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Besov Estimates for Sub-Elliptic Equations in the Heisenberg Group

Huimin Cheng, Feng Zhou*

School of Mathematics and Statistics, Shandong Normal University, Jinan, China
Email: 2995246312@qq.com, *zhoufeng@u.nus.edu

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Abstract

In this article, we deal with weak solutions to non-degenerate sub-elliptic equations in the Heisenberg group, and study the regularities of solutions. We establish horizontal Calderón-Zygmund type estimate in Besov spaces with more general assumptions on coefficients for both homogeneous equations and non-homogeneous equations. This study of regularity estimates expands the Calderón-Zygmund theory in the Heisenberg group.

Keywords

Heisenberg Group, Sub-Elliptic Equations, Regularity, Besov Spaces

1. Introduction

The main purpose of this article is to study Besov regularities of weak solutions to sub-elliptic equations of the type

$$\operatorname{div}_H A(x, \mathfrak{X}u) = 0 \tag{1.1}$$

and

$$\operatorname{div}_H A(x, \mathfrak{X}u) = \operatorname{div}_H \left(|F|^{p-2} F\right) \tag{1.2}$$

in Ω, respectively. Here Ω is an open and bounded sub-domain in the Heisenberg group $\mathbb{H}^n = \mathbb{R}^{2n+1}$ ($n \geq 1$). In the homogeneous Equation (1.1) and the non-homogeneous Equation (1.2), the unknown $u \in HW^{1,p}_{\text{loc}}(\Omega)$, where the sub-elliptic Sobolev space $HW^{1,p}(\Omega)$ will be introduced in Section 2. In both equations, the horizontal divergence operator div_H and the horizontal gradient $\mathfrak{X}$ are defined by

$$\operatorname{div}_H F = \sum_{i=1}^{2n} X_i F_i,$$

$$\mathfrak{X}u = \left(X_1u, X_2u, \cdots, X_{2n-1}u, X_{2n}u\right)$$

in the distributional sense. Moreover, $A:\Omega\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is assumed to be a Carathéodory vector field with general growth and uniformly elliptic conditions, that is, there exist constants $\nu, L, k > 0$ and $0<\mu<1$ such that

(A1) $\left[A(x,\xi)-A(x,\eta)\right]\cdot(\xi-\eta)\geq\nu\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|^2$,

(A2) $\left|A(x,\xi)-A(x,\eta)\right|\leq L\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|$,

(A3) $\left|A(x,\xi)\right|\leq k\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}$

for every $\xi,\eta\in\mathbb{R}^{2n}$ and for almost all $x\in\Omega$. In (1.2), $F:\Omega\to\mathbb{R}^{2n}$.

The regularity of solutions to elliptic equations in Euclidean spaces $\mathbb{R}^n$ has been well studied by Iwaniec [1], DiBenedetto and Manfredi [2]. Then this theory is extended to the case of general elliptic problems, see in relevant papers [3]-[6]. For the nonlinear Calderón-Zygmund estimate in the Heisenberg group, Goldstein and Zatorska-Goldstein [7] deal with the quadratic case $p=2$. Later on the $HW^{1,p}$ estimates for sub-elliptic equations on $\mathbb{H}^n$ are proved by Mingione, Zatorska-Goldstein and Zhong [8]. They consider the equation of the form

$$\mathrm{div}_H\left[b(x)a(\mathfrak{X}u)\right]=\mathrm{div}_H\left(|F|^{p-2}F\right)$$

with $b\in\mathrm{VMO}_{\mathrm{loc}}(\Omega)$.

At present, the studies are concerned with the regularity estimates of weak solutions in Besov spaces in both $\mathbb{R}^n$ and $\mathbb{H}^n$ ([9]-[11]). Besov spaces consist of a wide class of functions compared with the classical Sobolev spaces. Baisón [12] deals with nonlinear elliptic equations in divergence form, and obtains a regularity estimate of weak solutions in Besov spaces. Clop [13] and Lyaghfouri [14] extended the result in Besov spaces by establishing a higher integrability of weak solutions.

For the homogeneous case (1.1), we assume that there exists a function $g\in L^{\frac{Q}{\alpha}}(\Omega)$ ($0<\alpha<1$) such that

(A4) $\left|A(x,\xi)-A(y,\xi)\right|\leq\mathrm{dist}_{\mathrm{CC}}(x,y)^{\alpha}\left(g(x)+g(y)\right)\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}$

for almost every $x,y\in\Omega$ and all $\xi\in\mathbb{R}^{2n}$. Here $\mathrm{dist}_{\mathrm{CC}}(x,y)$ is the CC-distance between two points x and y in $\mathbb{H}^n$.

While for the non-homogeneous situation (1.2), we assume that there exists a sequence of measurable non-negative functions $g_k\in L^{\frac{Q}{\alpha}}(\Omega)(k\in\mathbb{N}, 0<\alpha<1)$ satisfying that

(A5) $\begin{cases}\sum_{k=1}^{\infty}\|g_k\|^q_{L^{\frac{Q}{\alpha}}(\Omega)}<\infty\ (1\leq q<\infty)\\ \left|A(x,\xi)-A(y,\xi)\right|\leq\mathrm{dist}_{\mathrm{CC}}(x,y)^{\alpha}\left(g_k(x)+g_k(y)\right)\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}\end{cases}$

for $\xi\in\mathbb{R}^{2n}$ and almost all $x,y\in\Omega$ such that $2^{-k}\leq\mathrm{dist}_{\mathrm{CC}}(x,y)<2^{-k+1}$.

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According to (A5), we write $\{g_k\}_k \in l^q\left(L^{\frac{Q}{\alpha}}(\Omega)\right)$ in short.

By introducing an auxiliary function

$$V(\xi)=\left(\mu^2+|\xi|^2\right)^{\frac{p-2}{4}}\xi \tag{1.3}$$

with $\xi \in \mathbb{R}^{2n}$, we present the main results of this article.

Theorem 1.1 *Let* $0<\alpha<1$ *and* $2\le p<4$. *Assume that A satisfies hypotheses* (A1)-(A4) *with* $0<\mu<1$. *If* $u\in HW^{1,p}_{\text{loc}}(\Omega)$ *is a weak solution to* (1.1), *then* $V(\mathfrak{X}u)\in B^{\alpha}_{2,\infty}(\Omega)$ *locally.*

Theorem 1.2 *Let* $0<\alpha<1$, $2\le p<4$, *and* $1\le q<\dfrac{2Q}{Q-2\alpha}$. *Assume that the hypotheses* (A1)-(A3) *and* (A5) *hold. If* $u\in HW^{1,p}_{\text{loc}}(\Omega)$ *is a weak solution to* (1.2) *with* $0<\mu<1$ *and* $|F|^{p-2}F\in B^{\alpha}_{2,q}(\Omega)$, *then* $V(\mathfrak{X}u)\in B^{\alpha}_{2,q}(\Omega)$ *locally.*

See Section 2 for the definitions of $HW^{1,p}(\Omega)$ and $B^{\alpha}_{2,q}(\Omega)$.

The contribution of the main results is to study a wide class of sub-elliptic equations in the Heisenberg group. Our aim is to obtain a Besov regularity estimate of weak solutions. The hypotheses (A1)-(A4) (or (A5)) shall be an extension of the VMO conditions.

This article is organized as follows. In Section 2 we give some definitions and tools such as classical inequalities, and we present two Lemmas relating to the reverse Hölder type inequalities of weak solutions. In Section 3 and Section 4, we present the proofs of Theorem 1.1 and Theorem 1.2, respectively.

2. Preliminary

2.1. Heisenberg Group

In this section, we collect some basic notations and preliminaries for the Heisenberg group (see [8] [15] for more details). We denote by $(x,t)=(x_1,x_2,\cdots,x_{2n},t)$ the coordinates of points of the Heisenberg group $\mathbb{H}^n$. The group structure on $\mathbb{H}^n$ is given by

$$\begin{aligned}&(x_1,x_2,\cdots,x_{2n},t)\circ(y_1,y_2,\cdots,y_{2n},s)\\&=\left(x_1+y_1,x_2+y_2,\cdots,x_{2n}+y_{2n},t+s+\frac{1}{2}\sum_{j=1}^{n}\left(x_jy_{n+j}-x_{n+j}y_j\right)\right).\end{aligned}$$

An anisotropic dilation induces a homogeneous norm (gauge) of (x,t) by $\left(|x|^2+t\right)^{\frac{1}{2}}$. For $j=1,\cdots,n$, we set

$$X_j=\frac{\partial}{\partial x_j}-\frac{x_{n+j}}{2}\frac{\partial}{\partial t},\quad X_{n+j}=\frac{\partial}{\partial x_{n+j}}+\frac{x_j}{2}\frac{\partial}{\partial t},\quad T=\frac{\partial}{\partial t},$$

which represent a basis of the space of left-invariant vector fields on $\mathbb{H}^n$. The vector field $X_1,X_2,\cdots,X_{2n}$ are called the horizontal vector fields. Then the length of the horizontal gradient is given by

$$|\mathfrak{X}u|^2=\sum_{j=1}^{2n}\left(X_jf\right)^2.$$

2.2. CC-Distance and CC-Balls

By considering the well-known Carnot-Carathéodory metric with CC-distance dist_{CC}, we define CC-balls by

$$B_R(x_0) = \{ y \in \mathbb{H}^n \mid \text{dist}_{CC}(x_0, y) < R \}$$

with the center x_0 and radius *R*. By introducing the homogeneous dimension $Q = 2n + 2$, one gets the Lebesgue measure of a CC-ball $|B_R(x_0)| \approx R^Q$.

2.3. Horizontal Sobolev Spaces and Besov Spaces

Let $L^p(\mathbb{H}^n)$ be the Lebesgue space in the Heisenberg group, then the dual space of $L^p(\mathbb{H}^n)$ is $L^{p'}(\mathbb{H}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. The horizontal Sobolev space with its norm is defined by

$$HW^{1,p}(\Omega) = \{ u \in L^p(\Omega) \mid \mathfrak{X} u \in L^p(\Omega) \},$$

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\mathfrak{X} u\|_{L^p(\Omega)}.$$

It is clear that a function $u \in HW^{1,p}_{\text{loc}}(\Omega)$, if $u \in HW^{1,p}(\Omega_0)$ for every $\Omega_0 \Subset \Omega$.

Let the parameters $0 < \alpha < 1$, $1 \le p < \infty$, $1 \le q \le \infty$. The Besov spaces $B^\alpha_{p,q}(\Omega)$ $(\Omega \subset \mathbb{H}^n)$ with its norm are defined via ([16])

$$\|u\|_{B^\alpha_{p,q}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{B^\alpha_{p,q}(\Omega)} < \infty,$$

$$[u]_{B^\alpha_{p,q}(\Omega)} = \begin{cases} \left(\int_\Omega \left(\int_\Omega \frac{|\Delta_h u|^p}{|h|^{\alpha p}} \mathrm{d}x \right)^{\frac{q}{p}} \frac{\mathrm{d}h}{|h|^Q} \right)^{\frac{1}{q}} < \infty, & 1 \le q < \infty, \\ \sup\limits_{h \in \Omega} \left(\int_\Omega \frac{|\Delta_h u|^p}{|h|^{\alpha p}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty, & q = \infty. \end{cases}$$

In this article, we shall write $\Delta_h u = u(x+h) - u(x)$ in short.

2.4. Basic Tools

For every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $s, t \ge 0$, there holds

$$st \le \varepsilon s^p + C(\varepsilon) t^{p'}, \tag{2.1}$$

which is the classical Young inequality. Here $\frac{1}{p} + \frac{1}{p'} = 1$. In particular,

$$ab \le \varepsilon a^2 + C(\varepsilon) b^2. \tag{2.2}$$

Let $B_R \Subset \mathbb{H}^n$ be a CC-ball, and *f* an integrable function on B_R, we define the average of *f* over the CC-ball B_R as

$$(f)_{B_R} = ⨍_{B_R} f(x) \mathrm{d}x = \frac{1}{|B_R|} \int_{B_R} f(x) \mathrm{d}x \approx R^{-Q} \int_{B_R} f(x) \mathrm{d}x. \tag{2.3}$$

We present the definition of weak solutions. If for any $\varphi \in C_0^\infty(\Omega)$, there holds

$$\int_{\Omega} A(x,\mathfrak{X}u)\cdot\mathfrak{X}\varphi \mathrm{d}x = \int_{\Omega} |F|^{p-2} F\cdot\mathfrak{X}\varphi \mathrm{d}x, \tag{2.4}$$

then $u \in HW^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to (1.2). Here we call φ is a test function.

2.5. Reverse Hölder Type Inequality

The higher integrability estimates for Laplace and p-Laplace equations are well known (see [1] and [2]). In the Heisenberg group, we have the following two results for homogeneous and non-homogeneous situations, see [8].

Lemma 2.1 *Let* $u \in HW^{1,p}(\Omega)$ *with* $2 \le p < 4$ *be a weak solution to* (1.1) *under the hypotheses* (A1)-(A4). *There exists a constant* $c(n,p,\nu,k,L)$, *but otherwise independent of* μ, *of the solution u, and of the vector field* $A(x,\nabla u)$, *such that the following inequalities hold for any CC-ball* $B_R \Subset \Omega$:

$$\sup_{B_{\frac{R}{2}}} |\mathfrak{X}u| \le c\left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.5}$$

Lemma 2.2 *Let* $u \in HW^{1,p}(\Omega)$ *with* $2 \le p < 4$ *be a weak solution to equation* (1.2). *Assume that* (A1)-(A3) *and* (A5) *hold. If* $F \in L^q_{\text{loc}}(\Omega)$, *then* $\mathfrak{X}u \in L^q_{\text{loc}}(\Omega)$, *where* $q \in (p,\infty)$. *Moreover, there exists a positive constant* $C(n,p,\nu,L,q,a)$ *such that*

$$\left(\fint_{B_{\frac{R}{2}}} |\mathfrak{X}u|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p \mathrm{d}x\right)^{\frac{1}{p}} + C\left(\fint_{B_R} |F|^q \mathrm{d}x\right)^{\frac{1}{q}} \tag{2.6}$$

for any CC-ball $B_R \Subset \Omega$.

3. Proofs of Theorem 1.1

In this section, we present the proofs of Theorem 1.1. Inspired by [13], for the vector field $A(x,\xi)$ appeared in (1.2), we introduce

$$A_B(\xi) = \fint_B A(x,\xi)\mathrm{d}x \tag{3.1}$$

for $\xi \in \mathbb{R}^{2n}$ and a CC-ball $B \subset \Omega$. Then we define

$$V(x,B) = \sup_{\xi\in\mathbb{R}^{2n}} \frac{|A(x,\xi) - A_B(\xi)|}{\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}}, \tag{3.2}$$

where $B \subset \Omega$ is a CC-ball and $x \in \Omega$. It follows that if $A:\Omega\times\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a Carathéodory vector field such that (A1)-(A4) hold, then A is locally uniformly in VMO, that is,

$$\lim_{R\to 0} \sup_{r(B)<R} \sup_{c(B)\in K} \fint_B V(x,B)\mathrm{d}x = 0, \tag{3.3}$$

where $K \subset \Omega$, $c(B)$ and $r(B)$ denote the center and the radius of the CC-ball B, respectively.

In order to prove Theorem 1.1, we mention that there exists a constant $\hat{C} > 0$

such that

$$\hat{C}^{-1}\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}} \le \frac{|V(\xi)-V(\eta)|^2}{|\xi-\eta|^2} \le \hat{C}\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}} \tag{3.4}$$

for any $\xi,\eta\in\mathbb{R}^{2n}$ and $|\xi-\eta|\neq 0$.

We are in a position to present the proof.

Proof of Theorem 1.1. We let $B_{3R}\Subset\Omega$ and select a test function $\varphi=\Delta_{-h}\left(\eta^2\Delta_h u\right)$ to (1.1), where $\eta\in C_0^\infty(B_{3R})$ is a cut-off function satisfying that

$$0\le\eta(x)\le 1,\ \eta(x)\equiv 1 \text{ for } x\in B_{\frac{R}{2}},\ \eta(x)\equiv 0 \text{ for } x\in B_{3R}\setminus B_R, \text{ and } |\mathfrak{X}\eta|\le\frac{C}{R}.$$

One gets that

$$\begin{aligned} G_1 &= \int_{B_{2R}}\left[A(x+h,\mathfrak{X}u(x+h))-A(x+h,\mathfrak{X}u)\right]\cdot\eta^2\Delta_h\mathfrak{X}u\,\mathrm{d}x \\ &= \int_{B_{2R}}\left[A(x,\mathfrak{X}u)-A(x+h,\mathfrak{X}u)\right]\cdot\eta^2\Delta_h\mathfrak{X}u\,\mathrm{d}x \\ &\quad+\int_{B_{2R}}\left[A(x+h,\mathfrak{X}u)-A(x+h,\mathfrak{X}u(x+h))\right]\cdot 2\eta\mathfrak{X}\eta\Delta_h u\,\mathrm{d}x \\ &\quad+\int_{B_{2R}}\left[A(x,\mathfrak{X}u)-A(x+h,\mathfrak{X}u)\right]\cdot 2\eta\mathfrak{X}\eta\Delta_h u\,\mathrm{d}x \\ &= G_2+G_3+G_4. \end{aligned} \tag{3.5}$$

We estimate each G_i ($1\le i\le 4$) in (3.5). By (A1), it is clear that

$$G_1\ge\nu\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\eta^2\mathrm{d}x. \tag{3.6}$$

For G_2, according to (A4) and (2.2), we obtain that

$$\begin{aligned} G_2 &\le \int_{B_{2R}}|h|^\alpha\left(g(x)+g(x+h)\right)\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-1}{2}}|\Delta_h\mathfrak{X}u|\eta^2\mathrm{d}x \\ &\le \varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\eta^2\mathrm{d}x \\ &\quad+C|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x \\ &\le \varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\eta^2\mathrm{d}x \\ &\quad+C|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x. \end{aligned} \tag{3.7}$$

where $\varepsilon>0$ will be chosen later. By (A2) and (2.2), one deduces that

$$\begin{aligned} G_3 &\le C\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2+|\mathfrak{X}u(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|\eta|\mathfrak{X}\eta||\Delta_h u|\mathrm{d}x \\ &\le \varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2+|\mathfrak{X}u(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\eta^2\mathrm{d}x \\ &\quad+C\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2+|\mathfrak{X}u(x+h)|^2\right)^{\frac{p-2}{2}}|\mathfrak{X}\eta|^2|\Delta_h u|^2\mathrm{d}x. \end{aligned}$$

By applying Lagrange Mean Value Theorem, we obtain

$$
\begin{aligned}
&C\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2+|\mathfrak{X}u(x+h)|^2\right)^{\frac{p-2}{2}}|\mathfrak{X}\eta|^2|\Delta_h u|^2\,\mathrm{d}x\\
&\le C|h|^2\int_{B_{2R+|h|}}\left(\mu^2+2|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\mathfrak{X}u|^2\,\mathrm{d}x\\
&\le C|h|^2\int_{B_{2R+|h|}}\left(\mu+|\mathfrak{X}u|\right)^p\,\mathrm{d}x.
\end{aligned}\tag{3.8}
$$

To estimate G_4, the hypothesis (A4) and (2.2) give us that

$$
\begin{aligned}
G_4&\le C\int_{B_{2R}}|h|^{\alpha}\left(g(x)+g(x+h)\right)\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-1}{2}}\eta|\mathfrak{X}\eta||\Delta_h u|\mathrm{d}x\\
&\le\varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}\eta^2|\Delta_h u|^2\,\mathrm{d}x\\
&\quad+C|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x.
\end{aligned}\tag{3.9}
$$

Here we notice that

$$
\varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}\eta^2|\Delta_h u|^2\,\mathrm{d}x\le C|h|^2\int_{B_{2R+|h|}}\left(\mu+|\mathfrak{X}u|\right)^p\,\mathrm{d}x.
$$

Combining the estimates of G_i and choosing ε small enough, we obtain that

$$
\begin{aligned}
&\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\eta^2\mathrm{d}x\\
&\le C|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\\
&\quad+C|h|^2\int_{B_{2R+|h|}}\left(\mu+|\mathfrak{X}u|\right)^p\,\mathrm{d}x.
\end{aligned}\tag{3.10}
$$

By the definition of V (1.3) and the property (3.4), one gets

$$
\begin{aligned}
|\Delta_h V(\mathfrak{X}u)|^2&=\left|\left(\mu^2+|\mathfrak{X}u(x+h)|^2\right)^{\frac{p-2}{4}}\mathfrak{X}u(x+h)-\left(\mu^2+|\mathfrak{X}u(x)|^2\right)^{\frac{p-2}{4}}\mathfrak{X}u(x)\right|^2\\
&\le C\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2.
\end{aligned}\tag{3.11}
$$

We integrate both sides of (3.11) on $B_{\frac{R}{2}}$ and apply the properties of η to get

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}}|\Delta_h V(\mathfrak{X}u)|^2\,\mathrm{d}x&\le C\int_{B_{\frac{R}{2}}}\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\eta^2\mathrm{d}x\\
&\le C|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\\
&\quad+C|h|^2\int_{B_{2R+|h|}}\left(\mu+|\mathfrak{X}u|\right)^p\,\mathrm{d}x
\end{aligned}\tag{3.12}
$$

Dividing both sides of (3.12) by $|h|^{2\alpha}$, it follows that

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V(\mathfrak{X}u)}{|h|^{\alpha}}\right|^2\mathrm{d}x&\le C\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\\
&\quad+C|h|^{2-2\alpha}\int_{B_{2R+|h|}}\left(\mu+|\mathfrak{X}u|\right)^p\,\mathrm{d}x\\
&=:P_1+P_2.
\end{aligned}\tag{3.13}
$$

Finally, we shall give the proof that P_i is bounded for each *i*. In view of Lemma 2.1, ones get $|\mathfrak{X}u|^p \in L^t(\Omega)$ with $t>1$. In particular, $|\mathfrak{X}u|^p \in L^{\frac{Q}{Q-2\alpha}}(\Omega)$. By choosing $0<|h|<\delta<R$ and (A4), we acquire

$$P_1 \le C\left(\int_{B_{2R}}\left(g(x)+g(x+h)\right)^{\frac{Q}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{Q}}\left(\int_{B_{2R}}\left[\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right]^{\frac{Q}{Q-2\alpha}}\mathrm{d}x\right)^{\frac{Q-2\alpha}{Q}}$$

$$\le C\left(\int_{B_{2R+|h|}} g(x)^{\frac{Q}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{Q}}\left(\int_{B_{2R}}\left[\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right]^{\frac{Q}{Q-2\alpha}}\mathrm{d}x\right)^{\frac{n-2\alpha}{Q}}<\infty.$$

Because $u\in HW^{1,p}_{\mathrm{loc}}(\Omega)$, we get $P_2<\infty$. It follows that

$$\sup_{|h|<\delta}\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V(\mathfrak{X}u)}{|h|^\alpha}\right|^2\mathrm{d}x<\infty$$ with $\delta<R$, that is, $V(\mathfrak{X}u)\in B^{\alpha}_{2,\infty}(\Omega)$ locally.

4. Proofs of Theorem 1.2

For the non-homogeneous case, we need the following lemma.

Lemma 4.1 *Let* $A:\Omega\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ *be a Carathéodory vector field such that* (A1)-(A3) *and* (A5) *hold. Then A is locally uniformly in VMO, that is,*

$$\lim_{R\to 0}\sup_{r(B)<R}\sup_{c(B)\in K}⨍_B V(x,B)\mathrm{d}x=0, \tag{4.1}$$

where $V(x,B)$ is given in (3.2), $K\subset\Omega$, $c(B)$ and $r(B)$ denote the center and the radius of the CC-ball *B*, respectively.

Proof. Given a point $x\in\Omega$, we let $A_k(x)=\left\{y\in\Omega:2^{-k}\le \mathrm{dist}_{CC}(x,y)<2^{-k+1}\right\}$. Ones get

$$⨍_B V(x,B)\mathrm{d}x \le ⨍_B \sup_{\xi\in\mathbb{R}^{2n}}⨍_B\frac{|A(x,\xi)-A(y,\xi)|}{\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}}\mathrm{d}y\mathrm{d}x$$

$$=⨍_B \sup_{\xi\in\mathbb{R}^{2n}}\frac{1}{|B|}\sum_k\int_{B\cap A_k(x)}\frac{|A(x,\xi)-A(y,\xi)|}{\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}}\mathrm{d}y\mathrm{d}x$$

$$\le\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}\mathrm{dist}_{CC}(x,y)^\alpha\left(g_k(x)+g_k(y)\right)\mathrm{d}y\mathrm{d}x$$

$$\le\left(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}\mathrm{dist}_{CC}(x,y)^{\frac{Q\alpha}{Q-\alpha}}\mathrm{d}y\mathrm{d}x\right)^{\frac{Q-\alpha}{Q}}$$

$$\times\left(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}\left(g_k(x)+g_k(y)\right)^{\frac{Q}{\alpha}}\mathrm{d}y\mathrm{d}x\right)^{\frac{\alpha}{Q}}$$

$$\le C(Q,\alpha)|B|^{\frac{\alpha}{Q}}\left(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}\left(g_k(x)+g_k(y)\right)^{\frac{Q}{\alpha}}\mathrm{d}y\mathrm{d}x\right)^{\frac{\alpha}{Q}}.$$

By Hölder inequality, we acquire that

$$
\begin{aligned}
&\left(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}\left(g_k(x)+g_k(y)\right)^{\frac{Q}{\alpha}}\mathrm{d}y\mathrm{d}x\right)^{\frac{\alpha}{Q}}\\
&\le C\left(\frac{1}{|B|^2}\sum_k\left|B\cap A_k(x)\right|\int_B g_k(x)^{\frac{Q}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{Q}}\\
&\le\frac{C}{|B|^{\frac{2}{q}}}\left(\sum_k\|g_k\|^q_{L^{\frac{Q}{\alpha}}(B)}\right)^{\frac{1}{q}}\frac{1}{|B|^{2\left(\frac{\alpha}{Q}-\frac{1}{q}\right)}}\left(\sum_k\left|B\cap A_k(x)\right|^{\frac{\alpha q}{\alpha q-Q}}\right)^{\frac{\alpha}{Q}\cdot\frac{\alpha q-Q}{\alpha Q}}\\
&\le C(Q,\alpha,q)|B|^{-\frac{\alpha}{Q}}\left(\sum_k\|g_k\|^q_{L^{\frac{Q}{\alpha}}(B)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

We choose $r>0$ small enough and observe that $x\to\|g_k\|_{l^q\left(L^{\frac{Q}{\alpha}}(B_r(x))\right)}$ is continuous on the set $\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>r\}$. Therefore, there is a point $x_r\in K$ for $r>0$ small enough such that

$$
\sup_{x\in K}\|g_k\|_{l^q\left(L^{\frac{Q}{\alpha}}(B_r(x))\right)}=\|g_k\|_{l^q\left(L^{\frac{Q}{\alpha}}(B_r(x))\right)}.
$$

We obtain that

$$
\lim_{r\to0}\|g_k\|_{l^q\left(L^{\frac{Q}{\alpha}}(B_r(x))\right)}=\left(\sum_k\lim_{r\to0}\left(\int_{B_r(x_r)}g_k^{\frac{Q}{\alpha}}\right)^{\frac{q\alpha}{Q}}\right)^{\frac{1}{q}}.
$$

Each of the limits on the right hand side equals to 0. Hence we complete the proof. □

With the help of preceding lemma, we have the following result.

Proof of Theorem 1.2. We assume that $B_{3R+1}\Subset\Omega$, and choose a test function $\varphi=\Delta_{-h}\left(\eta^2\Delta_h u\right)$ to (1.2), where $\eta\in C_0^\infty(\Omega)$ is a cut-off function satisfying that

$$
0\le\eta(x)\le1,\ \eta(x)\equiv1\text{ for }x\in B_{\frac{R}{2}},\ \eta(x)\equiv0\text{ for }x\in B_{3R+1}\setminus B_R,\text{ and }|\mathfrak{X}\eta|\le\frac{C}{R}.
$$

According to the definition of weak solution and choice of test function, we obtain

$$
\begin{aligned}
G_1&=\int_{B_{2R}}\left[A\left(x+h,\mathfrak{X}u(x+h)\right)-A\left(x+h,\mathfrak{X}u\right)\right]\cdot\eta^2\Delta_h\mathfrak{X}u\mathrm{d}x\\
&=\int_{B_{2R}}\left[A(x,\mathfrak{X}u)-A(x+h,\mathfrak{X}u)\right]\cdot\eta^2\Delta_h\mathfrak{X}u\mathrm{d}x\\
&\quad+\int_{B_{2R}}\left[A(x+h,\mathfrak{X}u)-A\left(x+h,\mathfrak{X}u(x+h)\right)\right]\cdot2\eta\mathfrak{X}\eta\Delta_h u\mathrm{d}x\\
&\quad+\int_{B_{2R}}\left[A(x,\mathfrak{X}u)-A(x+h,\mathfrak{X}u)\right]\cdot2\eta\mathfrak{X}\eta\Delta_h u\mathrm{d}x\\
&\quad+\int_{B_{2R}}\Delta_h\left[|F|^{p-2}F\right]\cdot2\eta\mathfrak{X}\eta\Delta_h u\mathrm{d}x+\int_{B_{2R}}\Delta_h\left[|F|^{p-2}F\right]\cdot\eta^2\Delta_h\mathfrak{X}u\mathrm{d}x\\
&=G_2+G_3+G_4+G_5+G_6.
\end{aligned}\tag{4.2}
$$

We have estimated the terms G_1 to G_4 in the proof of Theorem 1.1. Thus it remains to estimate G_5 and G_6. We apply (2.2) to get

$$G_5 \le C\int_{B_{2R}} \left|\Delta_h\left[|F|^{p-2}F\right]\right| |\Delta_h u| \eta \mathrm{d}x$$
$$\le C|h|^{2\alpha}\int_{B_{2R}} \left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^{\alpha}}\right|^2 \mathrm{d}x + C\int_{B_{2R}} |\Delta_h u|^2 \eta^2 \mathrm{d}x.$$

By applying the Lagrange Mean Value Theorem, the second term can be controlled by

$$C(\mu)\int_{B_{2R}} |\Delta_h u|^2 \eta^2 \mathrm{d}x \le C\int_{B_{2R}} \frac{\mu^p}{\mu^2} |\Delta_h u|^2 \eta^2 \mathrm{d}x$$
$$\le C|h|^2 \int_{B_{2R+|h|}} \frac{\left[\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{1}{2}}\right]^p}{\left[\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{1}{2}}\right]^2} |\mathfrak{X}u|^2 \mathrm{d}x$$
$$\le C|h|^2 \int_{B_{2R+|h|}} \left(\mu+|\mathfrak{X}u|\right)^p \mathrm{d}x.$$

For the estimate of G_6, it is apparent that

$$G_6 \le \int_{B_{2R}} \left|\Delta_h\left[|F|^{p-2}F\right]\right| |\Delta_h \mathfrak{X}u| \eta^2 \mathrm{d}x$$
$$\le C|h|^{2\alpha}\int_{B_{2R}} \left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^{\alpha}}\right|^2 \mathrm{d}x + \varepsilon\int_{B_{2R}} |\Delta_h \mathfrak{X}u|^2 \eta^2 \mathrm{d}x.$$

Similarly, one obtains that

$$\varepsilon\int_{B_{2R}} |\Delta_h \mathfrak{X}u|^2 \eta^2 \mathrm{d}x \le \frac{\varepsilon}{\mu^{p-2}}\int_{B_{2R}} \mu^{p-2} |\Delta_h \mathfrak{X}u|^2 \eta^2 \mathrm{d}x$$
$$\le \frac{\varepsilon}{\mu^{p-2}}\int_{B_{2R}} \left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \eta^2 \mathrm{d}x.$$

Combining the estimates of G_i, we evidently have

$$\left(\nu-2\varepsilon-\frac{\varepsilon}{\mu^{p-2}}\right)\int_{B_{2R}} \left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \eta^2 \mathrm{d}x$$
$$\le C|h|^{2\alpha}\int_{B_{2R}} \left(g_k(x)+g_k(x+h)\right)^2 \left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}} \mathrm{d}x \tag{4.3}$$
$$+C|h|^2\int_{B_{2R+|h|}} \left(\mu+|\mathfrak{X}u|\right)^p \mathrm{d}x + C|h|^{2\alpha}\int_{B_{2R}} \left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^{\alpha}}\right|^2 \mathrm{d}x.$$

By choosing $\varepsilon = \dfrac{\nu}{4+\dfrac{2}{\mu^{p-2}}}$, we obtain that

$$\int_{B_{2R}} \left(\mu^2+\left|\mathfrak{X}u(x+h)\right|^2+\left|\mathfrak{X}u\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h \mathfrak{X}u\right|^2 \eta^2 \mathrm{d}x$$
$$\le C|h|^{2\alpha}\int_{B_{2R}}\left(g_k(x)+g_k(x+h)\right)^2\left(\mu^2+\left|\mathfrak{X}u\right|^2\right)^{\frac{p}{2}}\mathrm{d}x \tag{4.4}$$
$$+C|h|^2\int_{B_{2R+|h|}}\left(\mu+\left|\mathfrak{X}u\right|\right)^p\mathrm{d}x+C|h|^{2\alpha}\int_{B_{2R}}\left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x.$$

Using (1.3) and (3.4) again, we obtain that

$$\left|\Delta_h V(\mathfrak{X}u)\right|^2\le C\left(\mu^2+\left|\mathfrak{X}u(x+h)\right|^2+\left|\mathfrak{X}u\right|^2\right)^{\frac{p-2}{2}}\left|\Delta_h\mathfrak{X}u\right|^2.$$

Using (4.4), it follows that

$$\int_{B_{\frac{R}{2}}}\left|\Delta_h V(\mathfrak{X}u)\right|^2\mathrm{d}x$$
$$\le C|h|^2\int_{B_{2R+|h|}}\left(\mu+\left|\mathfrak{X}u\right|\right)^p\mathrm{d}x+C|h|^{2\alpha}\int_{B_{2R}}\left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x \tag{4.5}$$
$$+C|h|^{2\alpha}\int_{B_{2R}}\left(g_k(x)+g_k(x+h)\right)^2\left(\mu^2+\left|\mathfrak{X}u\right|^2\right)^{\frac{p}{2}}\mathrm{d}x.$$

Dividing both sides of (4.5) by $|h|^{2\alpha}$ and applying the properties of η, one derives that

$$\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V(\mathfrak{X}u)}{|h|^\alpha}\right|^2\mathrm{d}x\le C|h|^{2-2\alpha}\int_{B_{2R+|h|}}\left(\mu+\left|\mathfrak{X}u\right|\right)^p\mathrm{d}x+C\int_{B_{2R}}\left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x \tag{4.6}$$
$$+C\int_{B_{2R}}\left(g_k(x)+g_k(x+h)\right)^2\left(\mu^2+\left|\mathfrak{X}u\right|^2\right)^{\frac{p}{2}}\mathrm{d}x.$$

By taking the power of $\frac{1}{2}$, one obtains

$$\left(\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V(\mathfrak{X}u)}{|h|^\alpha}\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\le C\left[\int_{B_{2R}}\left(g_k(x)+g_k(x+h)\right)^2\left(\mu^2+\left|\mathfrak{X}u\right|^2\right)^{\frac{p}{2}}\mathrm{d}x\right]^{\frac{1}{2}}$$
$$+C|h|^{1-\alpha}\left(\int_{B_{2R}}B_{2R+|h|}\left(\mu+\left|\mathfrak{X}u\right|\right)^p\mathrm{d}x\right)^{\frac{1}{2}} \tag{4.7}$$
$$+C\left(\int_{B_{2R}}\left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^\alpha}\right|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Restricting to B_δ with $0<|h|<\delta$ and taking the L^q norm with respect to the measure $\frac{\mathrm{d}h}{|h|^Q}$, it follows that

$$\begin{aligned}
&\left(\int_{B_\delta}\left(\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V(\mathfrak{X}u)}{|h|^\alpha}\right|^2 \mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
&\le C\left(\int_{B_\delta}\left(\int_{B_{2R}}\left(g_k(x)+g_k(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
&\quad + C\left(\int_{B_\delta}|h|^{(1-\alpha)q}\left(\int_{B_{2R+|h|}}\left(\mu+|\mathfrak{X}u|\right)^p \mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
&\quad + C\left(\int_{B_\delta}\left(\int_{B_{2R}}\left|\frac{\Delta_h\left[|F|^{p-2}F\right]}{|h|^\alpha}\right|^2 \mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
&=: P_1+P_2+P_3.
\end{aligned} \tag{4.8}$$

We shall show that each P_i $(1\le i\le 3)$ is bounded. Since $B^\alpha_{2,q}(\Omega)\subset L^{\frac{2Q}{Q-2\alpha}}(\Omega)$ with $1\le q<\frac{2Q}{Q-2\alpha}$, one has $|F|^{p-2}F\in L^{\frac{2Q}{Q-2\alpha}}(\Omega)$. By Lemma 2.2, we get $|\mathfrak{X}u|^{p-2}\mathfrak{X}u\in L^{\frac{2Q}{Q-2\alpha}}(\Omega)$. That is, $\mathfrak{X}u\in L^{\frac{2Q(p-1)}{Q-2\alpha}}(\Omega)$. Since

$$\frac{2Q(p-1)}{Q-2\alpha}\ge\frac{Qp}{Q-2\alpha},$$

then we get $|\mathfrak{X}u|^p\in L^{\frac{Q}{Q-2\alpha}}(\Omega)$.

To estimate P_1, we write the L^q norm in polar coordinates. There is no harm in supposing that $\delta=1$, so $h\in B_1\cap\mathbb{R}^{2n}$ is equivalent to $h=r\xi$ for $0\le r<1$ and ξ in the unit sphere $\mathbb{S}^{2n-1}$. Let $\mathrm{d}\sigma(\xi)$ be the surface measure on $\mathbb{S}^{2n-1}$. By letting $r_k=\frac{1}{2^k}$, we estimate P_1 as

$$\begin{aligned}
P_1&=\int_0^1\int_0^1\int_{\mathbb{S}^{2n-1}}\left(\int_{B_{2R}}\left(g_k(x+r\xi)+g_k(x)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{q}{2}}\mathrm{d}\sigma(\xi)\frac{\mathrm{d}r}{r}\mathrm{d}t\\
&=\int_0^1\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\mathbb{S}^{2n-1}}\left(\int_{B_{2R}}\left(g_k(x+r\xi)+g_k(x)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{q}{2}}\mathrm{d}\sigma(\xi)\frac{\mathrm{d}r}{r}\mathrm{d}t\\
&\le\int_0^1\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\mathbb{S}^{2n-1}}\left\|\left(\tau_{r\xi}g_k+g_k\right)\left(\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{2}}\right\|^q_{L^2(B_{2R})}\mathrm{d}\sigma(\xi)\frac{\mathrm{d}r}{r}\mathrm{d}t.
\end{aligned}$$

We note that $\tau_{r\xi}g_k(x)=g_k(x+r\xi)$. Since $|\mathfrak{X}u|^p\in L^{\frac{Q}{Q-2\alpha}}(\Omega)$ and

$g_k \in L^{\frac{Q}{\alpha}}(\Omega)$, one gets that

$$\left\| \left(\tau_{r\xi} g_k + g_k\right)\left(\left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{2}} \right\|_{L^2(B_{2R})}$$

$$\leq \left(\left[\int_{B_{2R}} \left(\tau_{r\xi} g_k + g_k\right)^{2\cdot\frac{Q}{2\alpha}} \mathrm{d}x \right]^{\frac{2\alpha}{Q}} \left[\int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}\cdot\frac{Q}{Q-2\alpha}} \mathrm{d}x \right]^{\frac{Q-2\alpha}{Q}} \right)^{\frac{1}{2}}$$

$$= \left\| \tau_{r\xi} g_k + g_k \right\|_{L^{\frac{Q}{\alpha}}(B_{2R})} \left\| \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \right\|^{\frac{1}{2}}_{L^{\frac{Q}{Q-2\alpha}}(B_{2R})}.$$

On the other hand, there holds

$$\left\| \tau_{r\xi} g_k + g_k \right\|_{L^{\frac{Q}{\alpha}}(B_{2R})} \leq \left\| g_k \right\|_{L^{\frac{Q}{\alpha}}((B_{2R})-r_k\xi)} + \left\| g_k \right\|_{L^{\frac{Q}{\alpha}}(B_{2R})} \leq 2\left\| g_k \right\|_{L^{\frac{Q}{\alpha}}(\varrho B_R)}$$

for each $\xi \in \mathbb{S}^{2n-1}$ and $r_{k+1} \leq r \leq r_k$, where $\varrho = 3 + \frac{1}{R}$. Therefore one gets

$$P_1 \leq C \left\| \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \right\|^{\frac{1}{2}}_{L^{\frac{Q}{Q-2\alpha}}(B_{2R})} \left\| \{g_k\}_k \right\|_{l^q\left(L^{\frac{Q}{\alpha}}(\varrho B_R)\right)} < \infty.$$

In the Heisenberg group, a direct calculation gives us that

$$\int_{B_\delta \cap \mathbb{H}^n} |h|^{(1-\alpha)q-Q} \mathrm{d}x = \int_{B_\delta \cap \mathbb{R}^{2n}} \left[\int_{B_\delta \cap \mathbb{R}} \left(|z|^2 + t\right)^{\frac{(1-\alpha)q-Q}{2}} \mathrm{d}t \right] \mathrm{d}z$$

$$= C(\alpha, q, Q) \int_{B_\delta \cap \mathbb{R}^{2n}} \left(|z|^2 + \delta^2\right)^{\frac{(1-\alpha)q-(2n+2)}{2}+1} \mathrm{d}z$$

$$= C(\alpha, q, Q)\, \omega_{2n-1} \int_0^\delta \left(\rho^2 + \delta^2\right)^{\frac{(1-\alpha)q-2n}{2}} \rho^{2n-1} \mathrm{d}\rho$$

$$\leq C(\alpha, q, Q)\, \omega_{2n-1} \int_0^\delta \rho^{(1-\alpha)q-1} \mathrm{d}\rho < \infty.$$

According to the fact that $u \in HW^{1,p}(\Omega)$, we deduce that

$$P_2 \leq C \left(\int_{B_\delta} |h|^{(1-\alpha)q-Q} \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{B_{2R+|h|}} \left(\mu + |\mathfrak{X}u|\right)^p \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_0^\delta \rho^{(1-\alpha)q-1} \mathrm{d}\rho \right)^{\frac{1}{q}} \left(\int_{B_{2R+\delta}} \left(\mu + |\mathfrak{X}u|\right)^p \mathrm{d}x \right)^{\frac{1}{2}} < \infty.$$

Because $|F|^{p-2} F \in B^{\alpha}_{2,q}(\Omega)$, it follows that

$$P_3 = C \left\| \frac{\Delta_h \left[|F|^{p-2} F \right]}{|h|^\alpha} \right\|_{L^q\left(\frac{\mathrm{d}h}{|h|^Q}; L^2(B_{2R})\right)} < \infty.$$

Therefore, we complete the proof of Theorem 1.2.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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