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A General Analytic Approximation Technique for the Modified Bessel Functions with Fractional Order $I_\nu(x)$, Applied to $I_{1/6}(x)$ and $I_{1/7}(x)$

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Abstract

A general technique to obtain simple analytic approximations for the first kind of modified Bessel functions. The general procedure is shown, and the parameter determination is explained through the applications to this particular case $I_{1/6}(x)$ and $I_{1/7}(x)$. In this way, it shows how to apply the technique to any particular order ν, in order to obtain an approximation valid for any positive value of the variable *x*. In the present method power series and asymptotic expansion are simultaneously used. The technique is an extension of the multipoint quasirational approximation method, MPQA. The main idea is to look for a bridge function between the power and asymptotic expansion of the $I_{1/6}(x)$, and similar procedure for $I_{1/7}(x)$. To perform this, rational functions are combined with hyperbolic ones and fractional powers. The number of parameters to be determined for each case is four. The maximum relative errors are 0.0049 for $\nu = 1/6$, and 0.0047 for $\nu = 7$. However, these relative errors decrease outside of the small region of the variables, wherein the maximum relative errors are reached. There is a clear advantage of this procedure compared with any other ones.

Keywords

Bessel Functions, Power Series, Asymptotic Expansion, MPQA Technique

1. Introduction

Bessel functions have become very important in Physics, Electrodynamics and other areas of science [1]-[7]. In order to simplify its calculations several procedures can be used. The power series and asymptotic expansions are well known. However, they are mainly useful for small or large values of the variable. In recent works, analytic approximations valid for all values of the variable have been determined with high accuracy for the modified Bessel functions $I_0(x)$ and $I_1(x)$, using the multipoint quasirational approximations technique, MPQA [8]-[11]. Approximations of polynomial type can also be found as [12] well as other interesting [13] [14]. Now a general expression to determine precise approximations for the modified Bessel functions of fractional order will be shown, and the procedure to determine the parameters explained by its applications to the particular case of the main $I_{1/6}(x)$ and $I_{1/7}(x)$ will be considered. The fact of working with not entire order involves some changes, which will be explained in Section 2, where the theoretical treatment will be exposed. The main idea of the MPQA technique is to determine a bridge function connecting the power and asymptotic expansion in such a way that part of both expansions can be reproduced. In order to do that, it is necessary to use rational functions combined with hyperbolic ones, as well as fractional powers. The new simple approximations functions here presented will be shown to be good for all real values of the variable x with good accuracy and easy to calculate. The parameter determination as well as the relative errors of the approximations will be considered in Section 3 for $I_{1/6}(x)$, and in Section 4 for $I_{1/7}(x)$. Finally, in the final section or Conclusion, the main results will be shown.

2. Theoretical Treatment

In this section, it will be first discussed in detail, how to determine the structure and parameters of the modified Bessel function of order ν, as a general case, and later the application to the particular case will be considered $I_{1/6}(x)$. The power series $I_\nu(x)$ is well known [9], and it is given by

$$I_\nu(x)=\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+1+\nu)}\left(\frac{x}{2}\right)^{2k} \tag{1}$$

The second series important in the kind of treatment here considered is the asymptotic expansion, given by

$$I_\nu(x)\sim\frac{\mathrm{e}^x}{\sqrt{2\pi x}}\left(1-\frac{4\nu^2-1}{8x}+\cdots\right) \tag{2}$$

In the present technique a bridge function connecting both series power and asymptotic, MPQA technique, must be defined. The way to determine this particular function is performed using several unknown parameters to be determined later. There are several ways to do this, depending on how many parameters are used, which also leads to approximations with high accuracy. Here we will consider a simple approximation producing an accuracy adequate to most of the applications. With this idea in mind, the bridge function will be defined as

$$I_\nu(x) = \frac{|x|^\nu \cosh(x)}{2^\nu \Gamma(\nu+1)\left(1+\lambda^2 x^2\right)^{(2\nu+1)/4}} \frac{p_{0\nu} + p_{1\nu} x^2}{1 + q_\nu x^2} \tag{3}$$

where the parameters $p_{0\nu}, p_{1\nu}, q_\nu$ and λ will be determined using the power series and the asymptotic expansions already known. It is interesting to point out, that all the functions in Equation (3) are even functions since in this way the symmetry of the function $I_\nu(x)$ present and the approximation will be good for positive values of *x*, and also for negative values.

The procedure to be followed is first to show the way to determine the equations for the parameters determination. This will be performed using the power series and the asymptotic expansions. Later we will proceed to the calculation of the parameters and the way to do this avoiding the usual problems of the defects in Padé technique, that is, a zero in the denominator with a nearby zero at the numerator [11]. These themes will be better explained in relation to the examples here treated in detail.

The procedure to determine the parameter will be better explained by application to particular cases, which in this work will be by particular case of $\nu = 1/6$ and $\nu = 1/7$.

In the case of $\nu = 1/6$, the power series will be written as

$$I_{1/6}(x) = \frac{x^{1/6}}{2^{1/6}\Gamma\left(\frac{7}{6}\right)}\left(1 + \frac{3}{14}x^2 + \frac{9}{728}x^4 + \cdots\right) \tag{4}$$

and in the case of the asymptotic expansion, this will be

$$I_{1/6}(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 + \frac{1}{9x} + \frac{5}{81x^2} + \cdots\right) \tag{5}$$

The idea is how to find a bridge function linking both expansions. It is clear that the rational function must be multiplied by fractional powers of *x*, in order to reproduce the structure of the power series. The power series shows also that after this factor, the series has only even powers. This kind of symmetry has to be also preserved in the rational function, and also new factors will be considered, in order to obtain the correct asymptotic structure. In this case, there is also the exponential factor e^x, but this factor is not even as required in the preceding lines, therefore this must be replaced by $\cosh(x)$, which has the exponential behavior at the infinite, and the symmetry of $I_{1/6}(x)$ is preserved. There is also the fractional power $1/\sqrt{x}$, in the asymptotic expansion, which has to be considered, as well as the fractional power $x^{1/6}$ coming from the power series considerations. In order to accomplish all the restrictions coming from the power and asymptotic expansions, the structure of the analytic approximations $I_{1/6}(x)$ will be written as

$$\tilde{I}_{1/6}(x) = \frac{x^{1/6}\cosh(x)}{2^{1/6}\Gamma\left(\frac{7}{6}\right)\left(1+\lambda^2 x^2\right)^{1/3}}\left(\frac{p_0 + p_1 x^2}{1 + q x^2}\right) \tag{6}$$

where the important point is that

$$\frac{x^{1/6}}{x^{2/3}}=\frac{1}{x^{1/2}} \tag{7}$$

as it is required. Now the parameters p_0, p_1, q and λ have to be determined, as it will be shown in the next section.

In the case of $I_{1/7}(x)$, the power series will be

$$I_{1/7}(x)=\frac{x^{1/7}}{2^{1/7}\Gamma\left(\frac{8}{7}\right)}\left(1+\frac{7}{32}x^2+\frac{49}{3840}x^4+\cdots\right) \tag{8}$$

and the asymptotic expansion

$$I_{1/7}(x)\sim\frac{e^x}{\sqrt{2\pi x}}\left(1+\frac{45}{392x}+\cdots\right) \tag{9}$$

Now, using an analysis in some ways similar to the previous one, the analytic approximation will have the structure

$$\tilde{I}_{1/7}(x)=\frac{x^{1/7}\cosh(x)}{2^{1/7}\Gamma\left(\frac{8}{7}\right)\left(1+\tilde{\lambda}^2x^2\right)^{9/28}}\left(\frac{\tilde{p}_0+\tilde{p}_1x^2}{1+\tilde{q}x^2}\right) \tag{10}$$

where now

$$2\frac{9}{28}-\frac{1}{7}=\frac{9-2}{14}=\frac{1}{2} \tag{11}$$

3. Parameters Determination and Results

In the case of the $I_{1/6}(x)$, two equations from the power series and one equation for the asymptotic expansion will be used. The fourth equation for the parameters will come from numerical calculations, as will be explained below. Before equalizing power terms from the series for $I_{1/6}(x)$ and $\tilde{I}_{1/6}(x)$, it is necessary to rationalize the equation, and in this way, nonlinear equations will be avoided. Thus the product will be,

$$\begin{aligned}&\left(1+\lambda^2x^2\right)^{1/3}\left(1+qx^2\right)I_{1/6}(x)\\&=\left(1+\frac{1}{3}\lambda^2x^2+\cdots\right)\left(1+qx^2\right)\left(1+\frac{3}{14}x^2+\cdots\right)\\&=1+\left(\frac{1}{3}\lambda^2+q+\frac{3}{14}\right)x^2+\cdots\\&\approx\left(p_0+p_1x^2\right)\left(1+\frac{1}{2}x^2+\cdots\right)\end{aligned} \tag{12}$$

Therefore

$$p_0=1 \tag{13}$$

$$\frac{1}{2}p_0+p_1=\frac{1}{3}\lambda^2+q+\frac{3}{14} \tag{14}$$

The third equation for this case, it is obtained from the asymptotic expansion as

$$\frac{p_1}{2\times 2^{1/6}\Gamma\left(\frac{7}{6}\right)\lambda^{2/3}q}=\frac{1}{\sqrt{2\pi}} \tag{15}$$

From these three equations, all the parameters are obtained as a function of λ. Thus

$$q=\frac{-\frac{1}{3}\lambda^2+\frac{2}{7}}{1-2^{1/6}\Gamma\left(\frac{7}{6}\right)\sqrt{\frac{2}{\pi}}\lambda^{2/3}} \tag{16}$$

$$p_1=2^{1/6}\Gamma\left(\frac{7}{6}\right)\sqrt{\frac{2}{\pi}}\lambda^{2/3}\left(\frac{-\frac{1}{3}\lambda^2+\frac{2}{7}}{1-2^{1/6}\Gamma\left(\frac{7}{6}\right)\sqrt{\frac{2}{\pi}}\lambda^{2/3}}\right) \tag{17}$$

Now we have to avoid the so-called de "*deffects*" in the Padé method, that is a zero at the denominator with another zero at the numerator near the first one, but not exactly equal. Our procedure to avoid this problem is to make a plot of q as a function of λ, and to avoid the negative values of q. It is clear that if q is negative, there real zeros at the denominator one positive and anothers negative. This means that there is the so-called deffect in the approximation which are want to avoid. Thus, **Figure 1** shows the values of λ, where q is positive.

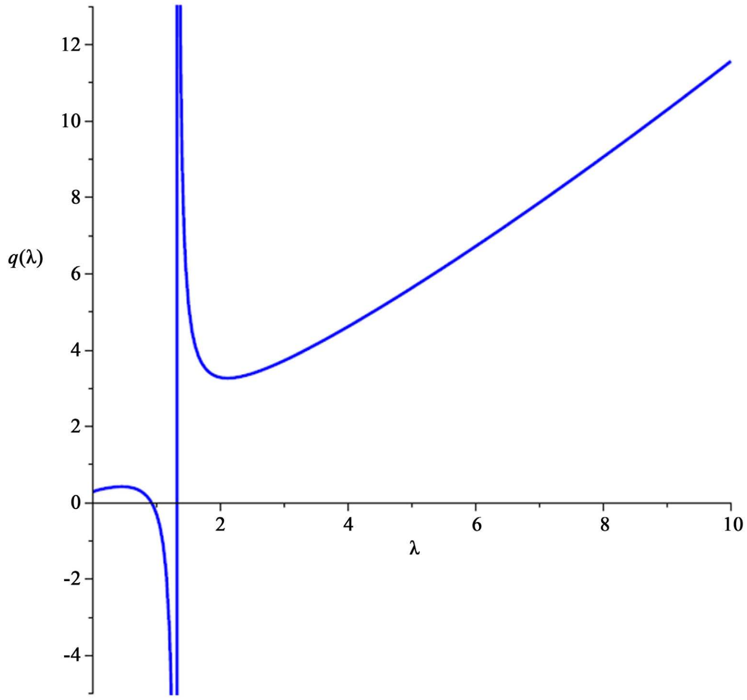

Figure 1. Plot of the curve $q(\lambda)$, in the regions $(0,0.92)\cup(1.38,+\infty)$ the values of $q(\lambda)$ are positive.

Now, it is obtained that for λ in the intervals $(0,0.92)\cup(1.38,+\infty)$, the values of q verify $q>0$. In order to determine the optimum value of λ, the procedure will be to give values to λ inside the right intervals, to determine the parameters for each value, and to calculate the maximum relative error $\epsilon_{\max}(\lambda)$ for that value. The optimum λ will be that with the lowest $\epsilon_{\max}(\lambda)$. In this case, this values is $\lambda_0=0.3675$, and the approximation will be

$$\tilde{I}_{1/6}=\frac{\cosh(x)x^{1/6}}{2^{1/6}\Gamma\left(\frac{7}{6}\right)\left(1+\lambda_0^2x^2\right)^{1/3}}\left(\frac{1+2^{1/6}\Gamma\left(\frac{7}{6}\right)\sqrt{\frac{2}{\pi}}\lambda_0^{2/3}x^2\left(\frac{-\frac{1}{3}\lambda_0^2+\frac{2}{7}}{1-2^{1/6}\Gamma\left(\frac{7}{6}\right)\sqrt{\frac{2}{\pi}}\lambda_0^{2/3}}\right)}{1+\left(\frac{-\frac{1}{3}\lambda_0^2+\frac{2}{7}}{1-2^{1/6}\Gamma\left(\frac{7}{6}\right)\sqrt{\frac{2}{\pi}}\lambda_0^{2/3}}\right)x^2}\right) \tag{18}$$

The relative error of this approximation is given in **Figure 2**. The maximum values of the errors are $\epsilon_{\max}(2.4)=0.0049$ for $x=2.4$, and $\epsilon_{\max}(11.1)=0.004$. However relative errors of the approximation out of the small regions around these values, are smaller than 0.002.

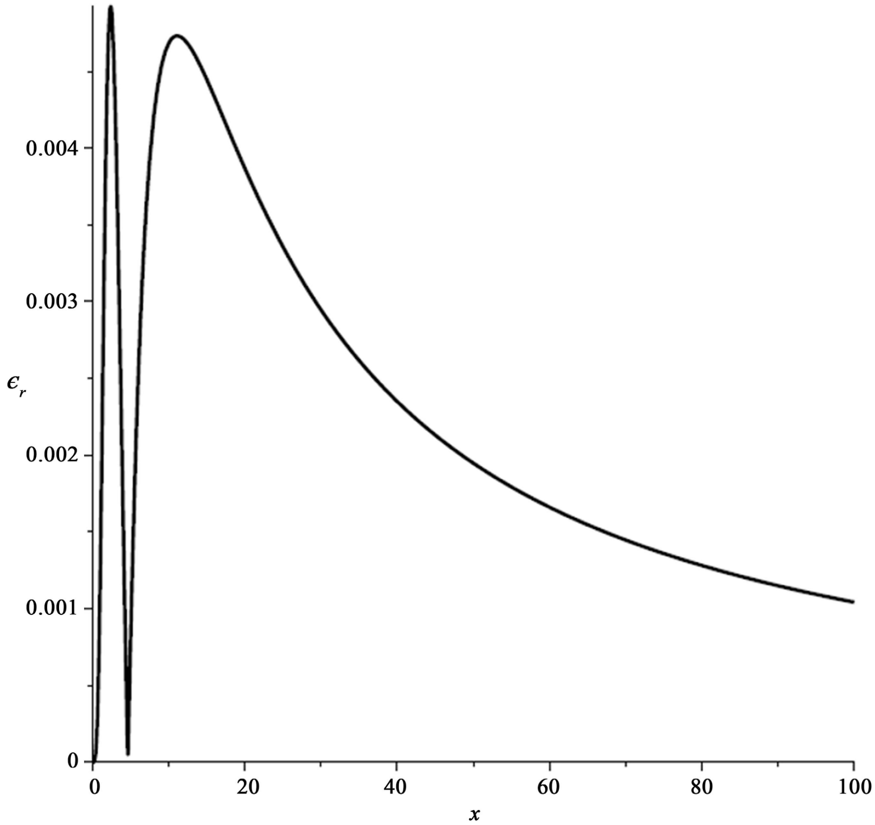

Figure 2. Relative errors of the approximation of $\tilde{I}_{1/6}(x)$ as a function of the variable *x*.

The same procedure applied to $I_{1/7}(x)$, leads to the equations

$$\begin{aligned}&\left(1+\tilde{\lambda}^2x^2\right)^{9/28}\left(1+\tilde{q}x^2\right)I_{1/7}(x)\\&=\left(1+\frac{9}{28}\tilde{\lambda}^2x^2+\cdots\right)\left(1+\tilde{q}x^2\right)\left(1+\frac{7}{32}x^2+\cdots\right)\\&=1+\left(\frac{9}{28}\tilde{\lambda}^2+\tilde{q}+\frac{7}{32}\right)x^2+\cdots\\&\approx\left(\tilde{p}_0+\tilde{p}_1x^2\right)\left(1+\frac{1}{2}x^2+\cdots\right)\end{aligned} \tag{19}$$

The equation to determine the parameters is now

$$\tilde{p}_0=1 \tag{20}$$

$$\frac{1}{2}\tilde{p}_0+\tilde{p}_1=\frac{9}{28}\tilde{\lambda}^2+\tilde{q}+\frac{7}{32} \tag{21}$$

$$\frac{\tilde{p}_1}{2\times 2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}^{9/14}\tilde{q}}=\frac{1}{\sqrt{2\pi}} \tag{22}$$

From these equations, it is obtained

$$\tilde{q}=\frac{-\frac{9}{28}\tilde{\lambda}^2+\frac{9}{32}}{1-2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}^{9/14}} \tag{23}$$

$$\tilde{p}_1=2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}^{9/14}\left(\frac{-\frac{9}{28}\tilde{\lambda}^2+\frac{9}{32}}{1-2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}^{9/14}}\right) \tag{24}$$

The plot of $\tilde{q}$ as a function of $\tilde{\lambda}$ is shown in **Figure 3**. The $\tilde{q}$ is positive for $\tilde{\lambda}\in(0,0.93)\cup(1.35,+\infty)$. Now looking for $\tilde{\lambda}$ optimum, it is found $\tilde{\lambda}=0.37$. With this value, the parameters are determined and the result will be,

$$\tilde{I}_{1/7}(x)=\frac{\cosh(x)x^{1/7}}{2^{1/7}\Gamma\left(\frac{8}{7}\right)\left(1+\tilde{\lambda}_0^2x^2\right)^{9/28}}\left(\frac{1+2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}_0^{9/14}x^2\left(\frac{-\frac{9}{28}\tilde{\lambda}_0^2+\frac{9}{32}}{1-2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}_0^{9/14}}\right)}{1+\left(\frac{-\frac{9}{28}\tilde{\lambda}_0^2+\frac{9}{32}}{1-2^{1/7}\Gamma\left(\frac{8}{7}\right)\sqrt{\frac{2}{\pi}}\tilde{\lambda}_0^{9/14}}\right)x^2}\right) \tag{25}$$

The relative errors for this approximation are shown in **Figure 4**. The maximum values of the relative errors will be in this case $\epsilon_{\max}(2.3)=0.0047$ and $\epsilon_{\max}(10.8)=0.005$. As in the case of $I_{1/6}(x)$, the relative errors are much smaller outside of the nearby regions of $x=2.3$ and $x=10.8$.

4. Conclusion

In this work a technique to obtain quasi-rational analytic approximations for the

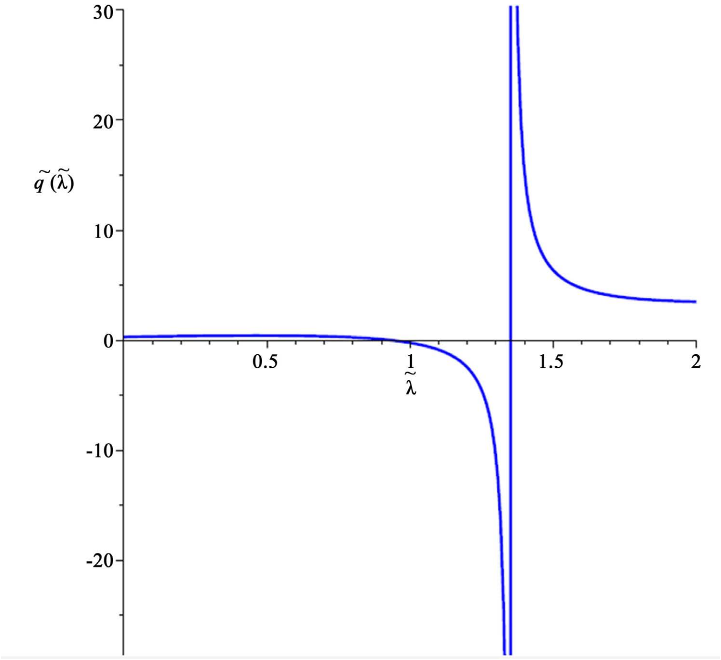

Figure 3. Plot of the curve $\tilde{q}(\tilde{\lambda})$, in the regions of $\tilde{\lambda} \in (0, 0.93) \cup (1.35, +\infty)$ the values of $\tilde{q}(\tilde{\lambda}) > 0$.

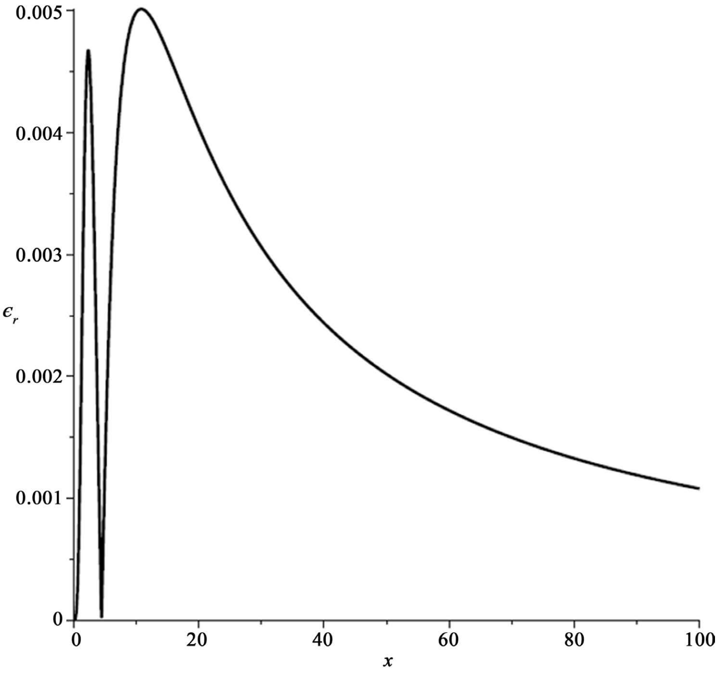

Figure 4. Relative errors of the approximation $\tilde{I}_{1/6}(x)$ as a function of the variable λ.

modified Bessel functions of general order ν is shown, and the structure of the function in the general case is shown. However, the procedure to determine the parameters of the approximation is shown by application to the particular cases of $\nu = 1/6$ and $\nu = 1/7$. The analytic approximations of $I_{1/6}(x)$ and $I_{1/7}(x)$ have been presented using rational functions with second order polynomials combined with the hyperbolic functions $\cosh(x)$ and the fractional powers $x^{1/6}$ and $x^{1/7}$, corresponding to each case. The numbers of parameters to be determined are four for each Bessel function. However, the accuracy is about as low as 0.005 for each case. The analytic approximations are very simple compared with their accuracy. The method used is an extension of the MPQA technique, and this is a clear advantage compared with other ones, since a unique approximation is valid for any value of the variable x, and the symmetry is preserved. The procedure here presented for $I_{1/6}(x)$ and $I_{1/7}(x)$ is also very general, and a general structure for any modified Bessel function of order ν is also described, therefore new approximations functions for any ν can be also determined, in agreement with the needs of other authors.

Conflicts of Interest

The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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