entry in the submatrix mentioned in (2) corresponds to a factor-pair $\left(q_j, q_{\hat{j}|i}\right)$, $i=1,2,\cdots,48$ with its generation $t_{j|i}$.

Figure 1 shows how to convert V_1 to B_1^0. For instance,

we set $b_1^0\left(1,0\right)=b_1^0\left(1,11\right)=1$, moreover,

since $t_{1|2}=9, b_1^0\left(2,9\right)=b_1^0\left(2,20\right)=1$;

since $t_{1|3}=5, b_1^0\left(3,5\right)=b_1^0\left(3,16\right)=1$;

since $t_{1|4}=3, b_1^0\left(4,3\right)=b_1^0\left(4,14\right)=1$.

Similarly, $V_2, V_3,\cdots$ and V_{48} is respectively converted into $B_2^0, B_3^0,\cdots$ and B_{48}^0.

3.2. Mapping Effect and Refined Binary Matrices

From Sections 2.1 and 2.2, we know a composite without factors of 2, 3, 5 and 7, which is rooted as r_i, can be expressed as

$\alpha=r_i+210k=\left(q_j+210\theta\right)\times\left(q_{\hat{j}|i}+210\hat{\theta}\right)$, and its generation is

$t_{j|i}=\dfrac{q_j q_{\hat{j}|i}-r_i}{210}\in N_0$.

Now if $\theta=0$, we observe that for $k=t_{j|i}+q_j\times\hat{\theta}$, where $\hat{\theta}\in N_0$, $r_i+210t_{j|i}$, $r_i+210\left(t_{j|i}+q_j\right)$, $r_i+210\left(t_{j|i}+2q_j\right)$, $r_i+210\left(t_{j|i}+3q_j\right),\cdots$, are all composites rooted at r_i. That means, there is a mapping effect between $r_i+210t_{j|i}$ and $r_i+210\left(t_{j|i}+m\times q_j\right)$ for any $m\in N_+$.

Similarly, if $\hat{\theta}=0$, there is a mapping effect between $r_i+210t_{j|i}$ and $r_i+210\left(t_{j|i}+m'\times q_{\hat{j}|i}\right)$ for $m'\in N_+$. Such an effect is very useful to quickly find the same-root composites based on the basic-binary-matrices, as described below:

Consider an integer interval from $11+210\times k^*$ to $211+210\times\left(k^*+\delta\right)$, where $k^*\in N_+$, $\delta\in N_0$, $\delta\le 10\left(<r_1=11\right)$. Define $B_j=\left[b_j\left(i,k^*+h\right)\right]_{48\times(\delta+1)}$, where $b_j\left(i,k^*+h\right)\in\{0,1\}$, as a refined binary matrix of composites for $i\in\{1,2,\cdots,48\}$, $h\in\{0,1,\cdots,\delta\}$. In other words, if $b_j\left(i,k\right)=1$, then the integer r_i+210k is a composite number. But if $b_j\left(i,k\right)=0$, we can't determine whether r_i+210k is a prime, as discussed later. Theorem 5 then shows the mapping effect from B_j^0 to B_j:

Theorem 5. If $k=w_j+r_j\times M$, where $M\in N_0$ and $w_j\in\{0,1,\cdots,2r_j-1\}$, then for any $i\in\{1,2,\cdots,48\}$, if $b_j^0\left(i,w_j\right)=1$, then r_i+210k is composite, *i.e.*, $b_j\left(i,k\right)=1$.

The value of each entry is determined accordingly. By setting

$w_j=k^*-r_j\left\lfloor k^*/r_j\right\rfloor$, we notice that $w_j+h\in\{0,1,\cdots,2r_j-1\}$. For any $h\in\{0,1,\cdots,\delta\}$ and $i\in\{1,2,\cdots,48\}$, if $b_j^0\left(i,w_j+h\right)=1$, then set $b_j\left(i,k^*+h\right)=1$; if $b_j^0\left(i,w_j+h\right)=0$, then set $b_j\left(i,k^*+h\right)=0$.

B_j has the following properties:

1) B_j is a submatrix of B_j^0, with dimensionality of $48\times(\delta+1)$.

2) Each non-zero entry in B_j corresponds to a composite number in the pre-specified interval.

The refined binary matrix B_j can quickly locate large composites in a pre-specified interval. However, it's not guaranteed that when $b_j(i,k^*+h)=0$, the corresponding integer is prime. The reason is that the mapping effect only locates composites with the form of $r_i+210(t_{j|i}+m\times q_j)=q_j\times(q_{\hat{j}|i}+210m)$ and $r_i+210(t_{\hat{j}|i}+m'\times q_{\hat{j}|i})=q_{\hat{j}|i}\times(q_j+210m')$. For composites with the smallest factor larger than 211, B_j may fail to detect them.

3.3. Composite Binary Matrix

A composite binary matrix $C=[c(i,k)]_{48\times(\delta+1)}$, where $i\in\{1,2,\cdots,48\}$ and $k\in\{k^*,k^*+1,\cdots,k^*+\delta\}$, can be formed based on $B_1,B_2,\cdots,B_{48}$. We want to assure that all composites within the interval $[11+210\times k^*,211+210\times(k^*+\delta)]$ are detected and located by C.

We first locate composite numbers detected by mapping effect. If $b_j(i,k)=1$ for some $j\in\{1,2,\cdots,48\}$, the number r_i+210k is a composite, *i.e.*, $c(i,k)=1$. After that, we only need to determine the primality of the remaining numbers.

The specific algorithm is as follows:

Algorithm 2:

Step 1: For each $c(i,k)$, where $i\in\{1,2,\cdots,48\}$ and $k\in\{k^*,k^*+1,\cdots,k^*+\delta\}$, check $b_j(i,k)$ for $j\in\{1,2,\cdots,48\}$:

If $b_j(i,k)=1$ for some j, then set $c(i,k)=1$;

otherwise, go to Step 2.

Step 2: Let $\alpha=r_i+210k$. Conduct a selected primality test algorithm for α:

If α is not a prime, set $c(i,k)=1$;

otherwise, set $c(i,k)=0$.

With matrix C, we can locate all composite numbers within the interval:

Theorem 6. In a prespecified interval $[11+210\times k^*,211+210\times(k^*+\delta)]$, where $k^*\in N_+$, $\delta\in[0,10]_N$, any integer number r_i+210k is composite if and only if $c(i,k)=1$.

3.4. Prime Binary Matrix

We are now able to form a binary matrix of primes $P=[p(i,k)]_{48\times(\delta+1)}$, where $i\in\{1,2,\cdots,48\}$ and $k\in\{k^*,k^*+1,\cdots,k^*+\delta\}$, within the pre-specified interval. Each entry of P is determined by $p(i,k)=1-c(i,k)$, $i=1,2,\cdots,48$ and $k=k^*,k^*+1,\cdots,k^*+\delta$.

In this way, we can list all primes within the interval as illustrated in the following theorem:

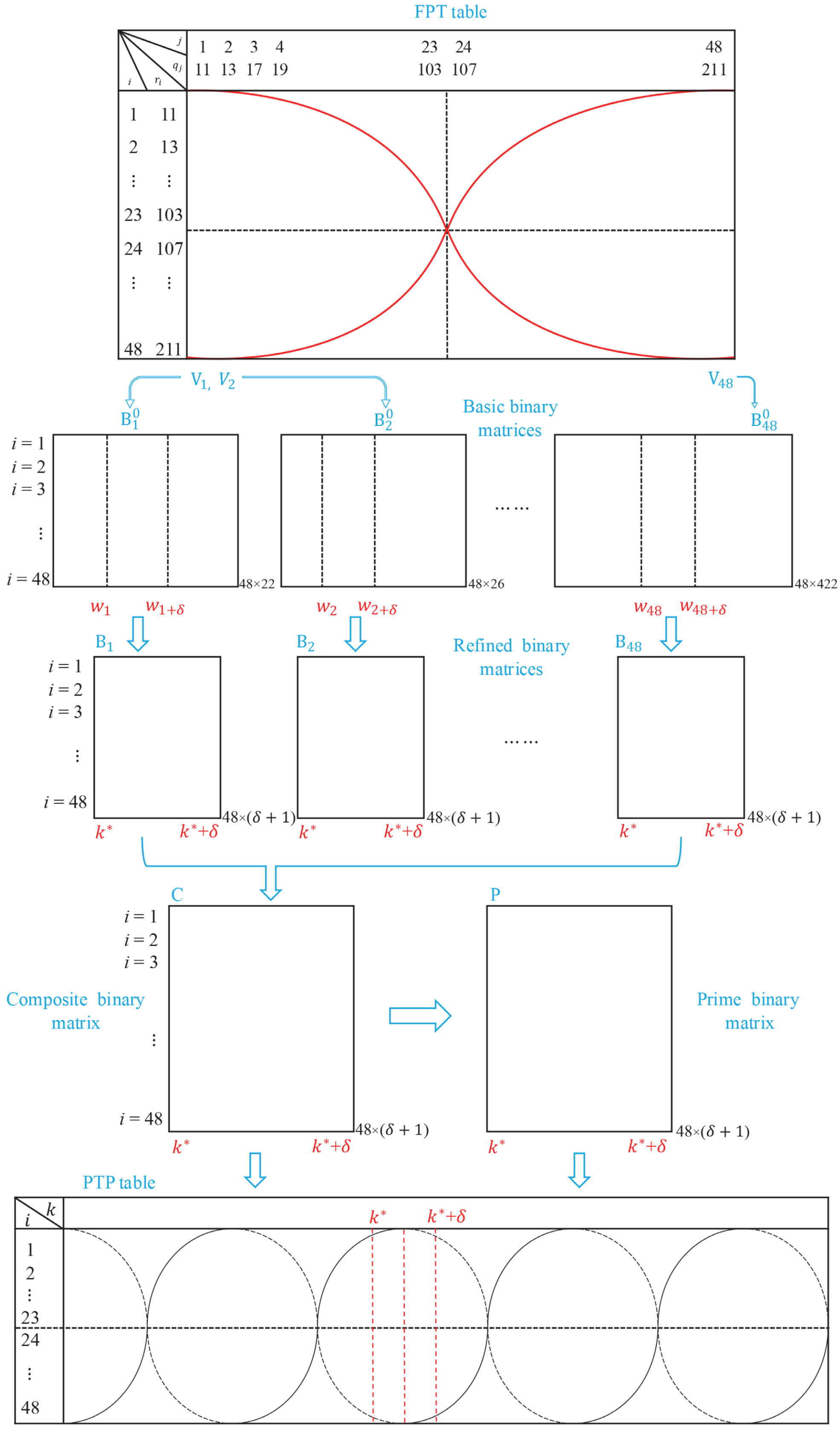


Figure 2. An outline of observations.

Theorem 7. In a pre-specified interval $\left[11+210\times k^{*}, 211+210\times\left(k^{*}+\delta\right)\right]$, where $k^{*}\in N_{+}$, $\delta\in[0,10]_{N}$, any integer number $r_{i}+210k$ is prime if and only if $p(i,k)=1$.

Figure 2 is an outline of our PTP method for listing all primes within the interval $\left[11+210\times k^{*}, 211+210\times\left(k^{*}+\delta\right)\right]$ for $\delta\in[0,10]_{N}$.

We first form a 48×48 Factor-Pair Table with generation information, then convert each column of it into a basic binary matrix B_{j}^{0}. After that, B_{j}^{0} is modified as a refined binary matrix B_{j}. By aggregating $B_{1}, B_{2},\cdots, B_{48}$, and testing the primality of the rest numbers, we obtain a composite binary matrix C. Based on $C=\left[c(i,k)\right]_{48\times(\delta+1)}$, we have a prime binary matrix $P=\left[p(i,k)\right]_{48\times(\delta+1)}$. The primes within the interval are listed as $\alpha=r_{i}+210k$ for $k\in\left\{k^{*}, k^{*}+1,\cdots,k^{*}+\delta\right\}$ such that $p(i,k)=1$ in the prime binary matrix P.

Moreover, all composites (without factors of 2, 3, 5 and 7) within the same interval can be predicted. Both primes and composites are illustrated on a PTP table.

4. Results

4.1. Formula of Primes

From Theorem 7, we have the following Formula of Primes:

Formula of Primes. All prime numbers within an integer interval $\left[11+210\times k^{*}, 211+210\times\left(k^{*}+\delta\right)\right]$, where $k^{*}\in N_{+}$, $\delta\in[0,10]_{N}$, have the form of $\alpha=r_{i}+210k$, where $k\in\left\{k^{*}, k^{*}+1,\cdots,k^{*}+\delta\right\}$ and $i\in\{1,2,\cdots,48\}$, such that $p(i,k)=1$ in the prime binary matrix P.

4.2. Algorithm of Prime Listing

Finally, an algorithm for listing all primes in a pre-specified interval is presented below.

Algorithm 3:

Part A: Data Preparation

Firstly, we need to construct a 48×48 Factor-Pair Table with generation information, $\left(V_{1}, V_{2},\cdots,V_{48}\right)$. Then we can convert each V_{j} into a basic binary matrix $B_{j}^{0}=\left(b_{j}^{0}(i,k)\right)_{48\times 2r_{j}}$, for $i=1,2,\cdots,48$ and $k=0,1,2,\cdots,2r_{j}-1$.

Part B: Prime Prediction (Given $k^{*}\in N_{+}$, $\delta\in[0,10]_{N}$)

Step 1: Generate refined binary matrices $B_{j}=\left(b_{j}(i,k)\right)_{48\times(\delta+1)}$ for $j=1,2,\cdots,48$ based on B_{j}^{0}.

Step 2: Generate a composite binary matrix $C=\left(c(i,k)\right)_{48\times(\delta+1)}$ with Algorithm 2.

Step 3: Generate a prime binary matrix $P=\left(p(i,k)\right)_{48\times(\delta+1)}$ based on C.

Step 4: The prime numbers within the interval $\left[11+210\times k^{*}, 211+210\times\left(k^{*}+\delta\right)\right]$ can be listed as

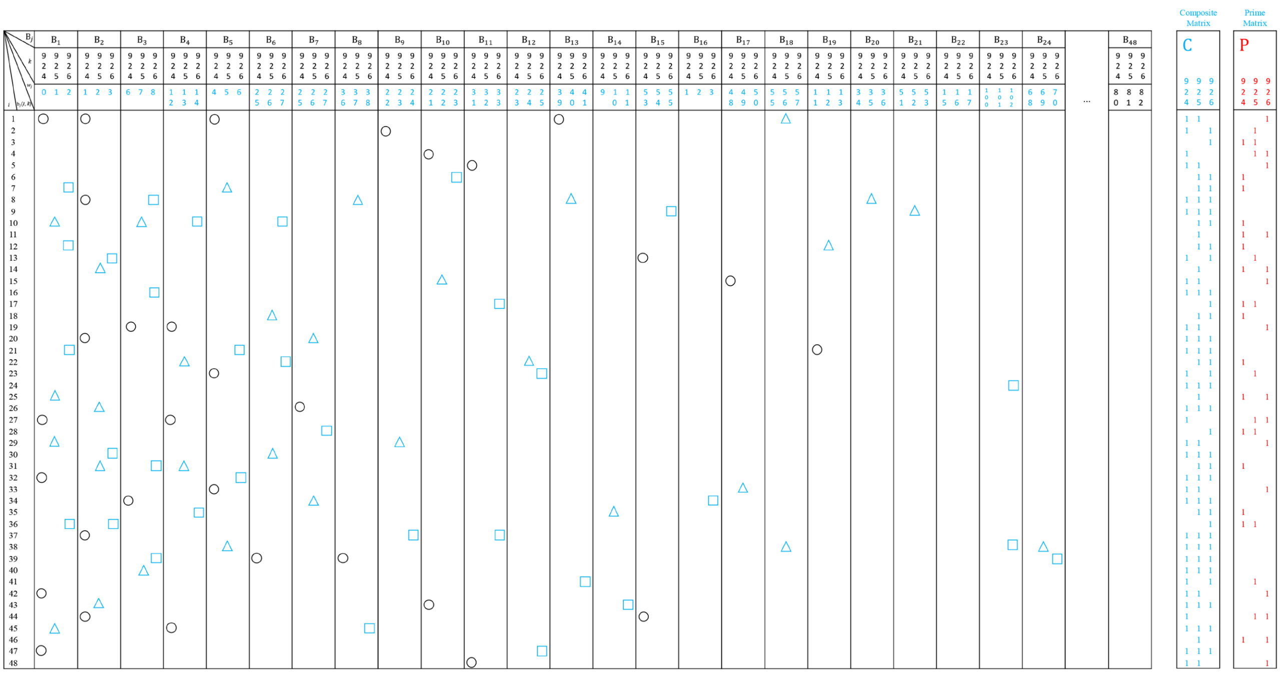
$$\left\{\alpha=r_{i}+210k \mid p(i,k)=1, i=1,2,\cdots,48, k=k^{*}, k^{*}+1,\cdots,k^{*}+\delta\right\}.$$

In the above algorithm, Part A and Step 1, Step 3 and Step 4 in Part B require constant time complexity. Step 2 in Part B involves a few primality tests, whose complexity can be reduced by choosing a more efficient primality test algorithm.

An example of Algorithm 3 can be found in Appendix A.

The procedure of Part B is presented in **Figure 3**. The top row shows $B_1, B_2, \cdots, B_{48}, C$ and P, respectively. The second row is for $k^* = 924$, $k^* + 1 = 925$ and $k^* + 2 = 926$. The third row is for $(\delta_j, \delta_j + 1, \delta_j + 2)$, where $j = 1, 2, \cdots, 48$. The $b_j(i, k)$ entry denotes the binary value for given B_j, r_i and k.

A table of listing all primes in $[11 + 210 \times 922, 211 + 210 \times 928]$ is shown in **Table 2**.



*○, △ and □ represent positions where $b_j(i, k) = 1$ in the table.

Figure 3. Process of predicting primes in [11 + 210 × 924, 211 + 210 × 926].

Table 2. Prediction of Primes within [11 + 210 × 922, 211 + 210 × 928].

i	r_i \ k	924	925	926
1	11			194,471
2	13		194,263	
3	17	194,057	194,267	
4	19		194,269	194,479
5	23			194,483
6	29	194,069		

Continued

7	**31**	194,071		
8	**37**			
9	**41**			
10	**43**	194,083		
11	**47**	194,087		194,507
12	**53**	194,093		
13	**59**		194,309	
14	**61**	194,101		194,521
15	**67**			194,527
16	**71**			
17	**73**	194,113	194,323	
18	**79**	194,119		
19	**83**			194,543
20	**89**			
21	**97**			
22	**101**	194,141		
23	**103**		194,353	
24	**107**			
25	**109**	194,149		194,569
26	**113**			
27	**121**		194,371	194,581
28	**127**	194,167	194,377	
29	**131**			194,591
30	**137**			
31	**139**	194,179		
32	**143**			
33	**149**			194,609
34	**151**			
35	**157**	194,197		
36	**163**	194,203	194,413	
37	**167**			
38	**169**			

Continued

39	**173**			
40	**179**			
41	**181**		194,431	
42	**187**			194,647
43	**191**			
44	**193**		194,443	194,653
45	**197**			
46	**199**	194,239		194,659
47	**209**			
48	**211**			194,671

4.3. The Periodic Table of Primes

The Periodic Table of Primes (PTP) is shown in **Figure 4**. Reconstructed from our prime listing algorithm, it illustrates that composites (without factors of 2, 3, 5 and 7) are distributed cyclically and primes are distributed periodically.

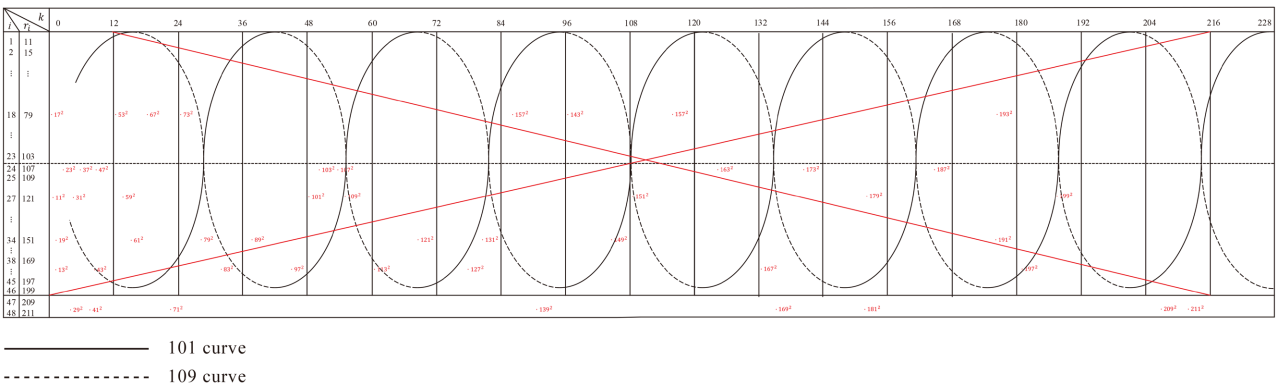


Figure 4. PTP [11, 211 + 210²].

5. Discussion

In this paper, we propose a novel way to list primes. Given an interval, without loss of generality, we first restrict the range of numbers to those having the form of $\alpha = r_i + 210k$. Then, by mapping the effect, we further eliminate a good amount of numbers and conduct primality checking on the rest of the numbers. Subsequently, we list all primes in the given interval.

Some further discussions are as follows:

1) With theoretical analysis and computer validation (in Appendix B), we can, on average, identify about 26 composite numbers in a column. This demonstrates that the proposed method largely reduces the number of primality checks required in the prime listing task over a pre-specified interval with simple mappings. The

details are found in Appendix B.

2) In practice, we can use bitwise operations to speed up the algorithm. We first encode each column of the basic binary matrices into a binary number, and the same applies to the columns of the refined binary matrices. Then, instead of checking individual entries, we perform the bitwise operation "OR" on all columns in the refined binary matrix corresponding to a certain column. With a significant improvement in efficiency, the resulting value implies the composites located in this column by Step 1 of Algorithm 2.

3) In this paper and all associated demonstrations, we consider composites without factors of 2, 3, 5 and 7 to generate a Factor-Pair Table which gives way to a designated and graphics-manageable period of $2\times 3\times 5\times 7=210$. To improve efficiency and to list a larger number of primes, we may consider composites without factors of 2, 3, 5, 7 and 11 which represent a larger period of $2\times 3\times 5\times 7\times 11=2310$. Further extension is also possible for even larger dimension by adding primes 13, 17, …

4) The proposed algorithm is obviously capable of listing prime numbers within a specific interval starting from a very large number. Unlike the conventional methods, the proposed algorithm does not require prior knowledge of smaller prime numbers. Furthermore, the mapping process has constant time and small space complexity, ensuring high efficiency.

5) We are investigating the application of the proposed method in various domains, such as cryptography-security, hash-data-base-design and DNA-sequence-identify.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A: An Example of Algorithm 3

Example: (List all primes within the interval $[11+210\times 924, 211+210\times 926]$)

Part A:

The Factor-Pair Table with generation information is shown in **Table 1** and the basic binary matrices B_j^0, $j=1,2,\cdots,48$, are converted accordingly.

Part B:

Step 1: Generate refined binary matrices $B_j=\left(b_j(i,k)\right)_{48\times 3}$ for $j=1,2,\cdots,48$ based on B_j^0.

Take $j=1$ for example, $B_1=\left(b_1(i,k)\right)_{48\times 3}$, where $i=1,2,\cdots,48$ and $k=924,925,926$, and $w_1=924-11\times\lfloor 924/11\rfloor=0$. From Theorem 5, $b_1(i,924+h)=b_1^0(i,w_1+h)=b_1^0(i,h)$. From B_1^0 in **Figure 1**, we know that

$$\because b_1^0(1,0)=b_1^0(27,0)=b_1^0(32,0)=b_1^0(42,0)=b_1^0(47,0)=1,$$

$$\therefore b_1(1,924)=b_1(27,924)=b_1(32,924)=b_1(42,924)=b_1(47,924)=1.$$

Similarly,

$$\because b_1^0(10,1)=b_1^0(25,1)=b_1^0(29,1)=b_1^0(45,1)=1,$$

$$\therefore b_1(10,925)=b_1(25,925)=b_1(29,925)=b_1(45,925)=1.$$

$$\because b_1^0(7,2)=b_1^0(12,2)=b_1^0(21,2)=b_1^0(36,2)=1,$$

$$\therefore b_1(7,926)=b_1(12,926)=b_1(21,926)=b_1(36,926)=1.$$

Step 2: Generate a composite binary matrix $C=\left(c(i,k)\right)_{48\times 3}$ with Algorithm 2.

Check $b_j(i,k)$ for all $j\in\{1,2,\cdots,48\}$, $i\in\{1,2,\cdots,48\}$ and $k\in\{924,925,926\}$.

For $i\in\{1,2,4,5,13,15,19,20,21,23,24,26,27,32,33,34,38,39,41,42,43,44,45,47\}$, set $c(i,924)=1$;

for other i, we run primality test and set $c(i,924)=1$ for $i\in\{8,9,16,29,30,37,40,48\}$.

For $i\in\{3,6,7,10,11,12,14,17,18,22,25,28,31,35,36,46\}$, set $c(i,924)=0$.

For $i\in\{1,5,6,7,8,9,10,12,14,15,18,20,21,22,26,29,30,31,33,34,35,38,39,40,42,43,45,47\}$, set $c(i,925)=1$;

for other i, we run primality test and set $c(i,925)=1$ for $i\in\{11,16,19,24,25,32,37,46,48\}$.

For $i\in\{2,3,4,13,17,23,27,28,36,41,44\}$, set $c(i,925)=0$.

For $i\in\{2,6,7,8,9,10,12,13,16,17,21,22,23,24,26,28,30,31,32,34,35,36,37,38,39,40,41,43,47\}$, set $c(i,926)=1$;

for other i, we run primality test and set $c(i,926)=1$ for $i\in\{3,18,20,45\}$.

For $i\in\{1,4,5,11,14,15,19,25,27,29,33,42,44,46,48\}$, set $c(i,926)=0$.

Step 3: Generate a prime binary matrix $P=\left(p(i,k)\right)_{48\times 3}$ based on C.

For $i\in\{1,2,\cdots,48\}$ and $k\in\{924,925,926\}$, set $p(i,k)=1-c(i,k)$.

Step 4: The prime numbers within the interval $[11+210\times 924, 211+210\times 926]$ can be listed as

$$\{\alpha = r_i + 210k \mid p(i,k) = 1, i = 1, 2, \cdots, 48, k = 924, 925, 926\}$$
$$= \{194471, 194263, 194057, 194267, 194269, 194479, 194483, 194069, 194071, 194083, 194087, 194507, 194093, 194309, 194101, 194521, 194527, 194113, 194323, 194119, 194543, 194141, 194353, 194149, 194569, 194371, 194581, 194167, 194377, 194591, 194179, 194609, 194197, 194203, 194413, 194431, 194647, 194443, 194653, 194239, 194659, 194671\}$$

Appendix B: Theoretical Analysis and Computer Validation of Algorithm 2

The formulation of the Composite Binary Matrix C is the most vital step in our proposed prime listing algorithm, and Step 1 in Algorithm 2 locates a good portion of composites in a pre-specified interval with constant time complexity. In the following analysis, we will conduct a comprehensive computation to see the expectation of the composites located in a column by Step 1. It is more reasonable to count all composites within the period than to analyze the performance of each column, as the latter is too time-consuming to study.

After conducting Step 1, the entries of column $c(i,k^*)$ are either 1 or 0. If $c(i,k_1) - c(i,k_2) = 0$ for $i \in \{1, 2, \cdots, 48\}$, we call the "result" of columns k_1 and k_2 identical. To analyze the effect of Step 1, we first elaborate that the "results" of Step 1 appear cyclically in the columns.

Theorem 5 implies that in a certain column k of matrix C, at which the entries are located as composite by Step 1 is determined by the result of

$$w_1 = k - r_1 \frac{k^*}{r_1}, w_2 = k - r_2 \frac{k^*}{r_2}, \cdots, w_{48} = k - r_{48} \frac{k^*}{r_{48}}, \tag{11}$$

which is identical to

$$k \equiv w_j \pmod{r_j}, w_j \in \{0, 1, \cdots, r_j - 1\}, j \in \{1, 2, \cdots, 48\} \tag{12}$$

in modular arithmetic. If there exists j such that $b_j^0(i, w_j) = 1$, then $b_j(i,k) = 1$, $c(i,k) = 1$ and $r_i + 210 \times k$ is a composite number.

We claim that there are infinite k's satisfying Equation (11) and they are of the same cyclic form. Denoting L as the least common multiple of $r_1, r_2, \cdots, r_{48}$, we have L = 1, 120, 700, 227, 446, 262, 311, 648, 514, 688, 739, 473, 686, 196, 699, 653, 055, 999, 250, 791, 866, 994, 591, 464, 755, 146, 178, 654, 953. Given $w_1, w_2, \cdots, w_{48}$, the Chinese Remainder Theorem says that there's a unique solution k^*, if any, in $[1, L]_N$ for Equation (11) and the general solution can be expressed as $k^* + L \times t$, where $t \in N_0$.

With the cyclic behavior detected, we can now limit our attention to the case where $k \in [1, L]_N$. However, the period L is too large to be counted directly, so we need some further analysis.

We should notice that for any $i \in \{1, 2, \cdots, 48\}$, there's one and only one w_j in $\{0, 1, \cdots, r_j - 1\}$ such that $b_j^0(i, w_j) = 1$, indicating $r_i + 210 \times k$ is composite. We denote this w_j as d_{ij} in the following discussion.

To avoid duplicated computation induced by the 5 composite numbers in S, we divide the set S into $S_1=\{11,13,143,187,209\}$, $S_2=\{17,19,121,169\}$ and $S_3=S\setminus(S_1\cup S_2)$. Accordingly, the index set $I=\{1,2,\cdots,48\}$ is divided into $I_1=\{1,2,32,42,47\}$, $I_2=\{3,4,27,38\}$ and $I_3=I\setminus(I_1\cup I_2)$.

The following Corollary demonstrates that $\{w_j \mid j\in I_2\cup I_3\}$ can uniquely determine a solution in $[1,L]_N$ for Equation (11). The reason is that $\{w_j \mid j\in I_1\}$ can be directly calculated given $\{w_j \mid j\in I_2\}$.

Corollary of the Chinese Remainder Theorem. Given $w_1,w_2,\cdots,w_{48}$, Equation (11) has a solution in $[1,L]_N$ if and only if

$$\begin{cases} w_1 \equiv w_{27} \pmod{11} \\ w_2 \equiv w_{38} \pmod{13} \\ w_{32} \equiv 78w_1+66w_2 \pmod{143} \\ w_{42} \equiv 34w_1+154w_3 \pmod{187} \\ w_{47} \equiv 133w_1+77w_4 \pmod{209} \end{cases} \tag{13}$$

and

$$k \equiv w_j \pmod{r_j}, w_j\in\{0,1,\cdots,r_j-1\}, j\in I_2\cup I_3 \tag{14}$$

has a solution in $[1,L]_N$.

The above Corollary is derived from the Chinese Remainder Theorem and can be easily verified. With this theorem, we can count how many composites are located by Step 1 in columns 1 to L.

Without loss of generality, we first focus on $i=1$.

Since each column in $[1,L]_N$ can be uniquely determined by $\{w_j \mid j\in I_2\cup I_3\}$, we first consider $\{w_j \mid j\in I_2\}=\{w_3,w_4,w_{27},w_{38}\}$. Notice that $\{w_j \mid j\in I_1\}$ is determined by Equation (13) for any fixed $\{w_j \mid j\in I_2\}$.

If there exists at least a $j\in I_2$ such that $w_j=d_{1j}$, then $r_i+210\times k$ is a composite number no matter the value of w_j, $j\in I_3$. The number of $\{w_3,w_4,w_{27},w_{38}\}$ satisfying this case is $r_3r_4r_{27}r_{38}-(r_3-1)(r_4-1)(r_{27}-1)(r_{38}-1)=798947$. In this case, $w_j, j\in I_3$ can be chosen arbitrarily from the domain. Therefore, there are $\prod_{j\in I_3} r_j$ composites located for each such $\{w_3,w_4,w_{27},w_{38}\}$ in $[1,L]_N$.

If $w_j\neq d_{1j}$ for any $j\in I_2$ but there exists at least a $j\in I_1$ such that $w_j=d_{1j}$, there are also $\prod_{j\in I_3} r_j$ composites located for each such $\{w_3,w_4,w_{27},w_{38}\}$. Here $w_j, j\in I_1$ is computed by Equation (13). The number of $\{w_3,w_4,w_{27},w_{38}\}$ satisfying this case is 864000 (Counted by computer, this value is the same when $i=2,3,\cdots,48$).

If $w_j\neq d_{1j}$ for any $j\in I_1\cup I_2$, then a composite number is located if and only
if there exists $w_j=d_{1j}$ for some $j\in I_3$. Therefore, there are $\prod_{j\in I_3} r_j-\prod_{j\in I_3}(r_j-1)$ composites located in $[1,L]_N$ for each such $\{w_3,w_4,w_{27},w_{38}\}$. The number of

$\{w_3, w_4, w_{27}, w_{38}\}$ satisfying this case is 4942080.

When $i = 2, 3, \cdots, 48$, the number of composites located by Step 1 in columns $[1, L]_N$ is the same as $i = 1$. Since

$$\prod_{j \in I_3} r_j = 169,673,829,864,171,987,737,296,863,243,628,479,671,120,141,228,$$
$$188,658,546,265,896,353,105,711,020,739$$

and

$$\prod_{j \in I_3} (r_j - 1) = 102,585,912,060,381,157,597,756,546,764,390,409,686,966,581,$$
$$524,267,485,974,696,558,592,000,000,000,000,$$

the total number of composites located by Step 1 in columns $[1, L]_N$ is

$$T = 48 \times \left[798947 \prod_{j \in I_3} r_j + 864000 \prod_{j \in I_3} r_j + 4942080 \left(\prod_{j \in I_3} r_j - \prod_{j \in I_3} (r_j - 1) \right) \right)$$
$$= 29,458,197,272,202,902,414,774,112,677,093,526,253,964,920,$$
$$792,154,274,897,889,854,062,645,442,967,016,575,437,744$$

The average number of composites located by Step 1 in a PTP's column is $T/L = 26.285528057158732$ in the first period $[1, L]_N$ and can be regarded as the overall expectation of the number of composites located by Step 1 in a PTP's column, which is over $26.28/48 = 54.75\%$ of the numbers in the column.

To further validate this result, the number of composites located by Step 1 in PTP's columns 1 to 10^{12} is on average 26.285525173295, which is quite close to our theoretical result.

The theoretical analysis and numerical experiment above illustrate that the proposed method largely reduces the number of primality checks in the prime listing task over a pre-specified interval.