Advances in Pure Mathematics, 2025, 15(3), 183-207
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Infinity of the Number of Pairs of Equidistant Primes around Any Integer over Three

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How to cite this paper: Doroszlai, P. and Keller, H. (2025) The Infinity of the Number of Pairs of Equidistant Primes around Any Integer over Three. *Advances in Pure Mathematics*, **15**, 183-207.
https://doi.org/10.4236/apm.2025.153008

Received: November 29, 2024
Accepted: March 4, 2025
Published: March 7, 2025

Abstract

From Goldbach's conjecture to Goldbach's law, the present proof is based on reflecting the series of the primes over any integer, giving the double density of occupation of integer positions. The remaining free positions are diads, equidistant primes to the point of reflection. Based on the symmetry due to the reflection different proofs are given on Goldbach's conjecture. One of the proofs requires the asymptotically exact number of primes up to a growing number of integers. Based on the prime-number-formula a correction as the best estimating function of the number of primes, the complete-prime-number-formula is evaluated. Similarly to the primes, a diads-number formula is evaluated. The dispersion of the effective number of primes and diads around the corresponding best estimation functions allows us to prove the asymptotic continuity of both functions. The asymptotic continuity of the best estimate functions the prime-number-formula and the diads-number formula may be proved as low limit functions of the effective number of primes and diads. This gives the first proof of Goldbach's conjecture. Two others follow. The theoretical evaluation is followed in annexes by numerical evaluation, demonstrating the theoretical results. The numerical evaluation results in different constants and relations, which represent inherent properties of the set of primes.

Keywords

Prime-Number-Formula, Complete-Prime-Number-Formula, Diads-Number-Formula, Goldbach's Conjecture, Twin Primes

1. Introduction

Goldbach's conjecture may be formulated stating that **any even greater than**

three has at least one pair of equidistant primes around it. To prove this conjecture, the following steps are shortly repeated, with references to more detailed evaluations.

The density of primes over the positive integers as distance is given with the fundamental identity of Riemann, by the identity of Euler as the Euler product. The integral of the density gives the number of primes at any distance. A first approximation of the integral of the density of primes by summation has an error. This error is compensated by a recursive formula, resulting in the complete-prime-number-formula (see Ref. [1]). A further simplification of the summation formula results in the prime-number-formula.

Next the double density of occupation by the straight and over an integer point reflected union of the series of multiples of the primes is defined (see Ref. [2]). Applying the same method of the prime-number-formula on the double density of occupation gives the diads-number-formula. Because of the symmetry of the double density of occupation, the remaining free positions around the point of reflection compose diads, equidistant primes to the point of reflection and there is no error to be compensated by a recursive formula.

The dispersion of the number of primes and of diads around their best estimate functions is evaluated. Both are proportional to the number of primes present to the square root of the distance of the point of reflection. Divided by this value, the standard deviation of the resulting relative values remains constant over the distance in both cases; by the primes and by the diads. But divided by the distance both relative dispersions are approaching zero. Thus, both functions are becoming smooth with growing distance.

The difference between the best estimate number of the primes and the prime-number-formula is the error of this last function. Similarly, the error of the diads-number-formula is the difference between the best estimate function of the diads and the diads-number-formula. Both errors are proportional to the power of nearly two of the number of primes present up to the square root of the distance, respectively of the distance of the point of reflection.

Therefore, with the best estimate functions approaching smooth functions, the prime-number-formula respectively the diads-number-formula are low limit functions of the number of primes, respectively of the number of diads. The low limit of the number of primes and of diads rising to infinity is the base of a first proof of Goldbach's conjecture. The share of primes at any distance is proportional to the inverse of the logarithm of the distance. This fact follows from the fundamental identity of Riemann, respectively from the Euler product. But this law is valid not only for the set of all positive integers but for any arithmetic progression as a subset of all integers. This is the base of the second proof of Goldbach's conjecture.

The double density of occupation results from the reflection of the union of the set of multiples of the primes over any integer greater than three. It is not possible to cover all reflected prime positions of the strait union of the multiples of the primes by the reflected union. This is the base of the third proof of Goldbach's conjecture.

2. The Best Estimate Number of Primes and of Diads

The series of multiples, arithmetic progression, of any prime covers the share ($\frac{1}{P_{(n)}}$) of all positive integer positions. The share of positions left free by these multiples is ($1-\frac{1}{P_{(n)}}$). The share of free positions is multiplicative. For all ($S(c)=\frac{c}{\ln(c)}$) primes the density of free positions left is given with the fundamental identity of Riemann for ($s=1$), by the identity of Euler as the Euler product, with the Mertens constant ($\delta_1 = \mathrm{e}^{\gamma}$):

$$\lim_{c\to\infty}\left[\delta_1 \cdot \prod_{j=1}^{S(c)}\left(1-\frac{1}{P_{(j)}}\right)\right]=\frac{1}{\ln(c)} \tag{1}$$

All components of the Euler product are positive and approach unity with the distance.

De la Valée Poussin établi 1899, (Ref. [3]), that the number of primes up to the distance (c) is given by the integral of the logarithmic density, which may be written, as part of a first simplification, as sum over all positive integers:

$$\pi(c)=Li(c)+O(c)\approx\int_2^c \frac{1}{\ln(c)}\cdot \mathrm{d}c\;;\quad \pi_{appr}(c)\approx\sum_{n=2}^{c}\frac{1}{\ln(n)} \tag{2}$$

This summary above may be written as summing up first over all integers within the sections of the length ($\sqrt{c}$) and then summing up over all the ($\sqrt{c}$) sections of the length ($\sqrt{c}$). Taking the average value over each section and summing up over the sections, as part of the first simplification, (see **Annex 2**) gives the **sum over all sections**:

$$\pi_{appr}(c)\approx\sum_{n=2}^{c}\frac{1}{\ln(n)}=\sum_{j=2}^{\sqrt{c}}\left[\sum_{n=(j-1)\cdot\sqrt{c}}^{j\cdot\sqrt{c}}\frac{1}{\ln(n)}\right]\approx\sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)} \tag{3}$$

The well-proven prime-number-formula **PNF** results from a second simplification of the above approximation by taking for each of the sections the smallest value of the density at the end of the sections ($j=\sqrt{c}$):

$$\sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)}>PNF(c)=\sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(\sqrt{c}\cdot\sqrt{c}\right)}=\frac{c}{\ln(c)}=S(c) \tag{4}$$

The systematic error of the sum over all sections may be corrected by recursive application of a correction algorithm, resulting in the function of the best estimate values of the number of primes, the **complete-prime-number-formula (CPNF)** below, evaluated and demonstrated in ref [1]:

$$\pi_{appr(c)} = \sum_{j=1}^{\sqrt{c}} \frac{\sqrt{c}}{\ln\left(j \cdot \sqrt{c}\right)} + \sum_{m=1}^{\sqrt{c}} \left[\left(\gamma_{sec}\right)^m \cdot \sum_{j=1}^{c^{\left(\frac{1}{2^{m+1}}\right)}} \frac{c^{\left(\frac{1}{2^{m+1}}\right)}}{\ln\left[j \cdot c^{\left(\frac{1}{2^{m+1}}\right)}\right]} \right] \tag{5}$$

This formula converges extremely fast: Two steps with ($m = 2$) are already sufficient.

The factor ($\gamma_{sec} = -0.041663$) is an inherent property of the set of primes. Further, this formula gives identical results (see **Annex 2**) as ($Li(c)$):

$$\pi(c) = Li(c) + O(c) = \pi_{appr\_}(c) + O(c); \quad \Delta\pi_{appr}(c) = \pi_{appr}(c) - \pi(c) \tag{6}$$

Reflecting the series of multiples of any prime over a point at the distance (c) from the origin results the double density of occupation of integer positions within the interval ($0 < c < 2 \cdot c$) by the series of multiples of the primes, if all the primes are relative primes to the point of reflection (c) (see left side of **Figure 1**). This is, because the positions covered by the straight and the reflected series of multiples are mutually exclusive for each prime if (c) is equal to a prime, greater than two. In the figure, empty circles mark diads and full circles mark positions covered by multiples of primes.

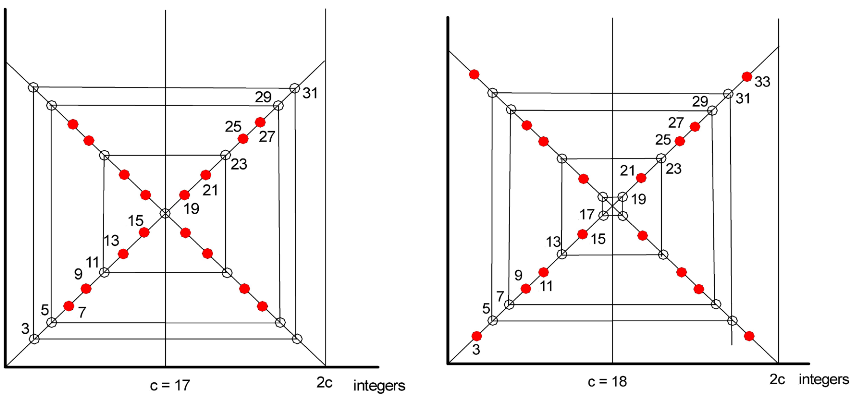

Figure 1. The double density of occupation in case of the point reflection being a prime or an even number.

Again, reflecting the union of the series of multiples of all primes over a point at the distance (c) from the origin results the double density of occupation of integer positions within the interval ($0 < c < 2 \cdot c$) by the union of the series of multiples of the primes (see left side of **Figure 1**). This is, because the positions covered by the straight and by the reflected series of multiples are mutually exclusive for each prime if (c) is equal to a prime greater than three. The integer positions remaining free by the double density of occupation represent equidistant primes to the point of reflection, composing **diads** (see ref. [2]).

The straight and the reflected series of multiples of any prime covers in this case the share ($\frac{2}{P_{(n)}}$) of all positive integer positions. The share of positions left free is ($1 - \frac{2}{P_{(n)}}$). The share of free positions is multiplicative. For all ($S(c) = \frac{1}{\ln(c)}$)

primes, with the constant ($\delta_2 = 1.320324$), the density of free positions left is given herewith again with the corresponding Euler product:

$$\lim_{c\to\infty}\left[\frac{1}{2}\cdot\prod_{n=2}^{S(c)}\left(1-\frac{2}{P_{(n)}}\right)\right]=\delta_2(c)\cdot\lim_{c\to\infty}\left[\frac{1}{2^2}\cdot\prod_{n=2}^{S(c)}\left(1-\frac{1}{P_{(n)}}\right)^2\right];\quad \delta_2=2\cdot\prod_{n=2}^{S_{eff}(\infty)}\left[\frac{1-\frac{2}{P_{(n)}}}{\left(1-\frac{1}{P_{(n)}}\right)^2}\right] \tag{7}$$

All components of the Euler product are positive and the product results in positive values for all distances.

Reflected series of multiples of prime divisors of (c) do not change the occupation of integer positions by such primes: Therefore, the number of remaining free integer positions has a minimum, if the point of reflection is a prime. This is shown on the right side of **Figure 1** and demonstrated in **Annex A2**, **Figures A1-3**, (see Ref. [2]). The quadrics with empty circles represent diads, with primes in all corners. Together with the central prime they are triads.

The prime positions within the interval ($0 < c < 2\cdot c$) reflected over the point of reflection are independent of the union of the series of multiples of primes covering positions within the lower half of the interval ($0 < c < 2\cdot c$): They cover the ($\frac{1}{\ln(c)}$)-th part of the integer positions.

The local density of free positions left by the density of occupation of the straight series of multiples of primes at the distance ($c-d$) from the point of reflection with (d , $1<d<c$) is ($\frac{1}{\ln(d)}$).

By the reflected series it is ($\frac{1}{\ln(2\cdot c-d)}$). The combined local density of free positions, the density of diads, is evaluated in ref. [2], [3] as product of the densities. The constant (7) has double the value of the twin prime constant (C2), defined by G. H. Hardy and John Littlewood, (see Ref. [4]):

$$D_{local}(c,d)=\frac{\delta_2}{\ln(d)\cdot\ln(2\cdot c-d)}>\frac{\delta_2}{\ln(c)^2} \tag{8}$$

Similarly to the evaluation of the number of primes as integral of the local logarithmic density of primes in (2), the number of the diads results as integral of the above density. Again, as part of a first simplification, the integral of the above density we may replace by the sum over all integers (This same generalization from the primes to the twins, respectively to the k-tuples made already by G. H. Hardy and John Littlewood):

$$\pi_{diads\_appr}(c)=\int_0^{x-2}\frac{1}{\ln(c-x)\cdot\ln(c+x)}\mathrm{d}x\approx\sum_{n=1}^{c}\frac{\delta_2}{\ln(c-n)\cdot\ln(c+n)} \tag{9}$$

In cases of the diads, similarly to (3) the sum of the diads (9) we may write as summing up first over all integers within the sections of the length ($\sqrt{c}$) and then summing up over all the ($\sqrt{c}$) sections. Taking the average value over each section, as part of the first simplification, and summing over the sections gives the best estimate value of the number of diads, respectively of the complete diads number formula **CDNF** and taking account of the density of the prime at the point of reflection of the complete triads number formula **CTNF:**

$$\pi_{diads\_appr}(c)=\sum_{j=1}^{\sqrt{c}}\sum_{n=(j-1)\cdot\sqrt{c}}^{j\cdot\sqrt{c}}\frac{\delta_2}{\ln(n)\cdot\ln(2\cdot c-n)}\approx\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-j\cdot\sqrt{c}\right)} \tag{10}$$

$$\pi_{triads\_appr}(c)=\sum_{j=1}^{\sqrt{c}}\sum_{n=(j-1)\cdot\sqrt{c}}^{j\cdot\sqrt{c}}\frac{\delta_2}{\ln(n)\cdot\ln(c)\cdot\ln(2\cdot c-n)}\approx\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln(c)\cdot\ln\left(2\cdot c-j\cdot\sqrt{c}\right)}$$

The dispersion of the effective number of diads around this function is symmetrical (see **Annex 3**), there is no correction of this function required, like at **CPNF**. The reason for this is the symmetricity of the set of diads.

Similarly, to the second simplification in cases of the primes in (4), the low limit of the best estimated number of diads results the **diads-number-formula (DNF)**, if the density is taken for all sections of the length ($\sqrt{c}$) at the upper limit of the sections at ($j=\sqrt{c}$):

$$\pi_{diads\_appr\_low}(c)=\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(\sqrt{c}\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-\sqrt{c}\cdot\sqrt{c}\right)}=\frac{\delta_2\cdot c}{\ln(c)^2} \tag{11}$$

3. The Dispersion around the Best Estimate Number of Primes and Diads

The remaining error of the number of primes is the symmetric dispersion around the **CPNF** (see (53) and **Figure A4**). It is proportional to the number of primes up to ($R(c)$), the series of multiples which are covering positions at (c). This is because each series of multiples of the first ($R(c)$) primes may contribute just one uncertainty to the number of primes at the position (c). The dispersion is additive, therefore the dispersion around the CDNF, in case of the double density of occupation, is proportional to ($R(c)$) as well:

$$\Delta\pi(c)=\pi_{appr}(c)-\pi(c);\quad \Delta\pi_{rel}(c)=\frac{\Delta\pi(c)}{R(c)} \tag{12}$$

$$\Delta\pi_{triads}(c)=\pi_{triads}(c)-\pi_{diads\_appr}(c);\quad \Delta\pi_{triads\_rel}(c)=\frac{\Delta\pi_{triads}(c)}{R(c)}$$

The standard deviation of the dispersion relative to ($R(c)=\frac{\sqrt{c}}{\ln\left(\sqrt{c}\right)}$) is approaching a constant value in both cases, see **Annex 3**, (53) and (58).

$$\lim_{c\to\infty}SD_{\Delta\pi\_rel}(c)=\lim_{c\to\infty}\sqrt{\frac{1}{c}\cdot\left[\sum_{c}\Delta\pi_{rel}(c)^2\right]}=F_{SD\_\Delta\pi}=0.160989 \tag{13}$$

$$\lim_{c\to\infty} SD_{\Delta\pi\_triads\_rel}(c) = \lim_{c\to\infty}\sqrt{\frac{1}{c}\cdot\left[\sum_c \Delta\pi_{triads\_rel}(c)^2\right]} = F_{SD\_\Delta\pi\_triads} = 0.16089$$

Therefore, the standard deviation of the relative dispersion of the diads approaches a constant value, similarly to the standard deviation of the dispersion of the effective number of the primes around the CPNF. The error term is limited and approaches zero.

$$SD_{\Delta\pi\_appr}(c) = \left(F_{SD\_\Delta\pi} \pm \Delta_{SD\_\pi}\right)\cdot R(c):\ \ \Delta_{SD\_\pi} < 0.02;\ \lim_{c\to\infty}\Delta\pi_{appr}(c) = O(c) = 0 \quad (14)$$

$$SD_{\Delta\pi\_triads\_appr}(c) = \left(F_{SD\_\Delta\pi\_triads} \pm \Delta_{SD\_\pi\_triads}\right)\cdot R(c);\ \ \Delta_{SD\_\pi\_triads} < 0.02;$$

$$\lim_{c\to\infty}\Delta\pi_{triads\_appr}(c) = O(c) = 0$$

4. The Prime-Number-Formula and the Diads-Number-Formula as Low Limits

The well-known and proven prime-number-formula PNF as well as the diads-number-formula DNF results from the sum over all sections formula (3) respectively (9), by replacing the local density of primes respectively diads by the density at the upper limit of the sections, at ($j = \sqrt{c}$). The error of each section due to this approximation is ($\frac{\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)} - \frac{\sqrt{c}}{\ln(c)}$) respectively

($\frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c - j\cdot\sqrt{c}\right)} - \frac{\delta_2\cdot\sqrt{c}}{\ln(c)^2}$). The total error is:

$$\sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)} - \frac{c}{\ln(c)} \quad \text{respectively} \quad \sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c - j\cdot\sqrt{c}\right)} - \frac{\delta_2\cdot c}{\ln(c)^2} \quad (15)$$

The difference between the sum over all sections of the number of primes and the value resulting from the **PNF** (see **Annex 4**), is proportional to the power near two of ($R(c)$), the near square of the number of primes up to the distance ($\sqrt{c}$). With (4) gives:

$$\sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)} - \sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(\sqrt{c}\cdot\sqrt{c}\right)} = \sum_{j=1}^{\sqrt{c}}\frac{\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)} - S(c) = \gamma_p\cdot R(c)^{\varepsilon_p} \quad (16)$$

From the two parameters (ε_p) and (γ_p) the first one, the exponent (ε_p), is evaluated in annex 4 on the condition, that the value of the second one (γ_p) remains, on average, constant over increasing distance. The second one is (γ_p) quickly and asymptotically converges with a recursive formula for a constant value:

$$\begin{aligned}\gamma_p &= \frac{\sqrt{c}}{\ln(c)\cdot R(c)^{\varepsilon_p}}\cdot\left(\sum_{j=1}^{\sqrt{c}}\frac{\ln(c)}{\ln\left(j\cdot\sqrt{c}\right)} - \sqrt{c}\right)\\ &= \frac{1}{2\cdot R(c)^{\varepsilon_p - 1}}\cdot\left(\sum_{j=1}^{\sqrt{c}}\frac{\ln(c)}{\ln\left(j\cdot\sqrt{c}\right)} - \sqrt{c}\right)\end{aligned} \quad (17)$$

The resulting error of the PNF is: ($\pi_{appr}(c)-\frac{c}{\ln(c)}=\gamma_p\cdot R(c)^{\varepsilon_p}$, $\gamma_p=0.32073756$), $\varepsilon_p=1.9816424$.

Thus, the error of the second simplification resulting in the **PNF** at the distance (c) is proportional to ($R(c)^{\varepsilon_p}$), close to the square of the number of primes present up to ($\sqrt{c}$).

The relation between the error of the PNF at (c) and the near square power of the number of primes up to ($\sqrt{c}$) is invariant. The power value (ε_p) and the constant factor of proportionality (γ_p) are inherent proprieties of the number of primes.

Herewith the difference between the CPNF and the PNF, the error of the PNF, is growing proportional to ($R(c)^{\varepsilon_p}$) while the width of the dispersion around the CPNF grows proportional to ($R(c)$). Therefore, the relation of the width of the dispersion around the CPNF to the error of the PNF is approaching zero with the distance (c) growing to infinity (see **Annex 4**):

$$\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi}\cdot R(c)}{\gamma_p\cdot R(c)^{\varepsilon_p}}\right)=\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi}}{\gamma_p\cdot R(c)^{\varepsilon_p-1}}\right)<\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi}}{\gamma_p\cdot\sqrt{R(c)}}\right)=0 \quad (18)$$

Herewith the PNF represents the low limit of the number of primes. The fact that the dispersion of the effective number of primes around the CPNF relative to the PNF approaches zero with increasing distance proves that the PNF represents the low limit of the number of primes and indicates that the set of primes is approaching continuity:

$$\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi}\cdot R(c)}{\pi_{appr\_low}(c)}\right)=\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi}\cdot\frac{\sqrt{c}}{\ln(\sqrt{c})}}{\frac{c}{\ln(c)}}\right)=\lim_{c\to\infty}\left(\frac{2\cdot F_{SD\_\Delta\pi}}{\sqrt{c}}\right)=0 \quad (19)$$

Therefore, PNF represents for each distance a monotonously to infinity growing low limit function.

Similarly, the difference between the number of diads CDNF and the value resulting from the DNF, is proportional to ($R(c)^{\varepsilon_d}$), the near square power of the number of primes up to the distance ($\sqrt{c}$).

Again, from the two parameters (ε_d) and (γ_d) the first one, the exponent ($\varepsilon_d=1.8350218$) is evaluated in **Annex 4** on the condition that the value of the second one remains, on average, constant over increasing distance. This second one quickly and asymptotically converges with a recursive formula to a constant value ($\gamma_d=0.04073037$), as shown in **Annex 4**.

$$\begin{aligned}&\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-j\cdot\sqrt{c}\right)}-\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(\sqrt{c}\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-\sqrt{c}\cdot\sqrt{c}\right)}\\&=\sum_{j=1}^{\sqrt{c}}\frac{\delta_2\cdot\sqrt{c}}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c-j\cdot\sqrt{c}\right)}-\frac{\delta_2\cdot c}{\ln(c)^2}\end{aligned} \quad (20)$$

Therefore, the error of the DNF is:

$$\pi_{diads\_appr}(c) - \frac{\delta_2 \cdot c}{\ln(c)^2} = \gamma_d \cdot R(c)^{\varepsilon_d} \tag{21}$$

Herewith the difference between the CDNF and the DNF, the error of the DNF, is growing proportional to ($R(c)^{\varepsilon_d}$). The width of the dispersion around the CDNF grows proportional to ($R(c)$). Therefore, the relation of the width of the dispersion around the CDNF, to the error of the DNF is approaching zero with growing distance (c):

$$\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi\_diads} \cdot R(c)}{\gamma_d \cdot R(c)^{\varepsilon_d}}\right) = \lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi\_diads}}{\gamma_d \cdot R(c)^{\varepsilon_d - 1}}\right) = 0 \tag{22}$$

As well as the relation of the error of the DNF to the number of diads:

$$\begin{aligned}\lim_{c\to\infty}\left(\frac{\gamma_d \cdot R(c)^{\varepsilon_d} \cdot \ln(c)^2}{\delta_2 \cdot c}\right) &= \lim_{c\to\infty}\left(\frac{\gamma_d \cdot \ln(c)^{2-\varepsilon_d} \cdot 2^{\varepsilon_d}}{\delta_2 \cdot c^{1-\frac{\varepsilon_d}{2}}}\right) \\ &= \lim_{c\to\infty}\left(\frac{\gamma_d \cdot \ln(c)^{0.164} \cdot 2^{1.836}}{\delta_2 \cdot c^{0.082}}\right) = 0\end{aligned} \tag{23}$$

5. The Infinity of the Number of Diads

First, we must consider the following statement: The best estimate values of the diads, the dispersion of the effective values around the best estimate values and the low limit functions evaluated with the same method as the prime-number-formula. Because of the well proven PNF for the set of primes, the DNF and the other results for the diads we may consider as proven on the same right.

The infinity of the number of diads around any integer which is rising to infinity proven additionally in the following in diverse ways for integers to rise to infinity. First, we formulate the following conjecture:

Conjecture:

Any integer over three has at least one pair of equidistant primes around it.

Proof 1: A first proof is based on the fact, that diads-number-formula is the low limit of the number of diads, like the prime-number-formula being the low limit of the number of primes. To prove this, we formulate the following lemma:

Lemma 1: The number of primes and the number of triads are approaching smooth analytical functions.

Proof: Taking account of the constancy of the relative dispersion of the number of primes around CPNF (5), the dispersion of the number of primes around CPNF (5) relative to the PNF approaches zero with growing distances:

$$\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi} \cdot R(c)}{\pi_{appr\_low}(c)}\right) = \lim_{c\to\infty}\left(F_{SD\_\Delta\pi} \cdot \frac{\sqrt{c} \cdot \ln(c)}{\ln(\sqrt{c}) \cdot c}\right) = \lim_{c\to\infty}\left(\frac{2 \cdot F_{SD\_\Delta\pi}}{\sqrt{c}}\right) = 0$$

Similarly, taking account of the constancy of the relative dispersion of the number of diads around CDNF (10), the dispersion of the diads around CDNF (10) relative to the DNF approaches zero with growing distances:

$$\lim_{c\to\infty}\left(\frac{F_{SD\_\Delta\pi\_diads}\cdot R(c)}{\pi_{diads\_appr\_low}(c)}\right)=\lim_{c\to\infty}\left(F_{SD\_\Delta\pi\_diads}\cdot\frac{\sqrt{c}\cdot\ln(c)^2}{\ln\left(\sqrt{c}\right)\cdot\delta_2\cdot c}\right)=\lim_{c\to\infty}\left(\frac{4\cdot F_{SD\_\Delta\pi\_diads}}{\delta_2\cdot R(c)}\right)=0$$

This proves that the set of primes and diads is approaching continuity **as stated in the lemma and concluding the proof.**

Lemma 2: The prime-number-formula respectively the diads-number-formula are the low limit functions of the number of primes, respectively diads.

Proof: The difference between the CPNF (5) and the PNF gives with (17) the error of the PNF:

$$\pi_{appr}(c)-\frac{c}{\ln(c)}=\gamma_p\cdot R(c)^{\varepsilon_p}$$

The difference between the CDNF (10) and the DNF gives with (21) the error of the DNF:

$$\pi_{diads\_appr}(c)-\frac{\delta_2\cdot c}{\ln(c)^2}=\gamma_d\cdot R(c)^{\varepsilon_d}$$

Both errors are growing to infinite with increasing distance. Therefore, PNF is the lowest function of the number of primes and the DNF is the low limit function of the number of diads. Because of Lemma 1 the dispersion around the CPNF respectively around the CDNF does not influence this low limit definition. This fact proves the lemma, **concluding the proof.**

With lemma 2 the number of diads has a low limit function growing to infinity. This function defined for all integers with distances over three, **proves Goldbach's conjecture.**

Proof 2: Any even greater than three has several equidistant primes around it that grows with increasing distance.

For the next proof, the following definitions are important:

Definition of subsets of all integers:

With the infinite number of integers ($m\in\mathbb{Z}$, $\mathbb{Z}=0,1,2,\cdots,\infty$) all integers are part of one of the following infinite subsets:

$$N_{sub(m.a)}=\left\{a+m\cdot P_{(1)}\cdot P_{(2)},s=0,1,2,3,4,5\right\}\tag{24}$$

All integers in the subsets for ($a=0,2,3,4$) are divisible either by ($P_{(1)}$), and/or by ($P_{(2)}$). Therefore, all primes ($P_{(n)},n=3,4,5,\cdots$) are members of one of the infinite subsets, of potential primes, corresponding to ($a=1$) or ($a=5$), having the distance ($\frac{N}{2}=1$) to the members of the sets corresponding to ($a=0$).

Members of the subset ($a=0$) with one of their neighboring positions equal to a prime compose the infinite subsets of the following subsets:

$$N_{sub(m,a,n_1)}=\left\{a+m\cdot P_{(1)}\cdot P_{(2)},s=P_{(n_1)};a=1;n_1\in\mathbb{Z}\right\}\tag{25}$$

$$N_{sub(m,a,n_2)}=\left\{a+m\cdot P_{(1)}\cdot P_{(2)},s=P_{(n_2)};a=1;n_2\in\mathbb{Z}\right\}$$

The section of the above sets defines an infinite subset of the subsets ($a=0$) with potential prime positions around in the subsets ($a=1$) and ($a=5$), having the distance between them equal to ($N=2$), meaning they are potential **twin primes**.

For the distance (c), the center point of the reflection, in the following we use **distance of the diads**. Similarly for the distance (N) between the potential prime positions composing the potential diads in the following we use **distance of the components**. The distance of the components may be any even number. The only condition is that the half-distance of any potential diad component must be smaller than the distance of the potential diad ($\frac{N}{2}<c$).

The components of the potential diads are ($d_{low}(n,m,a)=c(n,a)-\frac{1}{2}\cdot N(m,a)$) and ($d_{high}(n,m,a)=2\cdot c(n,a)-d_{low}(n,m,a)$).

In the following table we listed, the values, the distance of the components ($N(m,a)$), the distance of the potential diads and the components of the potential diads with the parameters ($m,n\in\mathbb{Z}$), belonging to one of the algebraic progressions as subsets ($a=0,2,3,4,5$) may take:

a	$N(m,a)$	$c(n,a)$	$d_{low}(n,m,a)$	$d_{high}(n,m,a)$	
0	$m\cdot 12+10$	$0+n\cdot 6$	$(n-m)\cdot 6+1$	$(n+1+m)\cdot 6-1$	(26)
0	$m\cdot 12+2$	$0+n\cdot 6$	$(n-m)\cdot 6-1$	$(n+1+m)\cdot 6+1$	(27)
1	$m\cdot 12+12$	$1+n\cdot 6$	$(n-m)\cdot 6+1$	$(n+1+m)\cdot 6+1$	(28)
2	$m\cdot 12+6$	$2+n\cdot 6$	$(n-m)\cdot 6-1$	$(n+1+m)\cdot 6-1$	(29)
3	$m\cdot 12+4$	$3+n\cdot 6$	$(n-m)\cdot 6+1$	$(n+1+m)\cdot 6-1$	(30)
3	$m\cdot 12+8$	$3+n\cdot 6$	$(n-m)\cdot 6-1$	$(n+1+m)\cdot 6+1$	(31)
4	$m\cdot 12+6$	$4+n\cdot 6$	$(n-m)\cdot 6+1$	$(n+1+m)\cdot 6+1$	(32)
5	$m\cdot 12+12$	$5+n\cdot 6$	$(n-m)\cdot 6-1$	$(n+1+m)\cdot 6-1$	(33)

The distances of the potential diads-components for ($a=1$) and ($a=5$) are equal.

The total number of potential diads-positions for both sets, for ($a=0$) and ($a=3$) is double the corresponding numbers for the other sets ($a=1$, $a=2$, $a=4$, $a=5$). Therefore, the effective number of diads for these positions must have double the value as well. This is effectively found in **Annex 2** and is demonstrated in Figures A2-A3.

The center points of potential diads ($c(n,a)$) includes all integers. The distance of the components ($N(m,a)$) includes all even numbers and grows to infinite with the distance of the diads ($c(n,a)$). Differently expressed: The number of diads, at growing distance, with the corresponding even number as distance of their components, is unlimited. With lemma 2 the **low limit of the number of diads, including twin primes, is given by the diads-number-formula and grows to infinity.**

The distances to the origin of the potential components of the diads (d_{low}) and (d_{high}) are algebraic progressions with (m). The density of the effective primes at the potential positions of the components of the diads are ($\frac{1}{\ln(d)}$) and ($\frac{1}{\ln(2\cdot c-d)}$). With (7) the density of the effective diads is therefore:

$$\frac{\delta_2}{\ln\left(d_{low}(n,m,a)\right)\cdot\ln\left(2\cdot c-d_{low}(n,m,a)\right)}=\frac{\delta_2}{\ln\left(c-\frac{N}{2}\right)\cdot\ln\left(c+\frac{N}{2}\right)}<\frac{\delta_2}{\ln(c)^2} \quad (34)$$

Lemma 3: Taking any even greater than three as point of reflection, there are two subsets of potential diad positions equidistantly placed around the point of reflection, which are growing to infinity with the distance of the point of reflection.

Proof: Taking any arbitrary even integer of the sets ($c_{ref}=c(n,a), a=0,2,4$), defined in (26), (27), (29) and (32) as points of reflection of the union of the series of multiples of primes, then to each point of reflection ($c(n,a)$) belong two sets algebraic progressions with (m) at equidistant distances as components of potential diads (see Figure A3), **as stated in the lemma and concluding the proof.**

Lemma 4: The section of two infinite sets of potential diads components result in an infinite set of effective diads.

Proof: The share of the effective diads with these infinite sets of potential diads is given by (34). The number of these diads grows to infinity with the distance ($c(n,a)$) **as stated in the lemma and concluding the proof of the lemma and** of proof 2.

Proof 3: Any even greater than three has with the distance growing number of equidistant primes around it.

This same fact is expressed as well by the following lemma leading to the next proof:

Lemma 5: It is not possible that none of the reflected positions of the primes up to an arbitrary distance is a prime.

Proof: Taking the arbitrary integer (c_{ref}) as distance of the point of reflection of the union of the series of multiples of primes, then in the impossible case, all reflected positions of the primes within the interval ($c_{ref}\cdots 2\cdot c_{ref}$):

$$P_{(n)};\ \ n=1,\cdots,S\left(c_{ref}\right);\ \ S(c)=\frac{c}{\ln(c)};\ \ c=P_{(n)}+2\cdot\left[c_{ref}-P_{(n)}\right]=2\cdot c_{ref}-P_{(n)};\ \ c_{ref}<c<2\cdot c_{ref}$$

must be divisible by at least one of the primes:

$$P_{(m)};\ \ m=1,\cdots,R(c);\ \ R(c)=\frac{\sqrt{c}}{\ln\left(\sqrt{c}\right)};\ \ \operatorname{mod}\left[c,P_{(m)}\right]=0$$

In this case, the product of the mod-values results zero, for all reflected prime-positions ($P_{(n)}$), called the **zero configuration:**

$$\prod_{m=1}^{R(c)}\operatorname{mod}\left[c,P_{(m)}\right]=\prod_{m=1}^{R(c)}\operatorname{mod}\left[\left(2\cdot c-P_{(n)}\right),P_{(m)}\right]=0;\quad \sum_{n=1}^{R\left(P_{(n)}\right)}\prod_{m=1}^{R(c)}\operatorname{mod}\left[\left(2\cdot c-P_{(n)}\right),P_{(m)}\right]=0$$

To prove that this configuration is possible, some way should be found to reduce the number of diads over the distance. Considering the primes ($P_{(n)} \le \sqrt{c_{ref}}$) it is known that at (c_{ref}) all composites have as dividend one of the primes ($P_{(n)} \le \sqrt{c_{ref}}$). Taking the next prime ($P_{(n+1)}$) it will cover positions up to ($\left[2+P_{(n)}\right]^2 \le P_{(n+1)}^2$). Equality is reached only when ($P_{(n)}$) and ($P_{(n+1)}$) are twin primes. In this case, the new prime as points of reflection will be the next smaller prime to ($c_{ref\_new} \le P_{(n+1)}^2$). The new upper half of the interval of double occupation will be larger by:

$$2 \cdot P_{(n+1)}^2 - P_{(n+1)}^2 \le \left(2+\sqrt{c_{ref}}\right)^2 = 4 + 4 \cdot \sqrt{c_{ref}} + c_{ref} = c_{ref\_new} \quad (35)$$

$$c_{ref\_new} - c_{ref} \ge 4 \cdot \left(1+\sqrt{c_{ref}}\right) = \Delta c_{ref} > 0$$

With the PNF the number of additional primes within the enlargement are:

$$\frac{2 \cdot c_{ref\_new}}{\ln\left(2 \cdot c_{ref\_new}\right)} - \frac{c_{ref\_new}}{\ln\left(c_{ref\_new}\right)} > \frac{2 \cdot c_{ref}}{\ln\left(2 \cdot c_{ref}\right)} - \frac{c_{ref}}{\ln\left(c_{ref}\right)} = \Delta\pi\left(c_{ref}\right) \quad (36)$$

The number of additional diads within the enlargement will be with the DNF and with (35):

$$4 \cdot \left(1+\sqrt{c_{ref}}\right) \ge 4 \cdot \sqrt{c_{ref}} > 0\,; \quad \frac{\delta_2 \cdot 4 \cdot \sqrt{c_{ref}}}{\ln\left(4 \cdot \sqrt{c_{ref}}\right)^2} = \frac{\delta_2 \cdot \Delta c_{ref\_new}}{\ln\left(\Delta c_{ref\_new}\right)^2} \ge 0\,; \quad \Delta\pi_{diads}\left(c_{ref}\right) \ge 0 \quad (37)$$

With lemma 1 this result is not influenced by the dispersion of the number of diads around its best estimate values: With (15) and (16) the dispersion is approaching zero with reference to the function-value, or to its low limits.

In case in (35) the first prime over ($P_{(n)} \le \sqrt{c_{ref}}$) is larger than a twin prime, the rise in (37) is larger than two ($\Delta\pi_{diads}\left(c_{ref}\right) > 0$). Since this is the average case with ($P_{(n)} < \sqrt{c_{ref}} < P_{(n+1)}$, $P_{(n+1)} - P_{(n)} > 2$), the number of diads rises on average. **Herewith the remaining free positions composing diads may remain unchanged, or will rise and never sink, in case the number of the series of multiples of primes covering positions at (** c_{ref} **) is rising by unity.**

That the number of diads will rise is proven the following way: It is known that up to any known distance (c_{ref}) there are at least **CDNF** diads present. Then because of symmetry within the last section of the length (c_{ref}) within the distance (c_{ref}^2) there will be the same number of diads. Total up to (c_{ref}^2) the double number. Thus, squaring the distance the number of diads at least doubles.

Thus, the zero configuration can never be reached, as stated in the lemma and concluding the proof.

From the symmetry of the set of diads follows the continuity of the set of primes and allows us to evaluate the integral of the inverse of the primes (see Ref. [5]).

6. Conclusion

The fact that the dispersion of the number of the primes at any distance is proportional to the number of the primes at the square root of the distance represents an art of congruency.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Doroszlai, P. and Keller, H. (2022) The Number of Primes. *Advances in Pure Mathematics*, **12**, 81-95. https://doi.org/10.4236/apm.2022.122008

[2] Doroszlai, P. and Keller, H. (2022) The Symmetric Series of Multiples of Primes. *Advances in Pure Mathematics*, **12**, 160-177. https://doi.org/10.4236/apm.2022.123014

[3] Titchmarsh, E.C. (1986) The Theory of the Riemann Zeta-Function. Oxford University Press.

[4] Hardy, G.H. and Littlewood, J.E. (1923) Some Problems of "partitio Numerorum"; III: On the Expression of a Number as a Sum of Primes. *Acta Mathematica*, **44**, 1-70. https://doi.org/10.1007/bf02403921

[5] Doroszlai, P. and Keller, H. (2023) The Integral of the Inverse of the Primes. *Advances in Pure Mathematics*, **13**, 250-266. https://doi.org/10.4236/apm.2023.135018

Appendix

Annex 1: Definition of vectors and variables for the numeric evaluation

First some general functions and values are defined: Based on the requirement of the constraint of non-divisibility by all smaller primes up to the square root, a set of consecutive primes is evaluated and written to a file. From this file they are read: ($P = \text{READPRN}(\text{"Primes\_large.prn"})$). The number of the primes in the set and their numbering are: ($N_P = lenght(P) - 1 = 5003713$, $n = 1, 2, \cdots, N_P$).

The complete-prime-number-formula CPNF, (5) is evaluated with the following routine (floor and ceil stand for round down and round up):

$$\exp\_(x) = e^x;\ \ \gamma_{sec} = -0.036765;\ \ c\_\exp(c, m) = \exp\_\left(\frac{1}{2^{m+1}}\right) \tag{38}$$

$$\pi_{\text{appr\_}}(c) := \left| \begin{array}{l} m \leftarrow 1 \\ S \leftarrow \displaystyle\sum_{j=1}^{\text{floor}(\sqrt{c})} \frac{\sqrt{c}}{\ln(j\sqrt{c})} \\ \text{while } m < \sqrt{c} \\ \quad \left| \begin{array}{l} \Delta_{(m)} \leftarrow (\gamma_{sec.})^m \cdot \displaystyle\sum_{j=1}^{\text{floor}(c\_\exp(c,m))} \text{floor}\left(\frac{c\_\exp(c,m)}{\ln(j \cdot c\_\exp(c,m))}\right) \\ \text{break if } \Delta_{(m)} < 1 \\ S \leftarrow S - \Delta_{(m)} \\ m \leftarrow m + 1 \end{array} \right. \\ S \leftarrow \text{ceil}(S) \end{array} \right.$$

For the evaluation of the number of the next smaller prime to the distance (c) the routine ($n_{next}(c, n_{last})$) resulting in the index (n) of the prime next to any integer is needed ($P_{(n)} \le c < P_{(n+1)}$).

The evaluation starts either at the last evaluated index (n_{last}), or at the index resulting from the prime-number-formula. This is to shorten the evaluation process. In case ($P_{(n)}$) is greater than the distance, the index is lowered. In case it is smaller, the index rises until the corresponding prime is just smaller, or equal to the distance:

$$n_{\text{next\_P}}(c, n_{last}) := \left| \begin{array}{l} \text{if } c > 0 \\ \quad \left| \begin{array}{l} n \leftarrow \text{floor}\left(\frac{c}{\ln(c)}\right) \text{ if } n_{last} = 1 \\ n \leftarrow n_{last} \text{ otherwise} \\ \text{while } P_{(n)} \le c \\ \quad n \leftarrow n + 1 \\ \text{Res} \leftarrow n - 1 \end{array} \right. \\ \text{Res} \leftarrow 0 \text{ otherwise} \\ \text{Res} \end{array} \right. \tag{39}$$

Further functions are the formula evaluating the index of the next smaller prime to any distance and to the square root of any distance:

$$S_{eff}(c) = n_{next\_P}(c,1);\quad R_{eff}(c) = n_{next\_P}\left(\sqrt{c},1\right);\quad S(c) = \frac{c}{\ln(c)};\quad R(c) = \frac{\sqrt{c}}{\ln\left(\sqrt{c}\right)} \tag{40}$$

For the visualization of the results of the analysis the functions will be taken at sparse values, at distances correspond to multiples of the square root of the largest prime considered. The index (kk) of the sparse values for the diads is limited to half the number of the index (k) in case of the primes.

$$\Delta c_{sp} = \sqrt{P_{(N_P)}};\quad k_{limit} = floor\left[\frac{P_{(N_P)}}{\Delta c_{sp}}\right] - 1 = 9277;\quad k = 1,2,\cdots,k_{limit};\quad c_{sp_{(k)}} = k \cdot \Delta c_{sp} \tag{41}$$

$$kk_{limit} = floor\left(\frac{k_{limit}}{2}\right) = 4637;\quad kk = 1,2,\cdots,kk_{limit}$$

The vector of the indexes of the primes next smaller to these sparse distances ($P_{\left[n_{sp(k)}\right]} < c_{sp_{(k)}} < P_{\left[n_{sp(k+1)}\right]}$) is evaluated as ($\pi_{sp}(c)$). It is evaluated once and written in a file. It is read from this file:

$$\pi_{sp_{(k)}} = n_{next}\left[c_{sp_{(k)}},1\right] \tag{42}$$

$$\text{WRITEPRN}(\text{"index\_distance\_sp.prn"}) = \pi_{sp};$$

$$\pi_{sp} = \text{READPRN}(\text{"index\_distance\_sp.prn"})$$

Annex 2: Evaluation of the best estimate number of primes and diads as sum over sections and of the factor of correction of the sum over all sections

The number of primes as sum over sections is evaluated with (3) as a first simplification:

$$\pi_{sec\_appr\_}(c) = \sum_{j=1}^{floor\left(\sqrt{c}\right)} \frac{\sqrt{c}}{\ln\left(\sqrt{c}\right)} \tag{43}$$

The approximating function is evaluated at sparse distances, respectively at the next smaller prime to these distances ($P_{\left(n_{sp(k)}\right)} < c_{sp_{(k)}} < P_{\left(n_{sp(k+1)}\right)}$) with (3). The evaluation at the next smaller prime corresponding to each distance assures that the evaluated numbers of the primes correspond exactly to the distances considered:

$$\pi_{sec\_appr_{(k)}} = \pi_{sec\_appr\_}\left[P_{\left(\pi_{sp(k)}\right)}\right] \tag{44}$$

They are evaluated once and written in files. They are read from these files:

$$\text{WRITEPRN}(\text{"pr\_sec\_appr\_sp\_t.prn"}) = \pi_{sec\_appr};$$

$$\pi_{sec\_appr} = \text{READPRN}(\text{"pr\_sec\_appr\_sp\_t.prn"})$$

The result of the first simplification (3) giving the sum over the sections of the density of primes has an error. This error is proportional to the number of primes up to ($\sqrt{c}$). The error relative to ($\pi\left(\sqrt{c}\right)$) results in the factor of correction. Assuming the factor of correction (γ_{sec}) is constant over the distance (c), it may be evaluated as relation of the average error to the effective number of primes

($\pi\left(\sqrt{c}\right) = R\left[P_{\left(\pi_{sp(k)}\right)}\right]$). The average error is:

$$\Delta\pi_{sec_{(k)}} = \pi_{sec\_appr_{(k)}} - \pi_{sp_{(k)}}; \quad \Delta\pi_{sec\_av_{(k)}} = \frac{1}{k}\cdot\sum_{j=1}^{k}\Delta\pi_{sec_{(j)}} \tag{45}$$

The value of the factor of correction is herewith:

$$\gamma_{sec}\left(c\right) = \frac{\Delta\pi_{sec\_av}\left(c\right)}{\pi\left(\sqrt{c}\right)}; \quad \gamma_{sec\_(k)} = \frac{\sqrt{2}\cdot\Delta\pi_{sec\_av_{(k)}}}{R\left[P_{\left(\pi_{sp(k)}\right)}\right]}; \quad \gamma_{sec} = \gamma_{sec\_(k_{limit})} = -0.040242 \tag{46}$$

Figure A1 shows the independence of the factor of correction (γ_{sec}) from the distance. The averaging process (A2.3) to evaluate the factor of correction is therefore justified. **This factor (** $\gamma_{sec} = -0.040242$ **) is invariant, an inherent property of the prime numbers**. It is important because it is applied in the recursive formula (7) of the complete-prime-number-formula **CPNF**, like the Euler-constant (γ).

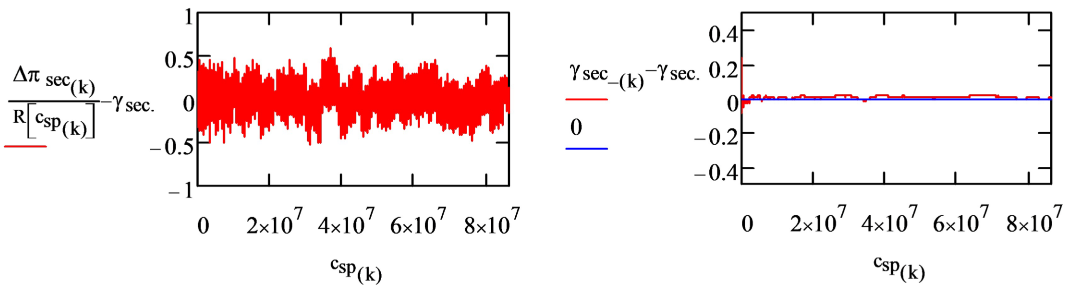

Figure A1. Convergence of the relation of the average relative error of the sum over all sections.

The results of the **CPNF**, (5) are evaluated with (38) once, at sparse values of the distance:

$$\pi_{appr_{(k)}} = \pi_{appr\_}\left[P_{\left(\pi_{sp(k)}\right)}\right] \tag{47}$$

They are written in a file and are read from this file:

$$\text{WRITEPRN}\left(\text{"pr\_sec\_appr\_sp\_t.prn"}\right) = \pi_{sec\_appr};$$

$$\pi_{sec\_appr} = \text{READPRN}\left(\text{"pr\_sec\_appr\_sp\_t.prn"}\right)$$

The approximate values of the number of diads (14) are evaluated once with ($\delta_2 = 1.320324$) for sparse distances ($c = P_{\left(\pi_{sp(kk)}\right)} \geq c_{sp_{(kk)}}$), with the center of the diads being primes, meaning triads. Additionally at integers next to primes, just below the sparse distances.

$$\pi_{diads\_appr\_}\left(c\right) = \sqrt{c}\cdot\sum_{j=1}^{ceil\left(\sqrt{c}\right)}\frac{\delta_2}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(2\cdot c - j\cdot\sqrt{c}\right)}; \quad \pi_{diads\_appr_{(kk)}} = \pi_{diads\_appr\_}\left[P_{\left(\pi_{sp(kk)}\right)}\right] \tag{48}$$

$$\pi_{triads\_appr\_}\left(c\right) = \sqrt{c}\cdot\sum_{j=1}^{ceil\left(\sqrt{c}\right)}\frac{\delta_2}{\ln\left(j\cdot\sqrt{c}\right)\cdot\ln\left(c\right)\cdot\ln\left(2\cdot c - j\cdot\sqrt{c}\right)};$$

$$\pi_{triads\_appr_{(kk)}} = \pi_{triads\_appr\_}\left[P_{\left(\pi_{sp_{(kk)}}\right)}\right]$$

The results are written in files and are read from these files:

$$\text{WRITEPRN}(\text{"diads\_appr.prn"}) = \pi_{diads\_appr};$$

$$\pi_{diads\_appr} = \text{READPRN}(\text{"diads\_appr.prn"})$$

$$\text{WRITEPRN}(\text{"triads\_appr.prn"}) = \pi_{triads\_appr};$$

$$\pi_{triads\_appr} = \text{READPRN}(\text{"triads\_appr.prn"})$$

The number of diads, respectively triads as sum of all diads and triads around the center point of the reflection being primes or other integers, at the limits of sparse distances, is evaluated as follows:

The routine evaluating the effective number of diads at (c) checks for each prime ($P_{(2)} < P_{(kk)} < c$), if the integer at de distance ($d = 2 \cdot c - P_{(kk)}$) were a prime too. If it is a prime, then the sum of the diads around the point of reflection raised by one. The checking is made by controlling the equality of the prime next to the distance (d) to the distance (d). For the evaluation of the index of the next smaller prime to a given distance the routine (39) is used. The routine below evaluates the number of triads corresponding to (28) and (33), or other odd numbers (30), (31) and diads at even numbers of points of reflection corresponding to (26), (27), (29) and (32).

The comparison of the number of diads in case of the points of reflection are a prime and multiple of three at ($6 \cdot n + 3$), is demonstrated in Figure A2: The number of diads in case of the point of reflection is a prime represents the low limit of all other points of reflection. Because the evaluation is time consuming, the results are written to files and read from these files.

```
πtriads_(c) := | S ← 0
               | n ← c − 1
               | klast ← 1
               | while n ≥ 2
               |   | d ← 2·P(c) − P(n)
               |   | k ← nnext_P(d, klast)
               |   | klast ← k
               |   | S ← S + 1 if d = P(k)
               |   | n ← n − 1
               | S

πdiads_3_(c) := | S ← 0
                | z ← P(c) + 2 if mod[P(c) − 1, 6] < 1
                | z ← P(c) − 2 if mod[P(c) + 1, 6] < 1
                | n ← 2
                | nlimit ← c − 1
                | while n ≤ nlimit
                |   | d ← 2·z − P(n)
                |   | k ← nnext_P(d, 1)
                |   | S ← S + 1 if d = P(k)
                |   | n ← n + 1
                | S
```
(49)

$$\pi_{triads_{(kk)}} = \pi_{triads\_}\left[\pi_{sp_{(kk)}}\right] \quad \pi_{diads\_3_{(kk)}} = \pi_{diads\_3\_}\left[\pi_{sp_{(kk)}}\right] \tag{50}$$

$$\text{WRITEPRN}(\text{"triads.prn"}) = \pi_{triads} \quad \text{WRITEPRN}(\text{"diads\_3.prn"}) = \pi_{diads\_3}$$

$$\pi_{triads} = \text{READPRN}(\text{"triads.prn"}) \quad \pi_{diads\_3} = \text{READPRN}(\text{"diads\_3.prn"})$$

Next the number of diads around points of reflection being even numbers at

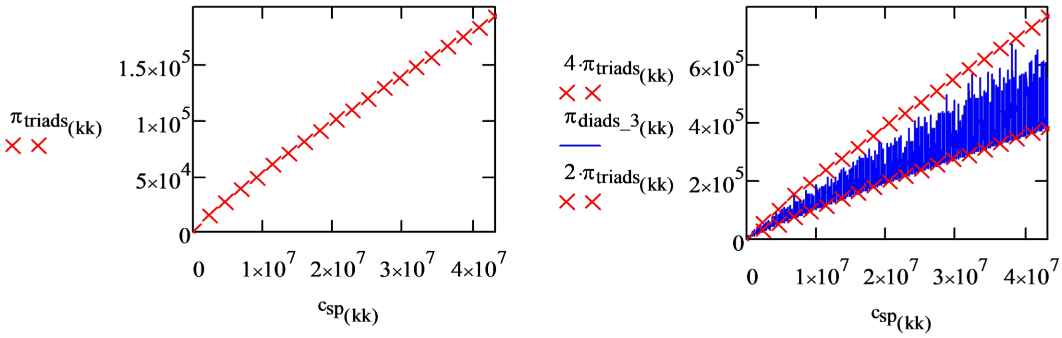

Figure A2. Distribution of the diads for points of reflection being even numbers or primes.

the distances ($n \cdot 6$) and ($n \cdot 6 + 2$) corresponding to (26), (27), (29) and (32) are evaluated:

```
π_diads_0_(c) :=  | S ← 0
                  | z ← P_(c) − 1  if  mod[P_(c) − 1, 6] < 1
                  | z ← P_c + 1  if  mod[P_(c) + 1, 6] < 1
                  | n ← c − 1
                  | n_limit ← 2
                  | k_last ← 1
                  | while  n ≤ n_limit
                  |   | d ← 2·z − P_(n)
                  |   | k ← n_next_P(d, k_last)
                  |   | k_last ← k
                  |   | S ← S + 1  if  d = P_(k)
                  |   | n ← n − 1
                  | S

π_diads_2_(c) :=  | S ← 0
                  | z ← P_(c) + 1  if  mod[P_(c) − 1, 6] < 1
                  | z ← P_(c) − 1  if  mod[P_(c) + 1, 6] < 1
                  | n ← c − 1
                  | n_limit ← 2
                  | k_last ← 1
                  | while  n > n_limit
                  |   | d ← 2·z − P_(n)
                  |   | k ← n_next_P(d, 1)
                  |   | k_last ← k
                  |   | S ← S + 1  if  d = P_(k)
                  |   | n ← n − 1
                  | S
```
(51)

$$\pi_{diads\_0_{(kk)}} = \pi_{diads\_0\_}\left[\pi_{sp_{(kk)}}\right] \quad \pi_{diads\_2_{(kk)}} = \pi_{diads\_2\_}\left[\pi_{sp_{(kk)}}\right] \tag{52}$$

$$\mathrm{WRITEPRN}(\text{"diads\_0.prn"}) = \pi_{diads\_0} \quad \mathrm{WRITEPRN}(\text{"diads\_2.prn"}) = \pi_{diads\_2}$$

$$\pi_{diads\_0} = \mathrm{READPRN}(\text{"diads\_0.prn"}) \quad \pi_{diads\_2} = \mathrm{READPRN}(\text{"diads\_2.prn"})$$

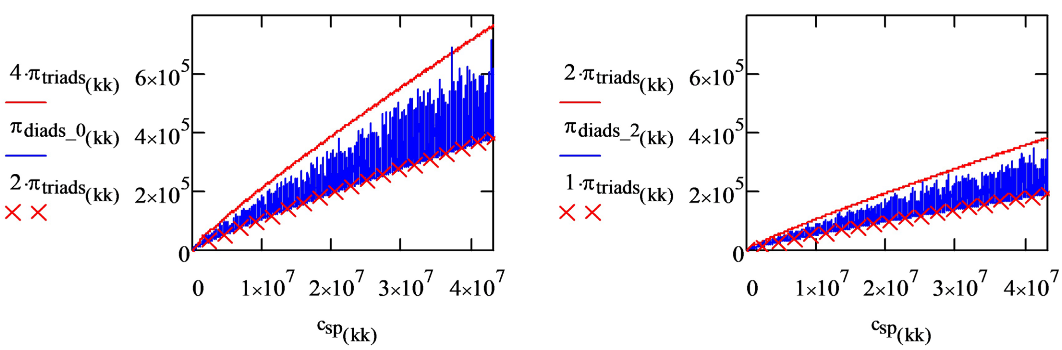

Figure A3. Distribution of the diads for points of reflection being even numbers.

ANNEX 3: Dispersion of the effective number of primes and diads around

the CPNF respectively around the CDNF

The standard deviation SD of the relative dispersion of the primes around their best approximation (5) is evaluated as follows:

$$\Delta\pi_{(k)} = \pi_{appr_{(k)}} - \pi_{sp_{(k)}}\ ;\quad \Delta\pi_{rel(k)} = \frac{\Delta\pi_{(k)}}{R\left[P_{\left(\pi_{sp_{(k)}}\right)}\right]}\ ;\quad SD_{\Delta\pi\_rel_{(k)}} = \sqrt{\frac{1}{k}\sum_{j=1}^{k}\left[\Delta\pi_{rel(j)}\right]^2} \quad (53)$$

The results are written in a file and are read from this file:

$$\text{WRITEPRN}(\text{"SD\_}\Delta\text{\_pi\_rel.prn"}) = SD_{\Delta\pi\_rel}\ ;$$

$$SD_{\Delta\pi\_rel} = \text{READPRN}(\text{"SD\_}\Delta\text{\_pi\_rel.prn"})$$

The average relation of the standard deviation converges to a final value, to the factor of proportionality ($F_{SD\_\Delta\pi}$). This factor is evaluated as follows:

$$SD_{\Delta\pi\_rel\_av_{(k)}} = \frac{1}{k}\cdot\sum_{j=1}^{k} SD_{\Delta\pi\_rel_{(j)}} \quad (54)$$

The results are written and read to and from a file:

$$\text{WRITEPRN}(\text{"SD\_}\Delta\text{\_pi\_rel\_av.prn"}) = SD_{\Delta\pi\_rel\_av}\ ;$$

$$SD_{\Delta\pi\_rel\_av} = \text{READPRN}(\text{"SD\_}\Delta\text{\_pi\_rel\_av.prn"})$$

The constant factor is equal to the final average value of the standard deviation at large distances. Figure A4 illustrates that the standard deviation is about constant over the distance. This fact rectifies taking the average over the whole distance for the evaluation:

$$F_{SD\_\Delta\pi} = SD_{\Delta\pi\_rel\_av_{(k_{limit}-2)}} = 0.162096\ ;\quad \Delta_{SD\_\pi} = 0.02 \quad (55)$$

Figure A4 indicates that the standard deviation of the dispersion of the effective number of primes around its approximation is rising proportionally to ($R(c)$), the number of the series of multiples of primes, which are covering integer positions at this distance (c). The factor of proportionality ($F_{SD\_\Delta\pi}$) is again an **inherent property of the prime numbers**, like the Euler-constant (γ).

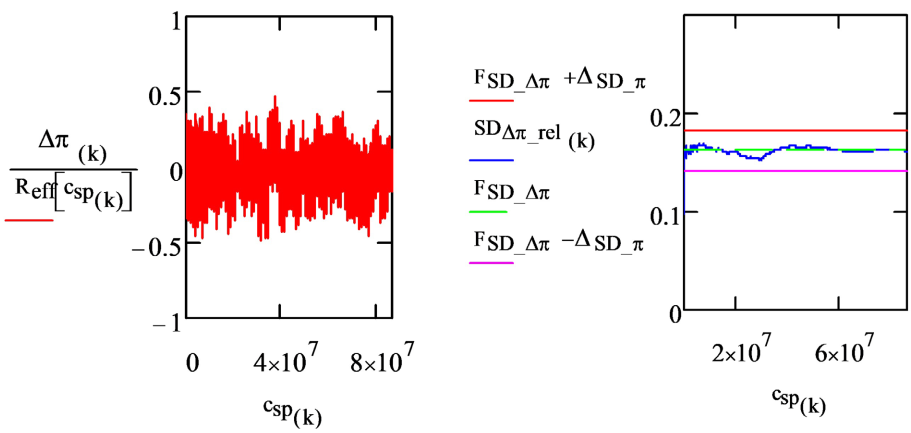

Figure A4. Dispersion of the standard deviation of the dispersion of the number of primes around its average, the resulting constant value ($F_{SD\_\Delta\pi}$).

The dispersion of the number of triads around their best estimate approximation and the dispersion relative to the number of primes at the square root of the distance are (see Figure A5):

$$\Delta\pi_{triads_{(kk)}} = \pi_{triads_{(kk)}} - \pi_{diads\_appr_{(kk)}} ; \quad \Delta\pi_{triads\_rel_{(kk)}} = \frac{\Delta\pi_{triads_{(kk)}}}{R\left[P_{\left(\pi_{sp_{(kk)}}\right)}\right]} \tag{56}$$

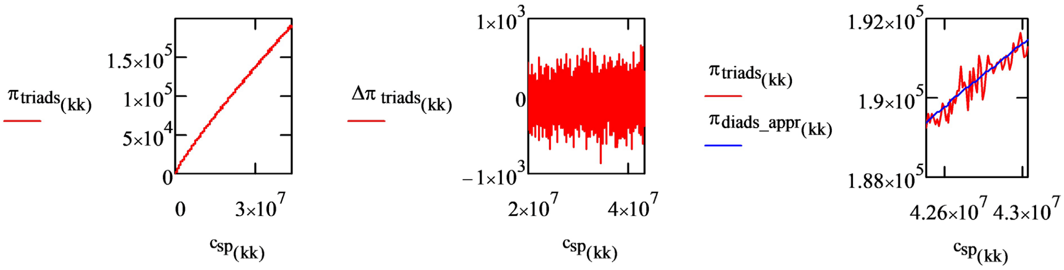

Figure A5. The number of triads evaluated for points of reflection being primes and their dispersion around the best estimate approximation.

The standard deviation SD of the relative dispersion of the diads around their best approximation (56) is evaluated as follows:

$$SD_{\Delta\pi\_triads\_rel_{(kk)}} = \sqrt{\frac{1}{kk}\sum_{j=1}^{kk}\left(\Delta\pi_{triads\_rel_{(j)}}\right)^2} \tag{57}$$

The results are written in a file and read from this file:

$$\text{WRITEPRN}(\text{"SD\_}\Delta\text{\_triads\_rel.prn"}) = SD_{\Delta\pi\_triads\_rel}$$

$$SD_{\Delta\pi\_triads\_rel} = \text{READPRN}(\text{"SD\_}\Delta\text{\_triads\_rel.prn"})$$

The constant factor is equal to the final average value of the standard deviation at large distances. Figure A6 illustrates that the standard deviation is about constant over the distance.

$$F_{SD\_\Delta\pi\_triads} = SD_{\Delta\pi\_triads\_rel_{(kk_{limit}-2)}} = 0.293491; \quad \Delta_{SD\_triads} = 0.04 \tag{58}$$

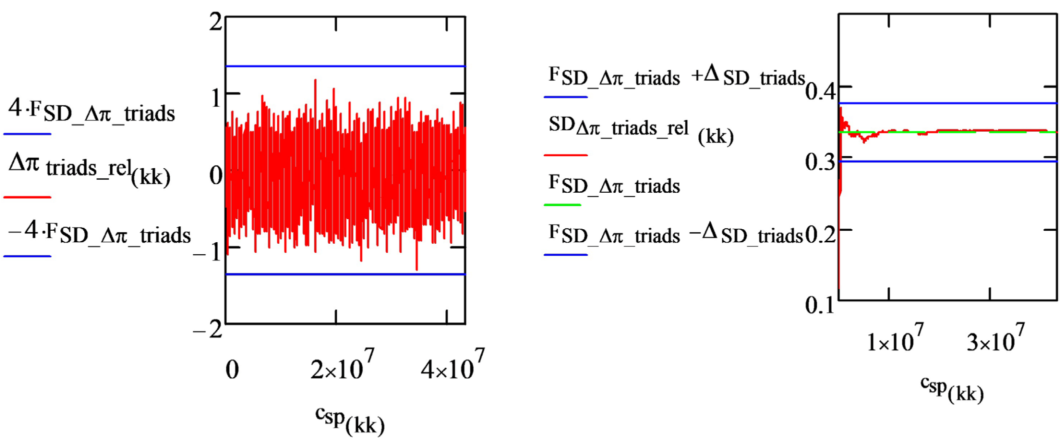

Figure A6. Standard deviation of the dispersion of the number of triads around the best estimate value, resulting in the constant value ($F_{SD\_\Delta\pi\_triads}$).

Figure A6 indicates that the standard deviation of the dispersion of the effective number of primes around its approximation is rising proportionally to ($R(c)$), the number of the series of multiples of primes, which are covering integer positions at this distance (c). The factor of proportionality ($F_{SD\_\Delta\pi\_triads}$) is again an **inherent property of the prime numbers,** like the Euler-constant (γ).

Annex 4: The error of the prime-number-formula and of the diads-number-formula

The number of primes resulting from the second simplification (4) results in the **PNF.** The difference between the **CPNF** and the **PNF** results in the error of the **PNF:**

$$\Delta\pi_{appr}(c) = \pi_{appr\_c\_}(c) - S(c); \quad \Delta\pi_{appr_{(k)}} = \pi_{appr_{(k)}} - S\left(c_{sp_{(k)}}\right) \tag{59}$$

At the distance (c) it is proportional to the near square of the number of primes covering positions at the distance ($\sqrt{c}$):

$$\Delta\pi_{appr}(c) = \gamma_p(c) \cdot R(c)^{\varepsilon_p(c)} \tag{60}$$

The exponent ($\varepsilon_p(c)$) and the factor of proportionality (γ_p) are variable over the distance. However, the exponent evaluated further down ($\varepsilon_p(c) = 1.9816428$) is about constant over the distance. Herewith the resulting relative error of the PNF about constant as well, as illustrated in Figure A7:

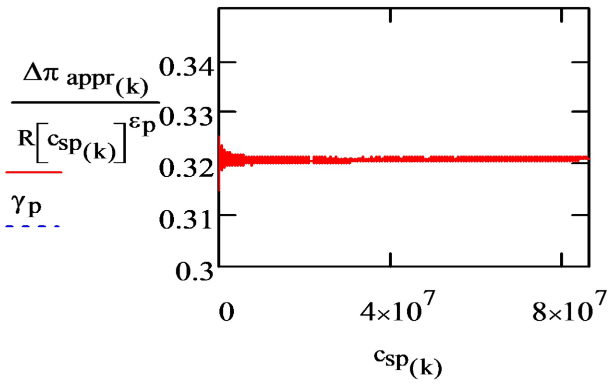

Figure A7. Relation of the difference between the best estimate number of primes (CPNF) and its value evaluated with the prime-number-formula (PNF) to ($R(c)^{\varepsilon_p(c)}$).

This fact allows us to take the average of the factor of proportionality over the whole distance ($\gamma_p(c); \; c = c_{max}$):

$$\gamma_{p\_av}(c_{max}) = \sum_{c_{max}} \frac{\Delta\pi_{appr}(c)}{R(c)^{\varepsilon_p}}; \quad \gamma_{p\_av\_(k_{limit})} = \frac{1}{k_{limit}} \cdot \sum_{j=1}^{k_{limit}} \frac{\Delta\pi_{appr_{(j)}}}{R\left(c_{sp_{(j)}}\right)^{\varepsilon_p}} \tag{61}$$

and express the exponent (ε_p) as function of this average:

$$\varepsilon_p(c_{max}) = \frac{\ln\left(\Delta\pi_{appr}(c_{max})\right) - \ln\left(\gamma_{p\_av}(c_{max})\right)}{\ln\left(R(c_{max})\right)}; \quad \varepsilon_{p\_(k_{limit})} = \frac{\ln\left(\Delta\pi_{appr_{(k_{limit})}}\right) - \ln\left(\gamma_{p\_av\_(k_{limit})}\right)}{\ln\left(R\left(c_{sp_{(k_{limit})}}\right)\right)} \tag{62}$$

This way the following routine results with an assumed value of the exponent a new value:

$$\varepsilon_{p\_new}(\varepsilon_{p\_old}) := \left| \begin{array}{l} \gamma_{p\_av\_}(k_{limit}) \leftarrow \frac{1}{k_{limit}} \cdot \sum_{j=1}^{k_{limit}} \frac{\pi_{appr(j)} - S[c_{sp(j)}]}{R[c_{sp(j)}]^{\varepsilon_{p\_old}}} \\ \frac{\ln\left[\pi_{appr}(k_{limit}) - S[c_{sp}(k_{limit})]\right] - \ln\left[\gamma_{p\_av\_}(k_{limit})\right]}{\ln\left[R[c_{sp}(k_{limit})]\right]} \end{array} \right. \quad (63)$$

Multiple calls of the same routine strongly converge for a final value of the exponent, as illustrated in **Figure A8**:

$$\varepsilon_{p\_p} = \varepsilon_{p\_p\_}(200) \quad (64)$$

With the final value of the exponent (ε_{p_p}) the final value of the factor of proportionality is evaluated. The results are evaluated once and written in files, for reading each time the annex is opened:

$$\text{WRITEPRN}(\text{"epsilon\_p\_p.prn"}) = \varepsilon_{p\_p};$$

$$\varepsilon_{p\_p} = \text{READPRN}(\text{"epsilon\_p\_p.prn"})$$

$$pp = 1,\cdots,length(\varepsilon_{p\_p}); \quad length(\varepsilon_{p\_p}) = 200; \quad \varepsilon_p = \varepsilon_{p\_p(length(\varepsilon_{p\_p})-1)} = 1.9816428 \quad (65)$$

$$\gamma_{p\_(k)} = \frac{\Delta\pi_{appr(k)}}{R\left(c_{sp(k)}\right)^{\varepsilon_p}}; \quad \gamma_{p\_av\_(k)} = \frac{1}{k} \cdot \sum_{j=1}^{k} \frac{\Delta\pi_{appr(j)}}{R\left(c_{sp(j)}\right)^{\varepsilon_p}}$$

$$\text{WRITEPRN}(\text{"gamma\_p\_av.prn"}) = \gamma_{p\_av\_};$$

$$\gamma_{p\_av\_} = \text{READPRN}(\text{"gamma\_p\_av.prn"})$$

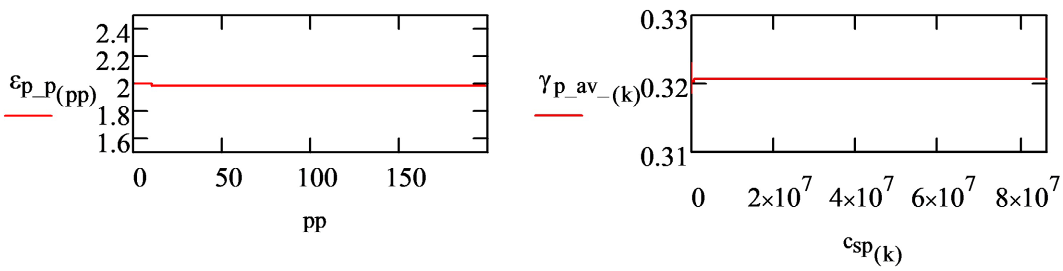

Figure A8. Relation of the difference between the best estimate number of primes (CPNF) and its value evaluated with the prime-number-formula (PNF) to ($R(c)$).

Therefore, the error of the PNF is:

$$\gamma_p = \gamma_{p\_av\_(k_{limit})} = 0.32073756; \quad \pi_{appr}(c) - \frac{c}{\ln(c)} = \gamma_p \cdot R(c)^{\varepsilon_p}; \quad \varepsilon_p = 1.9816428 \quad (66)$$

Similarly, the number of diads resulting from the second simplification (11) results in the **DNF.** The difference between the **CDNF**, (5) and the **DNF**, (11) results in the error of the **DNF.** At the distance (c) it is again proportional to the near square of the number of primes covering positions at the distance ($\sqrt{c}$):

$$\Delta\pi_{diads\_appr}(c) = \pi_{diads}(c) - \frac{\delta_2 \cdot c}{\ln(c)^2};\quad \Delta\pi_{diads_{(kk)}} = \pi_{diads}(c) - \frac{\delta_2 \cdot c}{\ln(c)^2} \tag{67}$$

$$\Delta\pi_{diads\_appr}(c) = \gamma_d \cdot R(c)^{\varepsilon_d}$$

The exponent ($\varepsilon_d(c)$) and the factor of proportionality ($\gamma_d(c)$) are again variables over the distance. However, for an assumed value of the exponent ($\varepsilon_d = 1.836$) the factor of proportionality ($\gamma_d = 0.04$) is about constant over the distance, as illustrated in **Figure A9**:

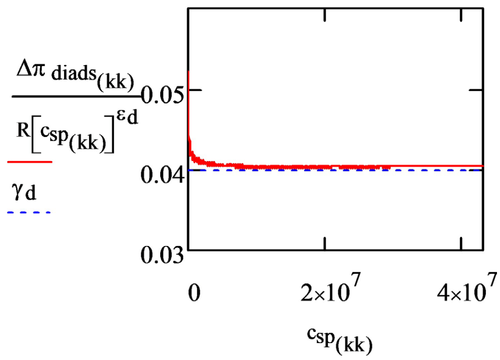

Figure A9. Relation of the difference between the best estimate number of diads (CDNF) and its value evaluated with the diads-number-formula (DNF) to ($R(c)$).

This fact allows us to take the average of the factor of proportionality over the whole distance ($\gamma_d(c)$, $c = c_{max}$):

$$\gamma_{d\_av}(c_{max}) = \frac{1}{c_{max}} \cdot \sum_{c_{max}} \frac{\Delta\pi_{diads}(c)}{R(c)^{\varepsilon_d}};\quad \gamma_{d\_av\_(k_{limit})} = \frac{1}{k_{limit}} \cdot \sum_{j=1}^{k_{limit}} \frac{\Delta\pi_{diads_{(j)}}}{R\left(c_{sp_{(j)}}\right)^{\varepsilon_d}} \tag{68}$$

and express the exponent (ε_d) as function of this average:

$$\varepsilon_d(c_{max}) = \frac{\ln\left(\Delta\pi_{diads}(c_{max})\right) - \ln\left(\gamma_{d\_av}(c_{max})\right)}{\ln\left(R(c_{max})\right)};$$

$$\varepsilon_{d\_(kk_{limit})} = \frac{\ln\left(\Delta\pi_{diads_{(kk_{limit})}}\right) - \ln\left(\gamma_{d\_av\_(kk_{limit})}\right)}{\ln\left(R\left(c_{sp_{(kk_{limit})}}\right)\right)}$$

This way the following routine results with an assumed value of the exponent a new value, Multiple calls of the same routine strongly converge for a final value of the exponent, as illustrated in **Figure A10**:

```
εd_d_(M) := | εd_old ← 2
            | m ← 1
            | while m < M
            |   | εd_d(m) ← εd_old
            |   | εd_old ← εd_new(εd_old)
            |   | m ← m + 1
            | εd_d
```

$$\varepsilon_{d\_new}(\varepsilon_{d\_old}) := \left| \begin{array}{l} \gamma_{d\_av\_(kk_{limit})} \leftarrow \dfrac{1}{kk_{limit}} \cdot \displaystyle\sum_{j=1}^{kk_{limit}} \dfrac{\Delta\pi_{diads_{(j)}}}{R\left[c_{sp_{(j)}}\right]^{\varepsilon_{d\_old}}} \\ \dfrac{\ln\left[\Delta\pi_{diads_{(kk_{limit})}}\right] - \ln\left[\gamma_{d\_av\_(kk_{limit})}\right]}{\ln\left[R\left[c_{sp_{(kk_{limit})}}\right]\right]} \end{array} \right. \tag{69}$$

$$\varepsilon_{d\_d} = \varepsilon_{d\_d\_}(200)$$

With the final value of the exponent (ε_{d_d}) the final value of the factor of proportionality is evaluated. The results are evaluated once and written in files, for reading each time the annex is opened:

$$\text{WRITEPRN}(\text{"epsilon\_d\_d.prn"}) = \varepsilon_{d\_d};$$

$$\varepsilon_{d\_p} = \text{READPRN}(\text{"epsilon\_d\_d.prn"})$$

$$pp = 1, \cdots, length(\varepsilon_{d\_d}); \quad length(\varepsilon_{p\_p}) = 200; \quad \varepsilon_d = \varepsilon_{d\_d_{\left(length(\varepsilon_{d\_d})-1\right)}} = 1.8350218 \quad (70)$$

$$\gamma_{d\_(kk)} = \frac{\Delta\pi_{diads_{(kk)}}}{R\left(c_{sp_{(kk)}}\right)^{\varepsilon_d}}; \quad \gamma_{d\_av\_(k)} = \frac{1}{kk} \cdot \sum_{j=1}^{kk} \frac{\Delta\pi_{diads_{(j)}}}{R\left(c_{sp_{(j)}}\right)^{\varepsilon_d}}; \quad \gamma_d = \gamma_{d\_av\_(kk_{limit})}$$

$$\text{WRITEPRN}(\text{"gamma\_d\_av.prn"}) = \gamma_{d\_av\_};$$

$$\gamma_{d\_av\_} = \text{READPRN}(\text{"gamma\_d\_av.prn"})$$

Therefore, the error of the DNF is:

$$\pi_{diads\_appr}(c) - \frac{\delta_2 \cdot c}{\ln(c)^2} = \gamma_d \cdot R(c)^{\varepsilon_d} \quad (71)$$

$$\gamma_d = \gamma_{d\_av\_(kk_{limit})} = 0.04073037; \quad \varepsilon_d = 1.8350218$$

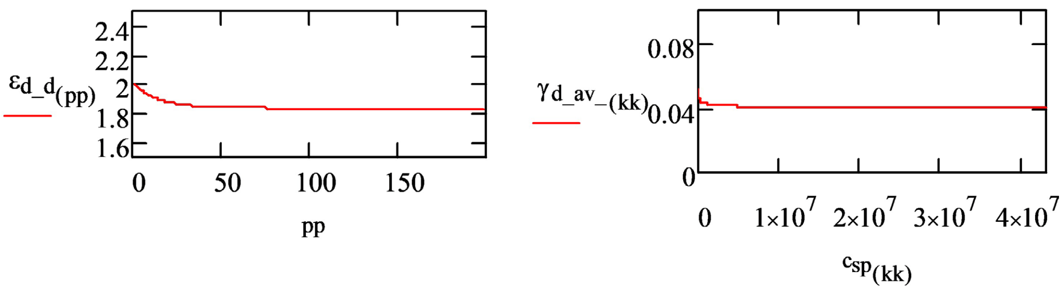

Figure A10. Relation of the difference between the best estimate number of diads (CDNF) and its value evaluated with the diads-number-formula (PDF) to ($R(c)$).