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Consequences of Invariant Functions for the Riemann Hypothesis

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Abstract

This paper attempts to form a bridge between a sum of the divisors function and the gamma function, proposing a novel approach that could have significant implications for classical problems in number theory, specifically the Robin inequality and the Riemann hypothesis. The exploration of using invariant properties of these functions to derive insights into twin primes and sequential primes is a potentially innovative concept that deserves careful consideration by the mathematical community.

Keywords

LambertW Function, Principal Branch, Riemann Hypothesis, Iterations, Robin Inequality, Robin Integers, Invariance, Gauss Gamma Function, Li-Function, Prime Counting Function, Sums of Divisors, Invariance, Primes, Twin Primes

1. Introduction

The Gamma-function, denoted as ($\Gamma(s)$), was first introduced by Swiss mathematician Leonhard Euler [1] in 1729. Euler's deep insights into Γ-function led to numerous results that provide key insights into many fields of mathematics including Probability theory and Statistics. Other major contributions to the development of the Γ-function used in this paper were developed by Carl Freidman Gauss [2]. Gauss's work led to the famous reflection formula of the ζ-function, and numerous new results that will be presented in this paper. A key insight into the Γ-function is its multiplicative nature. So far, there has been little development in the additive representation of the Γ-function. The form of the Γ-function ([3], p. 895):

$$\Gamma(s) \sim z^{s-\frac{1}{2}} e^{-s} \sqrt{2\pi} \left\{ 1 + \frac{1}{12z} + \frac{1}{288s^2} - \frac{139}{51840s^3} - \frac{571}{2488320s^4} + O\left(s^{-5}\right) \right\}, \tag{1}$$

$$\left[|\arg s| < \pi \right]$$

For s real and positive, the remainder of the series is less than the last term that is retained.

Similar series exists for $\ln(\Gamma(s))$. It will be significant if other forms of these series can be found.

The product form of the Γ-function due to Gauss, provides further insights into many relations that will be developed in this paper. The product form is given by, ([3], p. 896):

$$\Gamma(y \cdot n) = (2\pi)^{\frac{1-y}{2}} y^{(n \cdot y)-\frac{1}{2}} \prod_{k=0}^{y-1} \Gamma\left(n + \frac{k}{y}\right) \tag{2}$$

Certain invariant relations of the product Γ-function will be developed in this paper to show the connections of the Γ-function to other functions, particularly the Riemann-Zeta function, denoted by $\zeta(s)$. The Γ-function will not be complete without the mention of the ζ-function. The ζ-function, is defined by the series:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, \mathbb{R}(s) > 1 \tag{3}$$

The importance of the ζ-function is its relation to the distribution of primes and the Riemann hypothesis. There is a one-on-one correspondence between the non-trivial roots of the function and the primes. The ζ-function also has a product relation for primes p, given by ([3], p. 1037);

$$\zeta(s) = \prod_p \left(\frac{1}{1 - p^{-s}} \right), \quad \mathbb{R}(s) > 1 \tag{4}$$

Both the ζ-function, and the Γ-function are factorable. These two functions are related by the ζ-function reflection formula developed by Gauss given by ([3], p. 1038):

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{\frac{s-1}{2}} \zeta(1-s) \tag{5}$$

These relations are well studied, and they provide a wealth of information in Number theory and many disciplines in Mathematics.

2. Background

The Riemann Hypothesis, proposed by Bernhard Riemann [4] in 1859, conjectures about the distribution of prime numbers and their relation to the Riemann Zeta function, $\zeta(s)$. It correlates the non-trivial zeros of the ζ-function with the primes if the zeros of the ζ-function have a real part, $\frac{1}{2}$. This hypothesis (RH) is crucial to the understanding of the distribution of primes. The Robin criterion first specified Guy Robin [5] in 1984 relates the truth of the RH hypothesis to a direct correlation to the statement (Robin's inequality)

$$F_n = \frac{\sigma(n)e^{-\gamma}}{n\log\log(n)} < 1 \tag{6}$$

if and only if the set of numbers (I call them the Robin Integers, R_n):

$$R_n \in [3,4,5,6,8,9,10,12,16,18,20,24,30,36,48,60,72,84,\\ 120,180,240,360,720,840,2520,5040]$$

are the only integers that violate the inequality,

$$F_n = \frac{\sigma(n)e^{-\gamma}}{n\log\log(n)} < 1, \tag{7}$$

We will deal with this function in its various forms in this paper. This paper also studies these functions and their interrelations and hence provides an objective insight to the beauty of these relations to the Riemann Hypothesis. Further, the importance of functions such as the Γ-functions, the Bernoulli functions, B_m, and their relationships to the ζ-function is also explored.

The Bernoulli function is intimately related to the ζ-function by the relations ([3], p. 1048):

$$\zeta(2m) = \frac{2^{(2m-1)}\pi^{(2m)}|B_{2m}|}{(2m)!}, \qquad \zeta(1-2m) = -\frac{B_{2m}}{2m} \tag{8}$$

With this in mind, the present paper explores these relationships with some profound results. The distribution of primes and the confluence of the Robin criterion will be clearly shown. Results in prime numbers such as the twin prime conjecture will also be discussed. There are key relations for primes that will be highlighted in this article. In general, these relations stem from the ζ-function, and Γ-function and in particular, the Gauss's Γ-function product formula. The intriguing relations of the Sum of Divisors function $\sigma(m)$, with the ζ-function and the Riemann Hypothesis was developed by several pioneers including Guy Robin [5]. The Sums of Divisors is defined as

$$\sigma(m) = \sum_{d|n} d \tag{9}$$

The uniformity of the function for primes, p, $\sigma(p^x) = (p^{x+1}-1)/(p-1)$, provides a way to link the $\sigma(m)$ function to the Robin inequality and hence, the Riemann Hypothesis.

Note: If $\prod_{k=1}^{K} p_k^{a_k}$ is the prime factorization of m, where m is an integer with K factors, and a_k are particular numbers, then,

$$\sigma(m) = \begin{cases} \prod_{k=1}^{K} \frac{p_k^{(a_k+1)^r}}{p_k^r - 1} & r > 0 \\ \prod_{k=1}^{K} (a_k+1) & r = 0 \end{cases} \tag{10}$$

where, $\gamma = 0.5772156649\cdots$ is the Euler-constant.

Several attempts to refine the fundamental relations for F_n as a referendum on the Riemann hypothesis have been made with great success. P. Solé and M. Planat [6], showed in 1962 that, if N_n is the primordial number of index-n, defined as the product of the first n primes, with $N_n = \prod_{k=1}^{n} p_k$, then,

$$\lim_{n\to\infty} N_n = \frac{e^{\gamma}}{\zeta(2)} \sim 1.08, \tag{11}$$

J. B. Roser ([7], Theorem 15), proved that there are inequalities between Euler's totient function $\varphi(n)$, and $e^{\gamma} n \log\log(n)$,

$$\frac{n}{\varphi(n)} \le e^{\gamma} n \log\log(n), \quad n \ge 3 \tag{12}$$

In 1984 Heath-Brown [8], showed that there are infinitely many positive integers m, for which the divisor function has $d(m) = d(m+B)$, where B is an integer. C. G. Pinner [9] also demonstrated that there are repeated values of the divisor function. Y. J. Chloe, N. Lichiardopol, and P, Moree [10] also studied the Robin criteria as did Sayak Chakrabarty [11], and they found possible correlations of the repeated divisor function with Highly composite numbers. Jean-Louis Nicolas [12] proved that there are an infinite number of values of $n \ge 3$, that satisfy the inequality (12).

In recent times, I focused on iterations of number theoretic functions. In the paper the Towering Zeta Function [13], I show that the Zeta function can be iterated T-times, denoted as $\zeta^T(z)$ which tends to a constant when T approaches infinity:

$$\lim_{T\to\infty} \zeta^T(z) = \zeta\left(\lim_{T\to\infty} \operatorname{slog}_n(z)\right) = \zeta(z_w) = z_w = -0.29590500557213955\cdots$$

I also showed that the Zeta function only converge to the real constant when the complex part of conjugate Zeta function roots, T, are expressed in the exponential form:

$$e^{2i\theta} = \frac{\frac{1}{2} + iT}{\frac{1}{2} - iT}, \quad \text{where } \theta = \tan^{-1} 2T,$$

In 2024, I showed [14] that Perron's formula given by:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - x^{-2}\right) - \ln 2\pi \tag{13}$$

Involves the complex roots ρ, of the Zeta function can be iterated to condense to the k^{th} iterant:

$$x - \Psi^k(x) = k\left(x - \Psi(x)\right) \tag{14}$$

Such insights could give us a significant understanding of some of the most complex problems in mathematics.

3. The Invariance of the Gamma Function to Substitution $\sigma(m) \to \sigma(m+j)$

I first want to introduce the curious fact that any function with a relational product $\{n \cdot y\}$, can be represented by the Sums of Divisor function, $\sigma(m)$. Here is a simple example:

$$\log(n \cdot y) = \log n + \log y, \tag{15}$$

Then, if $n \cdot y = m$, we can put $n = \sigma(m)$, $y = \frac{m}{\sigma(m)}$, and so,

$$\log(m) = \log \sigma(m) + \log \frac{m}{\sigma(m)} \tag{16}$$

Here is another example:

If $n \cdot y = m$, we can put $n = \sigma(m)$, $y = \frac{m}{\sigma(m)}$, and so, applied to the formula ([3], p. 41):

$$\cos(n \cdot y) = \cos(y) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(y)}{\sin^2 \frac{(2k-1)\pi}{2n}} \right), \quad [n \text{ is even}] \tag{17}$$

$$\sin(n \cdot y) = n \sin(y) \cos(y) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(y)}{\sin^2 \frac{k\pi}{n}} \right), \quad [n \text{ is even}] \tag{18}$$

By using the sum of divisor function, $n = \sigma(m)$, $y = \frac{m}{\sigma(m)}$, we find the relation:

$$\cos(m) = \cos\left(\frac{m}{\sigma(m)}\right) \prod_{k=1}^{\frac{\sigma(m)-1}{2}} \left(1 - \frac{\sin^2\left(\frac{m}{\sigma(m)}\right)}{\sin^2 \frac{(2k-1)\pi}{2\sigma(m)}} \right), \quad [\sigma(m) \text{ is odd}] \tag{19}$$

Interestingly, (17) and (19) for example, differentiate between odd and even values of $\sigma(m)$! Since primes have $\sigma(p) = p+1$, an even number the relation (18) *does not apply to primes*. Since $p+1$ is always even except for the prime 2! Since $\sigma(2) = 3$,

$$\cos(2) = \cos\left(\frac{2}{3}\right) \prod_{k=1}^{1} \left(1 - \frac{\sin^2\left(\frac{2}{3}\right)}{\sin^2 \frac{(2k-1)\pi}{6}} \right), \quad [\sigma(2) \text{ is odd}] \tag{20}$$

$$\begin{aligned} -0.4161468365\cdots &= 0.7858872608\cdots\left(1 - \frac{0.3823812134\cdots}{0.2500000000}\right) \\ &= -0.4161468365\cdots \end{aligned} \tag{21}$$

The fact that the sum of divisor function $\sigma(m)$, can be manipulated this way leads to some interesting formulas that can produce significant and unexpected results.

For example, if we put:

$$n = \sigma(m),\ x = \frac{n}{\sigma(m)}, \tag{22}$$

in the following trigonometric relations, we get:

$$\left.\begin{array}{l} \sin(m) = \sigma(m)\sin\left(\dfrac{m}{\sigma(m)}\right)\cos\left(\dfrac{m}{\sigma(m)}\right)\displaystyle\prod_{k=1}^{\frac{\sigma(m)-2}{2}}\left(1-\dfrac{\sin^2\left(\dfrac{m}{\sigma(m)}\right)}{\sin^2\dfrac{\pi k}{\sigma(m)}}\right) \\ \cos(m) = \displaystyle\prod_{k=1}^{\sigma(m)}\left(1-\dfrac{\sin^2\left(\dfrac{m}{2\sigma(m)}\right)}{\sin^2\dfrac{(2k-1)\pi}{2\sigma(m)}}\right) \end{array}\right\}\left[\sigma(m)\text{ is even}\right] \tag{23}$$

$$\left.\begin{array}{l} \sin(m) = \sigma(m)\sin\left(\dfrac{n}{\sigma(m)}\right)\displaystyle\prod_{k=1}^{\frac{\sigma(m)-2}{2}}\left(1-\dfrac{\sin^2\left(\dfrac{m}{\sigma(m)}\right)}{\sin^2\dfrac{k\pi}{\sigma(m)}}\right) \\ \cos(m) = \cos\left(\dfrac{m}{\sigma(m)}\right)\displaystyle\prod_{k=1}^{\frac{\sigma(m)-1}{2}}\left(1-\dfrac{\sin^2\left(\dfrac{m}{\sigma(m)}\right)}{\sin^2\dfrac{(2k-1)\pi}{2\sigma(m)}}\right) \end{array}\right\}\left[\sigma(m)\text{ is odd}\right] \tag{24}$$

Any factorizable integer-function can be done this way. Of-course the construct restricts the functions we use to define the integers, m. Clearly, the relationship of integer functions to the Gauss Γ-function is clear. Again, we can represent the relationship of the Gauss gamma function ([3], p.896) as:

$$\Gamma(y\cdot n) = (2\pi)^{\frac{1-y}{2}}\, y^{(n\cdot y)-\frac{1}{2}}\prod_{k=0}^{y-1}\Gamma\left(n+\frac{k}{y}\right) \tag{25}$$

Let $g(z)$ be a function with k-factors, then for any given factor, $f_j(m), j<k$,

$$g(z) = f_j(z)\left(\prod\nolimits_{n=1}^{j-1} f_n(z)\prod\nolimits_{n=j+1}^{k} f_n(z)\right)$$

and, putting $f_j(z) = y, \left(\prod\nolimits_{n=1}^{j-1} f_n(z)\prod\nolimits_{n=j+1}^{k} f_n(z)\right) = n$,

$$\Gamma(g(z)) = (2\pi)^{\frac{1-f_j(z)}{2}}\, y^{g(z)-\frac{1}{2}}\prod_{k=0}^{f_j(z)-1}\Gamma\left(\frac{g(z)+k}{f_j(z)}\right) \tag{26}$$

The relation (26) is only true if $f_j(m)\in\mathbb{Z}(integers)$. Observe that there *may* exist multiple integer factors of $g(z)$ that satisfy (26). It is interesting to note that the product form of two variables of the gamma function is behaving like a *non-*

commutative function (I use that word loosely) since, $\Gamma(n\cdot y)\neq\Gamma(y\cdot n)$, when the roles of n, and y, are interchanged. This may provide a means to understand the sum of divisors function $\sigma(v)$.Suppose, we consider values of $n=\sigma(v)$, $y=\frac{m}{\sigma(v)}$ and, let $y\cdot n=m$, then,

$$\Gamma(m)=(2\pi)^{\frac{1-\sigma(v)}{2}}\sigma(v)^{m-\frac{1}{2}}\prod_{k=0}^{\sigma(v)-1}\Gamma\left(\frac{m+k}{\sigma(v)}\right) \tag{27}$$

The relation (23) *is invariant* to the substation $\sigma(v)\to\sigma(v+\mu)$. An Interchange of the variables, gives:

$$\Gamma(y\cdot n)=(2\pi)^{\frac{1-y}{2}}y^{(n\cdot y)-\frac{1}{2}}\prod_{k=0}^{y-1}\Gamma\left(n+\frac{k}{y}\right) \tag{28}$$

then,

$$\Gamma(m)=(2\pi)^{\frac{1-\sigma(v+\mu)}{2}}\left(\frac{m}{\sigma(v+\mu)}\right)^{m-\frac{1}{2}}\prod_{k=0}^{\sigma(v+\mu)-1}\Gamma\left(1+\frac{k}{\sigma(v+\mu)}\right) \tag{29}$$

A function $g(z)$, that has integers and complex factors can be represented as a product of its factors such that the following Theorem 1 applies.

Theorem 1. Let $f_j(z)>0$, represent one integer factor of $g(z)$, then,

$$\Gamma(g(z))=(2\pi)^{\frac{1-f_j(z)}{2}}(f_j(z))^{g(z)-\frac{1}{2}}\prod_{k=0}^{f_j(z)-1}\Gamma\left(\frac{g(z)+k}{f_j(z)}\right) \tag{30}$$

is invariant with respect to choices of any other factors of $g(z)$.

The significance of the Theorem 1 is its consequences for prime numbers, and their relations to functions like the ζ-function and the sum of divisors function, $\sigma(m)$ and primes.

PROOF:

Let $f_j(z)$ be some j^{th} integer factor of k-factors a real or complex function $g(z)$. Then,

$$g(z)=\prod_{n=1}^{k}f_n(z)=f_j(z)\left(\prod_{n=1}^{j-1}f_n(z)\prod_{n=j+1}^{k}f_n(z)\right) \tag{31}$$

The Gauss gamma product formula is a simple relation given by:

$$\Gamma(n\cdot y)=(2\pi)^{\frac{1-n}{2}}n^{(n\cdot y)-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(y+\frac{k}{n}\right) \tag{32}$$

Then, since $f_j(z)$, is an integer-factor of $g(z)$, we have, putting $n=f_j(z)$, $y=\prod_{n=1}^{j-1}f_n(z)\prod_{n=j+1}^{k}f_n(z)$ in $g(z)=f_j(z)\left(\prod_{n=1}^{j-1}f_n(z)\prod_{n=j+1}^{k}f_n(z)\right)$. Then,

$$\Gamma(g(z))=(2\pi)^{\frac{1-f_j(z)}{2}}(f_j(z))^{g(z)-\frac{1}{2}}\prod_{k=0}^{f_j(z)-1}\Gamma\left(\frac{g(z)+k}{f_j(z)}\right) \tag{33}$$

If there any other integer factor labelled here $f_v(z),\in Z$, then, the substitution

$f_j(z) \to f_v(z)$ leaves $\Gamma(g(z))$ invariant. However, this is also true for any integer factors, m of $f_v(z)$, then, for any m,

$$\Gamma(g(z)) = (2\pi)^{\frac{1-m}{2}} m^{g(z)-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{g(z)+k}{m}\right) \tag{34}$$

remains invariant to the substitutions $f_j(z) \to f_v(z) \to m$.

4. The Relation of the Product Gamma Function to Primes

From the Gauss Γ-product formula,

$$\Gamma(m) = (2\pi)^{\frac{1-\sigma(m)}{2}} \sigma(m)^{m-\frac{1}{2}} \prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right) \tag{35}$$

It is clear that the following relations are equal,

$$(2\pi)^{\frac{1-m}{2}} m^{m-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{m+k}{m}\right) = (2\pi)^{\frac{1-\sigma(m)}{2}} \sigma(m)^{m-\frac{1}{2}} \prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right) \tag{36}$$

Then, for all real numbers m,

$$(2\pi)^{\frac{\sigma(m)-m}{2}} \left(\frac{m}{\sigma(m)}\right)^{m-\frac{1}{2}} \frac{\prod_{k=0}^{m-1} \Gamma\left(\frac{m+k}{m}\right)}{\prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right)} = 1 \tag{37}$$

Since, for all primes, $m = p$, $\sigma(p) = p+1$, from (37), we get for all primes, p:

$$\sqrt{2\pi} \frac{p^{p-\frac{1}{2}} \prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}} \prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{p+k}{\sigma(p)}\right)} = 1 \tag{38}$$

and so, (38) is ***not invariant*** to the substitutions, $\sigma(p) \to \sigma(p+j)$, unless $\{p, p+j\} \in \text{primes}$.

The curious properties of the Γ-function can be made analogous to the divisor function when we study Heath Brown [15]. I will just highlight some relations that are of interest in mathematics. To further show how the Γ-function is related to primes, we consider a theorem by D.R. Heath Brown [15] on repeated values of the sums of the divisors, $\sigma(x)$.

Theorem 2. (D. R. Heath-Brown [15])

For any positive integer N there exist N distinct natural numbers a_n *with the following properties. For* $m \neq n$ *write* $|a_m - a_n| = d(x)$. *Then* $d(x)$ *divides* a_m *and* a_n. *Moreover,*

$$d(a_m) d\left(\frac{a_n}{d(x)}\right) = d(a_n) d\left(\frac{a_m}{d(x)}\right)$$

Theorem 2 is the first clue that there is a finite number of integers N in the Robin set. Let us explore this further. We can generalize Heath Brown to the Γ-function as follows using the sums of divisors, $\sigma(x)$ in the following lemma 1.

Lemma 1: *For* $m \neq n$ *write* $|a_m - a_n| = \sigma(x)$. *For any positive integer* $K_{\sigma(x),a_m}$, *there exist distinct natural numbers* $\sigma(x), a_m$ *with the following properties,* $\sigma(x) | \{a_m, a_n\}$, *moreover, the integers* $K_{\sigma(x),a_m}$ *are unique for values of* $\sigma(x)$, *and for the case* $\sigma(x) = 1$, *the integers* $a_n \in \text{primes}$, p, and $a_m = \sigma(p)$.

$$\frac{\Gamma(a_m)\Gamma\left(\frac{a_n}{\sigma(x)}\right)}{\Gamma(a_n)\Gamma\left(\frac{a_m}{\sigma(x)}\right)} = K_{a_n,a_m}, \quad K_{a_n,a_m} \in Z$$

Proof: Put $|a_m - a_n| = \sigma(x)$,

$$\frac{\Gamma(a_m)\Gamma\left(\frac{a_n}{\sigma(x)}\right)}{\Gamma(a_n)\Gamma\left(\frac{a_m}{\sigma(x)}\right)} = \frac{\Gamma(\sigma(x)+a_n)\Gamma\left(\frac{a_n}{\sigma(x)}\right)}{\Gamma(a_n)\Gamma\left(\frac{\sigma(x)+a_n}{\sigma(x)}\right)} \tag{39}$$

A study of (39) generates integer series for every a_n, and $\sigma(x)$, examples of which are given in **Table 1** below.

Table 1. It shows the various sequences that are obtained by (39).

	Representative functions of sequences	$K_{a_n,\sigma(x)}$, $\sigma(x) = 1,2,3,4,\cdots,k$.
$a_n = 1$	$\frac{(\sigma(x)-1)\sigma(x)!}{2}$	1, 4, 18, 96, 600, 4320…
$a_n = 2$	$\frac{(\sigma(x)-1)(\sigma(x)+1)!}{6}$	1, 6, 36, 240, 1800, 15120…
$a_n = 3$	$\frac{\sigma(x)(\sigma(x)+3)!}{24}$	1, 8, 60, 480, 4200, 40320…
$a_n = 4$	$\frac{(\sigma(x)+1)!(\sigma(x)+5)!}{5!((\sigma(x)+5)-5)!}$	1, 10, 90, 840, 8400, 90720...

All these integer sequences are well known. For example, the series for $a_n = 3$, is the total number of occurrences of the consecutive pattern 1324 in all permutations of $[a_n + 3]$. By computer analysis, (using Maple 2024), there are exactly N = 7 *distinct natural numbers* $a_n = [11,14,15,15,18,19,21]$, $a_m = [10,12,13,14,16,17,19]$ that satisfy the relation $a_m - a_n = \Gamma(x)$, *with the greatest common divisor being* $\Gamma(x) = 1$, $x = \{1,2\}$. It is clear that the Heath-Brown's Lemma 1 in [15] extends to a variety of relations including,

$$\frac{\Gamma(a_m)\Gamma\left(\frac{a_n}{\sigma(x)}\right)}{\Gamma(a_n)\Gamma\left(\frac{a_m}{\sigma(x)}\right)} = K_{a_n,a_m}, \quad K_{a_n,a_m} \in Z \tag{40}$$

and, the case of unity used by Heath Brown is just one example of Lemma 1. The trivial case is given by putting $a_m = \sigma(p)$, $a_n = p$, $\sigma(x) | a_m, a_n$, where p is a prime. Then, $\sigma(x) = a_m - a_n = 1$.

$$\frac{\Gamma(a_m)\Gamma\left(\frac{a_n}{\sigma(x)}\right)}{\Gamma(a_n)\Gamma\left(\frac{a_m}{\sigma(x)}\right)} = 1 \tag{41}$$

However, one must note that $\sigma(x)$, *is not a cover for all integer values, since there are values* $\sigma(x) \neq m, \{m \in \forall \mathbb{Z}\}$. To understand this, we can now use the general relation:

$$\Gamma(a_m) = (2\pi)^{\frac{1-\sigma(x)}{2}} (\sigma(x))^{a_m - \frac{1}{2}} \prod_{k=0}^{\sigma(x)-1} \Gamma\left(\frac{a_m}{\sigma(x)} + \frac{k}{\sigma(x)}\right) \tag{42}$$

$$\Gamma\left(\frac{a_m}{\sigma(x)}\right) = (2\pi)^{\frac{1-a_m}{2}} (a_m)^{\frac{a_m}{\sigma(x)} - \frac{1}{2}} \prod_{k=0}^{a_m - 1} \Gamma\left(\frac{1}{\sigma(x)} + \frac{k}{a_m}\right) \tag{43}$$

And dividing (42) by (43), we get:

$$\frac{\Gamma(a_m)}{\Gamma\left(\frac{a_m}{\sigma(x)}\right)} = \frac{(2\pi)^{\frac{1-\sigma(x)}{2}} (\sigma(x))^{a_m - \frac{1}{2}} \prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{a_m}{\sigma(x)} + \frac{k}{\sigma(x)}\right)}{(2\pi)^{\frac{1-a_m}{2}} (a_m)^{\frac{a_m}{\sigma(x)} - \frac{1}{2}} \prod_{k=0}^{a_m - 1} \Gamma\left(\frac{1}{\sigma(x)} + \frac{k}{a_m}\right)} \tag{44}$$

Expanding,

$$\begin{aligned} \frac{\Gamma(a_m)\Gamma\left(\frac{a_n}{\sigma(x)}\right)}{\Gamma(a_n)\Gamma\left(\frac{a_m}{\sigma(x)}\right)} &= (2\pi)^{\frac{\sigma(x)-a_n}{2}} (\sigma(x))^{\frac{(1+2a_m-2a_n)\sigma(x)-2a_m}{2\sigma(x)}} (a_n)^{\frac{2a_n-\sigma(x)}{2\sigma(x)}} \\ &\times \frac{\prod_{k=0}^{\sigma(x)-1} \Gamma\left(\frac{a_m+k}{\sigma(x)}\right) \prod_{k=0}^{a_n-1} \Gamma\left(\frac{a_n + k\sigma(x)}{a_n\sigma(x)}\right)}{\prod_{k=0}^{\sigma(x)-1} \Gamma\left(\frac{a_n+k}{\sigma(x)}\right) \prod_{k=0}^{\sigma(x)-1} \Gamma\left(\frac{a_m + k\sigma(x)}{(\sigma(x))^2}\right)} \end{aligned} \tag{45}$$

Then, since $a_m - a_n = \sigma(x)$, and if $a_n = p$, $a_m = \sigma(p)$, $\sigma(x) = 1$, then for primes p,

$$1 = (2\pi)^{\frac{1}{2} - \frac{p}{2}} p^{p - \frac{1}{2}} \frac{1}{\Gamma(p)} \prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right) \tag{46}$$

which can be obtained from the original Γ-product formula for primes.

$$\sqrt{2\pi} \frac{p^{p-\frac{1}{2}} \prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}} \prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{p+k}{\sigma(p)}\right)} = 1 \quad \{p \text{ is prime}\} \tag{47}$$

Combining the two relations (46) and (47), we get:

$$(2\pi)^{\frac{1}{2}-\frac{p}{2}}\frac{p^{p-\frac{1}{2}}}{\Gamma(p)}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)=\sqrt{2\pi}\frac{p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)} \tag{48}$$

$$\frac{1}{(2\pi)^{\frac{p}{2}}\Gamma(p)}=\frac{1}{\sigma(p)^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)} \tag{49}$$

$$\sigma(p)^{p-\frac{1}{2}}=\frac{(2\pi)^{\frac{p}{2}}\Gamma(p)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)} \tag{50}$$

$$\frac{\sigma(p)^{p-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}\Gamma(p)}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)=\begin{cases}1, & p \text{ is a prime}\\ K_p, & \text{otherwise}\end{cases} \tag{51}$$

Relation (51) follows a *new* relation for the Γ-product for primes:

$$\boxed{\Gamma(p)=(2\pi)^{\frac{-p}{2}}(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right),\quad p\in\text{prime}} \tag{52}$$

(52) only holds for *primes*. On the other hand, the general relation:

$$\Gamma(n)=(2\pi)^{-\frac{(1-\sigma(n))}{2}}(\sigma(n))^{n-\frac{1}{2}}\prod_{k=0}^{\sigma(n)-1}\Gamma\left(\frac{n+k}{\sigma(n)}\right) \tag{53}$$

holds for all integer despite the fact that $\sigma(n)-1\neq n$, when n is not a prime.

The significance of the Heath Brown [15] relation can be understood when one realizes that *not all integers* are sums of divisors. For example, the following integers are not sums of divisors, hence the series in **Table 1**, should not include these integers, as values of $\sigma(x)$.

$$\sigma(x)\notin[2,5,9,10,11,16,17,19,21,22,23,25,26,27,29,33,34,35,37,\\41,43,45,46,47,49,50,51,52,53,55,58,59,61,64,\cdots],$$

Hence, **Table 1** cannot be valid for all integers, and must be replaced by **Table 2**. The restriction to numbers that are sums of divisors is important. Consider the relation:

$$\Gamma(\sigma(v))=(2\pi)^{\frac{1-\sigma(v)}{2}}(\sigma(v))^{\sigma(v)-\frac{1}{2}}\prod_{k=0}^{\sigma(v)-1}\Gamma\left(1+\frac{k}{\sigma(v)}\right) \tag{54}$$

Surely, (54) does not cover the spectrum, $m\in\forall\mathbb{Z}$. Then, the solutions for which

$$\frac{\Gamma(\sigma(a_m))\Gamma\left(\frac{\sigma(a_n)}{\sigma(x)}\right)}{\Gamma(\sigma(a_n))\Gamma\left(\frac{\sigma(a_m)}{\sigma(x)}\right)}=1 \tag{55}$$

Table 2. It shows the valid integers for which the sigma function can be expressed as (39).

$\sigma(x)=1,3,4,7,6,12,8,15,13,18,\cdots,k.$	
$a_n=1$	$\frac{(\sigma(x)-1)\sigma(x)!}{2}=1,18,96,35280,4320,5748019200,322560,19615115520000,80951270400,\cdots$
$a_n=2$	$\frac{(\sigma(x)-1)(\sigma(x)+1)!}{6}=1,36,240,141120,15120,37362124800,1451520,156920924160000,566658892800,\cdots$
$a_n=3$	$\frac{\sigma(x)(\sigma(x)+3)!}{24}=1,60,480,423360,40320,174356582400,4838400,889218570240000,2833294464000,\cdots$
$a_n=4$	$\frac{(\sigma(x)+1)!(\sigma(x)+5)!}{5!((\sigma(x)+5)-5)!}=1,90,840,1058400,90720,653837184000,13305600,4001483566080000,11333177856000,\cdots$

holds true, belong to the set of numbers that are sums of divisors, *i.e.*, all integers excluding the set:

$$\sigma^{\nexists}=[2,5,9,10,11,16,17,19,21,22,23,25,26,27,29,33,34,35,$$
$$37,41,43,45,46,47,49,50,51,52,53,55,58,59,61,64,\cdots]$$

The significance of this exercise is to show that the integers are somehow classified into different groups that contain special properties. For example, the integers that relate to those that comply with the Robin inequality, and those that do not. The same applies in the case of the σ-function.

5. The π-Transformation

From the reflection formula,

$$2^{1-z}\Gamma(z)\zeta(z)\cos\left(\frac{\pi z}{2}\right)=\pi^{z}\zeta(1-z) \tag{56}$$

$$\Gamma(g(z))=\frac{(2\pi)^{g(z)}\zeta(1-g(z))}{2\zeta(g(z))\cos\left(\frac{\pi g(z)}{2}\right)} \tag{57}$$

Definition 1: Define the π-transformation as:

$$(2\pi)^{-g(z)}=\frac{\zeta(1-g(z))}{2\Gamma(g(z))\zeta(g(z))\cos\left(\frac{\pi(g(z))}{2}\right)} \tag{58}$$

Then, the left-hand-side is separable into real and complex parts. It is important to note that G.H. Hardy [16], found this function related to prime counting functions. If $g(z)=\sum_k(z_k)$, *i.e.*, the sum of k-terms, and functions, then, the π-

transformation gives:

$$(2\pi)^{-g(z_k)} = (2\pi)^{-\Sigma_k(z_k)} = \prod_k \left(\frac{\zeta(1-g(z_k))}{2\Gamma(g(z_k))\zeta(g(z_k))\cos\left(\frac{\pi(g(z_k))}{2}\right)} \right) \tag{59}$$

Example 1: Let $g(z) = \sigma + i\tau$, then, from (58),

$$(2\pi)^{-(\sigma+i\tau)} = \frac{\zeta(1-\sigma-i\tau)}{2\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)\cos\left(\frac{\pi(\sigma+i\tau)}{2}\right)} \tag{60}$$

And let $g(z) = -(\sigma+i\tau)y$, where y is any function, then, from (58),

$$(2\pi)^{(\sigma+i\tau)y} = \frac{\zeta(1+(\sigma+i\tau)y)}{2\zeta(-(\sigma+i\tau)y)\cos\left(\frac{-\pi(\sigma+i\tau)y}{2}\right)\Gamma(-(\sigma+i\tau)y)} \tag{61}$$

$$(2\pi)^{-(\rho+i\tau)+(\rho+i\tau)y} = \frac{\left[\dfrac{\zeta(1-\sigma-i\tau)}{2\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)\cos\left(\frac{\pi(\sigma+i\tau)}{2}\right)}\right]}{\left[\dfrac{\zeta(1+(\sigma+i\tau)y)}{2\zeta(-(\sigma+i\tau)y)\cos\left(\frac{-\pi(\sigma+i\tau)y}{2}\right)\Gamma(-(\sigma+i\tau)y)}\right]} \tag{62}$$

As an example, put $y = \rho - i\tau$,

$$(2\pi)^{\sigma^2+\tau^2-\sigma-i\tau} = \left[\frac{\zeta(-\sigma^2-\tau^2)\zeta(1-\sigma-i\tau)\Gamma(-\sigma^2-\tau^2)\cos\left(\frac{\pi(-\sigma^2-\tau^2)}{2}\right)}{\zeta(1+\sigma^2+\tau^2)\zeta(1+\sigma+i\tau)\Gamma(\sigma+i\tau)\cos\left(\frac{\pi(\sigma+i\tau)}{2}\right)}\right] \tag{63}$$

The relation (63) is again separable into real and complex parts for products of n-functions since the left-hand side is separable into products, $\prod_{k=1}^{n} F(\arg z_k)$. It is obvious that the powers of (2π) on the right-hand-side (LHS) determine completely the arguments on the left-hand-side (RHS), and in general, the form $(2\pi)^{g(z)}$ determines the RHS as having arguments

$$\prod_{k=1}^{n} F(z_k) = (2\pi)^{\sum_{n=1}^{n}(z_k)} \tag{64}$$

Example 2: Let $g(z) = \frac{1}{2} + iT, f(z) = \frac{1}{2} - iT$;

Then,

$$(2\pi)^{-\frac{1}{2}-iT} = 2\left[\frac{\zeta\left(\frac{1}{2}-iT\right)}{\Gamma\left(\frac{1}{2}+iT\right)\zeta\left(\frac{1}{2}+iT\right)\cos\left(\frac{\pi\left(\frac{1}{2}+iT\right)}{2}\right)}\right], \tag{65}$$

$$(2\pi)^{-\frac{1}{2}+iT} = 2\left[\frac{\zeta\left(\frac{1}{2}+iT\right)}{\Gamma\left(\frac{1}{2}-It\right)\zeta\left(\frac{1}{2}-iT\right)\cos\left(\frac{\pi\left(\frac{1}{2}-iT\right)}{2}\right)}\right]$$

Multiplying the two terms,

$$\frac{\pi}{2} = \left[\frac{\zeta\left(\frac{1}{2}-iT\right)}{\Gamma\left(\frac{1}{2}+iT\right)\zeta\left(\frac{1}{2}+iT\right)\cos\left(\frac{\pi\left(\frac{1}{2}+iT\right)}{2}\right)}\right]\left[\frac{\zeta\left(\frac{1}{2}+iT\right)}{\Gamma\left(\frac{1}{2}-It\right)\zeta\left(\frac{1}{2}-iT\right)\cos\left(\frac{\pi\left(\frac{1}{2}-iT\right)}{2}\right)}\right] \tag{66}$$

The product of the relation (66) reduces to a form that is independent of the ζ-function on the ½ line, and relies only on the relation:

$$\left\{\Gamma\left(\frac{1}{2}+iT\right)\Gamma\left(\frac{1}{2}-iT\right)\right\}\left(\cos\left(\frac{\pi\left(\frac{1}{2}+iT\right)}{2}\right)\cos\left(\frac{\pi\left(\frac{1}{2}-iT\right)}{2}\right)\right) = \frac{\pi}{2} \tag{67}$$

Example 3: Put $g(z) = \zeta(z)$, then,

$$(2\pi)^{-1-\sum_{k=2}^{\infty}\left(k^{-z}\right)} = \frac{1}{2\pi}\prod_{k=2}^{\infty}\left(\frac{\zeta\left(1-k^{-z}\right)}{2\Gamma\left(k^{-z}\right)\zeta\left(k^{-z}\right)\cos\left(\frac{\pi\left(k^{-z}\right)}{2}\right)}\right) = (2\pi)^{-\zeta(z)} \tag{68}$$

If $z = \rho$, is a root, then, $\zeta(\rho) = 0$, the non-trivial roots follow, and we get:

$$\prod_{k=2}^{\infty}\left(\frac{\zeta\left(1-k^{-\rho}\right)}{2\Gamma\left(k^{-\rho}\right)\zeta\left(k^{-\rho}\right)\cos\left(\frac{\pi\left(k^{-\rho}\right)}{2}\right)}\right)=2\pi \tag{69}$$

Example 4: Note that the conjugate relations on the 1/2 line give,

$$\left[\frac{\zeta\left(\frac{1}{2}-iT\right)}{\Gamma\left(\frac{1}{2}+iT\right)\zeta\left(\frac{1}{2}+iT\right)\cos\left(\frac{\pi\left(\frac{1}{2}+iT\right)}{2}\right)}\right]\left[\frac{\zeta\left(\frac{1}{2}+iT\right)}{\Gamma\left(\frac{1}{2}-It\right)\zeta\left(\frac{1}{2}-iT\right)\cos\left(\frac{\pi\left(\frac{1}{2}-iT\right)}{2}\right)}\right]=\frac{\pi}{2} \tag{70}$$

Then, using the fact that

$$(2\pi)^{-g(z)}=\frac{\zeta\left(1-g(z)\right)}{2\zeta\left(g(z)\right)\cos\left(\frac{\pi g(z)}{2}\right)\left[\Gamma\left(g(z)\right)\right]} \tag{71}$$

Put $g(z)=\zeta(z)-\frac{1}{z-1}$, and separate the terms for $\frac{1}{z-1}$, then,

$$(2\pi)^{-1-\sum_{k=2}^{\infty}\left(k^{-z}\right)+\frac{1}{z-1}}=\frac{\zeta\left(1-\zeta(z)+\frac{1}{z-1}\right)}{2\Gamma\left(\zeta(z)-\frac{1}{z-1}\right)\zeta\left(\zeta(z)-\frac{1}{z-1}\right)\cos\left(\frac{\pi\left(\zeta(z)-\frac{1}{z-1}\right)}{2}\right)} \tag{72}$$

Then when $z=\rho$, a root of the ζ-function,

$$(2\pi)^{-1-\sum_{k=2}^{\infty}\left(k^{-z}\right)+\frac{1}{z-1}}=\frac{1}{2\pi}\left[\frac{\zeta\left(\frac{\rho}{\rho-1}\right)}{\Gamma\left(\frac{1}{1-\rho}\right)\zeta\left(\frac{1}{1-\rho}\right)\cos\left(\frac{\pi}{2-2\rho}\right)}\right]\prod_{k=2}^{\infty}\left(\frac{\zeta\left(1-k^{-z}\right)}{2\Gamma\left(k^{-z}\right)\zeta\left(k^{-z}\right)\cos\left(\frac{\pi\left(k^{-z}\right)}{2}\right)}\right) \tag{73}$$

$$=\frac{\zeta\left(\frac{\rho}{\rho-1}\right)}{\Gamma\left(\frac{1}{1-\rho}\right)\zeta\left(\frac{1}{1-\rho}\right)\cos\left(\frac{\pi}{2-2\rho}\right)}$$

$$\prod_{k=2}^{\infty}\left(\frac{\zeta\left(1-k^{-\rho}\right)}{2\Gamma\left(k^{-\rho}\right)\zeta\left(k^{-\rho}\right)\cos\left(\frac{\pi\left(k^{-\rho}\right)}{2}\right)}\right)=2\pi \tag{74}$$

Example 5: Put $g(\rho)=\frac{1}{1-\rho}$,

$$(2\pi)^{\frac{1}{\rho-1}}=\frac{\zeta\left(\frac{\rho}{\rho-1}\right)}{2\Gamma\left(\frac{1}{\rho-1}\right)\zeta\left(\frac{1}{\rho-1}\right)\cos\left(\frac{\pi}{2}\left(\frac{1}{\rho-1}\right)\right)} \tag{75}$$

Also note that $\lim_{z\to\infty} g(z)=\lim_{z\to\infty}\left(\zeta(z)-\frac{1}{z-1}\right)=\gamma\approx 0.5772156649\cdots$, where γ is the Euler-Mascheroni constant [14]. Let's not separate the terms, then, the Cauchy principal value exists

$$\lim_{z\to\infty}\left((2\pi)^{-\zeta(z)+\frac{1}{z-1}}\right)=(2\pi)^{-\lambda} \tag{76}$$

$$(2\pi)^{-\gamma}=\lim_{z\to\infty}\left(\frac{\zeta\left(1-\zeta(z)+\frac{1}{z-1}\right)}{2\Gamma\left(\zeta(z)-\frac{1}{z-1}\right)\zeta\left(\zeta(z)-\frac{1}{z-1}\right)\cos\left(\frac{\pi\left(\zeta(z)-\frac{1}{z-1}\right)}{2}\right)}\right) \tag{77}$$

$$(2\pi)^{-\gamma}=\frac{\zeta(1-\gamma)}{2\Gamma(\gamma)\zeta(\gamma)\cos\left(\frac{\pi(\gamma)}{2}\right)}=0.34616095200041890772=C_{\infty} \tag{78}$$

It must be noted that to a large number of decimal places, the constant $C_{\infty}\approx\pi-e-\lambda+\frac{1}{2}$,

Then, we can conclude that for all the Stiltjes numbers λ_k, follow:

$$\boxed{(2\pi)^{-\lambda_k}=\frac{\zeta(1-\lambda_k)}{2\Gamma(\lambda_k)\zeta(\lambda_k)\cos\left(\frac{\pi(\lambda_k)}{2}\right)}} \tag{79}$$

6. The Prime Counting Function

Theorem 3. For any integer $x\geq 0$, the exact number of primes less than x is

$$R_x=\frac{1}{\pi}\sum_{n=2}^{x}\left(\left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}(2n+1)\right) \tag{80}$$

Proof: Given a multivalued variable, $\varnothing$, then,

$$\tan^{-1}\left(\tan(\varnothing)\right)=\varnothing-k\pi\in\left\{k\pi-\frac{\pi}{2}<\varnothing<k\pi+\frac{\pi}{2}\right\} \tag{81}$$

Let $\varnothing=\pi B_{2n}\Gamma(2n+1)$, were, B_{2n} are the Bernoulli numbers, and, n is an integer. Then, using von Staudt-Clausen theorem [17]-[19], the Bernoulli term could be expanded into the sum of reciprocals of primes, p, and integers such that for any n,

$$B_{2n}=A_n-\sum_{p|2n}\frac{1}{p} \tag{82}$$

where, A_n is an integer. Then, the sum, could be written as:

$$R_x=\frac{1}{\pi}\left\{\sum_{n=2}^{x}\left\{\tan^{-1}\left(\tan\left(\pi\left((2n)!A_n-\sum_{p|2n}\frac{(2n)!}{p}\right)\right)\right)\right\}(2n+1)\right\} \tag{83}$$

The first term $(2n)!A_n$, is an integer, and the second term is an integer except when $(2n+1)$ is a prime. Then, all other primes $p<(2n+1)$ are factored out in this sum term since every prime, $p=2n-m$, $m>1$, that satisfy von-Staudt Clausen primes and must be a divisor or $2n!$ It follows that for primes, the fractional expression:

$$\lfloor B_{2n}\Gamma(2n+1)\rfloor=\tan^{-1}\left(\tan\left(B_{2n}\Gamma(2n+1)\right)\right) \tag{84}$$

which leads us to the conclusion of the theorem:

$$R_x=\frac{1}{\pi}\sum_{n=2}^{x}\left(\left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}(2n+1)\right) \tag{85}$$

Since the prime counting function (PCF, R_x), (85) eliminates non-primes, one could write the ζ-function as the product of two PCFs,

$$\zeta(s)=\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\left\{\frac{1}{\pi^2}\left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}(2n+1)\left\{\tan^{-1}\left(\tan\left(\pi B_{2m}\Gamma(2m+1)\right)\right)\right\}(2m+1)\right\}\right]^{-s} \tag{86}$$

Putting $\omega_n=\frac{1}{\pi}\left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}$,
$\omega_m=\frac{1}{\pi}\left\{\tan^{-1}\left(\tan\left(\pi B_{2m}\Gamma(2m+1)\right)\right)\right\}$,

$$\zeta(s)=\sum_{n=2}^{\infty}\left[\sum_{m=2}^{n}\{\omega_n\omega_m\}\right]^{-s} \tag{87}$$

And in product form:

$$\zeta(s)=\prod_{j=1}^{\infty}\left(\frac{1}{1-\left(\frac{\omega_m}{2j+1}\right)^{-s}}\right) \tag{88}$$

The non-prime-counting function (NPCF) is given by:

$$Q_x = x - 1 - \sum_{n=2}^{x} \{\omega_n\} \tag{89}$$

One could now write the relationship between the counting functions as:

$$Q_x + R_x = x - 1 \tag{90}$$

Now, to understand the connection of R_x to prior work by Ramanujan ([16], p. 33), the number of primes less than x, was given by Ramanujan as:

$$R_x = \frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\ln x}{2\pi}\right) + \frac{4}{3B_4}\left(\frac{\ln x}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\ln x}{2\pi}\right)^5 + \cdots + \frac{2n}{p_n B_{2n}}\left(\frac{\ln x}{2\pi}\right)^{p_n} + \cdots\right\} \tag{91}$$

where, B_{2n} are the Bernoulli numbers. It is clear that (85) and (91) are related. Ramanujan's formula can be obtained using the Li-function as follows in section 6.0.

It is important to note that the form of the prime counting function (85) follows the relation for the 1/2 line:

$$\mathrm{e}^{2i\theta} = \frac{\frac{1}{2} + i\left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}}{\frac{1}{2} - i\left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}},$$

where $\theta = \left\{\tan^{-1}\left(\tan\left(\pi B_{2n}\Gamma(2n+1)\right)\right)\right\}$.

Hence the connection of the primes to the half-line.

7. The Primes and the Li-Function

I was curious as to the reason why Ramanujan's function for the number of primes less than x, as quoted in Hardy's 12 lectures [16],

$$g(x) = 1 + \sum_{n=1}^{\infty}\left(\frac{(\ln(x))^n}{n \cdot \Gamma(n+1)\zeta(n+1)}\right) \tag{92}$$

had the curious form of the π-transformation,

$$\boxed{(2\pi)^{-\lambda_k} = \frac{\zeta(1-\lambda_k)}{2\Gamma(\lambda_k)\zeta(\lambda_k)\cos\left(\frac{\pi(\lambda_k)}{2}\right)}} \tag{93}$$

Obviously, the π-transformation is just the ζ-reflection formula. The reason became evident when I analyzed the work. Consider the Li-function. Let $\{m,k\} \in Z$ (integers), $\{x\} \in \mathbb{R}(\text{reals})$.

$$\begin{aligned}\mathrm{Li}\left(\mathrm{e}^{\frac{\ln(x)}{m}(\sec(\pi k))}\right) &= -\ln(m) + \gamma + \ln(\ln(x)) + \ln(\sec(\pi k)) \\ &+ \sum_{n=1}^{\infty}\left(\frac{(\ln(x))^n}{n \cdot \Gamma(n+1) m^n (\cos(\pi k))^n}\right)\end{aligned} \tag{94}$$

Then, following Hardy's [16] on the work on Ramanujan's analytic theory on the distribution of primes, we construct the function $f(x)$, in place of the function

$g(x)$ used by Hardy [16]. The Möbius function $\mu(m)$ is defined as follows:

$$\mu(m)=\begin{cases}(-1)^k & \text{if } m \text{ contains } k \text{ different primes}\\ 0 & \text{if } m \text{ contains any quadratic prime factors}\end{cases} \tag{95}$$

Let $f(x,k)\in Z$ be the number of primes less than $x>0$. Then,

$$\begin{aligned} f(x,k) &= \sum_{m=1}^{\infty}\frac{\mu(m)}{m}\mathrm{Li}\left(\mathrm{e}^{\frac{\ln(x)}{m}(\sec(\pi k))}\right) \\ &= \sum_{m=1}^{\infty}\frac{\mu(m)}{m}(\gamma-\ln(m)+\ln(x))+\frac{\mu(m)}{m}\ln(\sec(\pi k)) \\ &\quad+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\mu(m)}{m^{n+1}}\frac{(\ln(x))^n}{n\cdot\Gamma(n+1)(\cos(\pi k))^n}\right) \end{aligned} \tag{96}$$

Noting that

$$\sum_{m=1}^{\infty}\frac{\mu(m)(\ln(m))}{m}=-1,\quad \sum_{m=1}^{\infty}\frac{\mu(m)}{m}=0,\quad \sum_{m=1}^{\infty}\frac{\mu(m)}{m^{n+1}}=\frac{1}{\zeta(n+1)}, \tag{97}$$

$$f(x,k)=1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\mu(m)}{m^{n+1}}\frac{(\ln(x))^n}{n\cdot\Gamma(n+1)(\cos(\pi k))^n}\right) \tag{98}$$

While Hardy did *not distinguish between odd and even* values, it is important that we separate the two for reasons that follow. Separating this into odd and even sums,

$$\begin{aligned} f(x,k) &= 1+\underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\mu(m)}{m^{2n+1}}\frac{(\ln(x))^{2n}}{2n\cdot\Gamma(2n+1)\zeta(2n+1)(\cos(\pi k))^{2n}}\right)}_{n,\text{even}} \\ &\quad+\underbrace{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\mu(m)}{m^{2n}}\frac{(\ln(x))^{2n-1}}{(2n-1)\cdot\Gamma(2n)(\cos(\pi k))^{2n-1}}\right)}_{n,\text{odd}} \end{aligned} \tag{99}$$

For n, odd, $(\cos(\pi k))^{2n-1}=(-1)^k$, and for n, even, $(\cos(\pi k))^{2n}=1$, where, $k\in Z$.

$$\begin{aligned} f(x,k) &= 1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\mu(m)}{m^{2n+1}}\frac{(\ln(x))^{2n}}{2n\cdot\Gamma(2n+1)}\right) \\ &\quad-(-1)^k\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\mu(m)}{m^{2n}}\frac{(\ln(x))^{2n-1}}{(2n-1)\cdot\Gamma(2n)}\right) \end{aligned} \tag{100}$$

$$\begin{aligned} f(x,k) &= 1+\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n}}{2n\cdot\Gamma(2n+1)\zeta(2n+1)}\right) \\ &\quad+(-1)^k\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n-1}}{(2n-1)\cdot\Gamma(2n)\zeta(2n)}\right) \end{aligned} \tag{101}$$

Since,

$$f(x,k_{even})=g(x)$$
$$=1+\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n}}{2n\cdot\Gamma(2n+1)\zeta(2n+1)}\right)+\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n-1}}{(2n-1)\cdot\Gamma(2n)\zeta(2n)}\right) \tag{102}$$

If k is even, we recover the formula due to Ramanujan, in G.H. Hardy ([16], p. 25),

$$f(x,k\in\text{even})=g(x)=1+\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{n}}{n\cdot\Gamma(n+1)\zeta(n+1)}\right) \tag{103}$$

If k is odd,

$$f(x,k_{odd})=1+\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n}}{2n\cdot\Gamma(2n+1)\zeta(2n+1)}\right)-\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n-1}}{(2n-1)\cdot\Gamma(2n)\zeta(2n)}\right) \tag{104}$$

Subtracting,

$$f(x,k_{even})-f(x,k_{odd})=2\sum_{n=1}^{\infty}\left(\frac{(\ln(x))^{2n-1}}{(2n-1)\cdot\Gamma(2n)\zeta(2n)}\right) \tag{105}$$

Hardy ([16], p. 24), determined that the function, $g(x)$ is twice its odd parts. Then, $f(x,k_{even})-f(x,k_{odd})$ is the same function used by Ramanujan [16], and as we can see, using the relation:

$$\zeta(2n)=\frac{2^{2n-1}\pi^{2n}B_{2n}}{(2n)!} \tag{106}$$

$$f(x,k_{even})-f(x,k_{odd})=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2n}{(2n-1)(B_{2n})}\left(\frac{\ln(x)}{2\pi}\right)^{2n-1}\right) \tag{107}$$

This is the same series as the series proposed by Ramanujan [16]:

$$R_x=\frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\ln x}{2\pi}\right)+\frac{4}{3B_4}\left(\frac{\ln x}{2\pi}\right)^3+\frac{6}{5B_6}\left(\frac{\ln x}{2\pi}\right)^5+\cdots+\frac{2n}{(2n-1)B_{2n}}\left(\frac{\ln x}{2\pi}\right)^{2n-1}+\cdots\right\} \tag{108}$$

It is clear that if we now take the integer values of $\ln(x)$, by making the substitution $y\in Z=\ln x$, in (105), we get,

$$\sum_{y=1}^{\infty}\left(f(x,k_{even})-f(x,k_{odd})\right)=2\sum_{n=1}^{\infty}\left(\frac{\zeta(1-2n)}{(2n-1)\cdot\Gamma(2n)\zeta(2n)}\right) \tag{109}$$

From the π-transformation,

$$\boxed{(2\pi)^{-\gamma_k} = \frac{\zeta(1-\gamma_k)}{2\Gamma(\gamma_k)\zeta(\gamma_k)\cos\left(\frac{\pi(\gamma_k)}{2}\right)}} \tag{110}$$

we get the relation:

$$\frac{2(2\pi)^{-2n}}{(2n-1)} = \frac{\zeta(1-2n)}{(2n-1)\Gamma(2n)\zeta(2n)\cos(n\pi)}, \tag{111}$$

$$\sum_{x=1}^{\infty}\left(f(x,k_{even}) - f(x,k_{odd})\right) = 2\sum_{n=1}^{\infty}\left(\frac{2(2\pi)^{-2n}}{(2n-1)}\right) \tag{112}$$

$$\begin{aligned}&\sum_{x=1}^{\infty}\left(f(x,k_{even}) - f(x,k_{odd})\right)\\ &= \frac{2}{\pi}\tanh^{-1}\left(\frac{1}{2\pi}\right) \approx 0.10218992416121490973\cdots\end{aligned} \tag{113}$$

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}, \quad \zeta(1-2n) = -\frac{B_{2n}}{2n} \tag{114}$$

Then, we find that Ramanujan's function for prime-counting is the same as

$$\begin{aligned}R_x &= f(x,k_{even}) - f(x,k_{odd})\\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2n}{(2n-1)B_{2n}(2\pi)^{2n-1}}(\ln(x))^{2n-1}\right)\end{aligned} \tag{115}$$

8. The Relation of the Product Gamma Function to Twin-Primes

We start with a theorem on twin primes.

Theorem 4. *There are an infinite number twin primes* $\{p, p+2\}$ *that have the restriction,*

$$\frac{\left(\frac{\sigma(p)}{\sigma(p+2)}\right)^2}{(2\pi)^{p+2-\frac{\sigma(p)}{2}-\frac{\sigma(p+2)}{2}}} \le 1, \quad \forall\{p, p+2\} \in \text{twin primes} \tag{116}$$

PROOF:

From the expression (47), put the prime relation for $p, p+2$,

$$\left[\sqrt{2\pi}\frac{p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\right]\left[\sqrt{2\pi}\frac{p^{p+2-\frac{1}{2}}\prod_{k=0}^{p+2-1}\Gamma\left(\frac{p+2+k}{p}\right)}{\sigma(p+2)^{p+2-\frac{1}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}\right] = 1 \tag{117}$$

Then,

$$(2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\frac{\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}=1 \tag{118}$$

It is clear that if $p<\sigma(p), p+2<\sigma(p+2)$, so if p is not a twin prime,

$$(2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\frac{\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)} \tag{119}$$

$$\ll 1,\ \text{if } \{p,p+2\}\notin \text{twin primes}$$

Hence for twin primes, $p, p+2, \frac{\sigma(p)}{\sigma(p+2)}=\frac{p+1}{p+3}<1$. Then, it follows that since (119) $\ll$ 1, then,

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{\left((2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\frac{\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}\right)}$$

$$\gg 1,\ \{p,p+2\}\notin \text{twin primes}$$

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{\left((2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\frac{\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}\right)}$$

$$<1,\ \{p,p+2\}\in \text{twin primes} \tag{120}$$

The relation (120) determines if *any* pair of integers, $\{p,p+2\}\in$ twin primes *exclusively*.

Hence,

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{\left((2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\frac{\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}\right)}<1 \tag{121}$$

$$\left(\frac{\sigma(p)}{\sigma(p+2)}\right)\frac{(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}{\left((2\pi)(p)^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)\right)}<1 \tag{122}$$

$$\left(\frac{\sigma(p)}{\sigma(p+2)}\right)\frac{\left\{(\sigma(p+2))^{p+\frac{3}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)\right\}\left\{(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\right\}}{\left(\left\{(2\pi)(p)^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\right\}\left\{(p+2)^{p+\frac{3}{2}}\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)\right\}\right)}<1 \tag{123}$$

$$\frac{\left(\frac{\sigma(p)}{\sigma(p+2)}\right)\left\{(2\pi)^{\left(\frac{1}{2}-\frac{\sigma(p+2)}{2}\right)}(\sigma(p+2))^{p+\frac{3}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)\right\}\left\{(2\pi)^{\left\{\frac{1}{2}-\frac{\sigma(p)}{2}\right\}}(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\right\}}{(2\pi)^{1-\left(\frac{1}{2}-\frac{p}{2}\right)-\left(\frac{1}{2}-\frac{p+2}{2}\right)+\left(\frac{1}{2}-\frac{\sigma(p+2)}{2}\right)+\left(\frac{1}{2}-\frac{\sigma(p)}{2}\right)}\left(\left\{(2\pi)^{\left(\frac{1}{2}-\frac{p}{2}\right)}p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\right\}\left\{(2\pi)^{\frac{1}{2}-\frac{p+2}{2}}(p+2)^{p+\frac{3}{2}}\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)\right\}\right)}<1 \tag{124}$$

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{(2\pi)^{1-\left(\frac{1}{2}-\frac{p}{2}\right)-\left(\frac{1}{2}-\frac{p+2}{2}\right)+\left(\frac{1}{2}-\frac{\sigma(p+2)}{2}\right)+\left(\frac{1}{2}-\frac{\sigma(p)}{2}\right)}}\frac{\{\Gamma(p+2)\}\{\Gamma(p)\}}{\{\Gamma(p)\}\{\Gamma(p+2)\}}<1 \tag{125}$$

All the terms in the curly bracket represent the Γ-function, and they cancel leaving us with the condition for $p, p+2$ to be twin primes:

$$\boxed{\frac{p+2}{\sigma(p+2)}\leq(2\pi)^{-\left\{\frac{\sigma(p+2)}{2}\right\}+\left(p+2-\frac{\sigma(p)}{2}\right)}\ \forall\{p,p+2\}\in\text{twin primes}} \tag{126}$$

Relation (126) is true for every known twin prime set. However, it is also satisfied by many integers p, that are not primes. It is possible that after a given number of integer solutions that involve primes, the remaining infinite number of solutions can be combinations of primes and non-primes.

The strategy is to first take $p+2$ as a known prime in the expression (126) and then see if p is a prime if we get a sequence of integer-relations relating $p, p+2$ that only survives when p is also a prime.

Statement 1: Let $p+2$ be a prime.

I use the strategy of finding the required relations p to be a twin prime when we accept that $p+2$, is a prime. *We don't know if* p *is* a prime yet, but we know that $p+2$ is a prime and so, $\sigma(p+2)=p+3$.

$\frac{\sigma(p)}{p+3}\leq(2\pi)^{-\left\{\frac{3}{2}+\frac{p}{2}\right\}+\left(p+2-\frac{\sigma(p)}{2}\right)}=M$, where M is some constant.

In terms of the π-transformations;

$$(2\pi)^{-g(z)} = \frac{\zeta(1-g(z))}{2\Gamma(g(z))\zeta(g(z))\cos\left(\frac{\pi(g(z))}{2}\right)} \tag{127}$$

$$\begin{aligned}&\left\{\frac{\zeta\left(1-\left\{\frac{3}{2}+\frac{p}{2}\right\}\right)}{2\Gamma\left(\left\{\frac{3}{2}+\frac{p}{2}\right\}\right)\zeta\left(\left\{\frac{3}{2}+\frac{p}{2}\right\}\right)\cos\left(\frac{\pi}{2}\left(\left\{\frac{3}{2}+\frac{p}{2}\right\}\right)\right)}\right\}\\&\times\left[\frac{\zeta\left(1+\left(p+2-\frac{\sigma(p)}{2}\right)\right)}{2\Gamma\left(\frac{\sigma(p)}{2}-p-2\right)\zeta\left(\frac{\sigma(p)}{2}-p-2\right)\cos\left(\frac{\sigma(p)}{2}-p-2\right)}\right]=M\end{aligned} \tag{128}$$

When $p+2$ is an odd prime, it is clear that p is odd, $\left\{\frac{3}{2}+\frac{p}{2}\right\}=k$, where $k \geq 3$ is an integer that can be either odd or even, and so, $p=2k-3$, $p+2=2k-1$.

$$\begin{aligned}&(2\pi)^{-k+\frac{\sigma(2k-3)}{2}-5}\\&=\left[\frac{\zeta(1-k)}{2\Gamma(k)\zeta(k)\cos\left(\frac{\pi}{2}(k)\right)}\right]\left[\frac{\zeta\left(\frac{\sigma(2k-3)}{2}-4\right)}{2\Gamma\left(5-\frac{\sigma(2k-3)}{2}\right)\zeta\left(5-\frac{\sigma(2k-3)}{2}\right)\cos\left(\frac{\pi}{2}\left(5-\frac{\sigma(2k-3)}{2}\right)\right)}\right]\\&\leq(2\pi)^{-\left\{\frac{3}{2}+\frac{p}{2}\right\}+\left(p+2-\frac{\sigma(p)}{2}\right)}=1\end{aligned} \tag{129}$$

Then, if k is an odd number, the expression vanishes, so k must be even and $p=4m-3$.

$$(2\pi)^{-2m+\frac{\sigma(4m-3)}{2}-5}=M \tag{130}$$

Statement 2: When $4m-3$ is a prime, $\sigma(4m-3)=4m-2$, hence,

$$M=(2\pi)^{-6}=\frac{1}{64\pi^6} \tag{131}$$

Hence for all twin primes, $p=4m-3$, $p+2=4m-1$, $M=\frac{1}{64\pi^6}$ only when $4m-1$ is a prime.

Applying Diritchlet's theorem [20] concludes the proof:

Theorem 5. *For any two positive coprime integers,* a,d, *there exists infinitely many primes of the form* $a+nd$, *where* n *is also a positive integer.*

9. The Relationship of the Zeta-Function, the Gamma-Function and the Li-Function

From (59),

$$\frac{\sigma(p)^{p-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}\Gamma(p)}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)=\begin{cases}1, & p \text{ is a prime}\\ K_p, & \text{otherwise}\end{cases} \tag{132}$$

Expression (132) *is invariant* to the substitution $\sigma(n)\to m$, $n\to\frac{1}{2}-\rho-i\tau$,

$$\frac{m^{-\rho-i\tau}}{(2\pi)^{\frac{m}{2}-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\rho-i\tau\right)}\prod_{k=0}^{m-1}\Gamma\left(\frac{\frac{1}{2}-\rho-i\tau+k}{m}\right)=1 \text{ for all } \rho,\tau\in R \tag{133}$$

$$m^{-\rho-i\tau}=\frac{(2\pi)^{\frac{m}{2}-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\rho-i\tau\right)}{\prod_{k=0}^{m-1}\Gamma\left(\frac{\frac{1}{2}-\rho-i\tau+k}{m}\right)} \text{ for all } \rho,\tau\in R \tag{134}$$

$$\sum_{m=1}^{\infty}m^{-\rho-i\tau}=\sum_{m=1}^{\infty}\left(\frac{(2\pi)^{\frac{m}{2}-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\rho-i\tau\right)}{\prod_{k=0}^{m-1}\Gamma\left(\frac{\frac{1}{2}-\rho-i\tau+k}{m}\right)}\right) \text{ for all } m\in Z,\ \rho,\tau\in R \tag{135}$$

$$\zeta(\rho+i\tau)=\sum_{m=1}^{\infty}\left(\frac{(2\pi)^{\frac{m}{2}-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\rho-i\tau\right)}{\prod_{k=0}^{m-1}\Gamma\left(\frac{\frac{1}{2}-\rho-i\tau+k}{m}\right)}\right) \tag{136}$$

Putting

$$\Gamma\left(\frac{1}{2}-\rho-i\tau\right)=(2\pi)^{\frac{1}{2}-\frac{m}{2}}m^{-\rho-i\tau}\prod_{k=0}^{m-1}\Gamma\left(\frac{\frac{1}{2}-\rho-i\tau+k}{m}\right) \tag{137}$$

gives,

$$\zeta(\rho+i\tau)=\sum_{m=1}^{\infty}m^{-\rho-i\tau} \tag{138}$$

Finally, I want to highlight the amazing relations developed by Robin [5] and many others over the many years of research. They have inspired me to devote many hours to the curiosities of Mathematics. In Section 8, we will take a small detour to the Li-function, and explore some properties that make the Robin inequality interestingly connected to counting functions, in much the same manner are the function $\pi(x)$, counts the primes.

10. The Robin Compliance Counting Function

The Riemann Hypothesis, proposed by Bernhard Riemann [4] in 1859, conjectures about the distribution of prime numbers and their relation to the Riemann Zeta function, $\zeta(s)$. It correlates the non-trivial zeros of the ζ-function with the primes if the zeros of the ζ-function have a real part, $\frac{1}{2}$. This hypothesis (RH) is crucial to the understanding of the distribution of primes. The Robin criterion first specified Guy Robin [5] in 1984 relates the truth of the RH hypothesis to a direct correlation to the statement (Robin's inequality)

$$F_n = \frac{\sigma(n)\mathrm{e}^{-\gamma}}{n\log\log(n)} < 1 \tag{139}$$

if and only if the set of numbers (I call them the Robin non-compliant integers, R_n):

$$R_n \in [3,4,5,6,8,9,10,12,16,18,20,24,30,36,48,60,72,\\ 84,120,180,240,360,720,840,2520,5040]$$

are the only integers that violate the inequality. It is important to note that Riemann Hypothesis complies with the Robin criteria if there exists no integer greater than 5040 that violates the criteria. The reason is that there exists a counting function governed by the Robin criterion [2] that implies the existence of a continuous run of integer values m, for the ζ-function,

$$\zeta(\rho + i\tau) = \sum_{m=1}^{\infty} m^{-\rho - i\tau} \tag{140}$$

To highlight this fact, we turn to the Li-function and see how it is related to the Robin criteria. Let n be an integer, then from the properties of the Li-function:

$$\mathrm{e}^{\mathrm{Li}(n) - \sum_{m=1}^{\infty}\frac{(\ln(n))^k}{k\,k!} + \ln\left(\left(\ln\left(\{\ln(n)\}^n\right)\right)\right) - \ln\left(\left\{\ln\left(n^n\right)\right\}\right)} = \mathrm{e}^{\mathrm{Li}(\ln(n)) - \sum_{m=1}^{\infty}\frac{(\ln(\ln(n)))^k}{k\,k!}} \tag{141}$$

This can be parametrized by putting $b = \mathrm{Li}(n) - \sum_{m=1}^{\infty}\frac{(\ln(n))^k}{kk!}$, and

$c = \mathrm{Li}(\ln(n)) - \sum_{m=1}^{\infty}\frac{(\ln(\ln(n)))^k}{kk!}$ in (145) to get:

$$\mathrm{e}^{b + \ln\left(\left(\ln\left(\{\ln(n)\}^n\right)\right)\right) - \ln\left(\left\{\ln\left(n^n\right)\right\}\right)} - \mathrm{e}^c = 0 \tag{142}$$

Solving for n, we get:

$$n = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{-W\left(k, -\mathrm{e}^{c-c}\right) + c - b}}} \tag{143}$$

where, $W(k,z)$, is the Lambert W function, with branch, $k = 0$. $W(k,z)$ is defined as

$$W(z)\mathrm{e}^{W(z)} = z \tag{144}$$

$W(z)$ has a series representation Corless, [18], for the primary branch, $k = 0$, as:

$$W(z) = \sum_{m=1}^{\infty}\frac{(-n)^{n-1}}{n!}z^n \tag{145}$$

For example, in Maple 2024, the branches of $W(k,z)$ are shown as in Figure 1.

The branches of the $W(k,z)$ (shown in Figure 1), play an important role in the Riemann-Hypothesis.

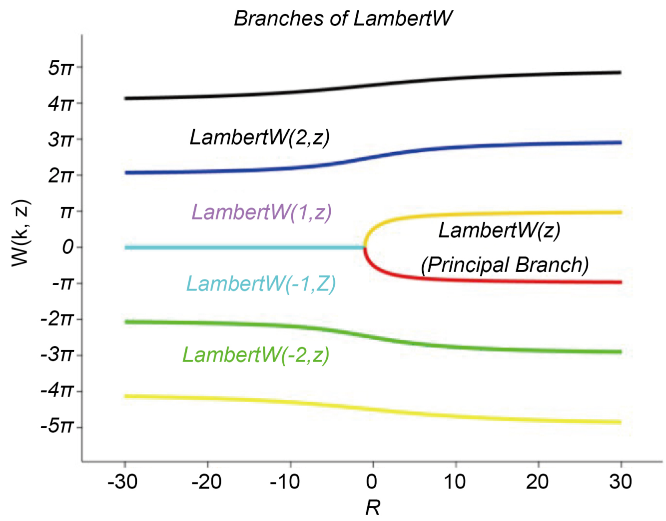

Figure 1. Branches of the LambertW Function.

The branch points can be represented as winding numbers [18],

$$K(z)=\left\lceil\frac{z-\ln e^{z}}{2\pi i}\right\rceil \tag{146}$$

As can be seen, from (149) the LambertW-function plays a role in separating the integers into classes based on the winding numbers. This is easy to see if we look at the class of numbers for which the Robin criterion holds. Simply choosing the winding number in (149) as

$$k=\left|\frac{\sigma(n)}{e^{\gamma}n\ln(\ln(n))}-\left\lfloor\frac{\sigma(n)}{e^{\gamma}n\ln(\ln(n))}\right\rfloor+1\right| \tag{147}$$

gives the distribution of integers,

$$n=e^{e^{e^{-W\left(\left|\frac{\sigma(n)}{e^{\gamma}n\ln(\ln(n))}-\left\lfloor\frac{\sigma(n)}{e^{\gamma}n\ln(\ln(n))}\right\rfloor+1\right|,-e^{c-c}\right)+c-b}}}\begin{cases}=n, & \frac{\sigma(n)}{e^{\gamma}n\ln(\ln(n))}<1\\ \neq n, & \frac{\sigma(n)}{e^{\gamma}n\ln(\ln(n))}>1\end{cases} \tag{148}$$

The integer part $[x]$ of a number x, is given by:

$$[x]-x+\frac{1}{2}=\sum_{m=1}^{\infty}\frac{\sin(2\pi nx)}{n\pi} \tag{149}$$

Hence the fractional part of $x>1$, is given by,

$$\lfloor x\rfloor=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{\sin\left(2\pi n\left(\lfloor x\rfloor+[x]\right)\right)}{n\pi} \tag{150}$$

$$\sin\left(2\pi n\left(\lfloor x\rfloor+[x]\right)\right)=\sin\left(2\pi n(x)\right) \tag{151}$$

$$x=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{\sin\left(2\pi n\left(\lfloor x\rfloor\right)\right)}{n\pi} \tag{152}$$

Now ([14], p. 46),

$$\sum_{n=1}^{\infty}\frac{\sin\left(2\pi n\left(\lfloor x\rfloor\right)\right)}{n}=\frac{\pi-2\pi\left(\lfloor x\rfloor\right)}{2}\quad[0<x<2\pi] \tag{153}$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(2\pi n\left(\lfloor x\rfloor\right)\right)}{n\pi}=\frac{1}{2}-\frac{\left(\lfloor x\rfloor\right)}{2}\quad[0<x<2\pi] \tag{154}$$

$$\lfloor x\rfloor=\frac{1}{2}-\left(\frac{\pi-(2\pi x)}{2}\right) \tag{155}$$

From the Li-function,

$$\mathrm{Li}(z)=\gamma+\ln\left(\ln(z)\right)+\sum_{k=1}^{\infty}\frac{\left(\ln(z)\right)^{k}}{kk!} \tag{156}$$

Putting for integer, n,

$$X=\mathrm{e}^{\left(\ln(n)\right)^{-\frac{\mathrm{e}^{\gamma}n}{\sigma(n)}\mathrm{e}^{-\gamma}}} \tag{157}$$

we get:

$$\mathrm{Li}(X)-\sum_{k=1}^{\infty}\frac{\{X\}^{k}}{kk!}=\frac{\mathrm{e}^{\gamma}n\ln\left(\ln(n)\right)}{\sigma(n)}=\gamma+\ln\left(\ln(X)\right)>1\quad[X>1] \tag{158}$$

Putting for integer, n,

$$Y=\mathrm{e}^{\mathrm{e}^{\left(\ln(n)\right)^{-\left(\gamma+\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln\left(\ln(n)\right)}\right)}}} \tag{159}$$

we get,

$$\mathrm{Li}(Y)=\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln\left(\ln(n)\right)}+\sum_{k=1}^{\infty}\frac{\left(\ln(Y)\right)^{k}}{kk!} \tag{160}$$

We note that the Robin criterion [2] is given by:

$$\mathrm{Li}(Y)-\sum_{k=1}^{\infty}\frac{\left(\ln(Y)\right)^{k}}{kk!}=\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln\left(\ln(n)\right)}=\gamma+\ln\left(\ln(Y)\right)<1\quad[Y>1] \tag{161}$$

Respectively,

$$X=\mathrm{e}^{\mathrm{e}^{\frac{\mathrm{e}^{\gamma}n\ln\left(\ln(n)\right)}{\sigma(n)}-\gamma}},\quad Y=\mathrm{e}^{\mathrm{e}^{\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln\left(\ln(n)\right)}-\gamma}},\quad \{X,Y\}>1 \tag{162}$$

We also note that

$$\begin{gathered}\left\{\mathrm{Li}(X)-\sum_{k=1}^{\infty}\frac{\{X\}^{k}}{kk!}\right\}\left\{\mathrm{Li}(Y)-\sum_{k=1}^{\infty}\frac{\left(\ln(Y)\right)^{k}}{kk!}\right\}\\ =\left\{\gamma+\ln\left(\ln(X)\right)\right\}\left\{\left\{\gamma+\ln\left(\ln(Y)\right)\right\}\right\}=1,\ \{X,Y\}>1\end{gathered} \tag{163}$$

$$\{\gamma + \ln(\ln(X))\}\{\gamma + \ln(\ln(Y))\} = 1, \quad \{X, Y\} > 1 \tag{164}$$

Figure 2 shows the growth of the function Y *with* n, and Figure 3 shows the growth of X *with* n.

With this preparatory work, I will explore the curiosities of the Robin criterion for integers.

Theorem 6. Let $p_{\max}$, be the maximum number of integers that do not comply with the Robin inequality, and let β be the minimum count for which the number integers that both comply and do not comply with the Robin inequality are equal, then,

$$\frac{\sigma(\beta)}{\beta} = e^{\lambda}(\ln(\ln(\beta))) + \frac{\left[\frac{7}{3} - e^{\lambda}\ln(\ln(12))\right]\ln(\ln(12))}{\ln(\ln(\beta))} \tag{165}$$

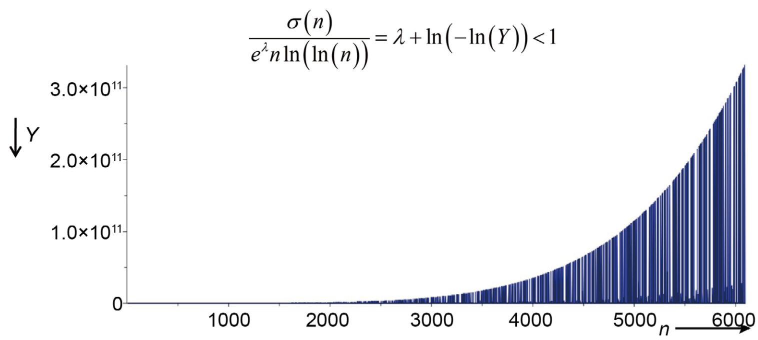

Figure 2. The growth of Y with n.

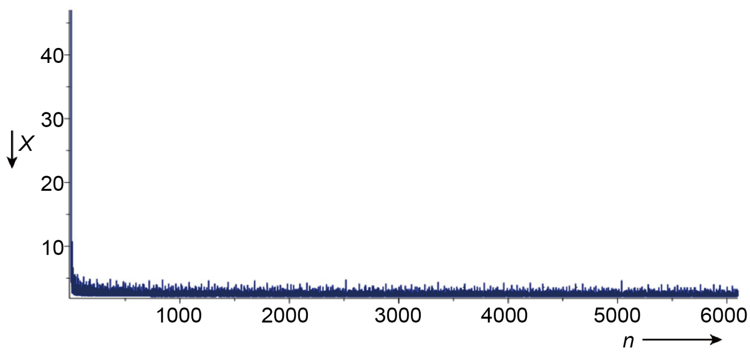

Figure 3. The growth of X with n.

If and only if $p_{\max} = 26$.

Proof: We start the proof of Theorem 6 with a theorem due to Robin [2].

Theorem 7. (G. Robin [2]): *Independent of the Riemann Hypothesis, except for* $n = 1, 2, 12$:

$$\frac{\sigma(n)}{n} < e^{\lambda}(\ln(\ln(n))) + \frac{\left[\frac{7}{3} - e^{\lambda}\ln(\ln(12))\right]\ln(\ln(12))}{\ln(\ln(n))} \tag{166}$$

Note that the form of the integer function (148)

$$n = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{-W\left(\left|\frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))} - \left\lfloor \frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))} \right\rfloor + 1\right|, -\mathrm{e}^{c-c}\right) + c - b}}} \begin{cases} = n, & \dfrac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))} < 1 \\ \neq n, & \dfrac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))} > 1 \end{cases} \tag{167}$$

is similar to the forms:

$$X = \mathrm{e}^{\mathrm{e}^{\frac{\mathrm{e}^{\gamma} n \ln(\ln(n)) - \gamma\sigma(n)}{\sigma(n)}}}, \quad Y = \mathrm{e}^{\mathrm{e}^{\frac{\gamma\, n \ln(\ln(n)) - \sigma(n)}{\mathrm{e}^{\gamma} \ln(\ln(n))}}}, \quad \{X, Y\} > 1 \tag{168}$$

in which case it is easy to see the equivalence of the LambertW function to the log function.

$$\begin{aligned} &-W\left(\left|\frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))} - \left\lfloor \frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))} \right\rfloor + 1\right|, -\mathrm{e}^{c-c}\right) \\ &\to \left\{\ln\left(\frac{\mathrm{e}^{\gamma} n \ln(\ln(n))}{\sigma(n)}\right), \ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))}\right)\right\} \end{aligned} \tag{169}$$

Indeed using (165) in Maple 2024, we find that the Robin number set

$$\begin{aligned} R_n \in [&3,4,5,6,8,9,10,12,16,18,20,24,30,36,48,60,72, \\ &84,120,180,240,360,720,840,2520,5040] \end{aligned}$$

do not obey (165)!

Indeed, the following theorem by A. F. Beardon on the LambertW function clears this relationship.

Theorem 8. [21] Let $\boldsymbol{W}_k$ be the branches of the Lambert function. Then,

$$\boldsymbol{W}_k(z) + \ln(\boldsymbol{W}_k(z)) = \begin{cases} \ln z & \text{if } k = -1 \text{ and } z \in \left[-\frac{1}{e}, 0\right] \\ \ln z + 2\pi i k & \text{otherwise} \end{cases}$$

Then, using the equivalence (169), we see that we can construct the relation:

$$\begin{aligned} &\boldsymbol{W}_k\left(\ln\left(\frac{\mathrm{e}^{\gamma} n \ln(\ln(n))}{\sigma(n)}\right)\right) + \ln\left(\boldsymbol{W}_k\left(\ln\left(\frac{\mathrm{e}^{\gamma} n \ln(\ln(n))}{\sigma(n)}\right)\right)\right) \\ &= \ln\left(\ln\left(\frac{\mathrm{e}^{\gamma} n \ln(\ln(n))}{\sigma(n)}\right)\right) \end{aligned} \tag{170}$$

$$\begin{aligned} &\boldsymbol{W}_k\left(\ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))}\right)\right) + \ln\left(\boldsymbol{W}_k\left(\ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))}\right)\right)\right) \\ &= \ln\left(\ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma} n \ln(\ln(n))}\right)\right) \end{aligned} \tag{171}$$

Adding the (170) to (171),

$$\begin{aligned}&\boldsymbol{W}_{\boldsymbol{k}}\left(\ln\left(\frac{\mathrm{e}^{\gamma}n\ln(\ln(n))}{\sigma(n)}\right)\right)+\ln\left(\boldsymbol{W}_{\boldsymbol{k}}\left(\ln\left(\frac{\mathrm{e}^{\gamma}n\ln(\ln(n))}{\sigma(n)}\right)\right)\right)\\&-\boldsymbol{W}_{\boldsymbol{k}}\left(\ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln(\ln(n))}\right)\right)-\ln\left(\boldsymbol{W}_{\boldsymbol{k}}\left(\ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln(\ln(n))}\right)\right)\right)\\&=\left\{\ln\left(\ln\left(\frac{\mathrm{e}^{\gamma}n\ln(\ln(n))}{\sigma(n)}\right)\right)-\ln\left(\ln\left(\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln(\ln(n))}\right)\right)\right\}\end{aligned} \tag{172}$$

It is worth noting that the sum of the difference, for $X=\mathrm{e}^{\mathrm{e}^{\frac{\mathrm{e}^{\gamma}n\ln(\ln(n))-\gamma\sigma(n)}{\sigma(n)}}}-Y=\mathrm{e}^{\mathrm{e}^{\frac{\gamma n\ln(\ln(n))-\sigma(n)}{\mathrm{e}^{\gamma}\ln(\ln(n))}}}$, is dominated by the prime 3 *for all values* of $p>3$ except when $p=2$, when the sum $S=0$.

$$\sum_{n=3}^{p>2}(Y-X)=C_R<2.05169589435448111952\times10^{698} \tag{173}$$

Let $a=\frac{\sigma(n)}{\mathrm{e}^{\gamma}n\ln(\ln(n))}$.

$$\begin{aligned}&\left\{\boldsymbol{W}_{\boldsymbol{k}}(-\ln(a))+\ln(\boldsymbol{W}_{\boldsymbol{k}}(-\ln(a)))\right\}-\left\{\boldsymbol{W}_{\boldsymbol{k}}(\ln(a))+\ln(\boldsymbol{W}_{\boldsymbol{k}}(\ln(a)))\right\}\\&=\left\{\ln\left(\ln\left(\frac{1}{a}\right)\right)-\ln(\ln(a))\right\}\end{aligned} \tag{174}$$

Now, if $a<1$,

$$\frac{1}{2}+\frac{\left[\ln\left(\ln\left(\frac{1}{a}\right)\right)-\ln(\ln(a))\right]i=\pi}{2\pi}=1 \tag{175}$$

And if $a>1$,

$$\frac{1}{2}+\frac{\left[\ln\left(\ln\left(\frac{1}{a}\right)\right)-\ln(\ln(a))\right]i=-\pi}{2\pi}=0 \tag{176}$$

Hence it is clear that the branch points of the LambertW function correspond to the Robin inequality.

It follows that we could generate a Robin compliant integer β, with $\frac{\sigma(\beta)}{\mathrm{e}^{\lambda}\beta\ln(\ln(\beta))}<1$, as follows:

$$\begin{aligned}&\beta\left(\frac{1}{2}+\frac{\left\{\ln\left(\ln\left(\frac{\mathrm{e}^{\lambda}\beta\ln(\ln(\beta))}{\sigma(\beta)}\right)\right)-\ln\left(\ln\left(\frac{\sigma(\beta)}{\mathrm{e}^{\lambda}\beta\ln(\ln(\beta))}\right)\right)\right\}i}{2\pi}\right)\\&=\begin{cases}\beta\\\text{otherwise }0\end{cases}\end{aligned} \tag{177}$$

Hence, using the same process, we can also develop a ***Robin compliant counting*** function $N_{RCC(p)}$, for up to the p^{th} integer element belonging to $N_{RCC(p)}$, and get:

$$N_{RCC(p)} = \sum_{\beta=3}^{p} \left(\frac{1}{2} + \frac{\left\{ \ln\left(\ln\left(\frac{e^{\lambda} \beta \ln\left(\ln\left(\beta\right)\right)}{\sigma\left(\beta\right)} \right)\right) - \ln\left(\ln\left(\frac{\sigma\left(\beta\right)}{e^{\lambda} \beta \ln\left(\ln\left(\beta\right)\right)} \right)\right) \right\} i}{2\pi} \right) \times \left(\frac{1}{2} + \frac{\left\{\left\{ \ln\left(\ln\left(\frac{e^{\lambda} n \ln\left(\ln\left(p\right)\right)}{\sigma\left(p\right)} \right)\right) - \ln\left(\ln\left(\frac{\sigma\left(p\right)}{e^{\lambda} p \ln\left(\ln\left(p\right)\right)} \right)\right) \right\}\right\} i}{2\pi} \right) \tag{178}$$

Note that the count is discontinuous, since there are "n-element counts", not belonging to $N_{RCC(p)}$, a count of N elements does not correspond to the value, n. **Table 3** below shows the Robin compliant integer counting function $N_{RCC(p)}$ as p increases.

Table 3. It shows the first 50 integers with the Robin Compliant Count $N_{RCC(p)}$.

0_{3}, 0_{4}, 0_{5}, 0_{6}, 1_{7}, 0_{8}, 0_{9}, 0_{10}, 2_{11}, 0_{12}, 3_{13}, 4_{14}, 5_{15}, 0_{16}, 6_{17}, 0_{18}, 7_{19}, 0_{20}, 8_{21}, 9_{22}, 10_{23}, 0_{24}, 11_{25}, 12_{26}, 13_{27}, 14_{28}, 15_{29}, 0_{30}, 16_{31}, 17_{32}, 18_{33}, 19_{34}, 20_{35}, 0_{36}, 21_{37}, 22_{38}, 23_{39}, 24_{40}, 25_{41}, 26_{42}, 27_{43}, 28_{44}, 29_{45}, 30_{46}, 31_{47}, 0_{48}, 32_{49}, 33_{50}, …
The suffix shows the number that contributes to the compliant count.

Indeed $N_{RC(p)}$ must be continuous for all *p that obey the Robin criterion*, otherwise one cannot define a ζ-function as:

$$\zeta\left(z\right) = \sum_{p=3}^{\infty}\left(N_{RCC(p)}\right) = \zeta\left(-1\right) = -\frac{1}{12} \tag{179}$$

Similarly, we can generate the counting function for the Robin non-compliant counting function for non-compliant integers, $\beta \in N_{RNC(p)}$. (**Table 4**)

$$N_{RNC(p)} = \sum_{n=3}^{p} \left(\frac{1}{2} - \frac{\left\{ \ln\left(\ln\left(\frac{e^{\lambda} \beta \ln\left(\ln\left(\beta\right)\right)}{\sigma\left(\beta\right)} \right)\right) - \ln\left(\ln\left(\frac{\sigma\left(\beta\right)}{e^{\lambda} \beta \ln\left(\ln\left(\beta\right)\right)} \right)\right) \right\} i}{2\pi} \right) \times \left(\frac{1}{2} - \frac{\left\{\left\{ \ln\left(\ln\left(\frac{e^{\lambda} n \ln\left(\ln\left(p\right)\right)}{\sigma\left(p\right)} \right)\right) - \ln\left(\ln\left(\frac{\sigma\left(p\right)}{e^{\lambda} p \ln\left(\ln\left(p\right)\right)} \right)\right) \right\}\right\} i}{2\pi} \right) \tag{180}$$

Table 4. It shows the first 50 integers with the count $N_{RNC(p)}$ for the non-compliant integers.

$1_3, 2_4, 3_5, 4_6, 0_7, 5_8, 6_9, 7_{10}, 0_{11}, 8_{12}, 0_{13}, 0_{14}, 0_{15}, 9_{16}, 0_{17}, 10_{18}, 0_{19}, 11_{20}, 0_{21}, 0_{22}, 0_{23}, 12_{24}, 0_{25}, 0_{26}, 0_{27}, 0_{28}, 0_{29}, 13_{30}, 0_{31}, 0_{32}, 0_{33}, 0_{34}, 0_{35}, 14_{36}, 0_{37}, 0_{38}, 0_{39}, 0_{40}, 0_{41}, 0_{42}, 0_{43}, 0_{44}, 0_{45}, 0_{46}, 0_{47}, 15_{48}, 0_{49}, 0_{50}, \ldots$

The suffix shows the number that contributes to the non-compliant count.

If by Robin's [2], *Theorem* 2, the Riemann Hypothesis is true, then, $N_{RNC(p)}$ should have a maximum value $N_{RNC(p_{\max})}$, with $p_{\max}=26$ elements. The Robin non-compliant integers can be counted for up to the "maximum element count" $n=p_{\max}{}^{th}$ term as follows:

$$N_{RNC(p_{\max})}=\sum_{n=3}^{p_{\max}}\left(\frac{1}{2}-\frac{\left\{\ln\left(\ln\left(\frac{e^{\lambda}\beta\ln(\ln(\beta))}{\sigma(\beta)}\right)\right)-\ln\left(\ln\left(\frac{\sigma(\beta)}{e^{\lambda}\beta\ln(\ln(\beta))}\right)\right)\right\}i}{2\pi}\right)\times\left(\frac{1}{2}-\frac{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}n\ln(\ln(p))}{\sigma(p)}\right)\right)-\ln\left(\ln\left(\frac{\sigma(p)}{e^{\lambda}p\ln(\ln(p))}\right)\right)\right\}\right\}i}{2\pi}\right) \tag{181}$$

As can be seen, the counting functions have stationary count-rise rates when we encounter the complimentary state integers. ***This is very much how the primes behave***, since the prime-counting function $\pi(n)$, repeats for values of n that are between primes. The Robin compliant numbers give:

$$\sum_{p=3}^{\infty}\left(N_{RCC(p)}\right)=\sum_{m=1}^{\infty}(m)=\zeta(-1)=-\frac{1}{12} \tag{182}$$

Let,

$$N_{RNC(p_{\max})}=N_{RCC(p_{\max})} \tag{183}$$

Then, for all integer n,

$$\frac{N_{RNC(p_{\max})}}{\left(\frac{1}{2}-\frac{\left\{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln(\ln(p_{\max}))}{\sigma(p_{\max})}\right)\right)-\ln\left(\ln\left(\frac{\sigma(p_{\max})}{e^{\lambda}p_{\max}\ln(\ln(p_{\max}))}\right)\right)\right\}i}{2\pi}\right)}=\sum_{n=3}^{p_{\max}}\left(\frac{1}{2}-\frac{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}n\ln(\ln(n))}{\sigma(n)}\right)\right)-\ln\left(\ln\left(\frac{\sigma(n)}{e^{\lambda}n\ln(\ln(n))}\right)\right)\right\}\right\}i}{2\pi}\right) \tag{184}$$

Similarly:

$$\frac{N_{RCC(p_{\max})}}{\left(\frac{1}{2}+\frac{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}{\sigma\left(p_{\max}\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(p_{\max}\right)}{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}\right)\right)i}{2\pi}\right)}=\sum_{n=3}^{p_{\max}}\left(\frac{1}{2}+\frac{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}n\ln\left(\ln\left(n\right)\right)}{\sigma\left(n\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(n\right)}{e^{\lambda}n\ln\left(\ln\left(n\right)\right)}\right)\right)\right\}\right\}i}{2\pi}\right) \tag{185}$$

Then,

$$\frac{N_{RNC(p_{\max})}}{\left(\frac{1}{2}-\frac{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}{\sigma\left(p_{\max}\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(p_{\max}\right)}{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}\right)\right)i}{2\pi}\right)}-\frac{N_{RCC(p_{\max})}}{\left(\frac{1}{2}+\frac{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}{\sigma\left(p_{\max}\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(p_{\max}\right)}{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}\right)\right)i}{2\pi}\right)}=\sum_{n=3}^{p_{\max}}\left(\frac{\left\{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}n\ln\left(\ln\left(n\right)\right)}{\sigma\left(n\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(n\right)}{e^{\lambda}n\ln\left(\ln\left(n\right)\right)}\right)\right)\right\}\right\}\right\}i}{\pi}\right) \tag{186}$$

$$\left\{\frac{N_{RNC(p_{\max})}\left(\frac{1}{2}+\frac{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}{\sigma\left(p_{\max}\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(p_{\max}\right)}{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}\right)\right)\right\}\right\}i}{2\pi}\right)-N_{RCC(p_{\max})}\left(\frac{1}{2}-\frac{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}{\sigma\left(p_{\max}\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(p_{\max}\right)}{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}\right)\right)\right\}\right\}i}{2\pi}\right)}{\left(\frac{1}{4}+\frac{\left(\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}{\sigma\left(p_{\max}\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(p_{\max}\right)}{e^{\lambda}p_{\max}\ln\left(\ln\left(p_{\max}\right)\right)}\right)\right)\right)^{2}}{4\pi^{2}}\right)}\right\}$$

$$=\sum_{n=3}^{p_{\max}}\left(\frac{\left\{\ln\left(\ln\left(\frac{e^{\lambda}n\ln\left(\ln\left(n\right)\right)}{\sigma\left(n\right)}\right)\right)-\ln\left(\ln\left(\frac{\sigma\left(n\right)}{e^{\lambda}n\ln\left(\ln\left(n\right)\right)}\right)\right)\right\}i}{\pi}\right)=0$$

(187)

$$N_{RCC(p_{\max})}=0 \qquad \text{if } p_{\max}\in R_m$$

$$\left(\frac{1}{2}-\frac{\left\{\left\{\ln\left(\ln\left(\frac{e^{\lambda}p_{\max}\ln(\ln(p_{\max}))}{\sigma(p_{\max})}\right)\right)-\ln\left(\ln\left(\frac{\sigma(p_{\max})}{e^{\lambda}p_{\max}\ln(\ln(p_{\max}))}\right)\right)\right\}\right\}i}{2\pi}\right)=0 \quad \text{if } p_{\max}\in R_m \tag{188}$$

Then numerator vanishes, for value of $p_{\max}$, and so,

$$\sum_{n=3}^{p_{\max}}\left(\frac{\left\{\ln\left(\ln\left(\frac{e^{\lambda}n\ln(\ln(n))}{\sigma(n)}\right)\right)-\ln\left(\ln\left(\frac{\sigma(n)}{e^{\lambda}n\ln(\ln(n))}\right)\right)\right\}i}{\pi}\right)=0 \tag{189}$$

$\{p_{\max}\}\in Z$, *i.e.*, they must both be integers at which the sum (189) vanishes and these points are exactly $\{p_{\max}=26\}$ as shown in **Figure 4**.

As can be seen from **Figure 5** below, the two functions intersect and become equal when $\{\beta=12, p_{\max}=26\}$.

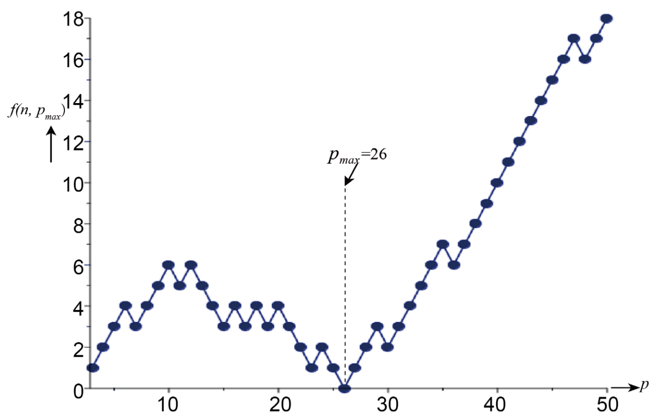

Figure 4. It shows the growth of $N_{RNC(p)}$.

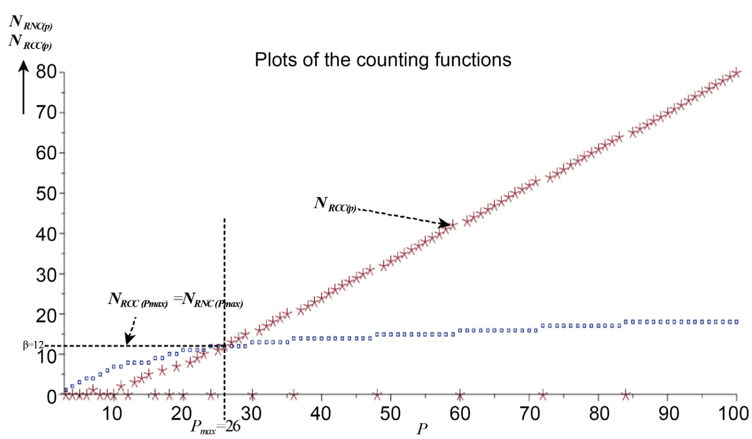

Figure 5. It shows the growth of $N_{RNC(p_{\max})}$ and $N_{RCC(p_{\max})}$, and the intersection of the two functions at $p_{\max}=26$ counts.

Using Theorem 6, we get exact equality when $\beta = 12$ for exactly 26 counting elements proving that the ***Riemann Hypothesis is true***.

11. Conclusion

It is clear that the number π seem to play a very important role in many mathematic relations, ranging from the prime numbers, invariant functions, and even the Robin numbers. Indeed, the reflection formula provides a means of relating any powers of 2π to Zeta and Gamma function. The amazing uses of π in this paper reflect the importance of primes, twin primes, the Gamma-function, the Zeta function, the prime counting functions, the Robin compliant and Robin non-compliant counting functions, and many more relations yet to be explored. A curious fact is that

$$\sum_{p=3}^{\infty}\left(N_{RCC(p)}\right) = \zeta(-1) = -\frac{1}{12} = -\frac{1}{\beta = \sum_{p=3}^{\infty}\left(N_{RCC(p)}\right) - \sum_{p=3}^{\infty}\left(N_{RCC(p)}\right)}.$$

Perhaps this relation is not coincidence and requires some more exploring. Note that this equality is a result of the fractional expressions of π. It is also clear that the Robin compliant counting function also depends on fractions of π.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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