Advances in Pure Mathematics, 2024, 14, 759-767
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Construction and Analysis of Linear Ring Spaces

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How to cite this paper: Jaffa, D.A. (2024) The Construction and Analysis of Linear Ring Spaces. *Advances in Pure Mathematics*, **14**, 759-767.
https://doi.org/10.4236/apm.2024.149040

Received: August 18, 2024
Accepted: September 21, 2024
Published: September 24, 2024

Abstract

This paper proposes the novel algebraic structure of a linear ring space. A linear ring space is an order triad consisting of two rings, and a linear map between the two rings. The definition of quasi-linearity is discussed, in addition to the examination of properties and classifications of linear ring spaces. Particularly, the ring of holomorphic functions on a region of the complex plane is examined, and the manner in which it generates an iterated linear ring space under the complex derivative operator. This notion is then generalized to all rings with nth order linear and surjective operators. Basic operator theory regarding the classifications of linear ring maps is also covered.

Keywords

Ring Theory, Group Theory, Commutative Algebra, Operator Theory, Holomorphic Functions

1. Introduction

The discovery of ring morphisms by Camille Jordan in 1870 enabled the establishment of comparability in the behavior of binary operators within algebraic structures. However, such morphisms seldom establish a similarity in the elements of each ring, rather, a similarity in structural behavior. This notion provides a basis for the proposition and construction of a further algebraic structure, which establishes a correlation between the elements of two rings through a linear map. Such a structure shall be referred to as a *linear ring space*.

This paper proposes the aforementioned algebraic structure, whilst examining various properties and particular examples of linear ring spaces. Specifically, this paper will cover the classification of linear ring spaces that possess certain properties, with an emphasis on the ring of holomorphic functions, and the iterated linear ring space generated by said ring.

2. Preliminaries

Definition 2.1. A *ring* is a set, often denoted throughout the paper by X, with two binary operations, addition ($\oplus$) and multiplication ($\otimes$), such that $\{X,\oplus\}$ forms an abelian group and $\{X,\otimes\}$ forms a semigroup. The two aforementioned binary operations are connected through the distributive laws present within the ring axioms (see [1], chapter I). The ring axioms are as follows:

1) Closed and Commutative Addition: $x+y=y+x\in X \quad \forall x,y\in X$.

2) Associative Addition: $x+(y+z)=(x+y)+z=x+y+z \quad \forall x,y,z\in X$.

3) Additive Identity: $\exists 0_X\in X$ s.t. $x+0_X=x \quad \forall x\in X$.

4) Additive Inverse: $\forall x\in X \quad \exists x^{-1}=-x\in X$ s.t. $x+(-x)=0_X$.

5) Closed and Associative Multiplication: $x(yz)=(xy)z=xyz\in X$ $\forall x,y,z\in X$.

6) Distributive Laws: $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$ $\forall x,y,z\in X$.

If $\{X,\otimes\}$ is an abelian semigroup, then $\langle X,\oplus,\otimes\rangle$ is said to be a *commutative ring*. A ring with multiplicative inverses for all non-zero elements is referred to as a *division ring* (see [2], chapter VII). Moreover, a ring with a unique multiplicative identity is referred to as a *ring with identity*. Commutative division rings with identity are *fields*.

Throughout this paper, the term *ring* refers to a *ring with identity*, unless otherwise stated. Additionally, the notation $\langle X,\oplus,\otimes\rangle$ will be condensed into X when referring to a ring with implied or evident binary operations that need not explicit definition.

Definition 2.2. Let X be a ring that need not have identity, and $\mathcal{M}$ an arbitrary set. The set $\mathcal{M}$ forms an *X-module*, denoted throughout the paper by $(\mathcal{M},X)$, if there exists a binary addition operation ($\oplus$) on $\mathcal{M}$ such that $\{\mathcal{M},\oplus\}$ forms an abelian group. Furthermore, there must exist an action map of X on $\mathcal{M}$, that functions as a multiplication operation, which adheres to the following axioms:

1) $(x+y)m=xm+ym \quad \forall x,y\in X$ and $m\in\mathcal{M}$.

2) $(xy)m=x(ym)=xym \quad \forall x,y\in X$ and $m\in\mathcal{M}$.

3) $x(m+n)=xm+xn \quad \forall x\in X$ and $m,n\in\mathcal{M}$.

4) $1_X m=m \quad \forall m\in\mathcal{M}$.

Note that axiom 4 is only imposed if X is a ring with identity, otherwise, the axiom is negated in the definition of a module. Additionally, the above axioms define a *left X-module*, however, the definition of a *right X-module* is comparable, with multiplication by elements in X on the right. If X is a commutative ring, then the above definition describes a generalized X-module.

It is crucial to note that the action map is defined as $\mu: X\times\mathcal{M}\to\mathcal{M}$, indicating that the set $\mathcal{M}$ must be closed under X-scalar multiplication (see [3], subsection 0.3).

3. Relatively and Absolutely Linear Ring Spaces

Definition 3.1. Let X and Y be two arbitrary rings, and let $\mathcal{L}: X\to Y$ be a map between the two rings. The map $\mathcal{L}$ is said to be a *quasi-linear* map between the two rings if $\forall x, y \in X$, $\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y) \in Y$. Whilst an absolutely linear operator must be closed under scalar field multiplication, the above definition of a quasi-linear map negates this property. Note that throughout this paper, the use of the term linear in reference to such maps truly refers to the term quasi-linear.

Definition 3.2. Let *X* and *Y* be two arbitrary rings. If there exists a linear map $\mathcal{L}$ between these two rings, then *X* and *Y* are *relatively linear* rings under $\mathcal{L}$. Note that if *X* is relatively linear to *Y*, this does not necessarily imply that the converse holds true.

Definition 3.3. Let *X* and *Y* be two relatively linear rings with respect to the quasi-linear map $\mathcal{L}$. The order triad $\langle X, Y, \mathcal{L} \rangle$ consisting of two relatively linear rings, and the linear map between the two rings forms a *linear ring space*, particularly, a *relatively linear ring space*.

To further comprehend the notion of relatively linear ring spaces, it is crucial to establish the definition of an absolutely linear ring space. As will be demonstrated further within this section, relatively linear ring spaces function as generalizations of absolutely linear ring spaces.

3.1. Absolutely Linear Ring Spaces

Definition 3.4. Let $\langle X, Y, \mathcal{L} \rangle$ be a linear ring space, and $\mathcal{R}$ an arbitrary ring of scalars. $\langle X, Y, \mathcal{L} \rangle$ is said to be a *left linear ring space under* $\mathcal{R}$ if $\forall x, y \in X$ and $r_1, r_2 \in \mathcal{R}$, $\mathcal{L}(r_1 x + r_2 y) = r_1 \mathcal{L}(x) + r_2 \mathcal{L}(y) \in Y$. Similarly, $\langle X, Y, \mathcal{L} \rangle$ is said to be a *right linear ring space under* $\mathcal{R}$ if $\forall x, y \in X$ and $r_1, r_2 \in \mathcal{R}$, $\mathcal{L}(x r_1 + y r_2) = \mathcal{L}(x) r_1 + \mathcal{L}(y) r_2 \in Y$. If $\mathcal{R}$ is a commutative ring, and either of the statements above hold true, then $\langle X, Y, \mathcal{L} \rangle$ is an *absolutely linear ring space under* $\mathcal{R}$, and *X* is *absolutely linear to Y under* $\mathcal{R}$.

Theorem 3.1. For commutative, relatively linear rings *X* and *Y*, paired with the linear operator $\mathcal{L}$, and arbitrary scalar ring $\mathcal{R}$, if $\langle X, Y, \mathcal{L} \rangle$ is a left linear ring space under $\mathcal{R}$, then $(X, \mathcal{R})$ and $(Y, \mathcal{R})$ form left $\mathcal{R}$-modules. Similarly, if $\langle X, Y, \mathcal{L} \rangle$ is a right linear ring space under $\mathcal{R}$, then $(X, \mathcal{R})$ and $(Y, \mathcal{R})$ form right $\mathcal{R}$-modules.

Proof. The above theorem shall be proven for the left module case, as the proof for the further cases follows. Let *X* and *Y* be relatively linear commutative rings with respect to the linear map $\mathcal{L}$, and let $\mathcal{R}$ be an arbitrary scalar ring. In order for the linear ring space $\langle X, Y, \mathcal{L} \rangle$ to be left linear under $\mathcal{R}$,
$\mathcal{L}(r_1 x + r_2 y) = r_1 \mathcal{L}(x) + r_2 \mathcal{L}(y) \in Y$, $\forall x, y \in X$ and $r_1, r_2 \in \mathcal{R}$.

Given that $\mathcal{L} : X \to Y$, and that $\mathcal{L}(r_1 x + r_2 y) \in Y$, it must hold that $r_1 x + r_2 y \in X$, which indicates that *X* is closed under $\mathcal{R}$-scalar multiplication. Moreover, noting that $\mathcal{L}(x) \in Y \quad \forall x \in X$, it additionally holds that $r_1 \mathcal{L}(x) + r_2 \mathcal{L}(y) \in Y$, which further implies that, similar to *X*, *Y* is closed under $\mathcal{R}$-scalar multiplication.

The module axioms follow as a result of the addition and multiplication binary operations in $\mathcal{R}$, and the closure of *X* and *Y* under $\mathcal{R}$-scalar multiplication. ■

Example. Consider the ring of real, single-variable, globally differentiable functions, denoted by $\mathfrak{D}$. Let $\mathfrak{D}'$ denote the ring of derivatives of the elements in $\mathfrak{D}$, and $\frac{\mathrm{d}}{\mathrm{d}x}$ the single-variable, real derivative operator.

The linear ring space $\left\langle \mathfrak{D}, \mathfrak{D}', \frac{\mathrm{d}}{\mathrm{d}x} \right\rangle$ is absolutely linear under the real field, $\mathbb{R}$. This stems from the derivative being an absolutely linear operator under $\mathbb{R}$, and the closure of $\mathfrak{D}$ and $\mathfrak{D}'$ under $\mathbb{R}$. It can also be verified that $(\mathfrak{D}, \mathbb{R})$ and $(\mathfrak{D}', \mathbb{R})$ are both $\mathbb{R}$-modules, however, the proof has been omitted due to its triviality.

3.2. Relatively Linear Ring Spaces

Absolutely linear ring spaces contain linear operators that map between two rings that are closed under a certain scalar ring, as previously discussed. Relatively linear ring spaces generalize this notion to form the broader algebraic structure of linear ring spaces.

Definition 3.5. Let $\langle X, Y, \mathcal{L} \rangle$ be a linear ring space. The rings X and Y form a *relatively linear ring space* under the map $\mathcal{L}$, if $\forall x, y \in X$, $\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y) \in Y$. If such a property holds, then X is relatively linear to Y.

By definition 3.5, it is evident that relatively linear ring spaces function as further abstractions of absolutely linear ring spaces.

Theorem 3.2. If X and Y are two rings with homomorphism $\phi : X \to Y$, then $\langle X, Y, \phi \rangle$ forms a relatively linear ring space. *i.e.* If $X \cong Y \Rightarrow X$ is relatively linear to Y.

Proof. Given that ϕ is a homomorphism, it holds that $\phi(x+y) = \phi(x) + \phi(y) \in Y$, $\forall x, y \in X$. This structure preserving property of homomorphisms is exactly the property of linear maps stated in definition 3.1, necessary for the establishment of relative linearity between rings. Hence, $\langle X, Y, \phi \rangle$ is a relatively linear ring space. ■

Corollary. If $\phi : X \to Y$ is an isomorphism, *i.e.* if X is isomorphic to Y, then X is relatively linear to Y, and Y is relatively linear to X. To restate this, both $\langle X, Y, \phi \rangle$ and $\langle Y, X, \phi^{-1} \rangle$ are relatively linear ring spaces. This is a direct result of the notion that the inverse of an isomorphism exists and is also an isomorphism.

All homomorphic rings are relatively linear, however, the converse does not necessarily hold true. Linear ring spaces, as such, function as generalizations of homomorphic rings, establishing a correspondence between the elements of distinct rings (see [4], chapter 5).

Proposition. Every ring is relatively linear to itself with respect to the identity map, $\mathcal{I} : x \mapsto x$. Additionally, all rings are absolutely linear to themselves, under themselves, with respect to the identity map. Theorem 3.1 is demonstrated by the fact that all rings form both left and right modules over themselves. Hence, for all rings X, $\langle X, X, \mathcal{I} \rangle$ is always an absolutely linear ring space under X.

4. Iterated Linear Ring Spaces

This section discusses the ring of holomorphic functions within a region of the complex plane, and the iterated linear ring space generated by said ring. This concrete example will be utilized in order to establish a generalized definition of iterated linear ring spaces generated by a particular parent ring.

4.1. Linear Ring Space of Holomorphic Functions

Recall that all holomorphic complex valued functions are infinitely complex differentiable, and that the sum, product, and quotient of two holomorphic functions is also holomorphic (see [5]).

Lemma 4.1. Let $\mathfrak{H}(\Omega) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic on } \Omega \}$, where $\Omega \subseteq \mathbb{C}$ is a region of the complex plane. $\mathfrak{H}(\Omega)$ forms a field.

Proof. As previously established, the sum, product, difference, and quotient of holomorphic functions is also holomorphic, as per the complex derivative rules. Therefore, $\mathfrak{H}(\Omega)$ is closed under addition and multiplication, and the set contains both additive and multiplicative inverses (with the exception of $f = 0$), as per its closure under differences and quotients.

The zero function behaves as the additive identity, as it is holomorphic over the entire complex plane, hence $0 \in \mathfrak{H}(\Omega)$ for every Ω. Similarly, the function $f = 1$ is in fact holomorphic, and resultantly functions as the multiplicative identity for $\mathfrak{H}(\Omega)$.

Furthermore, $\mathfrak{H}(\Omega)$ adopts the commutative and associative laws for addition and multiplication from the complex field, as both fields behave in a comparable manner, also adopting the distributive laws, as a result. It has thus been established that $\mathfrak{H}(\Omega)$ is a commutative division ring with identity, and, as per definition 2.1, a field. ■

Lemma 4.2. Let $\frac{\mathrm{d}}{\mathrm{dz}}$ denote the complex derivative operator, and $\tilde{\mathfrak{H}}(\Omega)$ the set of derivatives of elements in $\mathfrak{H}(\Omega)$. $\left\langle \mathfrak{H}(\Omega), \tilde{\mathfrak{H}}(\Omega), \frac{\mathrm{d}}{\mathrm{dz}} \right\rangle$ forms an absolutely linear ring space under the set of complex numbers.

Proof. By lemma 4.1, $\mathfrak{H}(\Omega)$ is a commutative ring, and, by a similar proof that has been omitted, $\tilde{\mathfrak{H}}(\Omega)$ also forms a commutative ring. The complex derivative operator is linear, as per the definition of linearity noted in definition 3.1. Moreover, the ring of holomorphic functions is closed under scalar multiplication by elements of $\mathbb{C}$, and the derivative operator is absolutely linear with respect to said scaling.

Let f and g be two arbitrary elements of $\mathfrak{H}(\Omega)$, with $\alpha, \beta \in \mathbb{C}$. By the linearity of the derivative operator under complex scaling, it must hold that:

$$\frac{\mathrm{d}}{\mathrm{dz}}[\alpha f + \beta g] = \frac{\mathrm{d}}{\mathrm{dz}}[\alpha f] + \frac{\mathrm{d}}{\mathrm{dz}}[\beta g] \in \tilde{\mathfrak{H}}(\Omega)$$

which indicates that:

$$\alpha \frac{\mathrm{d}}{\mathrm{d}z}[f] + \beta \frac{\mathrm{d}}{\mathrm{d}z}[g] \in \tilde{\mathfrak{H}}(\Omega)$$
$$\Rightarrow \alpha \frac{\mathrm{d}f}{\mathrm{d}z} \in \tilde{\mathfrak{H}}(\Omega) \text{ and } \beta \frac{\mathrm{d}g}{\mathrm{d}z} \in \tilde{\mathfrak{H}}(\Omega) \ \forall f, g \in \mathfrak{H}(\Omega) \text{ and } \alpha, \beta \in \mathbb{C}$$

This implies that $\tilde{\mathfrak{H}}(\Omega)$ is closed under complex scalar multiplication. Also note that the commutative nature of $\mathfrak{H}(\Omega)$, $\tilde{\mathfrak{H}}(\Omega)$, and $\mathbb{C}$ enables $\mathfrak{H}(\Omega)$ and $\tilde{\mathfrak{H}}(\Omega)$ to be closed under left and right $\mathbb{C}$-scaling. Hence, $\left\langle \mathfrak{H}(\Omega), \tilde{\mathfrak{H}}(\Omega), \frac{\mathrm{d}}{\mathrm{d}z} \right\rangle$ forms an absolutely linear ring space under $\mathbb{C}$. ∎

Lemma 4.3. Let $\tilde{\mathfrak{H}}_n(\Omega)$ denote the set of *n*th derivatives of elements in $\mathfrak{H}(\Omega)$, and $\frac{\mathrm{d}^n}{\mathrm{d}z^n}$ the *n*th order complex derivative operator. $\left\langle \mathfrak{H}(\Omega), \tilde{\mathfrak{H}}_n(\Omega), \frac{\mathrm{d}^n}{\mathrm{d}z^n} \right\rangle$ forms an absolutely linear ring space under $\mathbb{C}$ for all $n \in \mathbb{N}$.

Proof. This proof shall be completed through induction. The proof for the base case of $n = 1$ is presented above in lemma 4.2. Suppose that for $n = k$, $\left\langle \mathfrak{H}(\Omega), \tilde{\mathfrak{H}}_k(\Omega), \frac{\mathrm{d}^k}{\mathrm{d}z^k} \right\rangle$ forms an absolutely linear ring space under $\mathbb{C}$, *i.e.*:

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}[\alpha f + \beta g] = \alpha \frac{\mathrm{d}^k}{\mathrm{d}z^k}[f] + \beta \frac{\mathrm{d}^k}{\mathrm{d}z^k}[g] \in \tilde{\mathfrak{H}}_k(\Omega) \ \forall f, g \in \mathfrak{H}(\Omega) \text{ and } \alpha, \beta \in \mathbb{C}$$

For $n = k + 1$:

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}[\alpha f + \beta g] = \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}[\alpha f + \beta g]\right] = \frac{\mathrm{d}}{\mathrm{d}z}\left[\alpha \frac{\mathrm{d}^k}{\mathrm{d}z^k}[f] + \beta \frac{\mathrm{d}^k}{\mathrm{d}z^k}[g]\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}z}\left[\alpha \frac{\mathrm{d}^k}{\mathrm{d}z^k}[f]\right] + \frac{\mathrm{d}}{\mathrm{d}z}\left[\beta \frac{\mathrm{d}^k}{\mathrm{d}z^k}[g]\right] = \alpha \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}[f]\right] + \beta \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}[g]\right]$$
$$= \alpha \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}[f] + \beta \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}[g] \in \tilde{\mathfrak{H}}_{k+1}(\Omega) \ \forall f, g \in \mathfrak{H}(\Omega) \text{ and } \alpha, \beta \in \mathbb{C}$$

Therefore, the *n*th order complex derivative operator remains absolutely linear, and $\tilde{\mathfrak{H}}_n(\Omega)$ is closed under complex scaling for all $n \in \mathbb{N}$, which implies that $\left\langle \mathfrak{H}(\Omega), \tilde{\mathfrak{H}}_n(\Omega), \frac{\mathrm{d}^n}{\mathrm{d}z^n} \right\rangle$ forms an absolutely linear ring space under $\mathbb{C}$. ∎

Corollary. The ring $\mathfrak{H}(\Omega)$ generates infinitely many linear ring spaces by repeatedly applying the complex derivative operator. This indicates that $\left\langle \tilde{\mathfrak{H}}_n(\Omega), \tilde{\mathfrak{H}}_m(\Omega), \frac{\mathrm{d}^{m-n}}{\mathrm{d}z^{m-n}} \right\rangle$ is an absolutely linear ring space $\forall n < m \in \mathbb{N}$.

Proof. It has been proven in lemma 4.3 that the complex derivative operator retains its linearity under repeated composition. Hence, for all $n \in \mathbb{N}$, $\frac{\mathrm{d}^n}{\mathrm{d}z^n}$ is a linear operator. Since $n < m \in \mathbb{N}$, define $k = m - n \in \mathbb{N}$, so that $\frac{\mathrm{d}^k}{\mathrm{d}z^k}$ must be linear. Restating the definition of the two rings:

$$\tilde{\mathfrak{H}}_m(\Omega) := \left\{ f : \Omega \to \mathbb{C} \mid f = \frac{\mathrm{d}^m g}{\mathrm{d}z^m} = \frac{\mathrm{d}^{n+k} g}{\mathrm{d}z^{n+k}} \text{ for some } g \in \mathfrak{H}(\Omega) \right\}$$

$$\tilde{\mathfrak{H}}_n(\Omega) := \left\{ f : \Omega \to \mathbb{C} \mid f = \frac{\mathrm{d}^n g}{\mathrm{d}z^n} \text{ for some } g \in \mathfrak{H}(\Omega) \right\}$$

This indicates that in order to map $\tilde{\mathfrak{H}}_n(\Omega)$ to $\tilde{\mathfrak{H}}_m(\Omega)$, the derivative operator must be applied *k* times on the elements of $\tilde{\mathfrak{H}}_n(\Omega)$. As noted above, $\frac{\mathrm{d}^k}{\mathrm{d}z^k}$ is a linear operator, and, as per lemma 4.3, $\tilde{\mathfrak{H}}_m(\Omega)$ and $\tilde{\mathfrak{H}}_n(\Omega)$ are both closed under $\mathbb{C}$. Therefore, $\left\langle \tilde{\mathfrak{H}}_n(\Omega), \tilde{\mathfrak{H}}_m(\Omega), \frac{\mathrm{d}^{m-n}}{\mathrm{d}z^{m-n}} \right\rangle$ must be an absolutely linear ring space under $\mathbb{C}$ for all $n < m \in \mathbb{N}$. ■

In summary, the ring of holomorphic functions on a region Ω of the complex plane is absolutely linear under the complex field to the ring of complex derivatives of $\mathfrak{H}(\Omega)$, with respect to the complex derivative operator. As a matter of fact, $\mathfrak{H}(\Omega)$ is absolutely linear to $\tilde{\mathfrak{H}}_n(\Omega)$ for all $n \in \mathbb{N}$, with respect to the *n*th order complex derivative operator, as it retains its linearity under infinite composition.

Moreover, $\left\langle \tilde{\mathfrak{H}}_n(\Omega), \tilde{\mathfrak{H}}_m(\Omega), \frac{\mathrm{d}^{m-n}}{\mathrm{d}z^{m-n}} \right\rangle$ forms an absolutely linear ring space under $\mathbb{C}$ for all $n < m \in \mathbb{N}$, and this linear ring space is generated by the parent ring, $\mathfrak{H}(\Omega)$. The ring $\mathfrak{H}(\Omega)$ is said to *generate an infinitely iterated linear ring space* under the *infinitely linear* complex derivative operator, $\frac{\mathrm{d}}{\mathrm{d}z}$. This notion shall be further explored and generalized within the following subsection.

4.2. Finitely and Infinitely Iterated Linear Ring Spaces

Definition 4.1. Let $\mathcal{L} : X \to Y$ be a linear map between two rings, and $\mathcal{L}^n$ the composition of $\mathcal{L}$ *n* times. $\mathcal{L}$ is said to be an *nth order linear operator* if it retains its linearity under *n* compositions for some $n \in \mathbb{N}$, *i.e.* $\mathcal{L}^n$ and all lower order compositions are linear. If $n = \infty$, then $\mathcal{L}$ is said to be an *infinitely linear operator*.

Definition 4.2. Let $\mathcal{T} : A \to B$ be a surjective operator that maps between two arbitrary sets, and let $\mathcal{T}^n$ denote the composition of $\mathcal{T}$ *n* times. $\mathcal{T}$ is said to be an *nth order surjective operator* if it remains surjective after *n* compositions for some $n \in \mathbb{N}$, *i.e.* $\mathcal{T}^n$ and all lower order compositions are surjective. If $n = \infty$, then $\mathcal{T}$ is said to be an *infinitely surjective operator*.

Definition 4.3. Let $\langle X, Y, \mathcal{L} \rangle$ be a linear ring space, and let $\tilde{X}_n := codomain(\mathcal{L}^n)$. The ring *X* is said to *generate an nth order iterated linear ring space under* $\mathcal{L}$ if $\langle \tilde{X}_a, \tilde{X}_b, \mathcal{L}^{b-a} \rangle$ forms a linear ring space for all $a < b \le n \in \mathbb{N}$. If $n = \infty$, then *X* is said to *generate an infinitely iterated linear ring space under* $\mathcal{L}$.

Theorem 4.1. Let $\langle X, Y, \mathcal{L} \rangle$ be a linear ring space. If $\mathcal{L}$ is an *n*th order linear and surjective operator, then X generates an nth order iterated linear ring space, for some $n \in \mathbb{N}$.

Proof. This theorem shall be proven for the finite case, however, the proof for the case of $n = \infty$ follows in a comparable manner.

Given that $\mathcal{L}$ is an nth order linear and surjective operator, it must hold that $im(\mathcal{L}^m) = codomain(\mathcal{L}^m) = \tilde{X}_m$ for all $m \le n \in \mathbb{N}$, for some $n \in \mathbb{N}$, by definition 4.2. Moreover, it must also hold that $\mathcal{L}^m(x+y) = \mathcal{L}^m(x) + \mathcal{L}^m(y)$ for all $x, y \in X$ and $m \le n \in \mathbb{N}$. Hence, $\langle X, \tilde{X}_m, \mathcal{L}^m \rangle$ forms a linear ring space for all $m \le n$.

Consider the ring $\tilde{X}_a$ for some arbitrary $a < n$. Since $\mathcal{L}$ is an nth order surjective operator, describe the ring as $\tilde{X}_a := \{\mathcal{L}^a(x) \mid x \in X\}$. Pick some $b \in \mathbb{N}$, so that $a < b \le n$. The ring $\tilde{X}_b$ may be defined as $\tilde{X}_b := \{\mathcal{L}^b(x) \mid x \in X\}$.

Note that for all $x, y \in \tilde{X}_a$, $\mathcal{L}^{b-a}(x+y) = \mathcal{L}^{b-a}(x) + \mathcal{L}^{b-a}(y)$, by definition 4.1. Also recall that $\mathcal{L}$ is nth order surjective, so that x and y may be described as $x = \mathcal{L}^a(x')$ and $y = \mathcal{L}^a(y')$ for some $x', y' \in X$. Substituting yields:

$$\begin{aligned}\mathcal{L}^b(x') + \mathcal{L}^b(y') &= \mathcal{L}^{b-a}\left(\mathcal{L}^a(x') + \mathcal{L}^a(y')\right) \\ &= \mathcal{L}^{b-a}\left(\mathcal{L}^a(x')\right) + \mathcal{L}^{b-a}\left(\mathcal{L}^a(y')\right) \\ &= \mathcal{L}^{b-a}(x) + \mathcal{L}^{b-a}(y) = \mathcal{L}^{b-a}(x+y)\end{aligned}$$

which implies that $\tilde{X}_b$ may be defined as $\tilde{X}_b := \{\mathcal{L}^{b-a}(x) \mid x \in \tilde{X}_a\}$, and that in order to map from $\tilde{X}_a \to \tilde{X}_b$, the operator $\mathcal{L}$ must be applied $b-a$ times.

Since $b - a \in \mathbb{N} \le n$, $\mathcal{L}^{b-a}$ is a linear and surjective operator for all a and b that adhere to the aforementioned conditions. Therefore, $\langle \tilde{X}_a, \tilde{X}_b, \mathcal{L}^{b-a} \rangle$ forms a linear ring space for all $a < b \le n$. ■

Proposition. The ring of holomorphic functions $\mathfrak{H}(\Omega)$, as defined in lemma 4.1, generates an infinitely iterated linear ring space under the complex derivative operator, $\frac{\mathrm{d}}{\mathrm{d}z}$. The proof of this proposition is presented throughout subsection 4.1.

Theorem 4.2. Let $\mathcal{L}: A \to B$ be a linear, surjective operator between two arbitrary sets A and B. If $\mathcal{L}$ is idempotent, *i.e.*, $\mathcal{L}^2 = \mathcal{L}$, then $\mathcal{L}$ is an infinitely linear and surjective operator.

Proof. This proof shall be completed through induction. It is given that $\mathcal{L}^2 = \mathcal{L}$, which indicates that $\mathcal{L}^2$ is linear and surjective, as $\mathcal{L}$ is both linear and surjective. Suppose that for $n = k$, $\mathcal{L}^k = \mathcal{L}$. For $n = k+1$:

$$\mathcal{L}^{k+1} = \mathcal{L}^k \mathcal{L} = \mathcal{L}\mathcal{L} = \mathcal{L}^2 = \mathcal{L}$$

This indicates that $\mathcal{L}$ is surjective and linear for all $n \in \mathbb{N}$, which roots from the linearity and surjectivity of $\mathcal{L}$ itself. Therefore, $\mathcal{L}$ is an infinitely linear and surjective operator. ■

Corollary. Let $\mathcal{L}: X \to Y$ be an idempotent, linear and surjective map between two rings. X generates an infinitely iterated linear ring space under $\mathcal{L}$. Since $\mathcal{L}$ is idempotent, such an iterated ring space may be written as $\tilde{X}_a, \tilde{X}_b, \mathcal{L}$ for all $a < b \in \mathbb{N}$. The proof of this corollary has been omitted due to its triviality.

It is crucial to note that it is not necessary for a linear operator to be idempotent in order to be infinitely (or finitely) linear and surjective. This notion is prevalent in the example of the complex derivative operator, as demonstrated in lemma 4.3.

5. Discussion

A linear ring space is an ordered triad consisting of two rings and a linear map between the two rings. This algebraic structure establishes that two distinct rings share a comparability in elements, along with a similarity in the behavior of the addition binary operation in the two rings, functioning as a generalization of ring homomorphisms. This notion is demonstrated within theorem 3.2 and its corollary. Moreover, particular types of linear ring spaces possess properties that allow the formation of modules and iterations, as is prevalent in the ring of holomorphic functions. The concept of linear ring spaces provides a basis for applications to idempotent and Boolean algebras, and the structural preservation of quasi-linear operators is a powerful generalization of further structure preserving group morphisms.

Acknowledgements

I would like to greatly thank and acknowledge the American Community School of Beirut for consistently providing an excellent educational and research environment.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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