

Asymptotic Behaviors of Hankel Determinants Whose Entries Involve Regularly- or Rapidly-Varying Functions. Part II*

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Abstract

Here we complete our work on the asymptotics of Hankel determinants studying the case wherein the entries are “ultrarapidly”-varying functions in the sense that their logarithms are rapidly varying. Moreover, the last results in the paper highlight analogies between algebraic identities for Hankelians with special entries and asymptotic relations valid for large classes of entries.

Keywords

Asymptotic Behaviors of Hankel Determinants, Asymptotic Expansions in the Real Domain, Regularly-, Rapidly- and Exponentially-Varying Functions of Higher Order, Algebraic Identities for Hankelians

7. Introduction

This paper consists of two distinct but interrelated sections about Hankel determinants. The first section continues and completes the investigation of asymptotic behaviors of Hankel determinants in [1] examining the case wherein the entries are functions of type $\exp(R(x))$ with R rapidly varying. An exceptional case escaping the general method of investigation leads to discover an analogy between a quite special algebraic identity and a kind of, so to say, “asymptotic factorization relation” specific for Hankelians (but not for generic Wronskians) and valid for large classes of entries. In so doing, it will be highlighted an interplay between known closed formulas for special Hankelians and some of the previously-obtained asymptotic relations.

In the last section §11, containing final conclusions on this two-part work on

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Hankelians, we also point out the reason why the complicated theory of higher-order types of asymptotic variation brings about nice results notwithstanding the required long and involved proofs.

The results in this second part, as well as those in the first one, heavily depend on previously-developed theories; hence, for the reader's convenience, we insert an abundance of references to propositions scattered in many papers, some of them being reported in §2.

The numbering of the section continues that in [1] which the reader is referred to for the results and formulas therein and not rewritten here. As an exception, we copy in this introduction almost the whole content of §4.1 including formulas (4.2)-(4.5b), which must be constantly kept under the reader's eyes and are rewritten here *with the same numbering as in [1]*.

In fact, as in §§4-5 also in this second part we shall use the factorized expression:

$$\begin{aligned}
 H_n[\phi(x)] &= (\phi(x))^n \cdot W \left[\left(\frac{\phi'(x)}{\phi(x)} \right)', \left(\frac{\phi''(x)}{\phi(x)} \right)', \dots, \left(\frac{\phi^{(n-1)}(x)}{\phi(x)} \right)' \right] \\
 &\equiv (\phi(x))^n \cdot W(\psi_1'(x), \dots, \psi_{n-1}'(x)), \quad n \geq 2, \quad (\psi_k(x) := \phi^{(k)}(x)/\phi(x));
 \end{aligned}
 \tag{4.2}$$

looking for results concerning rapidly-varying functions of type

$$\phi(x) := \exp(R(x)) \quad \text{with } R(+\infty) = \pm\infty;
 \tag{4.3}$$

where also R is assumed to be rapidly varying. For such a function we have

$$\phi'(x)/\phi(x) = R'(x); \quad H_2[\phi(x)] = \exp(2R(x)) \cdot R''(x);
 \tag{4.4}$$

whereas, for $n \geq 3$, we resort to Faà Di Bruno's formula (1.40) which now takes on the following form for $k \geq 2$:

$$\psi_k(x) := \frac{\phi^{(k)}(x)}{\phi(x)} = (R'(x))^k + \sum_{\substack{0 \leq i_j \leq k; \\ i_1 \leq k-1}}^{i_1+2i_2+\dots+ki_k=k} a_{i_1, \dots, i_k} (R'(x))^{i_1} \cdot (R''(x))^{i_2} \dots (R^{(k)}(x))^{i_k},
 \tag{4.5a}$$

where a_{i_1, \dots, i_k} are suitable positive coefficients and the summation is taken over all possible ordered k -tuples of non-negative integers i_j such that " $i_1 + 2i_2 + \dots + ki_k = k$ ". Having isolated the term corresponding to the k -tuple " $i_1 = k, i_2 = \dots = i_k = 0$ ", it is essential to notice that:

$$\left\{ \begin{array}{l} \text{the exponents of the factors into the summation symbol satisfy:} \\ 1 \leq i_1 + \dots + i_k \leq k-1. \end{array} \right.
 \tag{4.5b}$$

As for the regularity assumptions on the involved function ϕ the reader is referred to the Remark at the end of §1 in [1]. Numbering of References in Part II is independent of that in Part I.

8. Results for the Exponential of a Rapidly-Varying Function

8.1. Preliminaries

Whereas the results in §3 are based on a theorem exhibiting the principal part of a Wronskian of smoothly-varying functions, ([2], Th. 9, p. 18), the results in this

section are based on claims in ([2], Th. 10-(III) and Th. 10-(IV), pp. 19-20) exhibiting the principal part of a Wronskian of rapidly-varying functions under special assumptions. We restate the needed claims correcting some misprints in the original statements.

Lemma 8.1. (Principal parts of Wronskians of rapidly-varying functions). *Let the functions $\phi_i \in C^{n-1}[T, +\infty)$, ($i = 1, \dots, n$), $n \geq 2$, satisfy the following properties:*

$$\begin{cases} \phi_i^{(k)}(x) \neq 0 \quad \forall x \text{ large enough and } 0 \leq k \leq n-1; \\ \lim_{x \rightarrow +\infty} x\phi_i'(x)/\phi_i(x) = \pm\infty \quad (\text{the sign } \pm \text{ depending on } i); \\ \phi_i^{(k)}(x)/\phi_i(x) \sim (\phi_i'(x)/\phi_i(x))^k, \quad x \rightarrow +\infty, \quad (1 \leq k \leq n-1, 1 \leq i \leq n); \end{cases} \quad (8.1)$$

and this means that each ϕ_i is rapidly varying at $+\infty$ and, for $n \geq 3$, it is rapidly varying at $+\infty$ of order $n-2$ in the strong restricted sense of Definition 1.3 in [1]. Then:

(I) Under the further assumption that

$$\frac{\phi_1'(x)}{\phi_1(x)} \gg \dots \gg \frac{\phi_n'(x)}{\phi_n(x)}, \quad x \rightarrow +\infty, \quad (8.2)$$

there exists a permutation $(\phi_{i_1}, \dots, \phi_{i_n})$ forming an asymptotic scale at $+\infty$ and

$$\begin{aligned} & W(\phi_1(x), \dots, \phi_n(x)) \\ & \sim (-1)^{n(n-1)/2} \left(\prod_{i=1}^n \phi_i(x) \right) \cdot \left(\prod_{i=1}^{n-1} \left(\frac{\phi_i'(x)}{\phi_i(x)} \right)^{n-i} \right), \quad x \rightarrow +\infty. \end{aligned} \quad (8.3)$$

(II) Under the further assumption that

$$\phi_i'(x)/\phi_i(x) = a_i\phi(x) + o(\phi(x)), \quad x \rightarrow +\infty, \quad 1 \leq i \leq n, \quad (8.4)$$

for some fixed function ϕ and arbitrary constants a_i (possibly $a_i = 0$ for some i), we have:

$$\begin{aligned} & W(\phi_1(x), \dots, \phi_n(x)) \\ & = \left(\prod_{i=1}^n \phi_i(x) \right) \cdot (\phi(x))^{n(n-1)/2} \cdot [V(a_1, \dots, a_n) + o(1)], \quad x \rightarrow +\infty. \end{aligned} \quad (8.5)$$

If ϕ is strictly positive and a_1, \dots, a_n are pairwise distinct then condition " $a_1 > \dots > a_n$ " implies that (ϕ_1, \dots, ϕ_n) is an asymptotic scale at $+\infty$ and

$$\begin{aligned} & W(\phi_1(x), \dots, \phi_n(x)) \\ & \sim V(a_1, \dots, a_n) \cdot \left(\prod_{i=1}^n \phi_i(x) \right) \cdot (\phi(x))^{n(n-1)/2}, \quad x \rightarrow +\infty; \end{aligned} \quad (8.6)$$

whereas " $a_1 < \dots < a_n$ " implies that (ϕ_n, \dots, ϕ_1) is an asymptotic scale at $+\infty$ and the foregoing relation is still true.

As in Part I our approach starts with a preliminary example.

Example 8.1. For the function

$$\phi(x) := \exp(c_1 e^{c_2 x}), \quad c_1 \neq 0, \quad c_2 > 0; \quad R(x) := c_1 e^{c_2 x}, \quad (8.7)$$

by grouping and reordering the terms of the sum in (4.5a), we get:

$$\begin{cases} R^{(k)}(x) = c_1 c_2^k e^{c_2 x}, \quad (k \geq 1); \\ \psi_k(x) := \frac{\phi^{(k)}(x)}{\phi(x)} = (c_1 c_2)^k \exp(kc_2 x) + \sum_{j=1}^{k-1} a_j \exp(c_2 jx), \\ \text{(with suitable coefficients } a_j \text{);} \end{cases} \quad (8.8)$$

whence:

$$\begin{cases} \psi'_k(x) \sim (c_1 c_2)^k kc_2 \exp(kc_2 x), \\ \psi''_k(x) \sim (c_1 c_2)^k (kc_2)^2 \exp(kc_2 x), \quad x \rightarrow +\infty, \quad (k \geq 1). \\ \psi''_k(x)/\psi'_k(x) \sim kc_2, \end{cases} \quad (8.9)$$

For the Wronskian $W(\psi'_1, \dots, \psi'_{n-1})$ in (4.2) this is the situation in Lemma 8.1-(II) with $\phi(x) \equiv 1$, and (8.6) yields:

$$W(\psi'_1, \dots, \psi'_{n-1}) \sim V(c_2, 2c_2, \dots, (n-1)c_2) \cdot \prod_{i=1}^{n-1} \psi'_i(x), \quad x \rightarrow +\infty;$$

whence, by ([2], formula (61)), the final formula follows:

$$\begin{aligned} H_n[\exp(c_1 e^{c_2 x})] &\sim \left(\prod_{i=1}^{n-1} i! \right) c_1^{n(n-1)/2} c_2^{n(n-1)} \\ &\times \exp(n(n-1)c_2 x/2) \cdot \exp(nc_1 e^{c_2 x}), \quad x \rightarrow +\infty, \quad (n \geq 2); \end{aligned} \quad (8.10)$$

with $c_1 \neq 0, c_2 > 0$, a formula to be commented on in Proposition 9.4 below.

8.2. Heuristic Sketch of the Procedure

(Let the reader keep in mind that a notation $f \in \mathcal{R}_{\pm\infty}(+\infty)$ contains the restriction “ $f(x) > 0$ for all x large enough”, whereas a notation $f \in \{\mathcal{R}_{\pm\infty}(+\infty)$ of order $k \geq 1\}$ implies that “ $f(x)$ has the same strict sign for all x large enough”.)

Let us consider a function of type:

$$\phi(x) := \exp(R(x)) \quad \text{where } R \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 2n-2\}, \quad (8.11)$$

where, for the argument’s sake, we restrict our attention to a function R positively divergent as $x \rightarrow +\infty$. The algebraic structure of “ $\phi = \exp \circ R$ ” implies, by ([3], Prop.7.6-(III), p. 827) that

$$\phi \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 2n-2\}, \quad (8.12)$$

whence, by (1.33) the following relations follow:

$$\begin{aligned} \psi_k(x) &:= \phi^{(k)}(x)/\phi(x) \\ &\sim (\phi'(x)/\phi(x))^k = (R'(x))^k, \quad x \rightarrow +\infty, \quad (k \geq 1), \end{aligned} \quad (8.13)$$

as in the situation of Theorem 5.1, formula (5.9). The special assumption on R implies that all the derivatives of its are positively divergent as R itself, as noticed in ([4], Def. 4.1, p. 807), so that the right-hand side in the representation (4.5a) is a linear combination, with positive coefficients, of functions in the class $\mathcal{R}_{+\infty}(+\infty)$ and, by Proposition 2.7, inference in (2.23), the function $\psi_k(x)$ and

its derivatives belong to this class, *i.e.*

$$\psi_k^{(i)} \in \mathcal{R}_{+\infty}(+\infty), \quad (i \geq 1), \quad (8.14)$$

We now want to apply Lemma 8.1 to the Wronskian in (4.2) and this procedure requires complete information on the order of rapid variation of the involved functions as well as on the asymptotic behaviors of the functions ψ_k''/ψ_k' , $1 \leq k \leq n-1$. In this heuristic sketch, we pay no attention to the order of rapid variation and try to guess the mentioned asymptotic behaviors. In the proof of Theorem 5.1 direct calculations were bypassed by the use of a preliminary result reported in §2; in the present situation, by formally differentiating twice the asymptotic relation in (8.13) and using (1.32) in estimating R'' , we obtain:

$$\left\{ \begin{array}{l} \psi_k'(x) \sim k(R'(x))^{k-1} R''(x) \sim k(R'(x))^{k-1} (R'(x))^2 (R(x))^{-1} \\ \quad \sim k(R'(x))^{k+1} (R(x))^{-1}, \quad x \rightarrow +\infty, \quad (k \geq 1); \\ \psi_k''(x) \sim k(k+1)(R'(x))^k R''(x)(R(x))^{-1} - k(R'(x))^{k+2} (R(x))^{-2} \\ \quad = k(k+1)(R'(x))^{k+2} (R(x))^{-2} [1+o(1)] - k(R'(x))^{k+2} (R(x))^{-2} \\ \quad \sim k^2 (R'(x))^{k+2} (R(x))^{-2}, \quad x \rightarrow +\infty, \quad (k \geq 1); \end{array} \right. \quad (8.15)$$

whence:

$$\psi_k''(x)/\psi_k'(x) \sim k(R'(x)/R(x)), \quad x \rightarrow +\infty, \quad (k \geq 1), \quad (8.16)$$

and applying Lemma 8.1 we get a final result. To make rigorous the preceding reasonings notice that formal differentiation of an asymptotic relation is not granted even between two functions having the same type of asymptotic variation, (rapid variation in the present case), as shown in ([5], §7.2, Counterexamples 2 and 3, p. 472). Hence, the very first steps in both chains of relations in (8.15) require proofs, whereas the subsequent steps are legitimate.

8.3. The Main Result

Theorem 8.2 *For a function of type:*

$$\phi(x) := \exp(R(x)) \quad \text{where } R \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 2n-2\}, \quad (8.17)$$

the following relation holds true:

$$\begin{aligned} H_n[\exp(R(x))] &\sim \left(\prod_{i=1}^{n-1} i! \right) \cdot (R(x))^{-n(n-1)/2} \cdot (R'(x))^{n(n-1)} \\ &\quad \times \exp(nR(x)), \quad x \rightarrow +\infty, \quad (n \geq 2), \end{aligned} \quad (8.18)$$

trivially checked for $n=2$ using both (4.4) (rewritten at the outset of this paper) and (8.20) below.

After the proof, we compare the two asymptotic relations (5.5) and (8.18)

Proof. Under the assumptions in (8.17) we have $\phi(x) \in \mathcal{R}_{\pm\infty}(+\infty)$ according as “ $R(+\infty) = \pm\infty$ ” by Proposition 1.3-(I). But, using a result about compositions of functions, ([3], Prop.7.6-(III), p. 827), we get, more precisely, that:
 $\phi(x) \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } 2n-2\}$.

First step. All the terms in the right-hand side in (4.5a) belong to the same class of R but with the pertinent order, namely $\{\mathcal{R}_{+\infty}(+\infty)$ of order $2n-2-k\}$, by a result on products of higher-order varying functions, ([3], Prop. 7.3-(II), p. 821), because the highest derivative appearing therein is $R^{(k)}$ which belongs to this class. This would imply that

$$\psi_k \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 2n-2-k\} \text{ (irrespective of } R(+\infty) = \pm\infty).$$

Second step. We shall now have recourse to Proposition 2.8-(I) to prove the following stronger relation:

$$\psi_k \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 2n-3\} \text{ (irrespective of } R(+\infty) = \pm\infty). \tag{8.19}$$

To this end, we need to estimate the growth-order of each term into the summation symbol in (4.5a) and we shall repeatedly use the basic relation (1.33), referred to R and written in the form:

$$R^{(j)}(x) \sim (R'(x))^j \cdot (R(x))^{1-j}, \quad x \rightarrow +\infty, \tag{8.20}$$

for all indexes $1 \leq i \leq 2n-1$. We have:

$$\begin{aligned} \prod_{1 \leq p \leq k} (R^{(p)}(x))^{i_p} &\sim \prod_{1 \leq p \leq k} (R'(x))^{p i_p} \cdot (R(x))^{(1-p)i_p} \\ &= (R'(x))^{i_1+2i_2+\dots+ki_k} (R(x))^{(1-1)i_1+(1-2)i_2+\dots+(1-k)i_k} \\ &= (R'(x))^k (R(x))^{i_1+i_2+\dots+i_k-k} \stackrel{\text{by (4.5b)}}{=} O\left((R'(x))^k (R(x))^{-1}\right) \\ &= o\left((R'(x))^k\right), \end{aligned} \tag{8.21}$$

and it follows that the first isolated term in (4.5a) dominates (as growth-order) each term into the summation symbol. Hence, no matter how the terms into the summation symbol in (4.5a) are ordered, Proposition 2.8-(I) can be applied inferring (8.19).

Third step. We now proceed to finding out the asymptotic behaviors of ψ'_k, ψ''_k . We rewrite (4.5a) as

$$\begin{cases} \psi_k(x) = (R'(x))^k + \sum \dots a_{i_1, \dots, i_k} P_k(x) \\ P_k(x) := \prod_{1 \leq p \leq k} (R^{(p)}(x))^{i_p}, \end{cases} \tag{8.22}$$

though using an imprecise notation as P_k depends on the k -tuple (i_1, \dots, i_k) , and get:

$$\begin{aligned} P'_k(x) &= \sum_{j=1}^k \left[i_j (R^{(j)}(x))^{i_j-1} R^{(j+1)}(x) \cdot \prod_{1 \leq p \leq k} (R^{(p)}(x))^{i_p} \right] \\ &= \sum_{j=1}^k \left[i_j (R^{(j)}(x))^{-1} R^{(j+1)}(x) \cdot \prod_{1 \leq p \leq k} (R^{(p)}(x))^{i_p} \right] \\ &\equiv \sum_{1 \leq j \leq k} \left[i_j (R^{(j)}(x))^{-1} R^{(j+1)}(x) \cdot P_k(x) \right]. \end{aligned} \tag{8.23}$$

For the factors in the last sum we repeatedly use (8.20) so getting:

$$\begin{aligned}
 P_k(x) &\sim \prod_{1 \leq p \leq k} (R'(x)/R(x))^{p i_p} (R(x))^{i_p} \\
 &\equiv (R'(x)/R(x))^{\dot{i}_1+2\dot{i}_2+\dots+k\dot{i}_k} R(x)^{\dot{i}_1+\dots+\dot{i}_k} \\
 &\equiv (R'(x)/R(x))^k R(x)^{\dot{i}_1+\dots+\dot{i}_k} \\
 &\stackrel{\text{by (4.5b)}}{=} O\left((R'(x))^k (R(x))^{-1}\right);
 \end{aligned}
 \tag{8.24}$$

and

$$\begin{aligned}
 &(R^{(j)}(x))^{-1} R^{(j+1)}(x) \cdot P_k(x) \\
 &\sim (R'(x))^{-j} \cdot (R(x))^{j-1} (R'(x))^{j+1} (R(x))^{-j} \cdot P_k(x) \\
 &= R'(x)(R(x))^{-1} \cdot O\left((R'(x))^k (R(x))^{-1}\right) \\
 &= O\left((R'(x))^{k+1} (R(x))^{-2}\right);
 \end{aligned}
 \tag{8.25}$$

both estimates depending solely on k and not on the many indexes. Replacing (8.25) into (8.23) gives:

$$\begin{cases} P'_k(x) = O\left((R'(x))^{k+1} (R(x))^{-2}\right); \\ \left(\sum_{\dots} a_{\dot{i}_1, \dots, \dot{i}_k} P_k(x)\right)' = O\left((R'(x))^{k+1} (R(x))^{-2}\right); \end{cases}
 \tag{8.26}$$

and from (8.22) the estimate for ψ'_k follows:

$$\begin{aligned}
 \psi'_k(x) &= k(R'(x))^{k-1} \cdot R''(x) + O\left((R'(x))^{k+1} (R(x))^{-2}\right) \\
 &\stackrel{\text{by (8.20)}}{=} k(R'(x))^{k+1} (R(x))^{-1} + O\left((R'(x))^{k+1} (R(x))^{-2}\right) \\
 &\sim k(R'(x))^{k+1} (R(x))^{-1}, \quad x \rightarrow +\infty, \quad (k \geq 1);
 \end{aligned}
 \tag{8.27}$$

the last relation depending on the fact that $|R(x)| = +\infty(1)$, $x \rightarrow +\infty$. As concerns ψ''_k we have from (8.23):

$$\begin{aligned}
 P''_k(x) &= \sum_{1 \leq j \leq k} \left\{ \left[-i_j (R^{(j)}(x))^{-2} (R^{(j+1)}(x))^2 + i_j (R^{(j)}(x))^{-1} R^{(j+2)}(x) \right] \cdot P_k(x) \right. \\
 &\quad \left. + i_j (R^{(j)}(x))^{-1} R^{(j+1)}(x) P'_k(x) \right\},
 \end{aligned}
 \tag{8.28}$$

wherein each single factor $(R^{(m)}(x))^h$ belongs to one of the classes $\mathcal{R}_{\pm\infty}(+\infty)$ with the appropriate order. For the quantity in square brackets we have the following estimate:

$$\begin{aligned}
 &\left[-i_j (R^{(j)}(x))^{-2} (R^{(j+1)}(x))^2 + i_j (R^{(j)}(x))^{-1} R^{(j+2)}(x) \right] \\
 &\stackrel{\text{by (8.20)}}{=} -i_j (R'(x)/R(x))^2 [1 + o(1)] + i_j (R'(x)/R(x))^2 [1 + o(1)] \\
 &= o\left((R'(x)/R(x))^2\right), \quad (\text{an estimate independent of } j);
 \end{aligned}
 \tag{8.29}$$

whereas $P_k(x)$ is estimated in (8.24), $P'_k(x)$ is estimated in (8.26) and

$$\begin{aligned} (R^{(j+1)}(x)/R^{(j)}(x))P'_k(x) &\stackrel{\text{by (8.20)}}{\sim} (R'(x)/R(x))P'_k(x) \\ &= O\left((R'(x))^{k+2}(R(x))^{-3}\right). \end{aligned} \tag{8.30}$$

Collecting together all these estimates we get:

$$\begin{cases} P''_k(x) = o\left((R'(x))^{k+2}(R(x))^{-3}\right) + O\left((R'(x))^{k+2}(R(x))^{-3}\right) \\ \quad = O\left((R'(x))^{k+2}(R(x))^{-3}\right); \\ \sum_{\dots} a_{i_1, \dots, i_k} P''_k(x) = O\left((R'(x))^{k+2}(R(x))^{-3}\right); \end{cases} \tag{8.31}$$

noticing once again that the estimates are independent of the indexes i_1, \dots, i_k , so that we may infer:

$$\begin{aligned} \psi''_k(x) &\stackrel{\text{by (8.22)}}{\equiv} \left(k(R'(x))^{k-1}R''(x)\right)' + \sum_{\dots} a_{i_1, \dots, i_k} P''_k(x) \\ &= k(k-1)(R'(x))^{k-2}(R''(x))^2 + k(R'(x))^{k-1}R'''(x) \\ &\quad + O\left((R'(x))^{k+2}(R(x))^{-3}\right) \\ &\stackrel{\text{by (8.20)}}{\sim} k^2(R'(x))^{k+2}(R(x))^{-2}. \end{aligned} \tag{8.32}$$

Summing up, both the asymptotic relations for ψ'_k, ψ''_k in (8.15) have been proved so that (8.16) holds true and we may legitimately apply Lemma 8.1-(II), namely relation (8.6), to the Wronskian in (4.2) so obtaining:

$$\begin{aligned} &W(\psi'_1(x), \dots, \psi'_{n-1}(x)) \\ &\sim V(1, \dots, n-1) \cdot \left(\prod_{i=1}^{n-1} \psi'_i(x)\right) \cdot (R'(x)/R(x))^{(n-1)(n-2)/2} \\ &\stackrel{\text{by (8.27)}}{\sim} \left(\prod_{i=0}^{n-2} i!\right)(n-1)! \cdot (R(x))^{-(n-1)-[(n-1)(n-2)]/2} \\ &\quad \times (R'(x))^{2+3+\dots+n+[(n-1)(n-2)]/2} \\ &\sim \left(\prod_{i=1}^{n-1} i!\right) \cdot (R(x))^{-n(n-1)/2} \cdot (R'(x))^{n(n-1)}, \quad x \rightarrow +\infty, \quad (n \geq 2); \end{aligned} \tag{8.33}$$

whence, by (4.2), one gets (8.18). □

A comparison between Theorem 5.1 and Theorem 8.2.

Comparing (5.5) and (8.18) we shall point out two meaningful facts.

1. Apart from the constants appearing in the two asymptotic relations, the structure of the comparison functions is the same because in (5.5) we have “ $R'(x)/R(x) \sim \gamma x^{-1}$ ” so that the principal part of the comparison function in (5.5) is

$$\begin{aligned} &x^{-n(n-1)} \cdot (R(x))^{n(n-1)/2} \cdot \exp(nR(x)) \\ &\sim \gamma^{-n(n-1)} \cdot (R'(x)/R(x))^{n(n-1)} \cdot (R(x))^{n(n-1)/2} \cdot \exp(nR(x)) \\ &\equiv \gamma^{-n(n-1)} \cdot (R(x))^{-n(n-1)/2} \cdot (R'(x))^{n(n-1)} \cdot \exp(nR(x)), \end{aligned} \tag{8.34}$$

coinciding with the comparison function in (8.18), a numerical constant apart.

2. We now show that the constant in (8.18) is the limit as $\gamma \rightarrow +\infty$ of the constant appearing in (5.5). In fact, using the comparison function in (8.18) we see that, by (8.34), the numerical constant in (5.5) is

$$\begin{aligned} C_\gamma &:= V_\gamma \cdot (n-1)! \gamma^{n(n-1)/2} \cdot (\gamma-1)^{n-1} \cdot \gamma^{-n(n-1)} \\ &\sim V_\gamma \cdot (n-1)! \gamma^{(2-n)(n-1)/2}, \quad \gamma \rightarrow +\infty, \end{aligned} \tag{8.35}$$

where $V_\gamma := V(\gamma-2, 2\gamma-3, \dots, (n-1)\gamma-n) \cdot (n-1)$. Now, from the explicit expression of a general Vandermondian,

$$\begin{aligned} V(c_1, \dots, c_n) &= (c_2 - c_1) [(c_3 - c_2)(c_3 - c_1)] \dots [(c_4 - c_3)(c_4 - c_2)(c_4 - c_1)] \\ &\quad \dots [(c_n - c_{n-1})(c_n - c_{n-2}) \dots (c_n - c_1)], \end{aligned}$$

containing $n(n-1)/2$ factors, one gets:

$$\begin{aligned} V(\gamma-2, 2\gamma-3, \dots, (n-1)\gamma-n) \\ \sim 1!2!3! \dots (n-2)! \cdot \gamma^{(n-1)(n-2)/2}, \quad \gamma \rightarrow +\infty, \end{aligned} \tag{8.36}$$

which, replaced into (8.35), gives:

$$C_\gamma \sim \left(\prod_{i=1}^{n-1} i! \right), \quad \gamma \rightarrow +\infty, \tag{8.37}$$

which is the numerical constant in (8.18).

This result parallels a formal link between higher-order regular variation (which is variation of a certain index $\gamma \in \mathbb{R}$) and higher-order rapid variation (which is variation of index $\pm\infty$) as we are going to highlight. If $f \in \{\mathcal{R}_\alpha(+\infty)$ of order $n\}$, $n \geq 1$, then we have the relations in (1.23), here rewritten:

$$\begin{aligned} f^{(k)}(x)/f(x) &= \alpha(\alpha-1) \dots (\alpha-k+1)x^{-k} + o(x^{-k}) \\ &\equiv \alpha^k x^{-k} + o(x^{-k}), \quad x \rightarrow +\infty, \quad (1 \leq k \leq n), \end{aligned} \tag{8.38}$$

for each fixed $\alpha \in \mathbb{R}$; and this is a characterization of the class $\{\mathcal{R}_\alpha(+\infty)$ of order $n\}$ if $\alpha^k \neq 0$: see ([4], Proposition 3.1-(II), p. 799) for a more precise statement. As, by the definition in (1.17),

$$x^{-k} \sim \alpha^{-k} (f'(x)/f(x))^k, \quad x \rightarrow +\infty, \quad (1 \leq k \leq n),$$

(8.38) may be rewritten in the form:

$$f^{(k)}(x)/f(x) \sim \alpha^k \alpha^{-k} (f'(x)/f(x))^k, \quad x \rightarrow +\infty, \quad (1 \leq k \leq n), \tag{8.39}$$

for each fixed α such that $\alpha^k \neq 0$. Obviously, $\lim_{\alpha \rightarrow \pm\infty} \alpha^k \alpha^{-k} = 1$ and, as $\alpha \rightarrow \pm\infty$, one formally gets from (8.39) the relations

$$f^{(k)}(x)/f(x) \sim (f'(x)/f(x))^k, \quad x \rightarrow +\infty, \quad (1 \leq k \leq n), \tag{8.40}$$

which, completed by the same relation for $k = n+1$, characterize the functions $f \in \{\mathcal{R}_{+\infty}(+\infty)$ of order $n\} \cup \{\mathcal{R}_{-\infty}(+\infty)$ of order $n\}$: see (1.33).

Example 8.2. For the function

$$\phi(x) := \exp\left(c_1 e^{c_2 x^\alpha}\right), \quad c_1 \neq 0, \quad c_2 > 0, \quad \alpha > 0, \tag{8.41}$$

we have the relation:

$$\begin{aligned} H_n \left[\exp\left(c_1 e^{c_2 x^\alpha}\right) \right] &\sim \left(\prod_{i=1}^{n-1} i! \right) \cdot (c_1)^{n(n-1)/2} (c_2 \alpha)^{n(n-1)} \cdot x^{(\alpha-1)n(n-1)} \\ &\times \exp\left(n(n-1)c_2 x^\alpha / 2\right) \cdot \exp\left(nc_1 e^{c_2 x^\alpha}\right), \quad x \rightarrow +\infty, \quad (n \geq 2), \end{aligned} \tag{8.42}$$

generalizing relation (8.10).

Example 8.3. For the function

$$\phi(x) := \exp\left(c_3 x^\gamma\right) \cdot \exp\left(c_1 e^{c_2 x^\alpha}\right), \quad c_1, c_3 \neq 0; \quad c_2 > 0, \quad \alpha > 0, \quad \gamma \in \mathbb{R}, \tag{8.43}$$

we shall show the relation:

$$\begin{aligned} H_n \left[\exp\left(c_3 x^\gamma\right) \cdot \exp\left(c_1 e^{c_2 x^\alpha}\right) \right] \\ \sim \exp\left(nc_3 x^\gamma\right) \cdot H_n \left[\exp\left(c_1 e^{c_2 x^\alpha}\right) \right], \quad x \rightarrow +\infty, \quad (n \geq 2), \end{aligned} \tag{8.44}$$

where the principal part of the last Hankelian is specified in (8.42). In fact, putting

$$R(x) := c_1 e^{c_2 x^\alpha}, \quad R_\gamma(x) := c_3 x^\gamma, \tag{8.45a}$$

we have:

$$\begin{cases} R(x) + R_\gamma(x) \sim R(x), \quad x \rightarrow +\infty; \\ R'(x) + R'_\gamma(x) \sim R'(x), \quad x \rightarrow +\infty; \end{cases} \tag{8.45b}$$

and from (8.18):

$$\begin{aligned} H_n \left[\phi(x) \right] &\equiv H_n \left[\exp\left(R(x) + R_\gamma(x)\right) \right] \\ &\sim \left(\prod_{i=1}^{n-1} i! \right) \cdot \left(R(x) + R_\gamma(x)\right)^{-n(n-1)/2} \cdot \left(R'(x) + R'_\gamma(x)\right)^{n(n-1)} \\ &\quad \times \exp\left(nR(x) + nR_\gamma(x)\right) \\ &\stackrel{\text{by (8.45b)}}{\sim} \left(\prod_{i=1}^{n-1} i! \right) \cdot \left(R(x)\right)^{-n(n-1)/2} \cdot \left(R'(x)\right)^{n(n-1)} \\ &\quad \times \exp\left(nR(x)\right) \cdot \exp\left(nR_\gamma(x)\right) \\ &= \exp\left(nR_\gamma(x)\right) \cdot H_n \left[\exp\left(R(x)\right) \right]. \end{aligned} \tag{8.46}$$

The structure of the asymptotic formula (8.44) will be commented on in §9.4.

8.4. An Application to Asymptotic Expansions

As an application to asymptotic expansions, we have the following result.

Theorem 8.3. *Let ϕ be the function in (8.11) and let L_n be the differential operator specified in Theorem 3.4, namely:*

$$\begin{cases} L_n u := u^{(n)} + \alpha_{n-1}(x)u^{(n-1)} + \dots + \alpha_0(x)u \\ \forall u \in AC^{n-1}[T, +\infty[; \quad \alpha_i \in L^1_{loc}[T, +\infty[, \quad 0 \leq i \leq n-1; \\ \ker L_n = \text{span}(\phi_1, \dots, \phi_n). \end{cases} \tag{8.47}$$

A function $f \in AC^{n-1}[T, +\infty[$ admits of an asymptotic expansion of type

$$f(x) = a_1 \phi^{(n-1)}(x) + a_2 \phi^{(n-2)}(x) + \dots + a_{n-1} \phi'(x) + a_n \phi(x) + o(\phi(x)), \quad x \rightarrow +\infty, \quad (8.48)$$

formally differentiable $n-1$ times in the sense of ([2], §6), provided that:

$$\int^{+\infty} |R(t)/R'(t)|^{n-1} \cdot \exp(-R(t)) \cdot |L[f(t)]| dt < +\infty. \quad (8.49)$$

Formal differentiability in the present context refers to the validity of the n expansions in (5.67), as $x \rightarrow +\infty$.

Proof. In this case the asymptotic scale is (1.3),

“ $\phi^{(n-1)}(x) \geq \phi^{(n-2)}(x) \geq \dots \geq \phi(x)$, $x \rightarrow +\infty$ ”, and as in Theorem 5.6-(II) we need to know the principal part of the ratio in (5.68), $H_{n-1}[\phi'(x)]/H_n[\phi(x)]$. The behavior of $H_n[\phi]$ is given by (8.18) and we may guess that the behavior of $H_{n-1}[\phi'(x)] \equiv H_{n-1}[R'(x)\exp(R(x))]$ can be determined by the same relation as in (5.69), i.e.

$$H_{n-1}[R'(x)\exp(R(x))] \sim (R'(x))^{n-1} \cdot H_{n-1}[\phi(x)], \quad x \rightarrow +\infty, \quad (8.50)$$

so that (8.49) follows from ([2], formula (198), p. 26). Though in §9.4 we shall give a general result about relations of the type in (8.50) the theory developed so far does not grant (8.50) and we must find a specific proof. In the present context, we may assume $R > 0$, hence $R' > 0$ as well as pointed out in the first remark in §1.4, and a possible way of doing this is writing “ $R'(x)\exp(R(x)) \equiv \exp(R(x) + \log R'(x))$ ” and showing, as a first step, the validity of the inference

$$\begin{cases} R \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } m\} \\ \Rightarrow r := R + \log R' \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } m-1\}, \quad m \geq 2, \end{cases} \quad (8.51)$$

by direct calculations of the higher derivatives. Remember that $R(+\infty) = R'(+\infty) = +\infty$. For the first derivative, we have:

$$\begin{aligned} r'(x) &= R'(x) + \frac{R''(x)}{R'(x)} \stackrel{\text{by (8.20)}}{=} R'(x) + \frac{R'(x)}{R(x)} [1 + o(1)] \\ &= R'(x) + o(R'(x)); \end{aligned} \quad (8.52a)$$

whence:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{xr'(x)}{r(x)} &= \lim_{x \rightarrow +\infty} \frac{xR'(x)}{R(x) + \log R'(x)} \\ &\equiv \lim_{x \rightarrow +\infty} \left(\frac{R(x)}{\underbrace{xR'(x)}_{o(1)}} + \frac{\log R'(x)}{\underbrace{xR'(x)}_{o(1)}} \right)^{-1} = +\infty, \end{aligned} \quad (8.52b)$$

where the first underbraced ratio is “ $o(1)$ ” by the definition of rapid variation, and the second ratio trivially is “ $o(1)$ ” as $x \rightarrow +\infty$. For the higher derivatives of $\log R'$ one may use, e.g., Faà Di Bruno’s formula (1.40) so obtaining:

$$\begin{aligned}
 (\log R'(x))^{(k)} &= \sum_{0 \leq i_j \leq k}^{i_1+2i_2+\dots+i_k=k} \bar{a}_{i_1, \dots, i_k} (R'(x))^{-(i_1+\dots+i_k)} \cdot \prod_{1 \leq p \leq k} (R^{(p+1)}(x))^{i_p} \\
 &\equiv \sum_{0 \leq i_j \leq k}^{i_1+2i_2+\dots+i_k=k} \bar{a}_{i_1, \dots, i_k} Q_k(x),
 \end{aligned}
 \tag{8.53}$$

with suitable coefficients (with variable signs) immaterial for our aims. Using (8.20), as in the proof of Theorem 8.2, one finds:

$$\begin{aligned}
 Q_k(x) &\sim (R'(x))^{-(i_1+\dots+i_k)} \cdot \prod_{1 \leq p \leq k} (R'(x))^{i_p(p+1)} \cdot (R(x))^{-p i_p} \\
 &\sim (R'(x)/R(x))^k, \quad \left(\text{because } \sum_{1 \leq p \leq k} p i_p = k\right),
 \end{aligned}
 \tag{8.54}$$

independently of the many indexes i_j ;

$$\begin{aligned}
 r^{(k)}(x) &= R^{(k)}(x) + (\log R'(x))^{(k)} \\
 &= R^{(k)}(x) + O\left(\left(R'(x)/R(x)\right)^k\right) \\
 &\stackrel{\text{by (8.20)}}{=} R^{(k)}(x) + O\left(R^{(k)}(x)/R(x)\right) \\
 &\sim R^{(k)}(x) \quad \forall k;
 \end{aligned}
 \tag{8.55}$$

whence “ $r^{(k+1)}(x)/r^{(k)}(x) \sim R^{(k+1)}(x)/R^{(k)}(x)$ ” for all the involved values of k , which imply (8.51) because the pertinent chain of relations for R in (1.31) is satisfied by assumption. As a second step, we try to apply Theorem 8.2. We shall need and prove the following two preliminary relations:

$$\log R'(x) = o(R(x)); \quad R''(x)/R'(x) = o(R'(x)).
 \tag{8.56}$$

The second one follows from (8.20) because:

$$R''(x)/(R'(x))^2 \sim (R(x))^{-2} = o(1);$$

and, moreover, this relation grants the following limit by L'Hospital's rule:

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} \log R'(x)/R(x) &\stackrel{H}{=} \lim_{x \rightarrow +\infty} \frac{R''(x)/R'(x)}{R'(x)} \\
 &= \lim_{x \rightarrow +\infty} R''(x)/(R'(x))^2 = 0.
 \end{aligned}
 \tag{8.57}$$

Now, using (8.56)-(8.57), applying (8.18) to the first Hankelian in (8.50) and denoting by c_{n-1} the constant therein, one gets:

$$\begin{aligned}
 &H_{n-1} \left[\exp(R(x) + \log R'(x)) \right] \\
 &\sim c_{n-1} (R(x) + \log R'(x))^{-(n-1)(n-2)/2} \\
 &\quad \times \left(R'(x) + \frac{R''(x)}{R'(x)} \right)^{(n-1)(n-2)} \cdot \exp((n-1)R(x) + (n-1)\log R'(x)) \\
 &\sim c_{n-1} \exp((n-1)R(x)) \cdot (R'(x))^{n-1} \\
 &\quad \times (R(x))^{-(n-1)(n-2)/2} \cdot (R'(x))^{(n-1)(n-2)} \\
 &\sim (R'(x))^{n-1} \cdot H_{n-1} \left[\exp(R(x)) \right],
 \end{aligned}
 \tag{8.58}$$

which is our thesis. Formula (8.50) is completely proved; it is an asymptotic relation similar to (5.22), a kind of asymptotic factorization to be investigated in §9.4. \square

A comparison between the integral conditions in (5.66) and in (8.49).

The reader may notice that these two integral conditions formally coincide. In fact, condition in (5.66) refers to a regularly-varying function R such that

$$|R| \in \mathcal{R}_\gamma(+\infty), \quad \gamma > 1; \quad |R'| \in \mathcal{R}_{\gamma-1}(+\infty), \quad (8.59)$$

so that $|R'(+\infty)| = +\infty$ and, by the definition of regular variation:

$$t \asymp |R(t)/R'(t)|, \quad t \rightarrow +\infty. \quad (8.60)$$

The quantity in (5.66) satisfies:

$$t^{2(n-1)} \cdot |R(t)|^{1-n} \cdot |R'(t)|^{n-1} \asymp |R(t)/R'(t)|^{2(n-1)-(n-1)}, \quad (8.61)$$

making the two integral conditions to coincide. However, this is a mere formal coincidence in so far as the logarithmic derivative $(R'(t)/R(t))$ satisfies (8.60) in the context of (5.66) whereas no a-priori estimate exists in the context of (8.49) where R is a generic rapidly-varying function with the order-regularity specified in (8.11).

Last, notice that relation in (8.69) yields a simpler restatement of condition in (5.66), not pointed out in [1], namely

$$\int^{+\infty} t^{n-1} \cdot \exp(-R(t)) \cdot |L[f(t)]| dt < +\infty, \quad (8.62)$$

and that, by an oversight, absolute values are missing in (5.63) and in the immediately-preceding line wherein R' and $R'(x)$ must be replaced by $|R'|$ and $|R'(x)|$ respectively: no sign-restriction on these two quantities is required.

9. A Reduction Formula, Some Closed Formulas and Factorization Identities for Hankelians. Their Asymptotic Counterparts

In all the results in §5, there was the restriction $\gamma \neq 1$ because this case is exceptional in so far as some of the pertinent Hankelians may be $\equiv 0$ as for the powers in (3.1). But the asymptotic study of this case becomes possible, and even elementary, by a reduction formula for Hankelians and a consequent lemma on a special factorization identity. This procedure leads us to discover a kind of “asymptotic factorization relation” for Hankelians useful in applications. The whole matter in this section highlights an interplay between algebraic and asymptotic relations. In this section, whenever an n -th order Hankelian appears, the involved function is assumed to be of class C^{2n-2} on an interval I wherein the various formulas are meant to hold true.

9.1. A Reduction Formula

For second-order Hankelians, we have the explicit expression

$$H_2[f(x)] = f(x)f''(x) - (f'(x))^2, \quad x \in I, \tag{9.1}$$

whereas for higher orders there is a classical reduction formula that can be derived from a more general Wronskian identity ([6], p. 72), or directly from Sylvester identity for determinants:

$$H_n[f(x)]H_{n-2}[f(x)] = H_2[H_{n-1}[f(x)]], \quad n \geq 3, \quad x \in I, \tag{9.2}$$

which we shall also use in the form:

$$H_n[f(x)] = (H_{n-2}[f(x)])^{-1} \cdot H_2[H_{n-1}[f(x)]], \quad n \geq 3, \tag{9.3}$$

under the restriction $H_{n-2}[f(x)] \neq 0$ for the values of x taken into consideration.

9.2. Factorization Identities

Motivated by the obvious property

$$H_n[cf(x)] = c^n H_n[f(x)], \quad (n \geq 1, c = \text{constant}), \tag{9.4}$$

we shall highlight a restricted class of non-constant functions w such that the analogous formula holds true for some values of x or on the whole interval I , with c replaced by $w(x)$. This is in contrast to the similar identity for Wronskians $W(v(x)u_1(x), \dots, v(x)u_n(x)) = (v(x))^n \cdot W(u_1(x), \dots, u_n(x))$ valid for any sufficiently-regular function v .

Lemma 9.1 (Identities for second-order Hankelians of products).

(I) For $n = 2$ we have the identity

$$H_2[f(x)w(x)] = (f(x))^2 \cdot H_2[w(x)] + (w(x))^2 \cdot H_2[f(x)], \tag{9.5}$$

with no restriction on the values of the involved quantities.

(II) For the Hankelians of the integer powers of a function we have the formulas

$$H_2[(w(x))^k] = k \cdot (w(x))^{2(k-1)} \cdot H_2[w(x)], \quad k \in \mathbb{N}, \quad k \geq 2, \tag{9.6}$$

with no restriction on the values of the involved quantities.

Proof. Formula (9.5) requires simple direct calculations. For $k = 2$, (9.6) follows from (9.5) with $f = w$. By induction on k , if (9.6) is assumed valid for some k then for $k+1$:

$$\begin{aligned} H_2[(w(x))^{k+1}] &\equiv H_2[w(x)(w(x))^k] \\ &\stackrel{(9.5)}{=} (w(x))^2 \cdot H_2[(w(x))^k] + (w(x))^{2k} \cdot H_2[w(x)] \\ &\dots \text{by the inductive assumption} \dots \\ &= k(w(x))^{2+2(k-1)} \cdot H_2[w(x)] + (w(x))^{2k} \cdot H_2[w(x)] \\ &= (k+1)(w(x))^{2k} \cdot H_2[w(x)]. \end{aligned} \tag{9.7}$$

Lemma 9.2 (Factorization identities for Hankelians involving e^{cx}). For any

function f sufficiently regular on I we have the factorization identities here specified.

(I) For second-order Hankelians:

$$\begin{cases} H_2[f(x)e^{cx}] = e^{2cx} \cdot H_2[f(x)], & x \in I, (c \in \mathbb{R}), \\ \text{with no restriction on the values of } f. \end{cases} \quad (9.8)$$

(II) For a fixed $n_0 \in \mathbb{N}$, $n_0 \geq 3$, the factorization identity holds true:

$$H_n[f(x)e^{cx}] = e^{ncx} \cdot H_n[f(x)], \quad x \in I, (1 \leq n \leq n_0, c \in \mathbb{R}), \quad (9.9)$$

provided that f satisfies the following conditions:

$$H_k[f(x)] \neq 0 \quad \forall x \in I \text{ and } 1 \leq k \leq n_0 - 2. \quad (9.10)$$

(III) The following identities hold true for each constant $c \in \mathbb{R}$, $n \in \mathbb{N}$ and α restricted as specified:

$$H_n[x^\alpha e^{cx}] \equiv \begin{cases} 0 \quad \forall x \in \mathbb{R} \text{ if } \alpha \in \{0, 1, \dots, n-2\}, & n \geq 2; (x^0 := 1 \quad \forall x \in \mathbb{R}); \\ (-1)^{n(n-1)/2} \left(\prod_{i=1}^{n-1} i! \right) \cdot \left(\prod_{i=1}^{n-1} \alpha^i \right) \cdot x^{n(\alpha-n+1)} e^{ncx}, & n \geq 2, \\ \text{for all the other real values of } \alpha \text{ and all the admissible values of } x. \end{cases} \quad (9.11)$$

Remarks. 1. By a classical result, ([6], Th. 7.1, p. 71), the first equality in (9.11) characterizes the circumstance " $H_n[\phi(x)] \equiv 0$ ".

2. The proof below would be valid with e^{cx} replaced by any function ϕ such that

$$\phi(x) \neq 0 \quad \forall x \in I, \quad H_2[\phi(x)] = 0 \quad \forall x \in I, \quad (9.12)$$

a condition characterizing the class of functions coinciding with ae^{cx} on a given interval as mentioned above. Hence the identities in the above lemma are the only possible ones.

3. Though the restrictions in (9.10) may be a hindrance to a practical use of (9.9), nevertheless they will reveal useful in our asymptotic investigations.

Proof. A rigorous proof requires a bit of attention. (I) For a fixed $c \neq 0$ formula (9.8) follows from (9.5) because $H_2[e^{cx}] \equiv 0$ and with no additional restriction on f apart from its regularity. (II) As for (9.9) and $n = 3$:

$$\begin{aligned} & H_3[f(x)e^{cx}] \cdot f(x)e^{cx} \\ & \stackrel{(9.2)}{=} H_2[H_2[f(x)e^{cx}]] \stackrel{(9.8)}{=} H_2[e^{2cx} H_2[f(x)]] \\ & \stackrel{(9.8)}{=} e^{4cx} H_2[H_2[f(x)]] \stackrel{(9.2)}{=} e^{4cx} H_3[f(x)] \cdot f(x), \end{aligned} \quad (9.13)$$

whence one gets (9.9) for $n = 3$ provided that $f(x) \neq 0$, otherwise no conclusion is legitimate. Now, for a fixed $n_0 \geq 4$, let us assume the validity of all the relations in (9.9) for $n \in \{1, 2, \dots, p\}$ and some fixed p , $1 \leq p < n_0$, and under restrictions in (9.10) for $1 \leq k \leq p-2$. If (9.10) is assumed valid also for $k = p-1$ we have:

$$\begin{aligned}
 H_{p+1} [f(x)e^{cx}] & \stackrel{(9.3)}{=} (H_{p-1} [f(x)e^{cx}])^{-1} \cdot H_2 [H_p [f(x)e^{cx}]] = \\
 & \dots \text{by the inductive assumption} \dots \\
 & = e^{(1-p)cx} \cdot (H_{p-1} [f(x)])^{-1} \cdot H_2 [e^{pcx} H_p [f(x)]] \\
 & \stackrel{(9.8)}{=} e^{(1-p)cx} \cdot (H_{p-1} [f(x)])^{-1} \cdot e^{2pcx} \cdot H_2 [H_p [f(x)]] \tag{9.14} \\
 & = e^{(p+1)cx} (H_{p-1} [f(x)])^{-1} \cdot H_2 [H_p [f(x)]] \\
 & \stackrel{(9.3)}{=} e^{(p+1)cx} H_{p+1} [f(x)].
 \end{aligned}$$

(III) Specializing (9.9) to the case $f(x) := x^\alpha$ we recall that the closed expressions of the Hankelians $H_n [x^\alpha]$ are given by formulas (3.1), one possible proof having been pointed out in ([1], §3.1); and $H_n [x^\alpha]$ equals the expression in (9.11) for $c = 0$. Let $\alpha \notin \{0, 1, \dots, n-2\}$ and restrict attention to the admissible values of $x \neq 0$. Because $H_n [x^\alpha] \neq 0$ on any interval not containing zero, the claim in part II may be applied and (9.11) follows. For $x = 0$, if this is an admissible value, the formulas hold true as well by the continuity of both sides with respect to the variable x . Last, identities for the exceptional values $\alpha \in \{0, 1, \dots, n-2\}$ cannot be proved by the above argument but they simply follow from the just-proved formulas by the continuity of both sides with respect to the parameter α . \square

9.3. Other Closed Formulas

Besides those in (9.11), other exact formulas are known for the Hankelians of some special exponentials and we report two of them pointing out their roles in our asymptotic context.

Proposition 9.3 *The following closed formulas hold true:*

$$H_n [\exp(cx^2)] = \left(\prod_{i=1}^{n-1} i! \right) (2c)^{n(n-1)/2} \cdot \exp(ncx^2), \quad (n \geq 2; c \in \mathbb{R}); \tag{9.15}$$

$$\begin{aligned}
 H_n [\exp(c_1 e^{c_2 x})] & = \left(\prod_{i=1}^{n-1} i! \right) c_1^{n(n-1)/2} c_2^{n(n-1)} \\
 & \times \exp[n(n-1)c_2 x/2] \cdot \exp[nc_1 e^{c_2 x}], \quad (n \geq 2; c_1, c_2 \in \mathbb{R}).
 \end{aligned} \tag{9.16}$$

From (9.4), (9.8) and (9.15) we get:

$$\begin{aligned}
 & H_n [\exp(c_0 + c_1 x + c_2 x^2)] \\
 & = \left(\prod_{i=1}^{n-1} i! \right) (2c_2)^{n(n-1)/2} \times \exp[n(c_0 + c_1 x + c_2 x^2)], \quad (n \geq 2; c_0, c_1, c_2 \in \mathbb{R}).
 \end{aligned} \tag{9.17}$$

In the context of the present work formulas (9.15)-(9.16) may be respectively used as a confirmation of our asymptotic formulas (4.12) and (8.10), the last having been proved for $c_2 > 0$.

Proofs. We report standard inductive proofs based on the identity (9.3) but using our previous asymptotic results as starting points. For (9.15) we look at relation (4.12) which, for $n = 2$, reduces to the elementary identity

$$H_2 \left[\exp(cx^2) \right] \equiv 2c \cdot \exp(2cx^2), \tag{9.18}$$

which simply is the product of a constant for the appropriate exponential according to (4.4). However, this special case does not permit by itself to guess the constant appearing in the hoped-for identity for $n \geq 3$ whereas formula (4.12) entitles us to conjecture that it is an identity rather than a mere asymptotic relation so that we try an inductive proof without checking the identity for more values of the index. Assuming the validity up to a certain index $n \geq 2$ we have:

$$\begin{aligned} H_{n+1} \left[\exp(cx^2) \right] &= \left(H_{n-1} \left[\exp(cx^2) \right] \right)^{-1} \cdot H_2 \left[H_n \left[\exp(cx^2) \right] \right] = \\ &\dots \text{by the inductive assumption (9.15) and using (9.4) and (9.18)} \dots \\ &= \left(\prod_{i=0}^{n-2} i! \right)^{-1} (2c)^{-(n-1)(n-2)/2} \cdot \exp \left[(1-n)cx^2 \right] \\ &\quad \times \left(\prod_{i=0}^{n-1} i! \right)^2 (2c)^{n(n-1)} \cdot 2nc \cdot \exp \left[2ncx^2 \right] \\ &= \left(\prod_{i=0}^{n-2} i! \right) \left[(n-1)! \right]^2 n (2c)^{[-(n-1)(n-2)/2] + n(n-1)+1} \times \exp \left[(n+1)cx^2 \right] \\ &= \left(\prod_{i=0}^n i! \right) (2c)^{(n+1)n/2} \cdot \exp \left[(n+1)cx^2 \right]. \end{aligned} \tag{9.19}$$

(In the products the index i runs starting from 0 instead of 1 for consistency with the case $n = 2$.)

For the proof of (9.16) we notice that for $n = 2$ it reduces to the easily-checked identity

$$H_2 \left[\exp(c_1 e^{c_2 x}) \right] = c_1 c_2^2 \cdot e^{c_2 x} \cdot \exp(2c_1 e^{c_2 x}), \tag{9.20}$$

and we conjecture that our asymptotic relation (8.10) is indeed an identity and try an inductive argument assuming its validity up to a certain index $n \geq 2$. In this case, we have:

$$\begin{aligned} H_2 \left[H_n \left[\exp(c_1 e^{c_2 x}) \right] \right] &= \dots \text{by the inductive assumption (9.16)} \dots \\ &= H_2 \left[\left(\prod_{i=0}^{n-1} i! \right) c_1^{n(n-1)/2} c_2^{n(n-1)} \cdot \exp \left[n(n-1)c_2 x / 2 \right] \cdot \exp \left[nc_1 e^{c_2 x} \right] \right] \\ &\stackrel{(9.4),(9.8)}{=} \left(\prod_{i=0}^{n-1} i! \right)^2 c_1^{n(n-1)} c_2^{2n(n-1)} \cdot \exp \left[n(n-1)c_2 x \right] \cdot H_2 \left[\exp \left[nc_1 e^{c_2 x} \right] \right] \\ &\stackrel{(9.20)}{=} \left(\prod_{i=0}^{n-1} i! \right)^2 c_1^{n(n-1)} c_2^{2n(n-1)} \cdot nc_1 c_2^2 e^{c_2 x} \cdot \exp \left[n(n-1)c_2 x \right] \cdot \exp \left[2nc_1 e^{c_2 x} \right]; \end{aligned} \tag{9.21}$$

$$\begin{aligned} H_{n+1} \left[\exp(c_1 e^{c_2 x}) \right] &= \left(\prod_{i=0}^{n-2} i! \right)^{-1} c_1^{-(n-1)(n-2)/2} c_2^{-(n-1)(n-2)} \\ &\quad \times \exp \left[-(n-1)(n-2)c_2 x / 2 \right] \cdot \exp \left[-(n-1)c_1 e^{c_2 x} \right] \\ &\quad \times \left(\prod_{i=0}^{n-1} i! \right)^2 c_1^{n(n-1)} c_2^{2n(n-1)} \cdot nc_1 c_2^2 e^{c_2 x} \cdot \exp \left[n(n-1)c_2 x \right] \cdot \exp \left[2nc_1 e^{c_2 x} \right] \\ &= \left(\prod_{i=0}^n i! \right) c_1^{n(n+1)/2} c_2^{n(n+1)} \cdot \exp \left[(n+1)nc_2 x / 2 \right] \cdot \exp \left[(n+1)c_1 e^{c_2 x} \right]. \end{aligned} \tag{9.22}$$

□

9.4. Asymptotic Counterpart of the Factorization Identity

For a pair of functions f, w never vanishing on a neighborhood of $+\infty$ the identity (9.1) may be written, as in (1.8), in the form

$$H_2[f(x)w(x)] = (f(x))^2 \cdot (w(x))^2 \cdot \left[\left(\frac{f'(x)}{f(x)} \right)' + \left(\frac{w'(x)}{w(x)} \right)' \right]. \tag{9.23}$$

Under the further restriction “ $(f'(x)/f(x))' \neq 0$ on a neighborhood of $+\infty$ ” or (equivalently in the present situation) “ $H_2[f(x)] \neq 0$ on a neighborhood of $+\infty$ ”, one gets the following result which may be read as an “asymptotic factorization”:

$$\begin{cases} (w'(x)/w(x))' = o\left((f'(x)/f(x))' \right), & x \rightarrow +\infty, \\ \Rightarrow H_2[f(x)w(x)] \sim (w(x))^2 \cdot H_2[f(x)], & x \rightarrow +\infty, \end{cases} \tag{9.24}$$

and which, from an asymptotic viewpoint, is quite more remarkable than the very special identity (9.8) valid only for the choice $w(x) = e^{cx}$. Now, for $n \geq 3$ formula (4.2) yields:

$$\begin{cases} H_n[f(x)w(x)] = (f(x))^n \cdot (w(x))^n \cdot W(\psi'_1(x), \dots, \psi'_{n-1}(x)), \\ \psi_i(x) := \left(\frac{(f(x)w(x))^{(i)}}{f(x)w(x)} \right)', \end{cases} \tag{9.25}$$

and our whole study on asymptotic behaviors of Wronskians suggests that suitable assumptions on f, w will grant

$$W(\psi'_1(x), \dots, \psi'_{n-1}(x)) \sim W\left(\left(\frac{f'(x)}{f(x)} \right)', \dots, \left(\frac{f^{(n-1)}(x)}{f(x)} \right)' \right) \tag{9.26}$$

hence the “asymptotic factorization relation” or, if preferred, “asymptotic multiplication formula”:

$$H_n[f(x)w(x)] \sim (w(x))^n \cdot H_n[f(x)], \quad x \rightarrow +\infty, \quad (n \geq 3), \tag{9.27}$$

wherein *the factored-out function is expected to be the one such that its logarithmic derivative has the least growth-order at $+\infty$* . Proposition 5.3 is a meaningful example of such a contingency and is included in the following

Theorem 9.4 (The “asymptotic factorization relation” for Hankelians involving functions in the classes $\mathcal{R}_\alpha(+\infty)$, $\alpha \in \mathbb{R}$, or exponentials of such functions). *The asymptotic relation (9.27), $n \geq 3$, holds true in each one of the following cases:*

(I) *If*

$$\begin{cases} f \in \{ \mathcal{R}_{\alpha_0}(+\infty) \text{ of order } 2n-2 \} \text{ with } \alpha_0 \in \mathbb{R} \setminus \{0, 1, \dots, n-2\}, \\ w \in \{ \mathcal{R}_0(+\infty) \text{ of order } 2n-2 \}, \end{cases} \tag{9.28}$$

a result based on the simple Theorem 3.2.

(II) If

$$\begin{cases} f(x) := \exp(\tilde{f}(x)) & \text{with } |\tilde{f}^{(k)}| \in \mathcal{R}_{\gamma-k}(+\infty), (\gamma > 0; 0 \leq k \leq 2n-3), \\ w(x) := \exp(\tilde{w}(x)) & \text{with } |\tilde{w}^{(k)}| \in \mathcal{R}_{-k}(+\infty), (0 \leq k \leq 2n-3), \end{cases} \quad (9.29)$$

which essentially is Proposition 5.3.

In the next theorem, there is a change of notations.

Theorem 9.5 (The “asymptotic factorization relation” for Hankelians involving exponentials of functions in the classes $\mathcal{R}_{\pm\infty}(+\infty)$).

(I) For a function of type:

$$\psi(x) := \exp(R_\gamma(x)) \cdot \exp(R(x)) \text{ where } \begin{cases} R \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 2n-2\}, \\ R_\gamma \in \{\mathcal{SR}_\gamma(+\infty) \text{ of order } 2n-3\}, \\ \gamma \in \mathbb{R}, \end{cases} \quad (9.30)$$

the following relation holds true for $n \geq 2$:

$$H_n[\exp(R_\gamma(x)) \cdot \exp(R(x))] \sim \exp(nR_\gamma(x)) \cdot H_n[\exp(R(x))], \quad x \rightarrow +\infty. \quad (9.31)$$

(II) For a function of type:

$$\psi(x) := \exp(R_\gamma(x)) \cdot \exp(R(x)) \text{ where } \begin{cases} R \in \mathcal{R}_{-\infty}(+\infty), 0 \leq k \leq 2n-3, \\ |R_\gamma^{(k)}| \in \mathcal{R}_{\gamma-k}(+\infty), 0 \leq k \leq 2n-3, \\ 0 < \gamma \neq 1, \end{cases} \quad (9.32)$$

the following relation holds true for $n \geq 2$:

$$H_n[\exp(R_\gamma(x)) \cdot \exp(R(x))] \sim \exp(nR(x)) \cdot H_n[\exp(R_\gamma(x))], \quad x \rightarrow +\infty. \quad (9.33)$$

Proof of Theorem 9.4. (I) For $n = 2$ the relations:

$$\begin{cases} f'(x)/f(x) \sim \alpha_0 x^{-1}, & f''(x)/f(x) \sim \alpha_0(\alpha_0 - 1)x^{-2} \quad (\alpha_0 \neq 0); \\ w'(x)/w(x) = o(x^{-1}), & w''(x)/w(x) = o(x^{-2}); \end{cases}$$

imply, as a special case of the equivalences in ([4], Prop. 3.2):

$$\begin{cases} \left(\frac{f'(x)}{f(x)} \right)' \equiv \left(\frac{f''(x)}{f(x)} \right) - \left(\frac{f'(x)}{f(x)} \right)^2 \sim -\alpha_0 x^{-2}; \\ \left(\frac{w'(x)}{w(x)} \right)' = o(x^{-2}); \end{cases}$$

whence (9.24). For greater values of n , condition $\alpha_0 \neq 0$ implies

$f w \in \{\mathcal{R}_{\alpha_0}(+\infty) \text{ of order } 2n-2\}$ by ([3], Prop. 7.3-(I)), and the other restrictions on α_0 stated in (9.32) imply, by Theorem 3.2-(II)-formula (3.7), the asymptotic relation

$$\begin{aligned} H_n[f(x)w(x)] &\sim C(\alpha_0, n) \cdot (f(x)w(x))^n \cdot x^{-n(n-1)}, \\ &\equiv (w(x))^n \cdot C(\alpha_0, n) \cdot (f(x))^n \cdot x^{-n(n-1)}, \quad x \rightarrow +\infty, \end{aligned} \quad (9.34)$$

where $C(\alpha_0, n)$ is a suitable constant depending only on α_0, n and not on the specific functions f, w . Hence, by the same formula (3.7), (9.34) coincides with

(9.27).

(II) Obviously $\tilde{w}(x) = o(\tilde{f}(x))$ and, by an elementary result in ([4], Prop. 2.1-(iii), first line after formula (2.27)), conditions (9.29) imply

$$|f^{(k)} + w^{(k)}| \in \mathcal{R}_{\gamma-k}(+\infty), \quad (0 \leq k \leq 2n-3),$$

Whence, by Theorem 5.1:

$$\begin{aligned} H_n[f(x)w(x)] &\equiv H_n[\exp(\tilde{f}(x) + \tilde{w}(x))] \\ &\sim C(\gamma, n) \cdot x^{-n(n-1)} \cdot (\tilde{f}(x) + \tilde{w}(x))^{n(n-1)/2} \cdot \exp(n\tilde{f}(x) + n\tilde{w}(x)) \\ &\sim C(\gamma, n) \cdot x^{-n(n-1)} \cdot (\tilde{f}(x))^{n(n-1)/2} \cdot \exp(n\tilde{f}(x)) \cdot \exp(n\tilde{w}(x)) \\ &\equiv \exp(n\tilde{w}(x)) \cdot H_n[\exp(\tilde{f}(x))] \equiv (w(x))^n \cdot H_n[f(x)], \quad x \rightarrow +\infty. \end{aligned} \tag{9.35}$$

□

Proof of Theorem 9.5. (I) We write

$$\psi(x) \equiv \exp(R(x) + R_\gamma(x)) \tag{9.36}$$

and notice the elementary relations, ([4], (2.19) and (2.41)), as $x \rightarrow +\infty$:

$$\begin{cases} R(x) \gg x^\beta, \quad \forall \beta > 0; \quad R_\gamma(x) \ll x^{\gamma+1}; \\ R'_\gamma(x)/R_\gamma(x) = o(R'(x)/R(x)) \quad (\text{by the very definition (1.17)}); \end{cases} \tag{9.37}$$

so that we may apply Proposition 2.8-(II), with $p=2$, and infer that $R + R_\gamma$ belongs to the same class of R . Proposition 8.2 in turn implies relation (8.18) with $R(x)$ replaced by $R(x) + R_\gamma(x)$ and (9.31) follows from the relations:

$$\begin{cases} R(x) + R_\gamma(x) \sim R(x), \quad x \rightarrow +\infty; \\ R'(x) + R'_\gamma(x) \sim R'(x), \quad x \rightarrow +\infty; \end{cases} \tag{9.38}$$

the last being implied by the first derivatives R', R'_γ satisfying relations analogous to those in the first line of (9.37).

(II) Proposition 2.8-(III) implies that $R_\gamma + R$ satisfies the same inclusion properties as R_γ in (9.32) and that “ $R_\gamma(x) + R(x) \sim R_\gamma(x)$ ”; hence we have to apply Proposition 5.1 to the function $\psi(x) \equiv \exp(R_\gamma(x) + R(x))$ so getting the asymptotic relation:

$$\begin{aligned} &H_n[\exp(R_\gamma(x)) \cdot \exp(R(x))] \\ &\sim V(\gamma - 2, 2\gamma - 3, \dots, (n-1)\gamma - n) \cdot (n-1)! \cdot \gamma^{n(n-1)/2} (\gamma - 1)^{n-1} \\ &\quad \times x^{-n(n-1)} \cdot (R_\gamma(x) + R(x))^{n(n-1)/2} \cdot \exp(nR_\gamma(x) + nR(x)) \\ &\sim \exp(nR(x)) \cdot H_n[\exp(R_\gamma(x))], \quad x \rightarrow +\infty, \quad (n \geq 2). \end{aligned} \tag{9.39}$$

10. Historical and Bibliographical Notes

-(References for §9.) Formulas (9.19), (9.20) and (9.21) are due to Radoux: formulas (9.19) and (9.21) essentially appear in [7] whereas formula (9.20) essentially appears in [8]; see also [9]. In each case, after noticing the immediately-checked formulas for $n = 1, 2$, the author starts doing the inductive proof. But he

must surely have carried out the explicit calculations for at least $n = 3$ to guess the structures of the general formulas which are quite different in the examined cases. In the present asymptotic context, the direct calculations for $n = 3$, and even for $n = 4$, have been replaced by the previously-proved asymptotic formulas.

We have no explicit reference for factorization identity (9.9) though it is related to the characterization of exponential polynomials as expounded, e.g., by Karlin ([6], Chap. 2, §7). This identity may have been of little interest from an algebraic viewpoint due to the severe restrictions in (9.10) but it reveals useful in asymptotic investigations.

-(*A complement to a previous historical note.*) In ([10], p. 66) some historical comments appear about the Wronskian identity

$$\left\{ \begin{array}{l} W(u_1(x), \dots, u_h(x), v_1(x), \dots, v_k(x)) \\ = [W(u_1(x), \dots, u_h(x))]^{1-k} \cdot W(w_1(x), \dots, w_k(x)) \\ \text{where } \begin{cases} w_i(x) := W(u_1(x), \dots, u_h(x), v_i(x)), \\ i = 1, 2, \dots, k; \quad h \geq 1; \quad k \geq 2. \end{cases} \end{array} \right. \quad (10.1)$$

Recently we happened to find another direct proof in a paper by Browne and Nilsen ([11], §3) who remarked that the first known proof goes back to Frobenius (1874), a proof requiring the restrictive assumption that all the Wronskians $W(u_1(x), \dots, u_i(x))$, $1 \leq i \leq h$, do not vanish on the given interval, as does the proof appearing in the book by Karlin ([6], p. 60). Hence we find in the literature four direct proofs of the above identity chronologically ordered as follows:

Frobenius (1874), Karlin (1968), Brunet (1975), Browne and Nilsen (1980) each author, save the first, seeming to be unaware of the proof by the immediate predecessor.

11. A Discussion about the Assumptions Granting the Presented Asymptotic Relations, and Collective Conclusions for Parts I and II

The present paper concludes the author's cycle of investigation about asymptotic behaviors of Wronskians of non-oscillatory functions of one-real variable. Here and there we have inserted pertinent applications to asymptotic expansions inferred from the summability of certain expressions involving ordinary differential operators, but further studies wait to be made concerning asymptotic properties of differential equations.

(I) At this point, a detailed discussion about the played role of the theory of "higher-order types of asymptotic variation" may be welcomed by those readers not too familiar with this theory and that might think that the involved classes of functions are too restricted.

The treatment of the case studied in §8 confirms the conclusion in ([1], §6) that: investigating the asymptotic behaviors of Wronskians requires the whole mentioned theory developed in previous papers: a lot of properties of and operations with such functions. Otherwise, one may only obtain rough asymptotic estimates with quite

a limited import for applications. Let us make this point clearer.

The reader must bear in mind that our study originated from a general analytic theory of asymptotic expansions in the real domain for strictly one-sided functions; more precisely from the need of practically applying certain integral conditions involving ratios of Wronskians. In order to establish the convergence of such integrals mere “ O ”- or “ o ”-estimates are useless: the exact order of growth is needed either in the weak sense of “ $W(x) \asymp \phi(x)$ ” or in the strong sense of “ $W(x) \sim \phi(x)$ ” with a strictly one-signed comparison function ϕ . Now, the theory of higher-order types of asymptotic variation (which by no means is an elementary theory if developed in great detail as in [3]-[5] [12]) and an abundance of pertinent examples and counterexamples show that the classes of regularly-, rapidly- and exponentially-varying functions practically include the majority of those functions which, they themselves and their derivatives up to a certain order are strictly one-signed on a neighborhood of some fixed point. In the framework of asymptotic analysis, these classes are not a limitation or a too-restricted class. On the contrary, these three types of asymptotic variation arrange in an organized system the essential types of non-oscillatory asymptotic behaviors and, as said above, they practically are the largest class of differentiable functions allowing nice and quite-precise asymptotic results. Effective elementary examples will convince the reader that this is no exaggerated claim.

One of the basic and historically-motivating results about regularly- and rapidly-varying functions is the precise description of the asymptotic behavior of their antiderivatives, the complete results being reported in ([12], Proposition 2.4). Let us consider, e.g., the case of functions f positive on an interval $[T, +\infty)$ and with divergent integrals for which the following results hold true. For a regularly-varying function:

$$f \in \mathcal{R}_\alpha(+\infty), -1 < \alpha < +\infty, \Rightarrow \int_T^x f \sim \frac{xf(x)}{\alpha+1}, x \rightarrow +\infty; \quad (11.1)$$

whereas two different results, a weak estimate and the exact asymptotic behavior, can be pointed out for a rapidly-varying function:

$$\left\{ \begin{array}{l} f \in \mathcal{R}_{+\infty}(+\infty) \Rightarrow \int_T^x f \in \mathcal{R}_{+\infty}(+\infty) \\ \text{i.e. } \int_T^x f = o(xf(x)), x \rightarrow +\infty; \end{array} \right. \quad (11.2)$$

$$\left\{ \begin{array}{l} f \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 1\} \\ \Rightarrow \int_T^x f \sim (f(x))^2 / f'(x) \equiv f(x) / D_\ell f(x), x \rightarrow +\infty. \end{array} \right. \quad (11.3)$$

Though the assumptions in (11.1) and (11.3), generally speaking, are not necessary for the validity of the respective claims they will be found the most natural ones.

The regularly-varying case. As far as (11.1) is concerned, after noticing that the result is trivially true for an f such that $f(x) \sim x^\alpha$, $x \rightarrow +\infty$, look at the following chain of trivial equivalences and one formal inference:

$$\begin{aligned}
\int_T^x f &\sim \frac{xf(x)}{\alpha+1} \Leftrightarrow \lim_{x \rightarrow +\infty} \frac{\int_T^x f}{xf(x)} = (\alpha+1)^{-1} \\
&\stackrel{\text{L'Hospital}}{\Leftrightarrow} \lim_{x \rightarrow +\infty} \frac{f(x)}{f(x) + xf'(x)} = (\alpha+1)^{-1} \\
&\Leftrightarrow \lim_{x \rightarrow +\infty} \left(1 + \frac{xf'(x)}{f(x)} \right) = \alpha+1 (> 0) \\
&\Leftrightarrow \lim_{x \rightarrow +\infty} \frac{xf'(x)}{f(x)} = \alpha (> -1).
\end{aligned} \tag{11.4}$$

These show that the first asymptotic relation obtains as soon as L'Hospital's rule is applicable, namely if $xf(x) \rightarrow +\infty$ which is granted by $\alpha > 1$ and one of the basic elementary asymptotic estimates associated to regular variation ([4], formula (2.19), p. 784) and if $f(x) + xf'(x) \neq 0$ for x large enough granted by the last limit in (11.4) as:

$$f(x) + xf'(x) = f(x) \left[1 + \frac{xf'(x)}{f(x)} \right];$$

and the reader might agree that these are quite natural circumstances. The following two counterexamples show the unreliability of conditions other than the last limit in (11.4) in order to get the exact asymptotic behavior of the integral.

First counterexample:

$$\left\{ \begin{array}{l} f(x) := 2x - \cos x \sim 2x, \quad x \rightarrow +\infty; \quad f'(x) = 2 + \sin x \asymp 1; \\ xf'(x)/f(x) = \frac{2 + \sin x}{2 + o(1)} \asymp 1 \text{ but with no definite limit at } +\infty; \\ \int_0^x f \sim x^2 \sim \frac{xf(x)}{2} \text{ as in (11.1).} \end{array} \right. \tag{11.5}$$

Second counterexample:

$$\left\{ \begin{array}{l} f(x) := (3 + \sin(\log x)) \cdot x; \quad f'(x) = 3 + \sin(\log x) + \cos(\log x); \\ xf'(x)/f(x) = \frac{3 + \sin(\log x) + \cos(\log x)}{3 + \sin(\log x)} \asymp 1, \quad x \rightarrow +\infty; \\ \int_1^x f = \frac{3x^2}{2} + \frac{2}{5}x^2 \sin(\log x) - \frac{1}{5}x^2 \cos(\log x) + \frac{13}{10} \\ \asymp x^2 \left[\frac{3}{2} + \frac{2}{5} \sin(\log x) - \frac{1}{5} \cos(\log x) \right] \\ \text{which is not the kind of relation in (11.1).} \end{array} \right. \tag{11.6}$$

By the way, these two counterexamples may be considered examples of the following weak result, not mentioned by the author in the four papers [3]-[5] [12]:

$$\begin{aligned}
&\left\{ \begin{array}{l} f \in AC^1[T, +\infty); \quad f > 0; \quad \int_T^{+\infty} f = +\infty; \\ 1 + xf'(x)/f(x) \asymp 1, \quad x \rightarrow +\infty; \end{array} \right. \\
&\Rightarrow \int_T^x f = O(xf(x)) \quad x \rightarrow +\infty.
\end{aligned} \tag{11.7}$$

In fact, the last assumption in (11.7) means

$$0 < M_1 \leq \left| 1 + \frac{xf'(x)}{f(x)} \right| \leq M_2 < +\infty \quad \forall x \text{ large enough,}$$

and the claim follows from a chain of inferences like that in (11.4) where one applies the most general version of L'Hospital's rule:

$$\underline{\lim}_{x \rightarrow +\infty} \frac{h'(x)}{g'(x)} \leq \overline{\lim}_{x \rightarrow +\infty} \frac{h(x)}{g(x)} \leq \overline{\lim}_{x \rightarrow +\infty} \frac{h'(x)}{g'(x)}. \tag{11.8}$$

The rapidly-varying case. The essential difference between the assumptions in (11.2) and in (11.3) is that f' satisfies

$$(f(x)/f'(x))' = o(1), \quad x \rightarrow +\infty.$$

Here again two counterexamples will show that, in order to get a precise result on the asymptotic behavior of the integral, it is not easy to get rid of a condition like this even if the implied condition " $f(x)/f'(x) = o(x)$, $x \rightarrow +\infty$," see (1.27), is replaced by the stronger one $f(x)/f'(x) \asymp 1$, $x \rightarrow +\infty$. These examples appear in ([12]; formulas (2.89) and (4.21)-(4.26)).

First counterexample:

$$\left\{ \begin{array}{l} f(x) := (3 + \cos x) \cdot \exp(3x + \sin x); \\ f'(x) = (9 + \cos^2 x + 6 \cos x - \sin x) \cdot \exp(3x + \sin x); \\ f(x)/f'(x) \asymp 1, \quad x \rightarrow +\infty; \quad (f(x)/f'(x))' \neq o(1), \quad x \rightarrow +\infty; \\ \int_0^x f = \exp(3x + \sin x) - 1; \end{array} \right. \tag{11.9}$$

a case wherein: $f \in \mathcal{R}_{+\infty}(+\infty)$; f, f' are strictly positive on $[1, +\infty)$; $F(x) := \int_0^x f_1$ is strictly positive, increasing and convex on $[1, +\infty)$ but the thesis in (11.3) does not hold.

Second counterexample:

$$\left\{ \begin{array}{l} f_c(x) := ce^x + \sin(e^{x/2}) = ce^x + O(1), \quad x \rightarrow +\infty; \quad c \neq 0; \\ f'_c(x) = ce^x + \frac{1}{2}e^{x/2} \cos(e^{x/2}) \sim ce^x; \\ f''_c(x) = ce^x \left[1 - \frac{1}{4c} \sin(e^{x/2}) + o(1) \right] \equiv \begin{cases} c \cdot b(x)e^x & \text{if } |c| > 1/4, \\ c \cdot \omega(x)e^x & \text{if } |c| \leq 1/4; \end{cases} \\ \int_0^x f \sim f(x), \quad x \rightarrow +\infty, \text{ as in (11.3);} \end{array} \right. \tag{11.10}$$

wherein ultimately " $0 < M_1 \leq b(x) \leq M_2 < +\infty$ ", whereas $\omega(x)$ changes sign infinitely often though being bounded. With calculations not reported here it is checked that for $|c| > 1/4$ we have an example of a function f such that:

$$\left\{ \begin{array}{l} f, f' \in \mathcal{R}_{+\infty}(+\infty) \text{ i.e. } \lim_{x \rightarrow +\infty} xf'(x)/f(x) = \lim_{x \rightarrow +\infty} xf''(x)/f'(x) = +\infty, \\ (f/f')' \text{ is bounded but not convergent at } +\infty; \\ \text{hence } f \notin \{ \mathcal{R}_{+\infty}(+\infty) \text{ of order 1} \} \text{ in the strong restricted sense of Definition 1.1.} \end{array} \right. \tag{11.11}$$

(II) The reader who has taken upon him/herself the cumbersome task of checking all the steps in one of the many proofs in this work or in one of the related works by the author, might have noticed that, though the calculations are quite long and complicated, nevertheless, as if by magic, everything goes to its place and the final result is a readable and, may be, useful one. These are the effects of the theory of higher-order types of asymptotic variation whose basic relations, see §1, reflect the harmonious structures of the differential-calculus formulas for the three elementary functions: powers, logarithms, exponentials. These structures are paralleled by those of the classical formulas for higher-order derivatives of products, composition and inversion, ([3], formulas (6.1) and (6.4)), both sides in each formula being obviously consistent with the rules of classical dimensional analysis, ([13], Chap. 1). The reader may easily check that the various asymptotic formulas highlighting the principal parts of Wronskians, in particular Hankelians, are consistent with dimensional analysis as well.

For instance, let us use the notations in [13] for the dimensions of physical quantities in a given class of systems of units:

$$\begin{cases} X := \text{dimension of the independent variable } x \equiv [x]; \\ \Phi_i := \text{dimension of the function } \phi_i(x) \equiv [\phi_i]; \\ [\phi_i'] = \Phi_i \cdot X^{-1}; \quad [\phi_i^{(k)}] = \Phi_i \cdot X^{-k}, \quad k \in \mathbb{N}. \end{cases}$$

Now, the Wronskian $W(\phi_1, \dots, \phi_n)$ being a sum of products of type $\pm \prod_{i=1}^n \phi_i^{(p_i)}(x)$ where (p_1, \dots, p_n) is a permutation of $(0, 1, \dots, n-1)$, its dimension is given by:

$$\begin{aligned} [W(\phi_1, \dots, \phi_n)] &= \prod_{i=1}^n \Phi_i \cdot X^{-p_i} = \left(\prod_{i=1}^n \Phi_i \right) \cdot X^{-(0+1+\dots+n-1)} \\ &= \left(\prod_{i=1}^n \Phi_i \right) \cdot X^{-n(n-1)/2}, \end{aligned}$$

consistently, e.g., with ([2], relation (131)) and ([2], relation (142)). In this last relation, we have $[\phi] \equiv [\phi/\phi'] = X^{-1}$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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