

On Algebraic Methods for the Riemann Hypothesis

Michael Mark Anthony

Department of Engineering, Enertron Inc., Hohenwald, TN, USA
Email: uinvent@aol.com

How to cite this paper: Anthony, M.M. (2025) On Algebraic Methods for the Riemann Hypothesis. *Advances in Pure Mathematics*, 15, 1-35.
<https://doi.org/10.4236/apm.2025.151001>

Received: October 7, 2024

Accepted: January 11, 2025

Published: January 14, 2025

Copyright © 2025 by author(s) and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Properties of the gamma function are examined with implications for the Riemann hypothesis. Some new relations are obtained for the roots of the Zeta function using the properties of the Gamma function, the Bernoulli function, and limitations imposed by the rational relations of the complex roots of the Riemann zeta function, and the rational relations of the complex Gamma functions.

Keywords

Gamma Function, Zeta Function, Bernoulli Numbers, Robin Inequality, Twin Primes, Primes

1. Introduction

The Riemann Zeta function, $\zeta(s)$, is defined by the series:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1 \quad (1)$$

The importance of the relation between the Riemann hypothesis and primes is highlighted by a one-on-one correspondence between the non-trivial roots of the function and the primes.

2. Background

The Riemann Hypothesis, proposed by Bernhard Riemann [1] in 1859, conjectures about the distribution of prime numbers and the relation to the Riemann Zeta function, $\zeta(s)$. It correlates the non-trivial zeros of the ζ -function with the primes if the zeros of the ζ -function have a real part, $\frac{1}{2}$. This hypothesis (RH) is crucial to the understanding of the distribution of primes.

The Robin criterion first specified Guy Robin [2] in 1984 relates the truth of the RH hypothesis to a direct correlation to the statement (Robin’s inequality)

$$F_n = \frac{\sigma(n)e^{-\gamma}}{n \log \log(n)} < 1 \tag{2}$$

if and only if the set of numbers (I call them the Robin Integers, R_n):

$$R_n \in [3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040]$$

are the only integers that violate the inequality,

$$F_n = \frac{\sigma(n)e^{-\gamma}}{n \log \log(n)} < 1, \tag{3}$$

This paper also studies these functions and their interrelations and hence provides an objective insight into the beauty of these relations to the Riemann Hypothesis. Further, the importance of functions such as the Γ -functions, the Bernoulli functions, B_m , and their relationships to the ζ -function is also explored. The Bernoulli function is intimately related to the ζ -function by the relations:

$$\zeta(2m) = \frac{2^{(2m-1)} \pi^{(2m)} |B_{2m}|}{(2m)!}, \quad \zeta(1-2m) = -\frac{B_{2m}}{2m} \tag{4}$$

It is also very clear that the Γ -function, ($\Gamma(z)$), is intimately to Riemann’s ζ -function. The Γ -function was pioneered round about 1729, by the Swiss mathematician Leonhard Euler [3]. Euler’s deep insights into Γ -function led to numerous results in probability theory and in statistics. The particular form of the Γ -function used in this paper was developed by Carl Freidman Gauss [4]. Gauss’s work led to the famous reflection formula of the ζ -function, and numerous new results that will be presented in this paper. Certain invariant relations of the Γ -function are developed in this paper to show the connections of the Γ -function to other functions.

With this in mind, the present paper explores these relationships with some profound results. The distribution of primes and the confluence of the Robin criterion is clearly shown in **Figures 1-3**, and **Figure 4**. Results in prime numbers such as the twin prime conjecture are also discussed.

The ζ -function for primes is given by the prime, p , relation:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \quad \Re(s) > 1 \tag{5}$$

There are key relations for primes that will be highlighted by this article. In general, these relations stem from the Γ -function and in particular, the Gauss’s Γ -function product formula. The intriguing relations of the Sum of Divisors function $\sigma(m)$, with the ζ -function and the Riemann Hypothesis, were developed by several pioneers including Guy Robin [2]. The Sums of Divisors is defined as

$$\sigma(m) = \sum_{d|m} d \tag{6}$$

The uniformity of the function for primes, p , $\sigma(p^x) = (p^{x+1} - 1)/(p - 1)$, provides a way to link the $\sigma(m)$ function to the Robin inequality and hence, the Riemann Hypothesis.

Note: If $\prod_{k=1}^K p_k^{a_k}$ is the prime factorization of m , where m , is an integer with K factors, and a_k are particular numbers, then,

$$\sigma(m) = \begin{cases} \prod_{k=1}^K \frac{p_k^{(a_k+1)^r}}{p_k^r - 1} & r > 0 \\ \prod_{k=1}^K (a_k + 1) & r = 0 \end{cases} \tag{7}$$

where, $\gamma = 0.5772156649\dots$ is the Euler-constant.

Several attempts to refine the fundamental relations for F_n as a referendum on the Riemann hypothesis have been made with great success. P. Solé and M. Planat [5], showed in 1962 that, if N_n is the primordial number of index- n , defined as the product of the first n primes, with $N_n = \prod_{k=1}^n p_k$, then,

$$\lim_{n \rightarrow \infty} N_n = \frac{e^\gamma}{\zeta(2)} \sim 1.08, \tag{8}$$

J. B. Roser ([6], Theorem 15), proved that there are inequalities between Euler’s totient function $\varphi(n)$, and $e^\gamma n \log \log(n)$,

$$\frac{n}{\varphi(n)} \leq e^\gamma n \log \log(n), \quad n \geq 3 \tag{9}$$

In 1984 Heath-Brown [7], showed that there are infinitely many positive integers m , for which the divisor function has $d(m) = d(m + B)$, where B is an integer. C.G. Pinner [8] also demonstrated that there are repeated values of the divisor function. Y.J. Chloe, N. Lichiardopol, and P. Moree [9] also studied the Robin criteria as did Sayak Chakrabarty [10], and they found possible correlations of the repeated divisor function with Highly composite numbers. Jean-Louis Nicolas [11], proved that there are an infinite number of values of $n \geq 3$, that satisfy the inequality (5).

In recent times, I focused on iterations of number theoretic functions. In the paper the Towering Zeta Function [12], I show that the Zeta function can be iterated T -times, denoted as $\zeta^T(z)$ which tends to a constant when T approaches infinity:

$$\begin{aligned} \lim_{T \rightarrow \infty} \zeta^T(z) &= \zeta\left(\lim_{T \rightarrow \infty} \text{slog}_n(z)\right) = \zeta(z_w) = z_w \\ &= -0.29590500557213955\dots \end{aligned}$$

I also showed that the Zeta function only converges to the real constant when the complex part of conjugate Zeta function roots, T , are expressed in the exponential form:

$$e^{2i\theta} = \frac{\frac{1}{2} + iT}{\frac{1}{2} - iT}, \text{ where } \theta = \tan^{-1} 2T,$$

In 2024, I showed [13] that Perron’s formula given by:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2} \ln(1 - x^{-2}) - \ln 2\pi \tag{10}$$

Involves the complex roots ρ , of the Zeta function can be iterated to condense to the k^{th} iterant:

$$x - \Psi^k(x) = k(x - \Psi(x)) \tag{11}$$

The rational functions of the roots of the Riemann zeta function are also expressed as rational Gamma functions. The behavior of the Gamma function for complex arguments restricts the validity of these rational arguments to arguments $\Re(s) > 0$. As will be shown, this restriction also applies to the roots.

The first method that shows that the roots of the function must lie on the half-line is obtained by means of the Gauss-gamma function. It is shown that the gamma function has properties that impose conditions for the functional relations on the half-line.

The second method also uses the complex form of the Γ -function in [14]. In both these methods, an analysis of the relations that involve the Zeta functions and the gamma functions is used to conditionally show that the half-line plays an important role in many functions. In a first demonstration of the relationship of the Γ -function to the $\frac{1}{2}$ -line, invariants of the Γ -function are used to demonstrate how the $\frac{1}{2}$ line features in a lot of results. In a second method, the relationship between ζ -function, the Bernoulli numbers, B_n and Γ -function is explored to precipitate unexpected results about the rational relations of complex Γ -function, and its specific influence on Riemann conjecture. The rational functions of the non-trivial roots of the ζ -function happen to be expressible as relations between the Bernoulli numbers, B_n and Γ -function and the complex roots. Both these methods produce indisputable verification that the non-trivial roots of the ζ -function lie on the half-line with no assumptions made. The process is a natural consequence of a special property of these functions.

3. The Invariance of the Gamma Function to Substitution

$$\sigma(m) \rightarrow \sigma(m + j)$$

Any function with a relational product $\{n \cdot y\}$, can be represented by the Sums of Divisor function, $\sigma(m)$. Here is a simple example:

$$\log(n \cdot y) = \log n + \log y, \tag{12}$$

Then, if $n \cdot y = m$, we can put $n = \sigma(m)$, $y = \frac{m}{\sigma(m)}$, and so,

$$\log(m) = \log \sigma(m) + \log \frac{m}{\sigma(m)} \tag{13}$$

Here is another example:

If $n \cdot y = m$, we can put $n = \sigma(m)$, $y = \frac{m}{\sigma(m)}$, and so, applied to the formula:

$$\cos(n \cdot y) = \cos(y) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(y)}{\sin^2 \frac{(2k-1)\pi}{2n}} \right), [n \text{ is even}] \tag{14}$$

By using the sum of divisor function, we find the relation:

$$\cos(m) = \cos\left(\frac{m}{\sigma(m)}\right) \prod_{k=1}^{\frac{\sigma(m)-1}{2}} \left(1 - \frac{\sin^2\left(\frac{m}{\sigma(m)}\right)}{\sin^2 \frac{(2k-1)\pi}{2\sigma(m)}} \right), [\sigma(m) \text{ is odd}] \tag{15}$$

Interestingly, (14) and (15) differentiate between odd and even values of $\sigma(m)$. Since primes have $\sigma(p) = p + 1$, an even number the relation (14) *does not apply to primes*. Since $p + 1$ is always even except for the prime 2! Since $\sigma(2) = 3$,

$$\cos(2) = \cos\left(\frac{2}{3}\right) \prod_{k=1}^1 \left(1 - \frac{\sin^2\left(\frac{2}{3}\right)}{\sin^2 \frac{(2k-1)\pi}{6}} \right), [\sigma(2) \text{ is odd}] \tag{16}$$

$$\begin{aligned} -0.4161468365\dots &= 0.7858872608\dots \left(1 - \frac{0.3823812134\dots}{0.2500000000} \right) \\ &= -0.4161468365\dots \end{aligned} \tag{17}$$

The fact that the sum of divisor function $\sigma(m)$, can be manipulated this way leads to some interesting formulas that can produce significant and unexpected results.

For example, if we put:

$$n = \sigma(m), x = \frac{n}{\sigma(m)}, \tag{18}$$

in the following trigonometric relations, we get:

$$\left. \begin{aligned} \sin(m) &= \sigma(m) \sin\left(\frac{m}{\sigma(m)}\right) \cos\left(\frac{m}{\sigma(m)}\right) \prod_{k=1}^{\frac{\sigma(m)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{m}{\sigma(m)}\right)}{\sin^2 \frac{\pi k}{\sigma(m)}} \right) \\ \cos(m) &= \prod_{k=1}^{\frac{\sigma(m)}{2}} \left(1 - \frac{\sin^2\left(\frac{m}{2\sigma(m)}\right)}{\sin^2 \frac{(2k-1)\pi}{2\sigma(m)}} \right) \end{aligned} \right\} [\sigma(m) \text{ is even}] \tag{19}$$

$$\left. \begin{aligned} \sin(m) &= \sigma(m) \sin\left(\frac{n}{\sigma(m)}\right) \prod_{k=1}^{\frac{\sigma(m)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{m}{\sigma(m)}\right)}{\sin^2\frac{k\pi}{\sigma(m)}}\right) \\ \cos(m) &= \cos\left(\frac{m}{\sigma(m)}\right) \prod_{k=1}^{\frac{\sigma(m)-1}{2}} \left(1 - \frac{\sin^2\left(\frac{m}{\sigma(m)}\right)}{\sin^2\frac{(2k-1)\pi}{2\sigma(m)}}\right) \end{aligned} \right\} [\sigma(m) \text{ is odd}] \quad (20)$$

Any factorizable integer function can be done this way. Of course, the construct restricts the functions we use to define the integers, m . Clearly, the relationship of integer functions to the Gauss Γ -function is clear. Again, we can represent the relationship of the Gauss gamma function ([14], p. 896) as:

$$\Gamma(y \cdot n) = (2\pi)^{\frac{1-y}{2}} y^{(n \cdot y) - \frac{1}{2}} \prod_{k=0}^{y-1} \Gamma\left(n + \frac{k}{y}\right) \quad (21)$$

Let $g(z)$ be a function with k -factors, then for any given factor, $f_j(m)$, $j < k$,

$$g(z) = f_j(z) \left(\prod_{n=1}^{j-1} f_n(z) \prod_{n=j+1}^k f_n(z) \right)$$

and,

$$\Gamma(g(z)) = (2\pi)^{\frac{1-\frac{f(z)}{f_j(m)}}{2}} \left(\frac{f(z)}{f_j(m)} \right)^{f(z) - \frac{1}{2} \frac{f(z)}{f_j(m)}} \prod_{k=0}^{\frac{f(z)}{f_j(m)} - 1} \Gamma\left(\frac{f(z) + k}{\frac{f(z)}{f_j(m)}} \right) \text{ for any } j. \quad (22)$$

The relation (22) is only true if $f_j(m) | g(z) \in \mathbb{Z}$. Observe that while $g(z)$, is uniquely defined as a real or complex function, $\frac{g(z)}{f_j(m)}$ is not uniquely defined

as an integer function since many functions $f_j(m)$, can satisfy the relation (22) for some $g(z)$. However, there *may* exist multiple functions for which

$\left\{ \frac{g(z)}{f_a(m)}, \frac{g(z)}{f_b(m)}, \frac{g(z)}{f_c(m)}, \dots \right\} \in \mathbb{Z}$. In such a case, $g(z)$ has *multiple integer-*

factors. It is interesting to note that the product form of two variables of the gamma function is behaving like a *non-commutative function* (I use that word loosely) since, $\Gamma(n \cdot y) \neq \Gamma(y \cdot n)$, when the roles of n , and y , are interchanged. This may provide a means to understand the sum of divisors function $\sigma(v)$. Suppose,

we consider values of $n = \sigma(v)$, $y = \frac{m}{\sigma(v)}$ and, let $y \cdot n = m$, then,

$$\Gamma(m) = (2\pi)^{\frac{1-\sigma(v)}{2}} \sigma(v)^{m - \frac{1}{2}} \prod_{k=0}^{\sigma(v)-1} \Gamma\left(\frac{m+k}{\sigma(v)}\right) \quad (23)$$

The relation (23) is *invariant* to the substitution $\sigma(v) \rightarrow \sigma(v + \mu)$. An

Interchange of the variables, gives:

$$\Gamma(y \cdot n) = (2\pi)^{\frac{1-y}{2}} y^{(n \cdot y) - \frac{1}{2}} \prod_{k=0}^{y-1} \Gamma\left(n + \frac{k}{y}\right) \tag{24}$$

then,

$$\Gamma(m) = (2\pi)^{\frac{1-m}{2}} \left(\frac{m}{\sigma(v)}\right)^{m - \frac{1}{2}} \prod_{k=0}^{\frac{m}{\sigma(v)} - 1} \Gamma\left(\sigma(v) + \frac{k}{\sigma(v)}\right) \tag{25}$$

Note that the multiplied variables $y \cdot n = m$, has not changed m , and, it is the multiplication operation that generates a distinct choice of $\sigma(v) | m$. There is a distinct difference between $\sigma(v)$, and $\sigma(v + \mu)$, since the expression is not invariant to substitution $\sigma(v) \rightarrow \sigma(v + \mu)$, unless $\left\{\frac{m}{\sigma(v)}, \frac{m}{\sigma(v + \mu)}\right\} \in \mathbb{Z}$ for any μ . A function $g(z)$, that may have factors that are integers and complex numbers can be represented as a product of its factors such that the following theorem applies.

THEOREM 1: Let $f_j(z) > 0$, represent an integer factor of $g(z)$, then,

$$\Gamma(g(z)) = (2\pi)^{\frac{1-f_j(z)}{2}} (f_j(z))^{g(z) - \frac{1}{2}} \prod_{k=0}^{f_j(z) - 1} \Gamma\left(\frac{g(z) + k}{f_j(z)}\right) \tag{26}$$

is invariant with respect to choices of any of the factors $f_v(z)$, of $g(z)$.

The significance of the THEOREM 1, is its consequences for prime numbers, and their relations to functions like the ζ -function and the sum of divisors function, $\sigma(m)$ and primes.

PROOF:

Let $f_j(z)$, be some j^{th} integer factor of k -factors a real or complex function $g(z)$. Then,

$$g(z) = \prod_{n=1}^k f_n(z) = f_j(z) \left(\prod_{n=1}^{j-1} f_n(z) \prod_{n=j+1}^k f_n(z)\right) \tag{27}$$

The Gauss gamma product formula is a simple relation given by:

$$\Gamma(n \cdot y) = (2\pi)^{\frac{1-n}{2}} n^{(n \cdot y) - \frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(y + \frac{k}{n}\right) \tag{28}$$

Then, since $f_j(z)$, is an integer-factor of $g(z)$, we have, putting $n = f_j(z)$,

$$\Gamma\left(f_j(z) \left(\prod_{n=1}^{j-1} f_n(z) \prod_{n=j+1}^k f_n(z)\right)\right) = (2\pi)^{\frac{1-f_j(z)}{2}} (f_j(z))^{g(z) - \frac{1}{2}} \prod_{k=0}^{f_j(z) - 1} \Gamma\left(\frac{g(z) + k}{f_j(z)}\right) \tag{29}$$

If there any other integer factor labelled here $f_v(z), \in \mathbb{Z}$, then, the substitution $f_j(z) \rightarrow f_v(z)$ leaves $\Gamma(g(z))$ invariant.

$$\Gamma(g(z)) = (2\pi)^{\frac{1-f_v(z)}{2}} (f_v(z))^{g(z) - \frac{1}{2}} \prod_{k=0}^{f_v(z) - 1} \Gamma\left(\frac{g(z) + k}{f_v(z)}\right) \tag{30}$$

THEOREM 2: If ρ is a non-trivial root of the ζ -function, then,

$$\prod_{k=2}^{\infty} \left(\frac{\zeta(1-k^{-\rho})}{2\Gamma(k^{-\rho})\zeta(k^{-\rho})\cos\left(\frac{\pi(k^{-\rho})}{2}\right)} \right) = 2\pi \tag{31}$$

PROOF:

From the reflection formula,

$$2^{1-z}\Gamma(z)\zeta(z)\cos\left(\frac{\pi z}{2}\right) = \pi^z \zeta(1-z) \tag{32}$$

$$\Gamma(g(z)) = \frac{(2\pi)^{g(z)} \zeta(1-g(z))}{2\zeta(g(z))\cos\left(\frac{\pi g(z)}{2}\right)} \tag{33}$$

DEFINITION 1: Define the transformation

$$(2\pi)^{-g(z)} = \frac{\zeta(1-g(z))}{2\Gamma(g(z))\zeta(g(z))\cos\left(\frac{\pi(g(z))}{2}\right)} \tag{34}$$

as the Pi-transformation.

Then, the left-hand-side is separable into real and complex parts.

If $g(z) = \sum_k z_k$, i.e., the sum of k -terms, and functions, then, the Pi-transformation gives:

$$(2\pi)^{-g(z_k)} = (2\pi)^{-\sum_k z_k} = \prod_k \left(\frac{\zeta(1-g(z_k))}{2\Gamma(g(z_k))\zeta(g(z_k))\cos\left(\frac{\pi(g(z_k))}{2}\right)} \right) \tag{35}$$

For example, let $g(z) = \sigma + i\tau$, then, from (35),

$$(2\pi)^{-(\sigma+i\tau)} = \frac{\zeta(1-\sigma-i\tau)}{2\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)\cos\left(\frac{\pi(\sigma+i\tau)}{2}\right)} \tag{36}$$

And let $g(z) = -(\sigma + i\tau)y$, where y is any function, then, from (33),

$$(2\pi)^{(\sigma+i\tau)y} = \frac{\zeta(1+(\sigma+i\tau)y)}{2\zeta(-(\sigma+i\tau)y)\cos\left(\frac{-\pi(\sigma+i\tau)y}{2}\right)\Gamma(-(\sigma+i\tau)y)} \tag{37}$$

$$(2\pi)^{-(\rho+i\tau)+(\rho+i\tau)y} = \frac{\frac{\zeta(1-\sigma-i\tau)}{2\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)\cos\left(\frac{\pi(\sigma+i\tau)}{2}\right)}}{\zeta(1+(\sigma+i\tau)y)} \tag{38}$$

$$\frac{2\zeta(-(\sigma+i\tau)y)\cos\left(\frac{-\pi(\sigma+i\tau)y}{2}\right)\Gamma(-(\sigma+i\tau)y)}{\zeta(1+(\sigma+i\tau)y)}$$

As an example, put $y = \rho - i\tau$

$$(2\pi)^{\sigma^2 + \tau^2 - \sigma - i\tau} = \frac{\zeta(-\sigma^2 - \tau^2) \zeta(1 - \sigma - i\tau) \Gamma(-\sigma^2 - \tau^2) \cos\left(\frac{\pi(-\sigma^2 - \tau^2)}{2}\right)}{\zeta(1 + \sigma^2 + \tau^2) \zeta(1 + \sigma + i\tau) \Gamma(\sigma + i\tau) \cos\left(\frac{\pi(\sigma + i\tau)}{2}\right)} \quad (39)$$

The relation (39) is again separable into real and complex parts for products of n -functions since the left-hand side is separable into products, $\prod_{k=1}^n F(\arg z_k)$. It is obvious that the powers of (2π) on the right-hand-side (LHS) determine completely the arguments on the left-hand-side (RHS), and in general, the form $(2\pi)^{g(z)}$ determines the RHS as having arguments

$$\prod_{k=1}^n F(z_k) = (2\pi)^{\sum_{k=1}^n z_k} \quad (40)$$

As another example, taking the product relation for the arguments, $g(z) = \frac{1}{2} + iT$, $f(z) = \frac{1}{2} - iT$.

Then,

$$(2\pi)^{\frac{1}{2} - iT} = 2 \left[\frac{\zeta\left(\frac{1}{2} - iT\right)}{\Gamma\left(\frac{1}{2} + iT\right) \zeta\left(\frac{1}{2} + iT\right) \cos\left(\frac{\pi\left(\frac{1}{2} + iT\right)}{2}\right)} \right], \quad (41)$$

$$(2\pi)^{\frac{1}{2} + iT} = 2 \left[\frac{\zeta\left(\frac{1}{2} + iT\right)}{\Gamma\left(\frac{1}{2} - iT\right) \zeta\left(\frac{1}{2} - iT\right) \cos\left(\frac{\pi\left(\frac{1}{2} - iT\right)}{2}\right)} \right]$$

$$2\pi = 4 \left[\frac{\zeta\left(\frac{1}{2} - iT\right)}{\Gamma\left(\frac{1}{2} + iT\right) \zeta\left(\frac{1}{2} + iT\right) \cos\left(\frac{\pi\left(\frac{1}{2} + iT\right)}{2}\right)} \right] \left[\frac{\zeta\left(\frac{1}{2} + iT\right)}{\Gamma\left(\frac{1}{2} - iT\right) \zeta\left(\frac{1}{2} - iT\right) \cos\left(\frac{\pi\left(\frac{1}{2} - iT\right)}{2}\right)} \right] \quad (42)$$

The product of the relation (42) is independent of the ζ -function, and relies only on the relation:

$$4 \left\{ \Gamma\left(\frac{1}{2} + iT\right) \Gamma\left(\frac{1}{2} - iT\right) \right\} \left\{ \cos\left(\frac{\pi\left(\frac{1}{2} + iT\right)}{2}\right) \cos\left(\frac{\pi\left(\frac{1}{2} - iT\right)}{2}\right) \right\} = 2\pi \quad (43)$$

Now,

$$(2\pi)^{-g(z)} = \prod_{k=1}^{\infty} \left(\frac{\zeta(1-g(z))}{2\Gamma(g(z))\zeta(g(z))\cos\left(\frac{\pi(g(z))}{2}\right)} \right) \quad (44)$$

Put $g(z) = \zeta(z)$, then,

$$(2\pi)^{-1-\sum_{k=2}^{\infty} k^{-z}} = \frac{1}{2\pi} \prod_{k=2}^{\infty} \left(\frac{\zeta(1-k^{-z})}{2\Gamma(k^{-z})\zeta(k^{-z})\cos\left(\frac{\pi(k^{-z})}{2}\right)} \right) = (2\pi)^{-\zeta(z)} \quad (45)$$

If $z = \rho$, is a root, then, $\zeta(\rho) = 0$, the non-trivial roots follow:

$$\prod_{k=2}^{\infty} \left(\frac{\zeta(1-k^{-\rho})}{2\Gamma(k^{-\rho})\zeta(k^{-\rho})\cos\left(\frac{\pi(k^{-\rho})}{2}\right)} \right) = 2\pi \quad (46)$$

Note that the conjugate relations on $\frac{1}{2}$ line give,

$$4 \left[\frac{\zeta\left(\frac{1}{2} - iT\right)}{\Gamma\left(\frac{1}{2} + iT\right)\zeta\left(\frac{1}{2} + iT\right)\cos\left(\frac{\pi\left(\frac{1}{2} + iT\right)}{2}\right)} \right] \left[\frac{\zeta\left(\frac{1}{2} + iT\right)}{\Gamma\left(\frac{1}{2} - It\right)\zeta\left(\frac{1}{2} - iT\right)\cos\left(\frac{\pi\left(\frac{1}{2} - iT\right)}{2}\right)} \right] = 2\pi \quad (47)$$

Then, using the fact that

$$(2\pi)^{-g(z)} = \frac{\zeta(1-g(z))}{2\zeta(g(z))\cos\left(\frac{\pi g(z)}{2}\right)\left[\Gamma(g(z))\right]} \quad (48)$$

Put $g(z) = \zeta(z) - \frac{1}{z-1}$, and separate the terms for $\frac{1}{z-1}$, then,

$$\begin{aligned}
 & (2\pi)^{-1-\sum_{k=2}^{\infty} k^{-z} + \frac{1}{z-1}} \\
 &= \frac{1}{2\pi} \left[\frac{\zeta\left(1 + \frac{1}{z-1}\right)}{2\Gamma\left(-\frac{1}{z-1}\right)\zeta\left(-\frac{1}{z-1}\right)\cos\left(\frac{\pi\left(\frac{1}{z-1}\right)}{2}\right)} \right] \prod_{k=2}^{\infty} \left(\frac{\zeta(1-k^{-z})}{2\Gamma(k^{-z})\zeta(k^{-z})\cos\left(\frac{\pi(k^{-z})}{2}\right)} \right) \quad (49) \\
 &= \frac{\zeta\left(1-\zeta(z) + \frac{1}{z-1}\right)}{2\Gamma\left(\zeta(z) - \frac{1}{z-1}\right)\zeta\left(\zeta(z) - \frac{1}{z-1}\right)\cos\left(\frac{\pi\left(\zeta(z) - \frac{1}{z-1}\right)}{2}\right)}
 \end{aligned}$$

Then when $z = \rho$, a root of the ζ -function,

$$\begin{aligned}
 & (2\pi)^{-1-\sum_{k=2}^{\infty} k^{-z} + \frac{1}{z-1}} \\
 &= \frac{1}{2\pi} \left[\frac{\zeta\left(\frac{\rho}{\rho-1}\right)}{\Gamma\left(\frac{1}{1-\rho}\right)\zeta\left(\frac{1}{1-\rho}\right)\cos\left(\frac{\pi}{2-2\rho}\right)} \right] \prod_{k=2}^{\infty} \left(\frac{\zeta(1-k^{-z})}{2\Gamma(k^{-z})\zeta(k^{-z})\cos\left(\frac{\pi(k^{-z})}{2}\right)} \right) \quad (50) \\
 &= \frac{\zeta\left(\frac{\rho}{\rho-1}\right)}{\Gamma\left(\frac{1}{1-\rho}\right)\zeta\left(\frac{1}{1-\rho}\right)\cos\left(\frac{\pi}{2-2\rho}\right)} \\
 & \quad \prod_{k=2}^{\infty} \left(\frac{\zeta(1-k^{-\rho})}{2\Gamma(k^{-\rho})\zeta(k^{-\rho})\cos\left(\frac{\pi(k^{-\rho})}{2}\right)} \right) = 2\pi \quad (51)
 \end{aligned}$$

Hence the relation is uniquely satisfied by the roots ρ , of the ζ -function when $g(\rho) = \frac{1}{1-\rho}$.

$$(2\pi)^{\frac{1}{\rho-1}} = \frac{\zeta\left(\frac{\rho}{\rho-1}\right)}{2\Gamma\left(\frac{1}{\rho-1}\right)\zeta\left(\frac{1}{\rho-1}\right)\cos\left(\frac{\pi\left(\frac{1}{\rho-1}\right)}{2}\right)} \quad (52)$$

Also note that $\lim_{z \rightarrow \infty} g(z) = \lim_{z \rightarrow \infty} \left(\zeta(z) - \frac{1}{z-1}\right) = \lambda \approx 0.5772156649\dots$, where λ is the Euler-Mascheroni constant [14]. Let us not separate the terms, then, the Cauchy principal value exists

$$\lim_{z \rightarrow \infty} \left((2\pi)^{-\zeta(z) + \frac{1}{z-1}} \right) = (2\pi)^{-\lambda} \tag{53}$$

$$(2\pi)^{-\lambda} = \lim_{z \rightarrow \infty} \left(\frac{\zeta \left(1 - \zeta(z) + \frac{1}{z-1} \right)}{2\Gamma \left(\zeta(z) - \frac{1}{z-1} \right) \zeta \left(\zeta(z) - \frac{1}{z-1} \right) \cos \left(\frac{\pi \left(\zeta(z) - \frac{1}{z-1} \right)}{2} \right)} \right) \tag{54}$$

$$(2\pi)^{-\lambda} = \frac{\zeta(1-\lambda)}{2\Gamma(\lambda)\zeta(\lambda)\cos\left(\frac{\pi(\lambda)}{2}\right)} = 0.34616095200041890772 = C_\infty \tag{55}$$

It must be noted that to a large number of decimal places, the constant $C_\infty \approx \pi - e - \lambda + \frac{1}{2}$:

Then, we can conclude that for all the Stieltjes numbers λ_k , follow:

$$\boxed{(2\pi)^{-\lambda_k} = \frac{\zeta(1-\lambda_k)}{2\Gamma(\lambda_k)\zeta(\lambda_k)\cos\left(\frac{\pi(\lambda_k)}{2}\right)}} \tag{56}$$

4. The Relation of the Product Gamma Function to Primes

From the Gauss Γ -product formula [14], (21),

$$\Gamma(m) = (2\pi)^{\frac{1-\sigma(m)}{2}} \sigma(m)^{m-\frac{1}{2}} \prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right) \tag{57}$$

It is clear that the following relations are equal,

$$(2\pi)^{\frac{1-m}{2}} m^{m-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{m+k}{m}\right) = (2\pi)^{\frac{1-\sigma(m)}{2}} \sigma(m)^{m-\frac{1}{2}} \prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right) \tag{58}$$

Then, for all real numbers m ,

$$(2\pi)^{\frac{\sigma(m)-m}{2}} \left(\frac{m}{\sigma(m)}\right)^{m-\frac{1}{2}} \frac{\prod_{k=0}^{m-1} \Gamma\left(\frac{m+k}{m}\right)}{\prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right)} = 1 \tag{59}$$

Since, for all primes, $m = p$, $\sigma(p) = p + 1$, from (59), we get for all primes, p ,

$$\sqrt{2\pi} \frac{p^{\frac{p-1}{2}} \prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}} \prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{p+k}{\sigma(p)}\right)} = 1 \tag{60}$$

and so, (60) is *not invariant* to the substitutions, $\sigma(p) \rightarrow \sigma(p + j)$, unless $\{p, p + j\} \in \text{Primes}$.

In 1999, Pierre Dusart [15] showed that every k^{th} prime p_k is greater than

$$p_k > k(\ln p + \ln \ln p - 1) \text{ for } k \geq 2.$$

Contributions by G. H. Hardy, Ramanujan [[16], §9, p. 141] also demonstrated bounds for primes. Leonhard Euler [17] presented to the St. Petersburg Academy on April 6, 1752. The paper “Observatio de summis divisorum, Novi Commentarii”. In that work, he presented a relation between the Triangular numbers and the Sums of Divisor function $\sigma(p)$. In this paper, I will show that the triangular numbers are indeed related to periods generated between integers as a result of Robin’s inequality.

THEOREM 3: *There are an infinite number twin primes $\{p, p + 2\}$ that have the restriction,*

$$\frac{\left(\frac{\sigma(p)}{\sigma(p+2)}\right)^2}{(2\pi)^{p+2-\frac{\sigma(p)}{2}-\frac{\sigma(p+2)}{2}}} \leq 1, \forall \{p, p+2\} \in \text{twin primes} \tag{61}$$

PROOF:

From the expression (61), put the prime relation for $p, p + 2$,

$$\left[\frac{\sqrt{2\pi} \cdot p^{p-\frac{1}{2}} \prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}} \prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{p+k}{\sigma(p)}\right)} \right] \left[\frac{\sqrt{2\pi} \cdot p^{p+2-\frac{1}{2}} \prod_{k=0}^{p+2-1} \Gamma\left(\frac{p+2+k}{p}\right)}{\sigma(p+2)^{p+2-\frac{1}{2}} \prod_{k=0}^{\sigma(p+2)-1} \Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)} \right] = 1 \tag{62}$$

Then,

$$\frac{\left(\frac{\sigma(p)}{\sigma(p+2)}\right)^2}{(2\pi)^{p+2-\frac{\sigma(p)}{2}-\frac{\sigma(p+2)}{2}}} \leq 1, \forall \{p, p+2\} \in \text{twin primes} \tag{63}$$

It is clear that if $p < \sigma(p), p + 2 < \sigma(p + 2)$, so if p is not a twin prime,

$$(2\pi) \left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}} \left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}} \frac{\prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{p+k}{\sigma(p)}\right)} \frac{\prod_{k=0}^{p+1} \Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p+2)-1} \Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)} \tag{64}$$

$\ll 1$, if $\{p, p + 2\} \notin \text{twin primes}$

Hence for twin primes, $p, p + 2, \frac{\sigma(p)}{\sigma(p+2)} = \frac{p+1}{p+3} < 1$. Then, it follows that since (64) $\ll 1$, then,

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{(2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}} \gg 1, \{p, p+2\} \notin \text{twin primes} \tag{65}$$

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{(2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}} < 1, \{p, p+2\} \in \text{twin primes}$$

The relation (65) determines if *any* pair of integers, $\{p, p+2\} \in$ twin primes *exclusively*.

Hence,

$$\frac{\frac{\sigma(p)}{\sigma(p+2)}}{(2\pi)\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)}} < 1 \tag{66}$$

$$\left(\frac{\sigma(p)}{\sigma(p+2)}\right)\frac{(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}{(2\pi)(p)^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\left(\frac{p+2}{\sigma(p+2)}\right)^{p+\frac{3}{2}}\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)} < 1 \tag{67}$$

$$\left(\frac{\sigma(p)}{\sigma(p+2)}\right)\frac{\left\{(\sigma(p+2))^{p+\frac{3}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)\right\}\left\{(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\right\}}{\left\{(2\pi)(p)^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\right\}\left\{(p+2)^{p+\frac{3}{2}}\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)\right\}} < 1 \tag{68}$$

$$\frac{\left(\frac{\sigma(p)}{\sigma(p+2)}\right)\left\{(2\pi)^{\frac{1}{2}}\frac{\sigma(p+2)}{2}(\sigma(p+2))^{p+\frac{3}{2}}\prod_{k=0}^{\sigma(p+2)-1}\Gamma\left(\frac{p+2+k}{\sigma(p+2)}\right)\right\}\left\{(2\pi)^{\frac{1}{2}}\frac{\sigma(p)}{2}(\sigma(p))^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)\right\}}{(2\pi)^{1-\left(\frac{1}{2}\frac{p}{2}\right)-\left(\frac{1}{2}\frac{p+2}{2}\right)+\left(\frac{1}{2}\frac{\sigma(p+2)}{2}\right)+\left(\frac{1}{2}\frac{\sigma(p)}{2}\right)}\left\{\left\{(2\pi)^{\frac{1}{2}}\frac{p}{2}p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)\right\}\left\{(2\pi)^{\frac{1}{2}}\frac{p+2}{2}(p+2)^{p+\frac{3}{2}}\prod_{k=0}^{p+1}\Gamma\left(\frac{p+2+k}{p+2}\right)\right\}\right\}} < 1 \tag{69}$$

$$\frac{\frac{\sigma(p)}{\sigma(p+2)} \{\Gamma(p+2)\} \{\Gamma(p)\}}{(2\pi)^{-\left(\frac{1}{2} \frac{p}{2}\right) - \left(\frac{1}{2} \frac{p+2}{2}\right) + \left(\frac{1}{2} \frac{\sigma(p+2)}{2}\right) + \left(\frac{1}{2} \frac{\sigma(p)}{2}\right)} \{\Gamma(p)\} \{\Gamma(p+2)\}} < 1 \tag{70}$$

All the terms in the curly bracket represent the Γ -function, and they cancel leaving us with the condition for $p, p+2$ to be twin primes:

$$\boxed{\frac{p+2}{\sigma(p+2)} \leq (2\pi)^{-\frac{\sigma(p+2)}{2} + \left(p+2 - \frac{\sigma(p)}{2}\right)} \forall \{p, p+2\} \in \text{twin primes}} \tag{71}$$

Relation (71) is true for every known twin prime set. However, it is also satisfied by many integers p , that are not primes. It is possible that after a given number of integer solutions that involve primes, the remaining infinite number of solutions can be combinations of primes and non-primes.

The strategy is to first take $p+2$ as a known prime in the expression (71) and then see if p is a prime if we get a sequence of integer-relations relating $p, p+2$ that only survives when p is also a prime.

STATEMENT 1: Let $p+2$ be a prime.

I use the strategy of finding the required relations p to be a twin prime when we accept that $p+2$, is a prime. *We don't know if p is a prime yet*, but we know that $p+2$ is a prime and so, $\sigma(p+2) = p+3$.

$$\frac{\sigma(p)}{p+3} \leq (2\pi)^{-\left\{\frac{3}{2} + \frac{p}{2}\right\} + \left(p+2 - \frac{\sigma(p)}{2}\right)} = M, \text{ where } M \text{ is a constant.}$$

In terms of the Pi-transformations;

$$(2\pi)^{-g(z)} = \frac{\zeta(1-g(z))}{2\Gamma(g(z))\zeta(g(z))\cos\left(\frac{\pi(g(z))}{2}\right)} \tag{72}$$

$$\left\{ \frac{\zeta\left(1 - \left\{\frac{3}{2} + \frac{p}{2}\right\}\right)}{2\Gamma\left(\left\{\frac{3}{2} + \frac{p}{2}\right\}\right)\zeta\left(\left\{\frac{3}{2} + \frac{p}{2}\right\}\right)\cos\left(\frac{\pi\left(\left\{\frac{3}{2} + \frac{p}{2}\right\}\right)}{2}\right)} \right\}$$

$$\times \left[\frac{\zeta\left(1 + \left(p+2 - \frac{\sigma(p)}{2}\right)\right)}{2\Gamma\left(\frac{\sigma(p)}{2} - p - 2\right)\zeta\left(\frac{\sigma(p)}{2} - p - 2\right)\cos\left(\frac{\sigma(p)}{2} - p - 2\right)} \right]$$

$$= M$$

When $p+2$ is an odd prime, it is clear that p is odd, $\left\{\frac{3}{2} + \frac{p}{2}\right\} = k$, where $k \geq 3$ is an integer that can be either odd or even, and so, $p = 2k - 3$, $p+2 = 2k - 1$.

$$\begin{aligned}
 & (2\pi)^{-k+\frac{\sigma(2k-3)}{2}-5} \\
 &= \left[\frac{\zeta(1-k)}{2\Gamma(k)\zeta(k)\cos\left(\frac{\pi}{2}(k)\right)} \right] \\
 & \times \left[\frac{\zeta\left(\frac{\sigma(2k-3)}{2}-4\right)}{2\Gamma\left(5-\frac{\sigma(2k-3)}{2}\right)\zeta\left(5-\frac{\sigma(2k-3)}{2}\right)\cos\left(\frac{\pi}{2}\left(5-\frac{\sigma(2k-3)}{2}\right)\right)} \right] \\
 &= 1
 \end{aligned} \tag{73}$$

Then, if k is an odd number, the expression vanishes, so k must be even and $p = 4m - 3$, $m \in \mathbb{Z}$.

$$(2\pi)^{-2m+\frac{\sigma(4m-3)}{2}-5} = M \in \mathbb{R} \tag{74}$$

STATEMENT 2: When $4m - 3$ is a prime, $\sigma(4m - 3) = 4m - 2$, hence,

$$M = (2\pi)^{-6} = \frac{1}{64\pi^6} \tag{75}$$

Hence for all twin primes, $p = 4m - 3$, $p + 2 = 4m - 1$, $M = \frac{1}{64\pi^6}$ only when $4m - 1$ is a prime.

Applying Dirichlet's theorem (20) concludes the proof:

THEOREM 5: For any two positive coprime integers, a , d , there exists infinitely many primes of the form $a + nd$, where n is also a positive integer.

5. The Relationship of the Divisor Function, the Gamma-Function, the Zeta Function and the Robin Inequality

From (57),

$$\frac{(\sigma(n))^{n-\frac{1}{2}}}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n)} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right) = 1 \text{ for all } n \in \mathbb{Z}. \tag{83}$$

Expression (83) is invariant to the substitution $\sigma(n) \rightarrow m$, $n \rightarrow \frac{1}{2} - \rho - i\tau$,

$$\frac{m^{-\rho-i\tau}}{(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{1}{2} - \rho - i\tau\right)} \prod_{k=0}^{m-1} \Gamma\left(\frac{\frac{1}{2} - \rho - i\tau + k}{m}\right) = 1 \text{ for all } \rho, \tau \in \mathbb{R} \tag{84}$$

$$m^{-\rho-i\tau} = \frac{(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{1}{2} - \rho - i\tau\right)}{\prod_{k=0}^{m-1} \Gamma\left(\frac{\frac{1}{2} - \rho - i\tau + k}{m}\right)} \text{ for all } \rho, \tau \in \mathbb{R} \tag{85}$$

$$\sum_{m=1}^{\infty} m^{-\rho-it} = \sum_{m=1}^{\infty} \left(\frac{(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{1}{2}-\rho-it\right)}{\prod_{k=0}^{m-1} \Gamma\left(\frac{1}{2}-\rho-it+k\right)} \right) \text{ for all } m \in \mathbb{Z}, \rho, \tau \in \mathbb{R} \quad (86)$$

$$\zeta(\rho+it) = \sum_{m=1}^{\infty} \left(\frac{(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{1}{2}-\rho-it\right)}{\prod_{k=0}^{m-1} \Gamma\left(\frac{1}{2}-\rho-it+k\right)} \right) \quad (87)$$

Putting

$$\Gamma\left(\frac{1}{2}-\rho-it\right) = (2\pi)^{\frac{1}{2}} m^{-\rho-it} \prod_{k=0}^{m-1} \Gamma\left(\frac{1}{2}-\rho-it+k\right) \quad (88)$$

gives,

$$\zeta(\rho+it) = \sum_{m=1}^{\infty} m^{-\rho-it} \quad (89)$$

Now, from elementary number theory, using the Li-function,

$$\text{Li}(n) = \gamma + \ln \ln(n) + \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) \quad (90)$$

One finds that:

$$e^{\text{Li}(n) - \text{Li}(\ln(n)) + \sum_{k=1}^{\infty} \left(\frac{\ln(\ln(n))^k}{k \cdot k!} \right) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))} = 1 \quad (91)$$

In an earlier draft-paper, I showed that the Zeta function vanishes for rational functions of the roots, when the gamma functions result in real arguments for which *real part* is given by

$$e^{-i\gamma(1-2\rho) + \log\left(\frac{\Gamma(\rho+1)}{\Gamma(2-\rho)}\right) + \sum_{k=1}^{\infty} \left(\frac{1-2\rho}{k} + \log\left(\frac{\rho+k}{1-\rho+k}\right) \right)} = 1 \quad (92)$$

Hence, from (57) and (92), for any, $n > 1$,

$$\frac{(\sigma(n))^{n-\frac{1}{2}}}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n) e^{\text{Li}(q) - \text{Li}(\ln(q)) + \sum_{k=1}^{\infty} \left(\frac{\ln(\ln(n))^k}{k \cdot k!} \right) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right) = 1 \quad (93)$$

and

$$\begin{aligned}
 & e^{-\text{Li}(\ln(n)) + \sum_{k=1}^{\infty} \left(\frac{\ln(\ln(n))^k}{k \cdot k!} \right)} \\
 &= \frac{(\sigma(n))^{n-\frac{1}{2}}}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n) e^{\text{Li}(n) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right) \tag{94}
 \end{aligned}$$

Multiplying both sides by $\frac{\sigma(n)}{n}$,

$$\begin{aligned}
 & \frac{\sigma(n) e^{-\text{Li}(\ln(n)) + \sum_{k=1}^{\infty} \left(\frac{\ln(\ln(n))^k}{k \cdot k!} \right)}}{p} \\
 &= \frac{(\sigma(n))^{n+\frac{1}{2}}}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n+1) e^{\text{Li}(n) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right) \tag{95}
 \end{aligned}$$

Again, using the Li-function relation,

$$\text{Li}(n) = \gamma + \ln \ln(n) + \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) \tag{96}$$

We find that both sides express the form of the Robin relation:

$$\begin{aligned}
 & \frac{\sigma(n) e^{-\gamma}}{n \ln \ln(n)} \\
 &= \frac{(\sigma(n))^{n+\frac{1}{2}}}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n+1) e^{\text{Li}(n) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right) \tag{97}
 \end{aligned}$$

6. The Relationship of the Robin Inequality to the Riemann Zeta Function

Consider the expression:

$$\frac{\sigma(n) e^{-\gamma}}{n \ln \ln(n)} = \frac{(\sigma(n))^{n+\frac{1}{2}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right)}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n+1) e^{\text{Li}(n) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \tag{98}$$

$$\frac{e^{-\gamma}}{n \ln \ln(n)} = \frac{(\sigma(n))^{n-\frac{1}{2}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right)}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n+1) e^{\text{Li}(n) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!} \right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \tag{99}$$

This is invariant to the substitutions, $\sigma(n) \rightarrow m, n \rightarrow \frac{1}{2} - s$.

$$\frac{e^{-\gamma}}{\left(\frac{1}{2}-s\right) \ln \ln \left(\frac{1}{2}-s\right)} \prod_{k=0}^{m-1} \Gamma\left(\frac{\left(\frac{1}{2}-s\right)+k}{m}\right) \tag{100}$$

$$= \frac{(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{3}{2}-s\right) e^{\operatorname{Li}\left(\frac{1}{2}-s\right)-\sum_{k=1}^{\infty} \frac{\ln\left(\frac{1}{2}-s\right)^k}{k \cdot k!}-\ln(\ln(z))+\ln(\ln(z))}}{m^{-s} e^{-\gamma} (2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{3}{2}-s\right) e^{\operatorname{Li}\left(\frac{1}{2}-s\right)-\sum_{k=1}^{\infty} \frac{\ln\left(\frac{1}{2}-s\right)^k}{k \cdot k!}-\ln\left(\ln\left(\frac{1}{2}-s\right)\right)+\ln\left(\ln\left(\ln\left(\frac{1}{2}-s\right)\right)\right)}} \tag{101}$$

$$\left(\frac{1}{2}-s\right) \ln \ln \left(\left(\frac{1}{2}-s\right)\right) \prod_{k=0}^{m-1} \Gamma\left(\frac{\left(\frac{1}{2}-s\right)+k}{m}\right)$$

Hence,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{e^{-\gamma} e^{\left\{\operatorname{Li}\left(\frac{1}{2}-s\right)-\sum_{k=1}^{\infty} \frac{\ln\left(\frac{1}{2}-s\right)^k}{k \cdot k!}-\ln\left(\ln\left(\frac{1}{2}-s\right)\right)\right\}+\ln\left(\ln\left(\ln\left(\frac{1}{2}-s\right)\right)\right)} (2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{1}{2}-s\right)}{\ln \ln \left(\left(\frac{1}{2}-s\right)\right) \prod_{k=0}^{m-1} \Gamma\left(\frac{\left(\frac{1}{2}-s\right)+k}{m}\right)} \tag{102}$$

$$\operatorname{Li}\left(\frac{1}{2}-s\right)-\sum_{k=1}^{\infty} \frac{\ln\left(\frac{1}{2}-s\right)^k}{k \cdot k!}-\ln \ln \left(\frac{1}{2}-s\right)=\gamma \tag{103}$$

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{1}{2}-s\right)}{\prod_{k=0}^{m-1} \Gamma\left(\frac{1}{2}-s+k\right)} \tag{104}$$

and since

$$\Gamma\left(\frac{1}{2}-s\right) = (2\pi)^{\frac{1-m}{2}} m^{-s} \prod_{k=0}^{m-1} \Gamma\left(\frac{1}{2}-s+k\right) \tag{105}$$

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s} \tag{106}$$

Obviously, the Robin inequality is related to the ζ -function!

Then, due to invariance of (104) to the $m \rightarrow f(m)$, were, $\{f(m), m\} \in Z$; and when $s = \rho + i\tau$ takes on certain values, and so:

$$\Gamma\left(\frac{1}{2}-s\right) = (2\pi)^{\frac{1-f(m)}{2}} (f(m))^{-s} \prod_{k=0}^{f(m)-1} \Gamma\left(\frac{\frac{1}{2}-s+k}{f(m)}\right) \tag{107}$$

Then, (107) can be put as:

$$\zeta(s) = \sum_{m=1}^{\infty} (f(m))^{-s} = \frac{(2\pi)^{\frac{m-f(m)}{2}} \prod_{k=0}^{f(m)-1} \Gamma\left(\frac{\frac{1}{2}-s+k}{f(m)}\right)}{\prod_{k=0}^{m-1} \Gamma\left(\frac{\frac{1}{2}-s+k}{m}\right)} \tag{108}$$

Obviously when $f(m) = m$, we recover the original ζ -function:

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s} \tag{109}$$

7. The Relationship of the Gamma-Function, to the Roots of the Zeta Function

Consider the rational forms of the ζ -function (Jensen formula [14], p. 1036),

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s-1}\right) \left(\frac{s}{s-1}\right) + \frac{2}{2^s-1} \int_0^{\infty} \left(\frac{\frac{1}{4}+t^2}{e^{2\pi t}-1}\right)^{\frac{s}{2}} \sin(s \tan^{-1} 2t) dt, \{ \Re(s) \neq 1 \} \tag{110}$$

Substituting $\tan \theta = 2t$, $dt = \frac{1}{2}(1+\tan^2 \theta) = \frac{1}{2}(\sec \theta)^2$, in (109), the tan function is valid for the range $\{0 \dots \infty\}$. The sin function is valid for the range $\left\{0 \dots \frac{\pi}{2}\right\}$.

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s-1}\right) \left(\frac{s}{s-1} + \int_0^{\frac{\pi}{2}} \left(\frac{\sin(s\theta) \left(\frac{1}{2}(\sec \theta)^{-s}\right)}{e^{\pi \tan \theta} - 1}\right) d\theta\right) \tag{111}$$

This reduces to

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s-1}\right) \left(\frac{s}{s-1} + \int_0^{\frac{\pi}{2}} \left(\frac{\sin(s\theta) \left((\sec \theta)^{2-s}\right)}{e^{\pi \tan \theta} - 1}\right) d\theta\right) \tag{112}$$

Using Chebyshev's polynomials of the first kind, ([14], p. 993), and putting $x = \cos \theta$, in

$$\sin(s \cos^{-1}(x)) = \sqrt{1 - \text{ChebyshevT}(s, \cos^{-1}(x))^2} \tag{113}$$

$$\begin{aligned} &\text{ChebyshevT}(s, \cos^{-1}(\cos \theta)) \\ &= \cos(s\theta) = \frac{1}{2} \left[(\cos \theta + i\sqrt{1 - \cos^2 \theta})^s + (\cos \theta - i\sqrt{1 - \cos^2 \theta})^s \right] \end{aligned} \tag{114}$$

$$\sin(s\theta) = \sqrt{1 - \left(\frac{1}{2} \left[(\cos \theta + i\sqrt{1 - \cos^2 \theta})^s + (\cos \theta - i\sqrt{1 - \cos^2 \theta})^s \right] \right)^2} \tag{115}$$

$$(\sec \theta)^{2-s} = (\cos \theta)^{s-2} \tag{116}$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} + \int_0^{\frac{\pi}{2}} \frac{(\cos \theta)^{s-2} \left(\sqrt{1 - \left(\frac{1}{2} (\cos \theta + i \sin \theta)^s + \frac{1}{2} (\cos \theta - i \sin \theta)^s \right)^2} \right)}{e^{\pi \tan \theta} - 1} d\theta \right] \tag{117}$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} + \int_0^{\frac{\pi}{2}} \frac{(\cos \theta)^{2s} \sqrt{\left((\cos \theta)^{-2s} - \left(\frac{1}{2} (1 + i \tan \theta)^s + \frac{1}{2} (1 - i \tan \theta)^s \right)^2 \right)}}{e^{\pi \tan \theta} - 1} d\theta \right] \tag{118}$$

Which can be reduced again by the substitution, $x = \tan \theta$, with $d\theta = \frac{dx}{1+x^2}$, in (118),

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left(\frac{s}{s-1} \right) + \int_0^\infty \frac{\left(\frac{1}{\sqrt{1+x^2}} \right)^{2s} \sqrt{\left(\frac{1}{\sqrt{1+x^2}} \right)^{-2s} - \left(\frac{1}{2} [(1+ix)^s + (1-ix)^s] \right)^2}}{e^{\pi x} - 1} dx \tag{119}$$

Simplifying,

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left(\frac{s}{s-1} \right) + \int_0^\infty \frac{\sqrt{\left(\frac{1}{1+x^2} \right)^s - \left(\frac{1}{2} \left[\frac{(1+ix)^s}{(1+x^2)^s} + \frac{(1-ix)^s}{(1+x^2)^s} \right] \right)^2}}{e^{\pi x} - 1} dx \tag{120}$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left(\frac{s}{s-1} \right) + \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\int_0^\infty \frac{\sqrt{2(1+x^2)^{-s} - (1-ix)^{-2s} - (1+ix)^{-2s}}}{e^{\pi x} - 1} dx \right] \tag{121}$$

Finally, one arrives at the Abel Plena form:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} + \left(\frac{2^{s-1}}{2^s - 1} \right) \int_0^\infty i \left(\frac{(1+ix)^{-s} + (1+ix)^{-s}}{e^{\pi x} - 1} \right) dx \right] \tag{122}$$

The function (121) can now be reduced to a series form, as follows:

$$(1+ix)^{-s} = \sum_{n=0}^{\infty} \frac{(ix)^n \Gamma(1-s)}{\Gamma(-s-n+1)n!}, \quad (1-ix)^{-s} = \sum_{n=0}^{\infty} \frac{(-ix)^n \Gamma(1-s)}{\Gamma(-s-n+1)n!} \tag{123}$$

Note that the odd terms vanish and the sums become real, and one is left with:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} + \int_0^{\infty} \left(2 \sum_{n=1}^{\infty} \frac{(x)^{2n-1} \Gamma(1-s)}{2\Gamma(-s-n+2)(2n-1)!e^{\pi x} - 1} \right) dx \right] \tag{124}$$

Using the relations,

$$\zeta(2n)\Gamma(2n) \left[\frac{1}{\pi^{2n}} \right] = \int_0^{\infty} \frac{x^{2n-1}}{e^{\pi x} - 1} dx, \quad \Gamma(2n) = (2n-1)!$$

We get the relation:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} + \sum_{n=1}^{\infty} \frac{2\Gamma(1-s)\zeta(2n)\Gamma(2n)}{\pi^{2n}\Gamma(-s-n+2)(2n-1)!} dx \right] \tag{125}$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} + \sum_{n=1}^{\infty} \frac{2\Gamma(1-s)\zeta(2n)}{\pi^{2n}\Gamma(-s-n+2)} dx \right] \tag{126}$$

Using the relation $\zeta(2n) = \frac{(-1)^{n+1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}$, one arrives at the desired form:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} - \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!} \right] \tag{127}$$

Now, a critical observation is that the series in (127) also determines an equal representation of the rational forms of the arguments by the observation that

$$\frac{s}{s-1} = - \sum_{n=0}^1 \frac{2^n B_n \Gamma(1-s)}{\Gamma(2-n-s)n!} \tag{128}$$

The function becomes:

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} - \left(\sum_{n=0}^{\infty} \frac{2^n B_n \Gamma(1-s)}{\Gamma(-s-n+2)n!} - \sum_{n=0}^1 \frac{2^n B_n \Gamma(1-s)}{\Gamma(2-n-s)n!} \right) \right] \tag{129}$$

$$\zeta(s) = \left(\frac{2^{s-1}}{2^s - 1} \right) \left[\frac{s}{s-1} - \left(\sum_{n=0}^{\infty} \frac{2^n B_n \Gamma(1-s)}{\Gamma(-s-n+2)n!} + \frac{s}{s-1} \right) \right] \tag{130}$$

Finally,

$$\zeta(s) = - \left(\frac{2^{s-1}}{2^s - 1} \right) \left(\sum_{n=0}^{\infty} \frac{2^n B_n \Gamma(1-s)}{\Gamma(2-n-s)n!} \right) \tag{131}$$

8. The Non-Trivial Roots of the Zeta Function

It is clear from (130) that when $s = \rho$ is a root of the ζ -function,

$$\frac{\rho}{\rho-1} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!} = - \sum_{n=0}^1 \frac{(2^n) B_n \Gamma(1-\rho)}{\Gamma(2-n-\rho)n!}, \tag{132}$$

and,

$$\sum_{n=0}^{\infty} \frac{2^n B_n \Gamma(1-\rho)}{\Gamma(-\rho-n+2)n!} = 0 \tag{133}$$

This is where the role of the Bernoulli numbers, B_n and the Gamma function come into play with the roots of the Riemann zeta function. Note that the expression (129) is the rational form of the Γ -function. Consider the Γ -function ([13], p. 894):

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{z}{k}}}{1 + \frac{z}{k}} \right), \Re(z) > 0 \tag{134}$$

here γ is Euler's constant [14]. The relation (134) is only valid for $\Re(z) > 0$, i.e., $\Re(z) = \Re(-\rho - 2n + 1) > 0$. The Bernoulli numbers and the Γ -function (134) interact only for values of $n \in \{0, 1\}$. These are precisely the values that truncate the relation:

$$\frac{\rho}{\rho-1} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(-s-2n+2)(2n)!} \rightarrow -\sum_{n=0}^1 \frac{2^n B_n \Gamma(1-\rho)}{\Gamma(2-n-s)n!} \tag{135}$$

Put $z = 1 - \rho$, and also, $z = 2 - 2n - \rho$ in (135), and since it is only valid for $n = 0, 1$, it can be seen that (135) produces the correct result:

$$-\frac{1}{\rho-1} - 1 + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!} = 0 \tag{136}$$

Which provides again for the relation,

$$\frac{\rho}{\rho-1} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!} \tag{137}$$

It follows that unless s is a root, the result (136) is not true. Let us expand $\Gamma(1-s)$ using the functional relation (134), and using (137)

$$\frac{\rho}{\rho-1} = \frac{1}{\rho-1} \left(e^{-\gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-\rho}{k}}}{1 + \frac{1-\rho}{k}} \right) \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!} \right) \tag{138}$$

It can be seen that from the numerator, a root s , can be expressed as

$$\rho = -e^{-\gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-\rho}{k}}}{1 + \frac{1-\rho}{k}} \right) \left[\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!} \right] \tag{139}$$

This relation must be satisfied by the substitution, $\rho \rightarrow 1 - \rho$ in (138); hence,

$$1 - \rho = -e^{-\gamma\rho} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{\rho}{k}}}{1 + \frac{\rho}{k}} \right) \left[\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(\rho-2n+1)(2n)!} \right] \tag{140}$$

Additionally, an equivalent representation of the rational form is obtained:

$$\frac{\rho}{\rho-1} = \frac{-e^{-\gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-\rho}{k}}}{1 + \frac{1-\rho}{k}} \right) \left[\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(-\rho-2n+2)(2n)!} \right]}{e^{-\gamma\rho} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{\rho}{k}}}{1 + \frac{\rho}{k}} \right) \left[\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{\Gamma(\rho-2n+1)(2n)!} \right]} \quad (141)$$

It is possible to simplify (141) as follows:

$$\frac{\rho}{\rho-1} = -e^{-\gamma(1-2\rho)} \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-2\rho}{k}} (k+\rho)}{k+1-\rho} \right) \left[\frac{\rho}{1-\rho} \right] \quad (142)$$

$$\frac{\rho}{\rho-1} = -e^{-\gamma(1-2\rho)} \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-2\rho}{k}} (k+\rho)}{k+1-\rho} \right) \left[\frac{\rho}{\rho-1} \right]^2 \quad (143)$$

Then,

$$\frac{\rho-1}{\rho} = -e^{-\gamma(1-2\rho)} \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-2\rho}{k}} (k+\rho)}{k+1-\rho} \right) \quad (144)$$

$$\frac{\rho-1}{\rho} = -e^{-\gamma(1-2\rho)} \frac{\Gamma(\rho)}{\Gamma(1-\rho)} \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-2\rho}{k}}}{\frac{k+\rho+(1-2\rho)}{k+\rho}} \right) \quad (145)$$

and finally,

$$\frac{\rho-1}{\rho} = -e^{-\gamma(1-2\rho)} \left[\frac{\Gamma(\rho)}{\Gamma(1-\rho)} \right] \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-2\rho}{k}}}{1 + \frac{1-2\rho}{k+\rho}} \right) \quad (146)$$

Now from (146), put $\rho = \sigma + i\tau$

$$\begin{aligned} & \frac{\sigma+i\tau-1}{\sigma+i\tau} \\ &= -e^{-\gamma(1-(2\sigma+2i\tau))} \left[\frac{\Gamma(\sigma+i\tau)}{\Gamma(1-\sigma-i\tau)} \right] \prod_{k=1}^{\infty} \left(\frac{e^{\frac{1-(2\sigma+2i\tau)}{k}}}{1 + \frac{1-(2\sigma+2i\tau)}{k+\sigma+i\tau}} \right) \end{aligned} \quad (147)$$

In this case, one uses the complex representation of the Γ -function ([14], p. 895):

$$\Gamma(\sigma+i\tau) = \Gamma(\sigma) \left[\frac{\sigma e^{-i\tau\gamma}}{\sigma+i\tau} \right] \prod_{k=1}^{\infty} \left(\frac{e^{\frac{i\tau}{k}}}{1 + \frac{i\tau}{\sigma+k}} \right), \sigma > 0, \quad (148)$$

$$\sigma + i\tau = \frac{\Gamma(\sigma)\sigma e^{-i\tau\gamma}}{\Gamma(\sigma + i\tau)} \prod_{k=1}^{\infty} \left(e^{\frac{i\tau}{k}} \frac{\sigma + k}{\sigma + k + i\tau} \right) \tag{149}$$

then, similarly,

$$1 - \sigma - i\tau = \frac{\Gamma(1 - \sigma)e^{i\tau\gamma}}{\Gamma(1 - \sigma - i\tau)} \prod_{k=1}^{\infty} \left(e^{\frac{-i\tau}{k}} \frac{1 - \sigma + k}{1 - \sigma - i\tau + k} \right) \tag{150}$$

One can now express the rational form of the roots as follows:

$$\frac{1 - \sigma - i\tau}{\sigma + i\tau} = \frac{\Gamma(1 - \sigma)e^{i\tau\gamma}}{\Gamma(\sigma + i\tau)} \frac{\prod_{k=1}^{\infty} \left(e^{\frac{-i\tau}{k}} \frac{1 - \sigma + k}{1 - \sigma - i\tau + k} \right)}{\prod_{k=1}^{\infty} \left(e^{\frac{i\tau}{k}} \frac{\sigma + k}{\sigma + i\tau + k} \right)} \tag{151}$$

$$\frac{1 - \sigma - i\tau}{\sigma + i\tau} = \frac{\Gamma(1 - \sigma)\Gamma(\sigma + i\tau)e^{i\tau\gamma}}{\Gamma(\sigma)\Gamma(1 - \sigma - i\tau)e^{-i\tau\gamma}} \prod_{k=1}^{\infty} \left(e^{\frac{-2i\tau}{k}} \frac{(1 - \sigma + k)(\rho + k + i\tau)}{(1 - \sigma - i\tau + k)(\sigma + k)} \right) \tag{152}$$

And so,

$$\frac{\sigma + i\tau}{1 - \sigma - i\tau} = \frac{\Gamma(\sigma)}{\Gamma(1 - \sigma)} \frac{\Gamma(1 - \sigma - i\tau)}{\Gamma(\sigma + i\tau)} e^{-2i\tau\gamma} \prod_{k=1}^{\infty} \left(e^{\frac{2i\tau}{k}} \left(\frac{\sigma + k}{1 - \sigma + k} \right) \left(\frac{1 - \sigma - i\tau + k}{\sigma + k + i\tau} \right) \right) \tag{153}$$

From (146), we also have,

$$\frac{\sigma + i\tau - 1}{\sigma + i\tau} = -e^{-\gamma(1 - (2\sigma + 2i\tau))} \left[\frac{\Gamma(\sigma + i\tau)}{\Gamma(1 - \sigma - i\tau)} \right] \prod_{k=1}^{\infty} \left(e^{\frac{1 - (2\sigma + 2i\tau)}{k}} \frac{k + \sigma + i\tau}{1 - \sigma + k - i\tau} \right) \tag{154}$$

Multiplying (153) by (154), it is possible to reach:

$$\frac{\Gamma(\sigma)}{\Gamma(1 - \sigma)} e^{-i\gamma(1 - 2\sigma)} \prod_{k=1}^{\infty} \left(e^{\frac{1 - 2\sigma}{k}} \frac{\sigma + k}{1 - \sigma + k} \right) = 1 \tag{155}$$

$$e^{-i\gamma(1 - 2\sigma) + \log \left(\frac{\Gamma(\sigma)}{\Gamma(1 - \sigma)} \prod_{k=1}^{\infty} \left(e^{\frac{1 - 2\sigma}{k}} \frac{\rho + k}{1 - \rho + k} \right) \right)} = 1 \tag{156}$$

$$e^{-i\gamma(1 - 2\sigma) + \log \left(\frac{\Gamma(\sigma)}{\Gamma(1 - \sigma)} \right) + \sum_{k=1}^{\infty} \left(\frac{1 - 2\sigma}{k} + \log \left(\frac{\sigma + k}{1 - \sigma + k} \right) \right)} = 1 \tag{157}$$

Now, the right-hand side of (157) is unity. However, the left-hand side of (157) can grow due to the fact that the sum

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \right) = \zeta(1) = \infty$$

Except when $\sigma = \frac{1}{2}$, and one finds that all the terms in (157) reduce to unity:

$$e^{-i\gamma(0) + \log(1) + \sum_{k=1}^{\infty} (\log(1))} = 1 \tag{158}$$

No assumptions have been made about σ .

9. A Sequence That Gives the Integers Obeying the Robin Inequality

THEOREM 2: Let p be an integer, such that

$$g(p) = \frac{\text{Li}\left(e^{\frac{\sigma(p)-\lambda}{p}}\right) - \sum_{k=1}^{\infty} \frac{e^{\frac{\sigma(p)-\lambda}{p}}}{k \cdot k!}}{e^{\text{Li}(\ln(p)) - \sum_{k=1}^{\infty} \frac{\ln(\ln p)^k}{k \cdot k!}}} \tag{159}$$

Then, for all $p > 5040$, the integer-function

$$\sum_{x=3}^p \left(\frac{1}{2} + \frac{\left(\ln(\ln(g(x))) - \ln\left(\ln\left(\frac{1}{g(x)}\right)\right) \right) i}{2\pi} \right) \tag{160}$$

$$\times \left(\frac{1}{2} + \frac{\left(\ln(\ln(g(p))) - \ln\left(\ln\left(\frac{1}{g(p)}\right)\right) \right) i}{2\pi} \right) = \begin{cases} n_p & p \in g(p) < 1 \\ 0 & p \in g(p) > 1 \end{cases}$$

maps p into the infinite sequence of integers, n_p , only if p satisfies the Robin inequality.

PROOF:

Using the Li-function, it is easy to show that:

$$g(p) = \frac{\lambda + \ln\left(\ln\left(e^{\frac{\sigma(p)-\lambda}{p}}\right)\right)}{e^{\lambda + \ln(\ln(p))}} \tag{161}$$

Note that from ([14], p. 46):

$$\sum_{k=1}^{\infty} \left(\frac{\sin(kx)}{k} \right) = \frac{\pi - x}{2} \tag{162}$$

Putting $x = \ln(\ln(g(x))) - \ln\left(\ln\left(\frac{1}{g(x)}\right)\right) i$,

$$\frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{\sin\left(k \left[\ln(\ln(g(x))) - \ln\left(\ln\left(\frac{1}{g(x)}\right)\right) i \right] \right)}{k} \right) \tag{163}$$

$$= \frac{1}{2} + \frac{\left(\ln\left(\ln\left(\frac{1}{g(x)}\right)\right) - \ln(\ln(g(x))) \right) i}{2\pi}$$

Then,

$$\frac{1}{\pi} \sum_{x=3}^p \left(\frac{1}{2} + \frac{\left(\ln(\ln(g(x))) - \ln\left(\ln\left(\frac{1}{g(x)}\right)\right) \right) i}{2\pi} \right) \tag{164}$$

$$\times \left(\frac{1}{2} + \frac{\left(\ln(\ln(g(p))) - \ln\left(\ln\left(\frac{1}{g(p)}\right)\right) \right) i}{2\pi} \right) = \begin{cases} n_p & p \in g(p) < 1 \\ 0 & p \in g(p) > 1 \end{cases}$$

$$\sum_{x=3}^p \left(\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin\left(k \left[\ln(\ln(g(x))) - \ln\left(\ln\left(\frac{1}{g(x)}\right)\right) \right] i\right)}{k} \right) \tag{165}$$

$$\times \left(\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin\left(k \left[\ln(\ln(g(p))) - \ln\left(\ln\left(\frac{1}{g(p)}\right)\right) \right] i\right)}{k} \right)$$

$$= \begin{cases} n_p & p \in g(p) < 1 \\ 0 & p \in g(p) > 1 \end{cases}$$

Note that

$$\ln\left(\ln\left(\frac{1}{g(x)}\right)\right) - \ln(\ln(g(x))) = i\pi \text{ for } g(x) > 1 \tag{166}$$

$$\ln\left(\ln\left(\frac{1}{g(x)}\right)\right) - \ln(\ln(g(x))) = -i\pi \text{ for } g(x) < 1$$

Then, if there exists a finite number n_p of integers p_j for which $g(p_j) < 1$, that survive, and $p - n_p$ integers p_m , for which $g(p_m) > 1$, that do not survive then, for the range, $3 \dots p$. The relation (158) is a continuous mapping of p to the sequence of the integers $n = \{1, 2, 3, 4, \dots, \infty\}$, with gaps, $G(n, n+1)$ between integers $n, n+1$, that satisfy the Robin inequality criterion. Then, if for $p > 5040$, the gap $G(n)$ between the sequence of integers, n must be zero.

The sequence of gaps that belong to $G(n, n+1) \in \{6, 3, 1, 0(\text{are these triangular numbers?})\}$. Note that these gaps are also related to the Leonhard Euler's σ -function. The question mark indicates that we do not know if for every $p > 5040$, $G(p > 5040) = 0$.

Note that the sequence of gaps are triangular numbers belonging to the set:

$$\left\{ 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, \right. \tag{167}$$

$$\left. 120, 136, 153, 171, 190, 210, \dots, \frac{k(k+1)}{2} \right\}$$

Now using the Li-function [14], we reconstruct the Robin inequality as follows:

$$\frac{\sigma(m)}{me^\gamma} - \text{Li}(m) + \gamma + \sum_{k=1}^{\infty} \frac{(\ln m)^k}{k \cdot k!} < 1, m \notin R_m \tag{168}$$

$$R_m \in [3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040] \tag{169}$$

It is clear that the general sum $\sum_{k=1}^{\infty} \frac{(\ln m)^k}{k \cdot k!}$, holds for all m in (168).

The understanding of the Robin Inequality has been stalled by the discovery of a function that can simulate its property and remain valid for all integers with the criteria that if, $m > 5040 \notin R_m$. To do this,

1) I create a new function, $F(m)$ that will be valid within the range of $m < 5041$, that generates all the Robin inequality conditions for all $m \leq 5041$, and is increasing for all higher $m > 5041$ values.

2) If the function is valid for the inequality for any $m < 5041$, and for a single value of $m > 5040$, it will be valid for all $m > 5040$.

To generate such a function, we consider the sum in the function $F(m)$ below. There exists a value n for which the function:

$$F(m) = \frac{\sigma(m)}{me^\gamma} - \text{Li}(m) + \gamma + \sum_{k=1}^n \frac{(\ln m)^k}{k \cdot k!} \begin{cases} > 1, & m \in R_m \\ < 1, & m \notin R_m \end{cases} \tag{170}$$

holds for all $m \in R_m$. A progressive determination of n , shows that the function, $F(m)$, (170) holds for $n = 22$, for $m \in R_m$, and therefore should hold for all m , if for $m > 5040$, it holds. This is so since if any integer greater than 5040 produces a positive value of $F(m)$ and does not follow Robin's inequality, and if $m = 5041$ also produces a negative result for $F(m)$ that obeys the Robin inequality, then it cannot produce a positive result for $m > 5040$ since the sum is increasing thereafter, and the maximum value of this increase is the infinite sum. **Table 1** shows the values of the two sums in the range 5035...5045.

Table 1. Difference between the infinite sum and the finite sum.

m	$\sum_{k=1}^{\infty} \frac{(\ln m)^k}{k \cdot k!}$	$\sum_{k=1}^{22} \frac{(\ln m)^k}{k \cdot k!}$	Difference in sums Δ_m	$\frac{\sigma(m)}{me^\gamma} - \left(\text{Li}(m) - \gamma - \sum_{k=1}^{22} \frac{(\ln m)^k}{k \cdot k!} = \ln(\ln(m)) \right)$
5035	685.66195184509884607000	685.66837027517579615000	0.00641843007695008000	-1.42673061028986065000,
5036	685.77923715916120116000	685.78565910254497690000	0.00642194338377574000	-1.16601623748948898860,
5037	685.89651974455881535000	685.90294520239490977000	0.00642545783609442000	-1.35751570774075386840,
5038	686.01379960195919260000	686.02022857539320316000	0.00642897343401056000	-1.22664046738100985820,
5039	686.13107673202953835000	686.13750922220716680000	0.00643249017762845000	-1.58786025843012142350,
5040	686.24835113543675930000	686.25478714350381195000	0.00643600806705265000	0.00547701121975959660,

Continued

5041	686.36562281284746401000	686.37206233994985156000	0.00643952710238755000	-1.58000600706458847760,
5042	686.48289176492796320000	686.48933481221170046000	0.00644304728373726000	-1.30698824063832039330,
5043	686.60015799234426936000	686.60660456095547561000	0.00644656861120625000	-1.38222149027422428420,
5044	686.71742149576209747000	686.72387158684699621000	0.00645009108489874000	-1.08052130523482364040,
5045	686.83468227584686490000	686.84113589055178396000	0.00645361470491906000	-1.47517274812773264550

It is clear that the infinite sum is slightly larger than the finite sum.

a) The maximum difference after 5050 for the sums is

$$\Delta_{5040} \approx 0.00643600806705265... \text{ corresponding to a positive value.}$$

b)

$$\frac{\sigma(5040)}{5040e^\gamma} - \left(\text{Li}(5040) - \gamma - \sum_{k=1}^{\infty} \frac{(\ln 5040)^k}{k \cdot k!} = \ln(\ln(5040)) \right) = 0.00547701121975959660$$

Therefore, there is no integer $m > 5040$ that generates a sum for which

$$\frac{\sigma(m)}{me^\gamma} - \left(\text{Li}(m) - \gamma - \sum_{k=1}^{\infty} \frac{(\ln m)^k}{k \cdot k!} = \ln(\ln(m)) \right) > 1$$

Then, the relation

$$\frac{\sigma(m)}{me^\gamma} - \text{Li}(m) + \gamma + \sum_{k=1}^{\infty} \frac{(\ln m)^k}{k \cdot k!} \begin{cases} > 1, & m \in R_m \\ < 1, & m \notin R_m \end{cases} \tag{171}$$

holds for all known numbers tested with the Robin inequality.

Figure 1 shows the function (170) for the integer range $m = \{3..6000\}$.

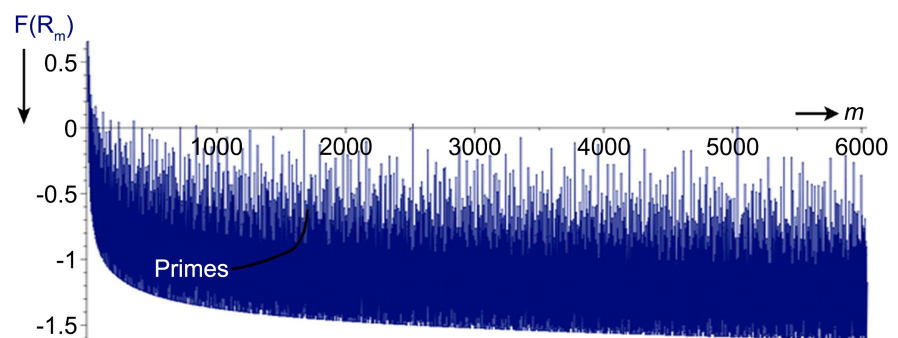


Figure 1. Shows the function $F(m)$ over the range $m = \{3..6000\}$ with the integer set that does not follow Robin’s criteria projecting above the horizontal axes.

It is visually clear from Figure 1, that the function (170) progressively becomes negative as m increases beyond 5040.

The value of the function (170) for $m = 5040$ is 0.005477031... The function (170) goes negative for all $m > 5040$, $k > 22$.

Now

$$\sum_{k=1}^{\infty} \frac{(\ln 5040)^k}{k \cdot k!} = 686.2547871\dots > \sum_{k=1}^{22} \frac{(\ln 5040)^k}{k \cdot k!} = 686.252636 \tag{172}$$

$$\gg \sum_{k=23}^{\infty} \frac{(\ln 5040)^k}{k \cdot k!} = 0.0064367327070\dots$$

And,

$$\sum_{k=1}^{\infty} \frac{(\ln 5040)^k}{k \cdot k!} - \sum_{k=1}^{22} \frac{(\ln 5040)^k}{k \cdot k!} = 686.2547871\dots - 686.252636 \tag{173}$$

$$= 0.0021511\dots < \sum_{k=23}^{\infty} \frac{(\ln 5040)^k}{k \cdot k!}$$

The function $\pi(m)$ repeats itself over a large range of integers, and it is an increasing function. After $\pi(49036) = 5040$, there always exists an integer for which $\pi(49036 + k) = 5040 + j$. Hence using the prime counting function, $\pi(m)$, we can determine if any value of $m > 49036$, will generate a number belonging to the set

$$\frac{\sigma(\pi(m))}{\pi(m)e^\gamma} - \text{Li}(\pi(m)) + \gamma + \sum_{k=1}^{\infty} \frac{(\ln \pi(m))^k}{k \cdot k!} > 1, \pi(m) \in R_{m+k} \tag{174}$$

Hence once the function (170) crosses 5040, it cannot become positive again!

$$\sum_{k=1}^{22} \frac{(\ln m)^k}{k \cdot k!} \gg \sum_{k=23}^{\infty} \frac{(\ln m)^k}{k \cdot k!} > 0 \tag{175}$$

The minimum values of the function (170) are at the prime numbers since $\sigma(p) = p + 1$, is the lowest bound of the function (170) for any given p .

As shown in **Figure 2**, the prime function $F(p)$ shows the fall of the function below the recovery range and the function remains in the negative region.

As shown in **Figure 3** the magnified graph shows the ascension to the last positive values of the function (170).

Figure 4 shows the perfect match of the infinite sum and the truncated sum in (170).

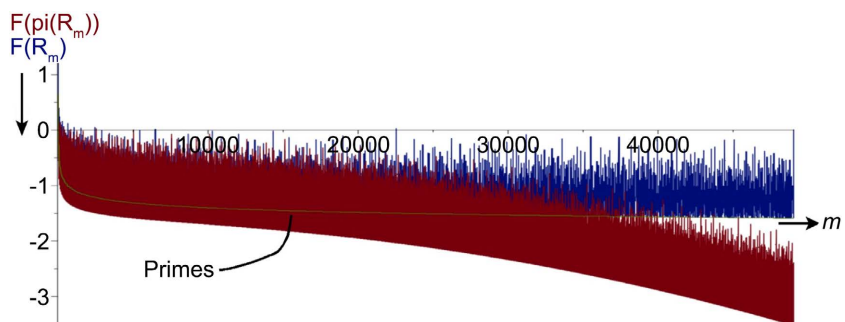


Figure 2. Shows the super-position graph of the prime counting function $F(\pi(p))$, and the integer function for the range $m = 3..48086$. The values $\pi(p) = 720$ and $\pi(49036) = 5040$ do not follow the Robin inequality and are shown to project above the horizontal axis. Asso visible is the fall of the range of $\pi(p)$ after 48086. The primes $\pi(\text{prime})$ are shown with the green line.

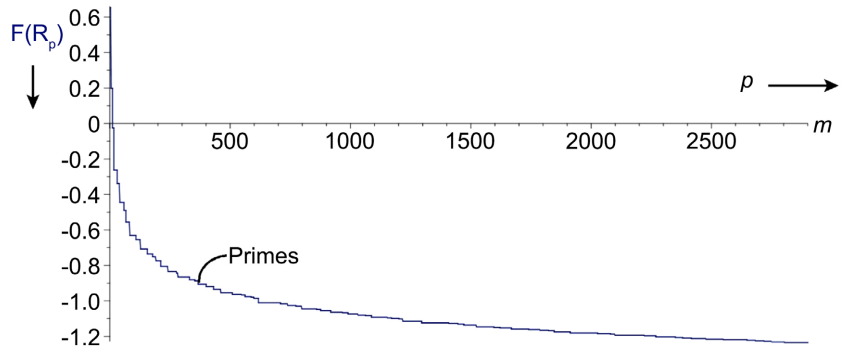
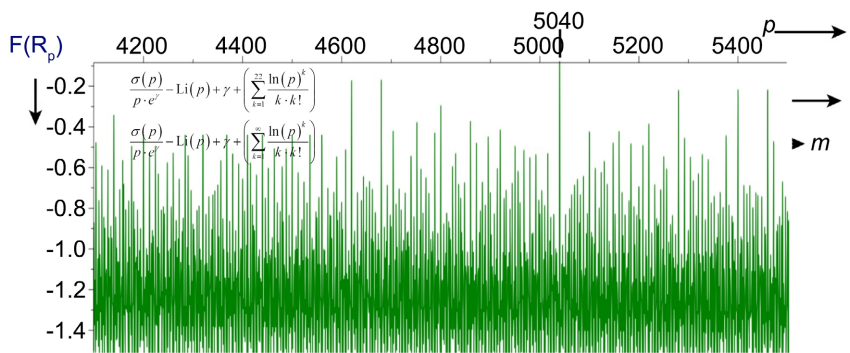


Figure 3. Shows the lower bounds of the function (170) as it progresses to higher values of the primes. Greater than 5040 will always give negative values, hence the Riemann Hypothesis is true.



The perfect match of the infinite sum to the truncated sum shows that the truncated sum cannot decrease the function for $m < 5040$

Figure 4. Shows the perfect match of the range of the function (171) when restricted to the maximum sum of 22, versus the infinite sum.

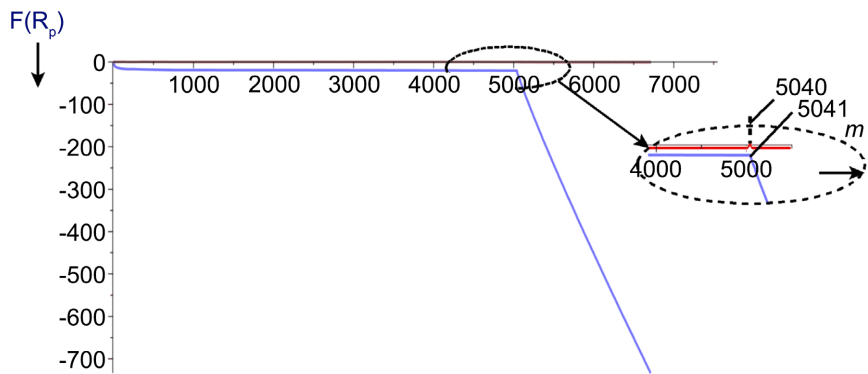


Figure 5. Shows the SHARP fall of the function (170)'s bound (primes are blue, and non-primes are red) after 5040 and the decline progresses ad infinitum.

It is clear that for finite range $3 \dots 5040$, the function is decreasing with higher primes. And if (170) is negative for any integer, one can see that the bounds of the function (170), are decreasing with larger primes and therefore should remain negative for higher sum terms. One sees the clear relationship between the prime counting function $F(\text{pi}(R_m))$ and $F(R_m)$. Finally, **Figure 5** shows the sharp decline of the function (170) after 5040, and progressive lowering of the bounds

and infinitum. The lowering of the integer relation for (170) is below the zero line, and the values are all negative after 5040.

Obviously, the flow of values of $F(m)$ and for the primes, $F(p_m)$ falls continuously as $m \rightarrow \infty$. The difference between

$$F(m) = \left[\frac{\sigma(m)}{me^\gamma} + \gamma + \sum_{k=1}^n \frac{(\ln m)^k}{k \cdot k!} \right] - \text{Li}(m),$$

remains negative beyond 5040 and does not recover again to a positive value. This is evidenced by the prime-function, (shown in blue) $F(p_m)$ (Figure 6).

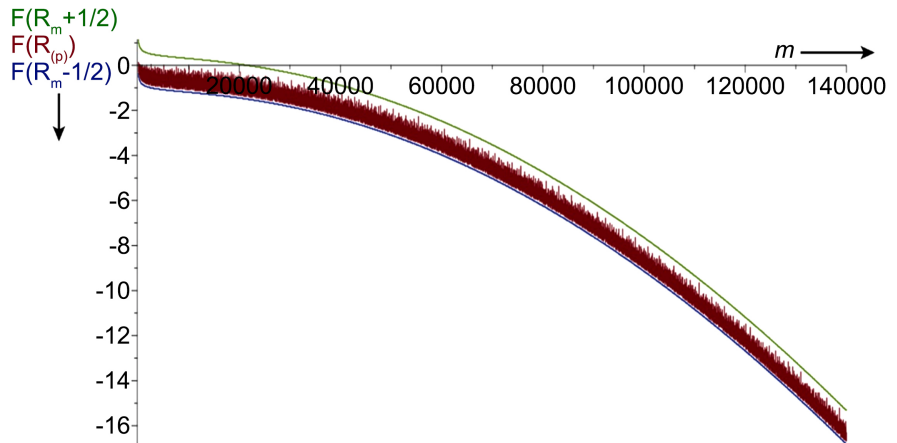


Figure 6. Shows that the range of the function $F(p_m)$ has at least the Upper and the Lower bounds $F(p_m) + \frac{1}{2} < F(m) < F(p_m) + 2$.

$$F(p_m) + \frac{1}{2} < F(m) < F(p_m) + 2 \tag{176}$$

Clearly, these bounds hold for all integer values $m > 5040$.

10. Conclusions

On the Algebra of the Riemann Zeta function:

The reduced algebra (158) is in the real domain, and does not violate any path integrals and also *does not violate any domain restrictions* since all these requirements cancel out to give a definite real result.

As I showed earlier, the 1/2-line is a natural result of the gamma function’s behavior. For example,

$$(2\pi)^{\frac{1-m}{2}} m^{-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{m+k}{m}\right) = (2\pi)^{\frac{1-\sigma(m)}{2}} \sigma(m)^{m-\frac{1}{2}} \prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right) \tag{177}$$

For all integers m ,

$$(2\pi)^{\frac{\sigma(m)-m}{2}} \left(\frac{m}{\sigma(m)}\right)^{m-\frac{1}{2}} \frac{\prod_{k=0}^{m-1} \Gamma\left(\frac{m+k}{m}\right)}{\prod_{k=0}^{\sigma(m)-1} \Gamma\left(\frac{m+k}{\sigma(m)}\right)} = 1 \tag{178}$$

For primes, p , $\sigma(p) = p + 1$,

$$(2\pi)^{\frac{1}{2}} \frac{p^{\frac{p-1}{2}} \prod_{k=0}^{p-1} \Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{\frac{p-1}{2}} \prod_{k=0}^{\sigma(p)-1} \Gamma\left(\frac{p+k}{\sigma(p)}\right)} = 1 \tag{179}$$

$$\frac{\sigma(n)e^{-\gamma}}{n \ln \ln(n)} = \frac{(\sigma(n))^{n+\frac{1}{2}} \prod_{k=0}^{\sigma(n)-1} \Gamma\left(\frac{n+k}{\sigma(n)}\right)}{(2\pi)^{\frac{\sigma(n)-1}{2}} \Gamma(n+1)e^{\text{Li}(n) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!}\right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))}} \tag{180}$$

$$(2\pi)^{\frac{1}{2} \frac{m}{2}} = \frac{\Gamma(g(z))}{(m)^{g(z)-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{g(z)+k}{m}\right)} \tag{181}$$

On the Robin Inequality:

The bounds of the Robin inequality have been elusive since we have always had no way to remove the $\ln \ln(m)$ part as a changing parameter while still getting a function that provides the same conditions as the inequality. However, by simply changing the range of the sum up to the bound $n = 1, 2, 3, \dots, 22$, one finds that for the integers in the range $m \in R_m$, are the only integers that do not obey the inequality. By studying the bounds

$$F(p_m) + \frac{1}{2} < F(m) < F(p_m) + 2$$

for the integer function

$$\frac{\sigma(m)}{me^{\gamma}} - \text{Li}(m) + \gamma + \sum_{k=1}^{\infty} \frac{(\ln m)^k}{k \cdot k!}$$

and by using the Li-function, we find that the continuity of the function for primes, and the integers > 5040 are bounded as stated. This satisfies the Robin hypothesis, and hence the Riemann Hypothesis is true.

On the algebra of prime numbers and twin primes:

The powers of 2π seem to play a very important role in many mathematic relations, ranging from the prime numbers to the Robin numbers. In deed the reflection formula provides a means of relating any powers of 2π to zeta and gamma function. The analysis of the Robin inequality suggests that the LambertW function plays an important role in separating the numbers that violate the Robin inequality from those that do not. Indeed, it is known that the LambertW function has branch points for which the transformations $g'(z) = z$ play a role for the branch points $k = 1$, and $k = 2$. However, this cannot be discovered without using the Li function to re-express the relation as exponential functions.

The twin prime conjecture seems to demand that for p to be a twin, the following integer relation must hold for all primes, p , and since an infinite number of primes, exists. Every one of them will obey the relation:

When $4m - 3$ is a prime, $\sigma(4m - 3) = 4m - 2$,

$$(2\pi)^{-2m + \frac{\sigma(4m-3)}{2} - 5} = \frac{1}{64\pi^6} \tag{173}$$

Hence for all twin primes, $p = 4m - 3$, $p + 2 = 4m - 1$, $M = \frac{1}{64\pi^6}$ only when $4m - 1$ is a prime.

11. Discussion

Why is the Riemann Hypothesis so difficult to solve? One of the reasons was highlighted by the advice of my Editor. I generated the series for which the function vanishes:

$$\frac{s}{s-1} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-s)}{\Gamma(-s-2n+2)(2n)!} = -\sum_{n=0}^1 \frac{2^n B_n \Gamma(1-s)}{\Gamma(2-n-s)n!}$$

in which the roots of the Zeta function are expressed involving the products of Bernoulli functions and Gamma products. In the initial review, it was pointed out that the, "I more importantly disregarded that the zeros of the ζ -function, which are not values of the series". It is important to note that the roots themselves are not what is computed by my work. The only object that was computed in this paper is the real part of the root of the Zeta function according to the reduction (157) which becomes real and valid only on the $\frac{1}{2}$ line. Since the Zeta function vanishes to generate the relations (138) to (159), the condition forces the roots to generate (157). It is clear that the restrictions for the validity of the Gamma function for complex variables lead to the consequences that restrict the real part of rational functions of the roots to the $\frac{1}{2}$ -line. The expression

$$e^{-i\gamma(1-2\rho) + \log\left(\frac{\Gamma(\rho+1)}{\Gamma(2-\rho)}\right) + \sum_{k=1}^{\infty} \left(\frac{1-2\rho}{k} + \log\left(\frac{\rho+k}{1-\rho+k}\right)\right)} = 1 \tag{183}$$

Is clearly derived without any assumptions but algebra using the axioms of multiplication, division and addition. It is surprising that the requirement that five expressions were reduced to the required result:

$$e^{-i\gamma(1-2\rho)} e^{-i\gamma(1-2\rho)} e^{\log\left(\frac{\Gamma(\rho+1)}{\Gamma(2-\rho)}\right)} e^{\sum_{k=1}^{\infty} \left(\frac{1-2\rho}{k}\right)} e^{\sum_{k=1}^{\infty} \left(\log\left(\frac{\rho+k}{1-\rho+k}\right)\right)} = 1 \tag{184}$$

It is important to note that Li function does the same:

$$e^{\text{Li}(n) - \text{Li}(\ln(n)) + \sum_{k=1}^{\infty} \left(\frac{\ln(\ln n)^k}{k \cdot k!}\right) - \sum_{k=1}^{\infty} \left(\frac{\ln(n)^k}{k \cdot k!}\right) - \ln(\ln(n)) + \ln(\ln(\ln(n)))} = 1 \tag{185}$$

Acknowledgements

I would like to pay respects to all the great mathematicians on whose shoulder I stand especially, Gauss, Euler, Ramanujan, G. Robin, J.L. Nicolas, J.H. Conway, Marc Prevost. I would like to thank the countless great mathematicians for the insights they have provided for this work over the years. My thanks also go to the Editors of this journal, in particular, Cindy Zhang.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Riemann, B. (1859) Über die Anzahl der Primzahlen unter einer gegebenen Grösse. [Monatsberichte der Berliner Akademie, November 1859]. 136-144.
- [2] Robin, G. (1984) Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *Journal de Mathématiques Pures et Appliquées*, **63**, 187-217.
- [3] Euler, L. (1755) Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum, Vol. 1.
- [4] Gauss, C.F. (1863) Theoria residuorum biquadraticorum, Commentatio secunda. Königlichem Gesellschaft der Wissenschaften zu Göttingen, 95-148.
- [5] Solé, P. and Planat, M. (2012) The Robin Inequality for 7-Free Integers. *Integers*, **12**, 301-308. <https://doi.org/10.1515/integ.2011.103>
- [6] Rosser, J.B. and Schoenfeld, L. (1962) Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics*, **6**, 64-94. <https://doi.org/10.1215/ijm/1255631807>
- [7] Heath-Brown, D.R. (1984) The Divisor Function at Consecutive Integers. *Mathematika*, **31**, 141-149. <https://doi.org/10.1112/s0025579300010743>
- [8] Pinner, C.G. (1997) Repeated Values of the Divisor Function. *The Quarterly Journal of Mathematics*, **48**, 499-502. <https://doi.org/10.1093/qmath/48.4.499>
- [9] Choie, Y.J., Lichiardopol, N., Moree, P. and Sol, P. (2007) On Robin's Criterion for the Riemann Hypothesis. *Tome*, **19**, 357-372.
- [10] Chakrabarty, S. (2019) The Repeated Divisor Function and Possible Correlation with Highly Composite Numbers. arXiv: 1704.00007.
- [11] Nicolas, J.L. (1983) Petites valeurs de la fonction d'Euler et hypothèse de Riemann, Séminaire de Théorie des nombres. *Progress in Mathematics*, **38**, 207-218.
- [12] Anthony, M.M. (2016) The Towering Zeta Function. *Advances in Pure Mathematics*, **6**, 351-392. <https://doi.org/10.4236/apm.2016.65026>
- [13] Anthony, M.M. (2024) On Perron's Formula and the Prime Numbers. *Advances in Pure Mathematics*, **14**, 487-494. <https://doi.org/10.4236/apm.2024.146027>
- [14] Gradshteyn, I.S. and Ryzhik, I.M. (2007) Table of Integrals, Series, and Products. 7th Edition, Academic Press.
- [15] Dusart, P. (1999) The k^{th} Prime Is Greater than $k(\ln k + \ln \ln k - 1)$ for $k > 2$. *Mathematics of Computation*, **68**, 411-415. <https://doi.org/10.1090/s0025-5718-99-01037-6>
- [16] Hardy, G.H. (1999) Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work. AMS Chelsea Publishing.
- [17] Euler, L. (1760) Observation de summis divisorum, Novi Commentarii. *Academiae scientiarum Imperialis Petropoli tanae*, **5**, 59-74.