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Lychrel Numbers in Base 10: A Probabilistic Approach

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Abstract

For decades, Lychrel numbers have been studied on many bases. Their existence has been proven in base 2, 11 or 17. This paper presents a probabilistic proof of the existence of Lychrel number in base 10 and provides some properties which enable a mathematical extraction of new Lychrel numbers from existing ones. This probabilistic approach has the advantage of being extendable to other bases. The results show that palindromes can also be Lychrel numbers.

Keywords

Probabilistic Approach, Palindromes, Lychrel Numbers, Iteration Function, Digits

1. Introduction

On the set of natural numbers, the **reverse number** of a natural number $a = a_1 \cdots a_{l(a)}$ (with $l(a)$ the number of digits of *a*), denoted by $Rev(a)$, is the natural number such that $Rev(a) = a_{l(a)} a_{l(a)-1} \cdots a_1$. When *a* is equal to its reverse number ($a = Rev(a)$), we say that *a* is a **palindrome**. It is the case with 11, 22, 33, 44, 55, 66, 77, 121 or 985,589. If a natural number *a* is not a palindrome, one can eventually obtain a palindrome after the iterative process of reversion and addition with its reverse number. The example with $a = 145$ yields $145 + Rev(145) = 145 + 541 = 686$ which is a palindrome. With $a = 159$, we have $159 + Rev(159) = 159 + 951 = 1110$, and $1110 + Rev(1110) = 1110 + 0111 = 1221$. Hence, 159 leads to a palindrome after two iterations. The natural number 89 leads to a palindrome after 24 iterations; 903,282,208 produces a palindrome after 33 iterative reversions and additions. With the same process, a palindrome is obtained after 45 iterative reversions and additions of 90,914,509. However, there are natural numbers that do not seem to form a palindrome after the computation of the iterative process of reversing and adding their reverse numbers. These numbers are usually called **Lychrel Numbers**. A more detailed definition of Palindromes and Lychrel numbers can be found in [1].

They are often used for the encryption and decryption of messages in cryptography, especially in cryptography on elliptic curves ([2]), and for the search of palindromes which are important in biotechnology, notably in DNA and RNA sequences, as they help to stabilize these types of molecules ([3]).

To the best of our knowledge, the problem of Lychrel numbers was first raised in [4] when the author mentioned the number of natural numbers for which no palindrome is reached after 100 iterations. Since then, the existence of Lychrel numbers has been proved in many bases, among which the bases 2, 11, 17 or 26, and presented in [1].

In base 10, some works have been done to verify whether 196 is a Lychrel number: in 1987, the author from [5] presented an algorithm to verify that 196 is a Lychrel number, and this algorithm ran for many years and stopped after 2416 million iterations, without reaching a palindromic number. In 1995, the author from [6] obtained a non-palindromic number with 2 million digits with a super computer. Jason Doucette took over the research in the years 2000 and reached a number with 12.5 million digits, which is still not a palindrome. In 2006, Wade VanLandingham continued with the computation and reached a non-palindromic number with more than 300 million digits. Romain Dolbeau took over the work and after more than 2.4 billion iterations, he obtained in 2015 a non-palindromic number with a billion digits. The summary of works done on 196 and other Lychrel candidates can be found in [7] and [5]. In 2017, the authors from [8] established that the existence of Lychrel numbers in base 10 is due to the symmetry breaking in the space of the natural numbers. However, their demonstration remains a global one because they were not able to provide proof for a particular natural number.

The probabilistic approach has been used in many domains, notably in the study of the reliability of cementless hip prostheses in the presence of mechanical uncertainties and its application to the investigation of the influence of bone-implant interface properties ([9]). Also, the implementation of non-recursive base conversion is presented in [10] with a probabilistic approach, notably the deterministic Markov process.

In this work, we use a simple probability calculation.

The rest of the paper is organized as follows: in section 2 we present some tools which enable the explanation of how to obtain the length and the expression of an iteration. In section 3 we discuss some blocs of consecutive iterations and present some properties of the iteration function which enable a probabilistic characterization of Lychrel numbers. Section 4 presents some properties for extracting new Lychrel numbers from others. Section 5 addresses the notion of Palindrome-Lychrel numbers and section 6 concludes the paper.

2. Length and Expression of an Iteration

In this section, we define the notion of iteration and provide some tools that help to obtain the length and the expression of an iteration.

Let $a = a_1 \cdots a_{l(a)}$ be a natural number, where $l(a)$ denotes the number of digits of *a* and a_i a digit of *a* for all $i \in \{1, \cdots, l(a)\}$. An **iteration** of *a* is the operation that consists of adding *a* to $Rev(a) = a_{l(a)} \cdots a_1$. To express it conveniently, we define the **unity function** $\mathcal{U}$ which corresponds to its last digit, as follows:

$$\mathcal{U} : N \to N$$
$$a \mapsto a_{l(a)}$$

Some examples in base 10 are contained in Table 1 below.

Table 1. The unity of some numbers.

a	0	3	25	1268	1000	7960	1997	40,961	20,000	178,997
$l(a)$	1	1	2	4	4	4	4	5	5	6
$\mathcal{U}(a)$	0	3	5	8	0	0	7	1	0	7

The unity function $\mathcal{U}$ verifies some elementary properties:

Lemma 1. Let $a \neq 0$ be a digit of a natural number, and $\rho \in \{0,1\}$.

1) For all $x, y \in N$, $\mathcal{U}(x+y) = \mathcal{U}(\mathcal{U}(x) + \mathcal{U}(y))$;

2) For all $x \in N$, if $\mathcal{U}(x) \in \{0, \cdots, 8\}$ then $\mathcal{U}(x+\rho) = \mathcal{U}(x) + \rho$;

3) For all $i \in \{1, \cdots, 10\}$, if $a + i \geq 10$, then $\mathcal{U}(a+i) = a - 10 + i$.

Proof. The properties are straightforward. □

Hence, if $a \neq 0$ is a digit of a natural number, then $\mathcal{U}(a+10) = a$ and $\mathcal{U}(a+10-1) = a-1$.

Also, we consider the **iteration function** ϕ, which is actually the one defined in [8], as follows:

$$\phi : N \to N$$
$$a \mapsto a + Rev(a)$$

Then, the first iteration of *a* will be denoted by $\phi_1(a) = \phi(a)$, the second iteration will be $\phi_2(a) = \phi(\phi_1(a)) = \phi(\phi(a))$ and for all $n \in N$, $\phi_n(a) = \phi(\cdots(\phi(a)))$ (*n* factors). For example, in base 10, if $a = 75$, then $\phi_2(75) = \phi(\phi(75)) = \phi(75) + Rev(\phi(75)) = 75 + Rev(75) + Rev(75 + Rev(75))$ $= 75 + 57 + Rev(75+57) = 132 + Rev(132) = 132 + 231 = 363$.

Now let's consider $\phi(a) = a + Rev(a)$. Then the n^{th} iteration of *a* can be expressed as a function of *a* as $\phi_n(a) = a + \sum_{i=1}^{n-1} Rev(\phi_i(a))$. As for the first iteration, its length is given by $l(\phi(a)) = l(a) + \alpha$, where $\alpha \in \{0,1\}$. $\alpha = 1$ if and only if $a_1 + a_{l(a)} \geq 10$ or there exists $i \in \{1, \cdots, l(a)\}$ such that $a_i + a_{l(a)-i+1} \geq 10$ and $a_j + a_{l(a)-j+1} = 9$ for all $j \leq i-1$. For example, in base 10, if $a = 4006$, then we have $l(a) = 4$ and $a_1 + a_{l(a)} = 4 + 6 = 10 \geq 10$. Hence, $\alpha = 1$ and

$l(\phi(a)) = l(a)+1 = 4+1 = 5$. In the second example, we consider the case where $a = 456745$. Then for $j = 3$, $a_3 + a_4 = 6+7 = 13 \geq 10$ and $a_j + a_{7-j} = 9$ for all $j \leq 2$. Hence, $\alpha = 1$ and then $l(\phi(a)) = l(a)+1 = 6+1 = 7$.

In order to successfully add *a* to its reverse number to obtain the first iteration of *a*, we define the vector parameter $\epsilon(a)$ (or $\epsilon^1(a)$) which depends both on the length and the digits of *a*, as follows:

$$\epsilon_j(a) = \begin{cases} 0 & \text{if } j = l(a)+\alpha \\ 1 & \text{if } j < l(a)+\alpha \text{ and } a_j + a_{l(a)-j+1} + \epsilon_{j+1}(a) \geq 10 \\ 0 & \text{else} \end{cases}$$

with $\epsilon_j(a)$ being calculated in the decreasing order.

Then the following are the expressions, depending on *a*, of the first iteration of *a*.

If $\alpha = 1$, then:

- $l(\phi(a)) = l(a)+1$,
- $\phi(a)_1 = 1$,
- $\phi(a)_{l(\phi(a))} = \mathcal{U}(a_{l(a)} + a_1)$,
- For every $j \in \{2,\cdots,l(a)\}$, $\phi(a)_j = \mathcal{U}(a_{j-1} + a_{l(a)-j+2} + \epsilon_j(a))$.

If $\alpha = 0$, then:

a) $l(\phi(a)) = l(a)$,

b) $\phi(a)_1 = \mathcal{U}(a_1 + a_{l(a)} + \epsilon_1(a))$,

c) $\phi(a)_{l(\phi(a))} = U(a_{l(a)} + a_1)$,

d) For every $j \in \{2,\cdots,l(a)\}$, $\phi(a)_j = \mathcal{U}(a_j + a_{l(a)-j+1} + \epsilon_j(a))$.

For example, we consider the case where $a = 974$, we have $l(a) = 3$, $a_1 + a_{l(a)} = a_1 + a_3 = 9+4 = 13 \geq 10$. Therefore $l(\phi(a)) = l(a)+1 = 3+1 = 4$. We then determine the epsilons in the decreasing order as follows: $\epsilon_4(a) = 0$, $\epsilon_3(a) = 1$ because $a_3 + a_1 \geq 10$, $\epsilon_2(a) = 1$ because $a_2 + a_2 + \epsilon_3(a) \geq 10$ and $\epsilon_1(a) = 1$ because $a_1 + a_3 + \epsilon_2(a) \geq 10$. Hence, we can determine the digits of $\phi(a)$ as follows:

1) $\phi(a)_1 = 1$,

2) $\phi(a)_{l(\phi(a))} = \phi(a)_4 = \mathcal{U}(a_{l(a)} + a_1) = \mathcal{U}(a_3 + a_1) = \mathcal{U}(4+9) = \mathcal{U}(13) = 3$,

3) $$\begin{aligned}\phi(a)_3 &= U(a_2 + a_{l(a)-1} + \epsilon_3(a)) = \mathcal{U}(7+7+\epsilon_3(a)) = \mathcal{U}(14+\epsilon_3(a)) \\ &= \mathcal{U}(14+1) = \mathcal{U}(15) = 5\end{aligned},$$

4) $$\begin{aligned}\phi(a)_2 &= \mathcal{U}(a_1 + a_{l(a)} + \epsilon_2(a)) = \mathcal{U}(9+4+\epsilon_2(a)) = \mathcal{U}(13+\epsilon_2(a)) \\ &= \mathcal{U}(13+1) = U(14) = 4\end{aligned}.$$

Thus, $\phi(a) = \phi(a)_1\phi(a)_2\phi(a)_3\phi(a)_4 = 1453$.

In the more general way, if *a* and *m* are two natural numbers with $l(a)$ and $l(m)$ their respective lengths. Let's say that our aim is to make the addition of the two numbers. Then the length of $a+m$ is given by $\max(l(a),l(m))+\alpha$ (with $\alpha \in \{0,1\}$). $\alpha = 1$ if and only if $a_1 + b_1 \geq 10$ or there exists $i \in \{1,\cdots,\max(l(a),l(m))\}$ such that $a_i + m_i \geq 10$ and $a_j + m_j = 9$ for all $j \leq$

$i-1$. For example, if $a=900$ and $m=105$, then we have $l(a)=l(m)=3$ and $a_1+m_1=9+1=10$. Hence, $\alpha=1$ and $l(a+m)=\max(l(a),l(m))+1=l(a)+1=3+1=4$. The second example presents the case where $a=4545$ and $m=5456$, then for $j=4$, $a_4+m_4=11\geq 10$ and $a_j+m_j=9$ for all $j\leq 3$. Hence, $\alpha=1$ and $l(a+m)=\max(l(a),l(m))+1=l(a)+1=4+1=5$.

In the above examples, $l(a)=l(m)$. If $l(a)\neq l(m)$, then we make the following adjustments: if $l(m)<l(a)$, then 0 is added $l(a)-l(m)$ times to m and before the digit m_1. For example, if $a=105$ and $m=9$, then $l(a)-l(m)=2$. Hence 0 is added 2 times to m and before 9. m then becomes $d=009$ with a greater length $l(d)=3=l(a)$ and $a+m=a+d$. The same situation occurs with a if $l(a)<l(m)$.

We then define the vector parameter $\sigma(a,m)$ which depends both on the digits of a and m as follows:

$$\sigma_j(a,m)=\begin{cases}0 & \text{if } j=\max(l(a),l(m))+\alpha\\ 1 & \text{if } j<\max(l(a),l(m))+\alpha \text{ and } a_j+m_j+\sigma_{j+1}(a,m)\geq 10\\ 0 & \text{else}\end{cases}$$

The $\sigma_j(a,m)$ are also calculated in a decreasing order. The digits of $a+m$ are then obtained as follows:

If $\alpha=1$, then:

- $l(a+m)=l(a+d)=l(d)+1$,
- $(a+m)_1=(a+d)_1=1$,
- $(a+m)_{l(a+d)}=(a+d)_{l(a+d)}=\mathcal{U}(a_{l(a)}+d_{l(d)})$,
- For every $j\in\{2,\cdots,l(d)+1\}$,
  $(a+m)_j=(a+d)_j=\mathcal{U}(a_{j-1}+d_{j-1}+\sigma_j(a,m))$.

If $\alpha=0$, then:

1) $l(a+m)=l(a+d)=l(d)$,
2) $(a+m)_1=\mathcal{U}(a_1+d_1+\sigma_1(a,m))$,
3) $(a+m)_{l(a+m)}=(a+d)_{l(a+d)}=\mathcal{U}(a_{l(a)}+d_{l(d)})$,
4) For every $j\in\{2,\cdots,l(d)\}$, $(a+m)_j=(a+d)_j=\mathcal{U}(a_j+d_j+\sigma_j(a,m))$.

3. A Probabilistic Characterization of Lychrel Numbers

Let $a=a_1a_2\cdots a_{l(a)}$ be a natural number. If a is a palindrome, then the couples of digits $(a_i,a_{l(a)-i+1})$ of a most be chosen among the set of couples $\{(0,0),(1,1),\cdots,(9,9)\}$, with $(a_1,a_{l(a)})\neq(0,0)$. Therefore the probability for a to be a palindrome is given by

$$p(a)=\begin{cases}\dfrac{C_9^1 10^{\frac{l(a)}{2}-1}}{C_{90}^1 100^{\frac{l(a)}{2}-1}}=\left(\dfrac{1}{10}\right)^{\frac{l(a)}{2}} & \text{if } l(a) \text{ is even}\\[2ex] \dfrac{C_9^1 C_{10}^1 10^{\frac{l(a)-1}{2}-1}}{C_{90}^1 C_{10}^1 100^{\frac{l(a)-1}{2}-1}}=\left(\dfrac{1}{10}\right)^{\frac{l(a)-1}{2}} & \text{else}\end{cases}$$

This probability then depends on the length $l(a)$ of *a*. The more $l(a)$ increases, the more this probability decreases.

If *a* is not a palindrome, then a palindrome can eventually be reached after several consecutive iterations of *a*. We suppose that 5 consecutive iterations are computed, then several situations can be noticed:

- The length of *a* cannot increase by 0 or by 5;
- The length of *a* can increase by 1. It is the case with most natural numbers such that $a_1 = 1$ and $a_{l(a)} = 0$;
- The length of *a* can increase by 3. It is the case with most natural numbers *a* verifying $\left(a_1, a_{l(a)}\right) \subseteq \{(9,8),(8,8)\}$ and $a_2 + a_{l(a)-1} \neq 0$, or such that $a_1 = a_{l(a)} = 9$ and $a_2 + a_{l(a)-1} = 3$;
- The length of *a* can increase by 4, like in most of the cases where $a_1 = a_{l(a)} = 9$ and $a_2 = a_{l(a)-1} = 0$;
- The length of *a* can increase by 2. It happens with most natural numbers, especially when $\left\{a_1, a_{l(a)}\right\} \subseteq \{8,7\}$.

3.1. Analyzing Some Particular Blocks of Consecutive Iterations

For the *n* first iterations of *a* ($n \in N$ and $n \geq 5$), its numerical sequence can be decomposed in several blocks of 5 consecutive iterations. For each block of iterations, we name the **starting point** or **origin** of the block, the element (out of the block) that generates the 5 iterations; and the **destination**, the last iteration of the block. Two blocks of consecutive iterations are said to be **consecutive** when the starting point of one of them is the destination of the other. For example, if $n = 13$, we have the numerical sequence *a*, $\phi_1(a), \cdots, \phi_{13}(a)$. The first block of 5 consecutive iterations is $\phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a), \phi_5(a)$ and its **starting point** or **origin** is *a* and its **destination** is $\phi_5(a)$. The second block of 5 consecutive iterations is $\phi_6(a), \phi_7(a), \phi_8(a), \phi_9(a), \phi_{10}(a)$ and its **starting point** or **origin** is $\phi_5(a)$ and its **destination** is $\phi_{10}(a)$. $\phi_{11}(a), \phi_{12}(a), \phi_{13}(a)$ is the residual block with less than 5 iterations. The 2 blocks of 5 consecutive iterations are **consecutive** because $\phi_5(a)$ which is the destination of the first block, is also the starting point of the second block.

Now assume that *n* ($n \geq 5$) iterations are computed from *a*. We split the *n* iterations into blocks of 5 consecutive iterations. We set $A_n(a)$ the set of blocks for which the length of the starting point increases by 4; $B_n(a)$ the set of blocks for which the length of the origin increases by 3; $C_n(a)$ the set of blocks for which the length of the starting point increases by 2 and $D_n(a)$ the set of blocks for which the length of the origin increases by 1. Then we have $B_n(a) = B_{1n}(a) \cup B_{2n}(a) \cup B_{3n}(a)$, where $B_{3n}(a)$ is the set of blocks from $B_n(a)$ such that their destinations generate their next blocks of 5 consecutive iterations also from $B_n(a)$, $B_{2n}(a)$ is the set of blocks from $B_n(a)$ such that their destination produces their next blocks of 5 consecutive iterations from $C_n(a)$ and $B_{1n}(a)$ is the set of blocks from $B_n(a)$ such that their destination generate their next blocks of 5 consecutive iterations from $D_n(a)$.

We find the probability for a natural number *m* to be the starting point of the blocks from $A_n(a)$, $B_{2n}(a)$ and $B_{3n}(a)$. To get there, we consider each natural number $m = m_1 m_2 \cdots m_{l(m)}$ as a set of $\frac{l(m)}{2}$ couples of digits $(m_i, m_{l(m)-i+1})$ if $l(m)$ is even, and as a set of $\frac{l(m)-1}{2}$ couples of digits $(m_i, m_{l(m)-i+1})$ and a middle digit in $\{0,1,\cdots,9\}$ if $l(m)$ is odd. The total number of possible couples of digits is 100 because each digits belongs to $\{0,1,\cdots,9\}$. Moreover, for each number, the first couple $(m_1, m_{l(m)})$ must not belong to the set $\{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9)\}$. Hence, the first couple must be chosen among the total of 90 cases instead of 100. If $l(m)$ is odd, the digit in the middle $m_{\frac{l(m)-1}{2}+1}$ must be chosen among $\{0,1,\cdots,9\}$, leading us to the total number of 10 possible cases. Now we study the different situations.

Case of $A_n(a)$

For *m* to be the origin of a block from $A_n(a)$, it is necessary in the very large majority of cases that its digits verify $m_1 = m_{l(m)} = 9$ and $m_2 = m_{l(m)-1} = 0$. Hence, $\phi(m) = 1\mathcal{U}(8 + \epsilon_1^1(m))\epsilon_2^1(m) B\mathcal{U}(m_3 + m_{l(m)-2})18$ where B is the rest of digits of $\phi(m)$.

It implies that $\phi(m) = 18\epsilon_2^1(m) B\mathcal{U}(m_3 + m_{l(m)-2})18$ because $\epsilon_1^1(m) = 0$.

$$\phi^2(m) = \mathcal{U}(9 + \epsilon_1^2(m))\mathcal{U}(9 + \epsilon_2^2(m))\mathcal{U}(\epsilon_2^1(m) + \mathcal{U}(m_3 + m_{l(m)-2}) + \epsilon_3^2(m))$$
$$C\mathcal{U}(1 + U(m_3 + m_{l(m)-2}) + \epsilon_{l(m)-2}^2(m))99$$

where C is the rest of digits of $\phi^2(m)$. We impose that $\epsilon_1^2(m) = 0$. That is the case if $\epsilon_2^2(m) = 0$, which implies that $\epsilon_2^1(m) + U(m_3 + m_{l(m)-2}) + \epsilon_3^2(m) \le 9$. For that to be possible, it requires at least that $\mathcal{U}(m_3 + m_{l(m)-2}) \le 7$ or $\mathcal{U}(m_3 + m_{l(m)-2}) = 8$ and $(\epsilon_3^2(m), \epsilon_3^2(m)) \in \{(0,1),(1,0)\}$ or $\mathcal{U}(m_3 + a_{l(m)-2}) = 9$ and $(\epsilon_3^2(m), \epsilon_3^2(m)) = (0,0)$. The last two cases are minor cases and depend on others digits $m_4, m_5, \cdots$ of *m*. Hence, for simplification, we consider only the case $\mathcal{U}(m_3 + m_{l(m)-2}) \le 7$. That leads us to $0 \le m_3 + m_{l(m)-2} \le 7$ or $10 \le m_3 + m_{l(m)-2} \le 17$. It gives us 80 possible couples $(m_3, m_{l(m)-2})$. The probability of *m* to be the starting point in a block of iterations from $A_n(a)$ is then given by

$$P(A_n(a))(m) \approx \frac{C_1^1 C_1^1 C_{80}^1 100^{\frac{l(m)}{2}-3}}{C_{90}^1 100^{\frac{l(m)}{2}-1}} = \frac{80}{900000} = \frac{8}{90000} \quad \text{if } l(m) \text{ is even and}$$

$$P(A_n(a))(m) \approx \frac{C_1^1 C_1^1 C_{80}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-4}}{C_{90}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-2}} = \frac{80}{900000} = \frac{8}{90000} \quad \text{if } l(m) \text{ is odd.}$$

Case of $B_{3n}(a)$

For *m* to be the starting point of a block from $B_{3n}(a)$, it is necessary in the very large majority of cases that its digits verify $m_1 = m_{l(m)} = 9$ and $m_2 + m_{l(m)-1} = 3$ and $U(m_3 + m_{l(m)-2}) \ne 0$.

Hence, the length of m increases by 3 after 3 iterations and also by 3 after 5 iterations. Since $m_2 + m_{l(m)-1} = 3$ and $\mathcal{U}\left(m_3 + m_{l(m)-2}\right) \neq 0$, the first and last digits of the 7th iteration will also be equal to 9. Hence, from the 6th to the 10th iterations, the length also increases by 3. Therefore we obtain that for m to be the starting point of the blocks in B_{3n}, it is necessary at least that $\left(m_1, m_{l(m)}\right) = (9,9)$ and $m_2 + m_{l(m)-1} = 3$. The number of possible couples $\left(m_2, m_{l(m)-1}\right)$ such that $m_2 + m_{l(m)-1} = 3$ is 4. The probability of m to be the starting point in a block of iterations from $B_{3n}(a)$ is then given by

$$P\left(B_{3n}(a)\right)(m) \approx \frac{C_1^1 C_4^1 100^{\frac{l(m)}{2}-2}}{C_{90}^1 100^{\frac{l(m)}{2}-1}} = \frac{4}{9000} \text{ if } l(m) \text{ is even and}$$

$$P\left(B_{3n}(a)\right)(m) \approx \frac{C_1^1 C_4^1 C_{10}^1 100^{\frac{l(m)-1}{2}-3}}{C_{90}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-2}} = \frac{4}{9000} \text{ if } l(m) \text{ is odd.}$$

Case of $B_{2n}(a)$

For m to be the starting point of a block of iterations from $B_{2n}(a)$, it requires at least that $\left(m_1, m_{l(m)}\right) \in \{(9,8),(8,8)\}$. Moreover, If $\left(m_1, m_{l(m)}\right) = (9,8)$, then in the very large majority of cases, $\left(m_2, m_{l(m)-1}\right)$ has to verify $m_2 + m_{l(m)-1} \in N \cap \left([3,9] \cup [12,18]\right)$, which gives us a total of 67 possible cases for couples $\left(m_2, m_{l(m)-1}\right)$. If $\left(m_1, m_{l(m)}\right) = (8,8)$, then in the very large majority of cases, $\left(m_2, m_{l(m)-1}\right)$ has to verify $m_2 + m_{l(m)-1} \in N \cap \left([4,9] \cup [13,18]\right)$, which gives us a total of 56 possible cases for couples $\left(m_2, m_{l(m)-1}\right)$. The probability of m to be the starting point of a block of iterations from $B_{2n}(a)$ is then given by

$$P\left(B_{2n}(a)\right)(m) \approx \frac{C_1^1 C_{67}^1 100^{\frac{l(m)}{2}-2}}{C_{90}^1 100^{\frac{l(m)}{2}-1}} + \frac{C_1^1 C_{56}^1 100^{\frac{l(m)}{2}-2}}{C_{90}^1 100^{\frac{l(m)}{2}-1}} = \frac{123}{9000} \text{ if } l(m) \text{ is even and}$$

$$P\left(B_{2n}(a)\right)(m) \approx \frac{C_1^1 C_{67}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-3}}{C_{90}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-2}} + \frac{C_1^1 C_{56}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-3}}{C_{90}^1 C_{10}^1 100^{\frac{l(m)-1}{2}-2}} = \frac{123}{9000} \text{ if } l(m) \text{ is odd.}$$

3.2. Some Properties of the Iteration Function

The blocks of consecutive iterations studied in the above subsection lead us to some properties of the iteration function. These properties relate the progression of the length of iterations to the number of iterations, and present the link with the detection of Lychrel numbers.

Proposition 2. Let $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)} \in N$ be *a* natural number in base 10 such that $l(a) \geq 3$. For all $n \geq 1$, we denote by $\phi_n(a)$ the n^{th} iteration of *a*

$$\lim_{n \to +\infty} \frac{l\left(\phi_n(a)\right)}{n} = \kappa \approx 0.414$$

Proof. Let consider $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)} \in N$. Then after the first iteration, we have $l(\phi(a)) = l(a) + \alpha_1(a)$, where $\alpha_1(a) \in \{0,1\}$ ($\alpha_1(a) = 1$ if the length of a increases and $\alpha_1(a) = 0$ if not). After the second iteration, $l(\phi_2(a)) = l(\phi(a)) + \alpha_2(a)$ (with $\alpha_2(a) \in \{0,1\}$), which implies that $l(\phi_2(a)) = l(a) + \alpha_1(a) + \alpha_2(a)$. Hence, after n iterations, $l(\phi_n(a)) = l(a) + \sum_{i=1}^{n} \alpha_i(a)$, where $\alpha_i(a) \in \{0,1\}$ for all $i \in \{1, \cdots, n\}$.

Notice that after 5 consecutive iterations, we have $1 \le \sum_{i=1}^{5} \alpha_i(a) \le 4$. Hence we split the n iterations into blocks of 5 consecutive iterations. Then we will have $E\left(\frac{n}{5}\right)$ blocks of 5 consecutive iterations, where $E(n)$ stands for the floor of n. Setting

$$A_n(a) = \left\{ k \in \left\{1, \cdots, E\left(\frac{n}{5}\right)\right\} : \sum_{i_k=1}^{5} \alpha_{i_k}(a) = 4 \right\};$$

$$B_n(a) = \left\{ k \in \left\{1, \cdots, E\left(\frac{n}{5}\right)\right\} : \sum_{i_k=1}^{5} \alpha_{i_k}(a) = 3 \right\};$$

$$C_n(a) = \left\{ k \in \left\{1, \cdots, E\left(\frac{n}{5}\right)\right\} : \sum_{i_k=1}^{5} \alpha_{i_k}(a) = 2 \right\};$$

$$D_n(a) = \left\{ k \in \left\{1, \cdots, E\left(\frac{n}{5}\right)\right\} : \sum_{i_k=1}^{5} \alpha_{i_k}(a) = 1 \right\}.$$

Then, we have

$$|A_n(a)| + |B_n(a)| + |C_n(a)| + |D_n(a)| = E\left(\frac{n}{5}\right) \tag{1}$$

Moreover, any block of 5 iterations from $B_n(a)$ produces the next block of 5 iterations either from $B_n(a)$, or from $C_n(a)$ or $D_n(a)$. Then we have $|B_n(a)| = |B_{1n}(a)| + |B_{2n}(a)| + |B_{3n}(a)|$ where $B_{3n}(a)$ is the set of blocks that yield blocks from $B_n(a)$, $B_{2n}(a)$ the set of blocks that yield blocks from $C_n(a)$ and $B_{1n}(a)$ the set of blocks that produce blocks from $D_n(a)$. Also, any block from $A_n(a)$ produces the next block of 5 iterations from $D_n(a)$. Hence,

$$|D_n(a)| = |A_n(a)| + |B_{1n}(a)| \tag{2}$$

moving further, we have

$$\sum_{i=1}^{n} \alpha_i(a) = 4|A_n(a)| + 3|B_n(a)| + 2|C_n(a)| + |D_n(a)| + \beta_n(a), \tag{3}$$

with $\beta_n(a)$ the cardinality of a possible block of less than 5 iterations $\beta_n(a) \le 4$.

From (1), we get $|C_n(a)| = E\left(\frac{n}{5}\right) - |A_n(a)| - |B_n(a)| - |D_n(a)|$. Hence,

$$\begin{aligned} l(\phi_n(a)) &= l(a) + \sum_{i=1}^{n} \alpha_i(a) \\ &= l(a) + 4|A_n(a)| + 3|B_n(a)| + 2|C_n(a)| + |D_n(a)| + \beta_n(a) \\ &= l(a) + 4|A_n(a)| + 3|B_n(a)| + 2\left(E\left(\frac{n}{5}\right) - |A_n(a)| - |B_n(a)| - |D_n(a)| \right) \\ &\quad + |D_n(a)| + \beta_n(a) \end{aligned}$$

$$= 2E\left(\frac{n}{5}\right)+l(a)+2|A_n(a)|+|B_n(a)|-|D_n(a)|+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+2|A_n(a)|+|B_n(a)|-\left(|A_n(a)|+|B_1n(a)|\right)+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_n(a)|+|B_{1n}(a)|+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_{2n}(a)|+|B_{3n}(a)|+\beta_n(a)$$
$$\leq 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_{2n}(a)|+|B_{3n}(a)|+4.$$

In the same way, we have

$$l(\phi_n(a)) = l(a)+\sum_{i=1}^{n}\alpha_i(a)$$
$$= l(a)+4|A_n(a)|+3|B_n(a)|+2|C_n(a)|+|D_n(a)|+\beta_n$$
$$= l(a)+4|A_n(a)|+3|B_n(a)|+2\left(E\left(\frac{n}{5}\right)-|A_n(a)|-|B_n(a)|-|D_n(a)|\right)$$
$$+|D_n(a)|+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+2|A_n(a)|+|B_n(a)|-|D_n(a)|+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+2|A_n(a)|+|B_n(a)|-\left(|A_n(a)|+|B_{1n}(a)|\right)+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_n(a)|-|B_{1n}(a)|+\beta_n(a)$$
$$= 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_{2n}(a)|+|B_{3n}(a)|+\beta_n(a)$$
$$\geq 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_{2n}(a)|+|B_{3n}(a)| \text{ since } \beta_n(a)\geq 0.$$

Hence,

$$2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_{2n}(a)|+|B_{3n}(a)| \leq l(\phi_n(a))$$
$$\leq 2E\left(\frac{n}{5}\right)+l(a)+|A_n(a)|+|B_{2n}(a)|+|B_{3n}(a)|+4$$

Which implies that

$$\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}\leq\frac{l(\phi_n(a))}{n}$$
$$\leq\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}+\frac{4}{n}$$

Hence, from relation (1), the numerical sequences $\frac{|A_n(a)|}{n}$, $\frac{|B_{2n}(a)|}{n}$ and $\frac{|B_{3n}(a)|}{n}$ are convergent. Therefore,

$$\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n} \text{ and}$$

$$\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}+\frac{4}{n}$$ are convergent with the same limit and then from the gendarme theorem, we have

$$\lim_{n\to+\infty}\frac{l(\phi_n(a))}{n}=\frac{2}{5}+\lim_{n\to+\infty}\left(\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}\right)=\kappa$$

We approximate $\lim\frac{|A_n(a)|}{n}$ by the probability of a given natural number m to be the starting point of a block of iterations from $A_n(a)$, which is $\frac{8}{90000}$ We approximate $\lim\frac{|B_{3n}(a)|}{n}$ by the probability of a given natural number m to be the starting point of a block of iterations from $B_{3n}(a)$, which is $\frac{4}{9000}$. Also, $\lim\frac{|B_{2n}(a)|}{n}$ is approximated by the probability of a given natural number m to be the starting point of a block of iterations from $B_{2n}(a)$, which is $\frac{123}{9000}$.

Hence,

$$\kappa=\frac{2}{5}+\lim\frac{|A_n(a)|}{n}+\lim\frac{|B_{2n}(a)|}{n}+\lim\frac{|B_{3n}(a)|}{n}\approx\frac{2}{5}+\frac{8}{90000}+\frac{123}{9000}+\frac{4}{9000}\approx 0.414\,.$$

□

We now provide a link between the above result and Lychrel numbers.

Let a be an element of N such that $l(a)\geq 3$. Then we consider the map defined as follow:

$$\omega_a : N^* \to N$$

$$n \mapsto \begin{cases}\frac{l(\phi_n(a))}{n} & \text{if } \phi_i(a)\neq 2\phi_{i-1}(a) \text{ for all } i\in\{1,2,\cdots,n\}\\ 0 & \text{else}\end{cases}$$

Assume that a is a Lychrel number. Then $\omega_a(n)=\frac{l(\phi_n(a))}{n}$ for all $n\in N^*$. Hence,

$$\lim_{n\to+\infty}\omega_a(n)=\lim_{n\to+\infty}\frac{l(\phi_n(a))}{n}=\kappa\approx 0.414$$

Conversely, suppose that

$$\lim_{n\to+\infty}\omega_a(n)=\kappa\approx 0.414$$

Then for all $\epsilon\in\mathbb{R}^*$, there exists $n_\epsilon\in N$ such that $|\omega_a(n)-\kappa|<\epsilon$ for all

$n > n_\epsilon$. For $0 < \epsilon_0 < \kappa$, there exists $n_{\epsilon_0} \in N$ such that $|\omega_a(n) - \kappa| < \epsilon_0$ for all $n > n_{\epsilon_0}$. Assume there is $n_0 > n_{\epsilon_0}$ such that $\phi_{n_0}(a) = 2\phi_{n_0 - 1}(a)$. Then $n_0 + 1 > n_0 > n_{\epsilon_0}$ and then $\omega_a(n_0 + 1) = 0$, which implies that $\kappa = |0 - \kappa| = |\omega_a(n_0 + 1) - \kappa| < \epsilon_0$, which is absurd because $\epsilon_0 < \kappa$. Hence, $\phi_n(a) \neq 2\phi_{n-1}(a)$ for all $n > n_{\epsilon_0}$. Moreover, $\omega_a(n_{\epsilon_0} + 1) = \dfrac{l(\phi_{n_{\epsilon_0}+1}(a))}{n_{\epsilon_0} + 1}$, which implies that for all $p < n_{\epsilon_0} + 1$, $\phi_p(a) \neq 2\phi_{p-1}(a)$, and then that for all $p \le n_{\epsilon_0}$, $\phi_p(a) \neq 2\phi_{p-1}(a)$. It thus comes that $\phi_n(a) \neq 2\phi_{n-1}(a)$ for all $n \in N^*$.

That leads us to the following result:

Theorem 3. Let *a* be a natural number such that $l(a) \ge 3$. Then the following assertions are equivalent:

1) *a* is a Lychrel number

2) $\lim_{n \to +\infty} \omega_a(n) = \kappa \approx 0.414$

3.3. A Probabilistic Detection of Lychrel Numbers

The last result of the above subsection does not enable us to point a specific natural number as a Lychrel number, since the evaluation of $\omega_a(n)$ any given $a \in N$ remains difficult. In order to solve this particular problem, we propose a probabilistic definition of Lychrel numbers and associate to this result a probability that enables the detection of Lychrel numbers.

Hence, a **natural number *a* will be qualified to be a Lychrel number when its probability $P(a)$ to reach the first palindrome after several iterations is less than $\left(\frac{1}{10}\right)^{330}$**.

The choice of $\left(\frac{1}{10}\right)^{330}$ is explained by the principle that for *a* to be a lychrel number, $P(a)$ should be equal to 0. But computationally, $\left(\frac{1}{10}\right)^{330}$ is equivalent to 0 (in many softwares like Excel, R, Anaconda,…). Another reason is that this probability is small enough to give the assurance that $\omega_a(n)$ is close to $\kappa \approx 0.414$, which gives us sufficient power to consider *a* as a Lychrel number according to theorem 3.

It means that for a natural number *a* to be a Lychrel number, the minimum number of iterations n_0 to be computed must be as enormous as possible for $\omega_a(n_0)$ to be closed to κ, let say $|\omega_a(n_0) - \kappa| < \epsilon_0 = 10^{-5}$.

Having established that

$$\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}} + \frac{l(a)}{n} + \frac{|A_n(a)|}{n} + \frac{|B_{2n}(a)|}{n} + \frac{|B_{3n}(a)|}{n} \le \frac{l(\phi_n(a))}{n}$$

$$\le \frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}} + \frac{l(a)}{n} + \frac{|A_n(a)|}{n} + \frac{|B_{2n}(a)|}{n} + \frac{|B_{3n}(a)|}{n} + \frac{4}{n}$$

for all $n \in N$, let

κ_0 be the limit of $\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}$. Then there exists $r_0 \in N$ such that for all $n > r_0$, $\left|\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}-\kappa_0\right| < \epsilon_0$. It implies that for all $n > r_0$, $\kappa_0-\epsilon_0 < \frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}$. We then conclude that $\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\kappa_0-\epsilon_0 < \frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n}$ for all $n > r_0$.

Also, there exists $r_1 \in N$ such that for all $n > r_1$, $\left|\frac{l(\phi_n(a))}{n}-\kappa\right| < \epsilon_0$, which implies that $\frac{l(\phi_n(a))}{n} < \kappa+\epsilon_0$ for all $n > r_1$.

Setting $r_2 = \max\{r_0, r_1\}$, we have

$$\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\kappa_0-\epsilon_0 < \frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\frac{|A_n(a)|}{n}+\frac{|B_{2n}(a)|}{n}+\frac{|B_{3n}(a)|}{n} \le \frac{l(\phi_n(a))}{n} < \kappa+\epsilon_0$$

For all $n > r_2$, which implies that

$$\frac{2}{5}\frac{E\left(\frac{n}{5}\right)}{\frac{n}{5}}+\frac{l(a)}{n}+\kappa_0-\epsilon_0 < \kappa+\epsilon_0 \text{ for all } n > r_2.$$

Taking into consideration the fact that $E\left(\frac{n}{5}\right) \le \frac{n}{5} \le E\left(\frac{n}{5}\right)+1$ and $\kappa = \kappa_0+\frac{2}{5}$, we conclude that $\frac{l(a)-2}{2*\epsilon_0} < n$.

Hence, $\frac{l(a)-2}{2*\epsilon_0} < n$ and then for $\frac{l(\phi_{n_0}(a))}{n_0}$ to be near κ, at least n_0 iterations has to be produced without reaching a palindrome.

For all $n \in N$, let $P_n(a)$ be the probability of reaching the first palindrome at the n^{th} iteration.

From an iteration $\phi_{i-1}(a)$ ($2 \le i \le n_0+1$) to $\phi_i(a)$, there are two possibilities: $\phi_i(a)$ is a palindrome or $\phi_i(a)$ is not a palindrome. We set A_i the event which makes $\phi_i(a)$ a palindrome and B_i the event for which $\phi_i(a)$ is not a palindrome. Also, we set $q_i(a) = p(A_i)$ the probability for $\phi_i(a)$ to be a palindrome and $p_i(a) = p(B_i) = 1-q_i(a)$ the probability for $\phi_i(a)$ not to be a palindrome.

Then $P_{n_0+1}(a) = p\left(A_{n_0+1} \cap B_1 \cap B_2 \cap B_3 \cap B_4 \cap \cdots \cap B_{n_0}\right) \le p\left(A_{n_0+1}\right)$. Hence, $0 \le P_{n_0+1}(a) \le p\left(A_{n_0+1}\right)$.

However

$$p\left(A_{n_0+1}\right)=\begin{cases}\left(\frac{1}{10}\right)^{\frac{l\left(\phi_{n_0+1}(a)\right)}{2}} & \text{if } l\left(\phi_{n_0+1}(a)\right) \text{ is even}\\ \left(\frac{1}{10}\right)^{\frac{l\left(\phi_{n_0+1}(a)\right)-1}{2}} & \text{else}\end{cases}$$

For, $u=\frac{l(a)-2}{2*\epsilon_0}$, $l\left(\phi_u(a)\right)=E(\kappa*u)\approx E(0.414*u)=E\left(0.414*\frac{l(a)-2}{2*\epsilon_0}\right)$.

Hence,

if $l\left(\phi_u(a)\right)$ is even, then

$$P_u(a)\le\left(\frac{1}{10}\right)^{\frac{E\left(0.414*\frac{l(a)-2}{2*\epsilon_0}\right)}{2}}\le\left(\frac{1}{10}\right)^{\frac{E\left(0.414*\frac{l(a)-2}{2*\epsilon_0}\right)-1}{2}}$$

if $l\left(\phi_u(a)\right)$ is odd, then

$$P_u(a)\le\left(\frac{1}{10}\right)^{\frac{E\left(0.414*\frac{l(a)-2}{2*\epsilon_0}\right)-1}{2}}$$

Therefore,

$$P(a)\le P_{n_0+1}(a)\le P_u(a)\le\left(\frac{1}{10}\right)^{\frac{E\left(0.414*\frac{l(a)-2}{2*\epsilon_0}\right)-1}{2}},$$

We then have the following result:

Theorem 4. Let $a=a_1a_2\cdots a_{l(a)-1}a_{l(a)}\in N$ be a natural number such that $l(a)\ge 3$, n_0 the minimum number of iterations to be computed without the reach of a palindrome and such that $\left|\omega_a(n_0)-\kappa\right|<\epsilon_0$ ($\kappa\approx 0.414$, $0<\epsilon_0<1$), $P(a)$ the probability of reaching de first palindrome and for all $n\in N$, $P_n(a)$ the probability to reach the first palindrome at the n^{th} iteration of *a*.

Then

1) $\frac{l(a)-2}{2*\epsilon_0}<n_0$;

2) $P(a)\le P_{n_0+1}(a)\le P_u(a)\le\left(\frac{1}{10}\right)^{\frac{E(0.414*u)-1}{2}}$, with $u=\frac{l(a)-2}{2*\epsilon_0}$.

Now we test the above properties on 196, 879, 1997 and 7059.

For the test to be carried out successfully, we have written a code in Anaconda Navigator software, which takes a natural number *a*, executes the iterative process of reversing and adding the reverse number, returns the last value W of $\frac{l\left(\phi_n(a)\right)}{n}$, C of $\left|\frac{l\left(\phi_n(a)\right)}{n}-0.414\right|$ and P of $p\left(A_{n+1}\right)$ in both cases where *a* is not a Lychrel number and where *a* is a Lychrel number. The code can be seen in **Figure 1** below:

```
In [127]: i=0
     ...: o=0
     ...: d=0
     ...: P=1
     ...: u=0
     ...: C=0
     ...: e=0
     ...: t=0
     ...: W=0
     ...: z=0
     ...: a=1675
     ...: rev_num_str=str(a)[::-1]
     ...: if type(a)==float:
     ...:     reverse_num=float(rev_num_str)
     ...: elif type(a)==int:
     ...:     reverse_num=int(rev_num_str)
     ...: nu=a+reverse_num
     ...: m=int(str(nu)[0])
     ...: if m==2:
     ...:    d=d+1
     ...: while (o!=nu) and (0.1**330<P):
     ...:     i=i+1
     ...:     nu=nu+o
     ...:     W=len(str(nu))/i
     ...:     t=W-0.414
     ...:     C=abs(t)
     ...:     u=(len(str(a))-2)/(2*C)
     ...:     z=0.414*u
     ...:     e=(z-1)/2
     ...:     P=(0.1)**e
     ...:     m=int(str(nu)[0])
     ...:     if m==2:
     ...:        d=d+1
     ...:     rev_num_str=str(nu)[::-1]
     ...:     if type(nu)==float:
     ...:         reverse_num=float(rev_num_str)
     ...:     elif type(nu)==int:
     ...:         reverse_num=int(rev_num_str)
     ...:     o=reverse_num
     ...: print(C)
     ...: print(W)
     ...: print(P)
```

Figure 1. A programme for testing Lychrel numbers.

The code is globally conceived in a R.A.I (Reverse, Add and Iterate) process, and is therefore described in three mean steps: firstly, a natural input number a is taken and reversed. Secondly, the reverse number obtained (*reverse-num*) is added to the input and the first iteration (nu) is obtained. These two steps are described from line 1 to line 20 of the above code. Thirdly, the process described in the first and the second steps is iterated. At the end, P, C and W are returned. If P is less than $\left(\frac{1}{10}\right)^{330}$, then the number is concluded to be a Lychrel number. The results of the test are presented in the following Table 2.

Table 2. Lychrel number test on some natural numbers.

a	196	879	1997	7059	16,909,736,969,870,700,090,800
$l(a)$	3	3	4	4	23
$\omega_a(n)$	0.41409	0.41429	0.41464	0.41429	0
$\lvert\omega_a(n)-0.414\rvert$	9.69 × 10^{-05}	0.00029	0.00064	0.00029	0.414
$p(A_{n+1})$	0	0	0	0	2.51 × 10^{-28}

The 0s in the last line of the table above represent the values of $p(A_{n+1})$ which are inferior to $\left(\frac{1}{10}\right)^{330}$. Hence, From the result in this table, $p(A_{n+1})$ is less than $\left(\frac{1}{10}\right)^{330}$ for 196, 879, 1997 and 7059. Therefore,

$P(a) < P_{n+1}(a) < P_u(a) < p(A_{n+1}) < \left(\frac{1}{10}\right)^{330}$. We conclude that 196, 879, 1997 and 7059 are Lychrel numbers according to the probabilistic definition and to theorem 4. However 16,909,736,969,870,700,090,800, which comes from [11], is not a Lychrel number since $2.51\times 10^{-28} \geq \left(\frac{1}{10}\right)^{330}$.

4. Extracting New Lychrel Numbers from Others

Having presented a (probabilistic) characterization of Lychrel numbers, we now answer the question of how to extract new Lychrel numbers from others. In this section, we especially extend our results to bases *b* such that ($3 \leq b \leq 10$).

Let *l* be a natural number and N_l the set of all natural numbers of length *l* in base *b* ($3 \leq b \leq 10$). Then we define on N_l the binary relation R_l as follows: $(a,m) \in R_l$ if and only if for all $i \in \{1,\cdots,l\}$, $a_i + a_{l-i+1} = m_i + m_{l-i+1}$. Then R_l is reflexive, symmetric and transitive and is then an equivalence relation on the set N_l. In other words, $(a,m) \in R_l$ if and only if $\phi(a) = \phi(m)$. The following properties help us to search, for a given number *a*, the number *m* verifying $\phi(a) = \phi(m)$ under some conditions.

4.1. Some Properties of Construction of New Lychrel Numbers

The following results hold:

Proposition 5. Let $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)} \in N$ be a natural number in base *b* such that $3 \leq b \leq 10$ and $l(a) \geq 2$.

1) If $1 \leq a_1 \leq b-2$ and $a_{l(a)} > 0$, then $\phi(a) = \phi\left(a + 10^{l(a)-1} - 1\right)$;

2) If $l(a)$ is even, $1 \leq a_{\frac{l(a)}{2}} \leq b-2$ and $a_{\frac{l(a)}{2}+1} > 0$, then $\phi(a) = \phi\left(a + (b-1)\cdot 10^{\frac{l(a)}{2}-1}\right)$;

3) If $l(a)$ is even, $(a_1, a_{l(a)}) \neq (b-1, b-1)$, $a_1 + a_{l(a)} \geq 10$,

$a_{\frac{l(a)}{2}-1}+a_{\frac{l(a)}{2}+2}\ge 10$, $1\le a_{\frac{l(a)}{2}}\le b-2$ and $a_{\frac{l(a)}{2}+1}=0$, then
$\phi_2(a)=\phi_2\left(a+(b-1)\cdot 10^{\frac{l(a)}{2}-1}\right)$;

4) If $l(a)\ge 5$, $l(a)$ is odd, $a_1\ne b-1$, $a_2\ne b-1$ and $a_{l(a)-1}\ne 0$, then $\phi(a)=\phi\left(a+\sum_{k=1}^{l(a)-3}(b-1)\cdot 10^k\right)$.

Proof. Let consider such number $a=a_1a_2\cdots a_{l(a)-1}a_{l(a)}\in N$.

1) We set $x=a+\left(10^{l(a)-1}-1\right)$. Then $l(x)=l(a)$ because $a_1\le b-2$ and $l\left(10^{l(a)-1}-1\right)=l(a)-1$. Moreover, setting $m=10^{l(a)-1}-1$, $l(m)=l(a)-1$. We then consider *d* the number obtained by adding 0 before the first digit of *m*. Then $l(a)=l(d)$ and $a+m=a+d$. Hence, $\sigma_{l(x)}(a,d)=0$ by definition and since in base *b*, all the digits of *d* are equal to $b-1$ except the first one and $a_{l(a)}>0$, we have $\sigma_i(a,d)=1_{\{a_{i+1}+10-1+\sigma_{i+1}(a,d)\ge 10\}}=1$ for all $i\in\{l(a)-1,\cdots,1\}$ (the $\sigma_i(a,d)$ are calculated in a decreasing order). Hence, the digits of *x* are then given by $x_1=\mathcal{U}(a_1+0+\sigma_1(a,d))=\mathcal{U}(a_1+1)=a_1+1$ (since $a_1\le b-2$), $x_{l(x)}=x_{l(a)}=\mathcal{U}\left(a_{l(a)}+10-1\right)=a_{l(a)}-1$ (by lemma 1), and for all $i\in\{2,\cdots,l(a)-1\}$, $x_i=\mathcal{U}(a_i+10-1+\sigma_i(a,d))=\mathcal{U}(a_i+10-1+1)=\mathcal{U}(a_i+10)=a_i$ (also by lemma 1). Hence, whether $l(\phi(x))=l(\phi(a))=l(a)=l(x)$ or $l(\phi(x))=l(\phi(a))=l(a)+1=l(x)+1$, we have $\phi(x)_i=\phi(a)_i$ for all $i\in\{1,\cdots,l(x)\}$.

2) Let consider such number $a=a_1a_2\cdots a_{l(a)-1}a_{l(a)}\in N$. We set $x=a+(b-1)\cdot 10^{\frac{l(a)}{2}-1}$. Then $l(x)=l(a)$ because $a_{\frac{l(a)}{2}}\le b-2$ and $l\left((b-1)\cdot 10^{\frac{l(a)}{2}-1}\right)=\frac{l(a)}{2}$. Moreover, setting $m=(b-1)\cdot 10^{\frac{l(a)}{2}-1}$, $l(m)=\frac{l(a)}{2}\in N$ since $l(a)$ is even. We then consider *d* the number obtained by adding 0 to *m* $\frac{l(a)}{2}$ times and just before the first digit of *m*. Then $l(a)=l(d)$ and $a+m=a+d$. The digits of *d* are all equal to 0, except the one in the position $\frac{l(a)}{2}+1$ which is equal to $b-1$. Hence, we have $\sigma_{l(x)}(a,d)=0$, $\sigma_i(a,d)=1_{\{a_{i+1}+0+\sigma_{i+1}(a,d)\ge 10\}}=0$ for all $i\in\left\{l(a)-1,\cdots,\frac{l(a)}{2}+1\right\}$ (Note that the $\sigma_i(a,d)$ are calculated in a decreasing order) and since $a_{\frac{l(a)}{2}+1}>0$, $\sigma_{\frac{l(a)}{2}}(a,d)=1$. Also, for all $i\in\left\{\frac{l(a)}{2}-1,\cdots,1\right\}$, $\sigma_i(a,d)=1_{\{a_{i+1}+0+\sigma_{i+1}(a,d)\ge 10\}}=0$, since $a_{\frac{l(a)}{2}}\le b-2$. Hence, the digits of *x* are then expressed as the function of the digits of *a* as follows: for all $i\in\left\{1,\cdots,\frac{l(a)}{2}-1\right\}$,

$x_i = \mathcal{U}\left(a_i + 0 + \sigma_i(a,d)\right) = \mathcal{U}(a_i + 0) = a_i$.

$x_{\frac{l(a)}{2}} = \mathcal{U}\left(a_{\frac{l(a)}{2}} + 0 + \sigma_{\frac{l(a)}{2}}(a,d)\right) = \mathcal{U}\left(a_{\frac{l(a)}{2}} + 1\right) = a_{\frac{l(a)}{2}} + 1$ because $1 \le a_{\frac{l(a)}{2}} \le b-2$. $x_{\frac{l(a)}{2}+1} = \mathcal{U}\left(a_{\frac{l(a)}{2}+1} + 10 - 1 + \sigma_{\frac{l(a)}{2}+1}(a,d)\right) = \mathcal{U}\left(a_{\frac{l(a)}{2}+1} + 10 - 1\right) + \sigma_{\frac{l(a)}{2}+1}(a,d)$ (because of lemma 1) $= a_{\frac{l(a)}{2}+1} - 1 + \sigma_{\frac{l(a)}{2}+1}(a,d)$ (also by lemma 1) $= a_{\frac{l(a)}{2}+1} - 1$. for all $i \in \left\{\frac{l(a)}{2} + 2, \cdots, l(a)\right\}$,
$x_i = \mathcal{U}\left(a_i + 0 + \sigma_i(a,d)\right) = (a_i + 0) = a_i$. its comes that $i \in \{1, \cdots, l(x)\}$, $x_i + x_{l(x)-i+1} = a_i + a_{l(a)-i+1}$. We then conclude that $l(\phi(x)) = l(\phi(a))$, and that $\phi(x)_i = \phi(a)_i$ for all $i \in \{1, \cdots, l(a)\}$.

3) Let consider such number $a = a_1 a_2 \cdots a_{l(a)} \in N$. We set $x = a + (b-1)\cdot 10^{\frac{l(a)}{2}-1}$. Then $l(x) = l(a)$ because $1 \le a_{\frac{l(a)}{2}} \le b-2$ and $l\left((b-1)\cdot 10^{\frac{l(a)}{2}-1}\right) = \frac{l(a)}{2}$. Moreover, setting $m = (b-1)\cdot 10^{\frac{l(a)}{2}-1}$, $l(m) = \frac{l(a)}{2} \in N$ since $l(a)$ is even. We then consider *d* the number obtained by adding 0 to *m* $\frac{l(a)}{2}$ times and just before the first digit of *m*. Then $l(a) = l(d)$ and $a + m = a + d$. The digits of *d* are all equal to 0, except the one

in position $\frac{l(a)}{2} + 1$ which is equal to $b-1$. Hence, we have $\sigma_{l(x)}(a,d) = 0$ and $\sigma_i(a,d) = 0$ for all $i \in \{l(a)-1, \cdots, 1\} \setminus \left\{\frac{l(a)}{2}\right\}$ (Note that the $\sigma_i(a,d)$ are calculated in a decreasing order) and since $a_{\frac{l(a)}{2}+1} = 0$, we also have $\sigma_{\frac{l(a)}{2}}(a,d) = 0$. We then conclude that for all $i \in \{1, \cdots, l(x)\} \setminus \left\{\frac{l(a)}{2} + 1\right\}$, $x_i = \mathcal{U}\left(a_i + 0 + \sigma_i(a,d)\right) = \mathcal{U}(a_i + 0) = a_i$. Moreover,
$x_{\frac{l(a)}{2}+1} = \mathcal{U}\left(a_{\frac{l(a)}{2}+1} + 10 - 1 + \sigma_{\frac{l(a)}{2}+1}(a,d)\right) = \mathcal{U}(0 + 10 - 1 + 0) = 10 - 1$. Now we search for the digits of $\phi(x)$. In fact, $\phi(x) = \phi(x)_1 \phi(x)_2 \cdots \phi(x)_{l(\phi(x))}$. Since $x_1 + x_{l(x)} = a_1 + a_{l(a)} \ge 10$, then $l(\phi(x)) = l(x) + 1 = l(a) + 1$. Hence, $\phi(x)_1 = 1 = \phi(a)_1$ and $\phi(x)_{l(\phi(x))} = \mathcal{U}\left(x_{l(x)} + x_1\right) = \mathcal{U}\left(a_{l(a)} + a_1\right) = \phi(a)_{l(\phi(a))}$. Also, for all $i \in \left\{2, \cdots, \frac{l(a)}{2} - 1\right\}$, $\phi(x)_i = \phi(a)_i$, $\phi(x)_{\frac{l(x)}{2}} = \phi(a)_{\frac{l(a)}{2}} + 1$, (using lemma 1) $\phi(x)_{\frac{l(x)}{2}+1} = \phi(a)_{\frac{l(a)}{2}+1}$, $\phi(x)_{\frac{l(x)}{2}+2} = \phi(a)_{\frac{l(a)}{2}+2} - 1$ (also because of lemma 1) and for all $i \in \left\{\frac{l(a)}{2} + 3, \cdots, l(a) + 1\right\}$, $\phi(x)_i = \phi(a)_i$. After finding the digits of $\phi(x)$, it is quite clear that $\phi(x) \ne \phi(a)$. As for the digits of $\phi_2(x)$, it can be seen

that $\phi_2(x)_i = \phi_2(a)_i$ for all $i \in \{1,\cdots,l(x)+1\}$.

4) Let suppose $l(a)$ is odd. We consider the natural number $m = \sum_{k=1}^{l(a)-3}(b-1)\cdot 10^k$. For example, if $b=10$ and $l(a)=5$, $m = \sum_{k=1}^{5-3} 9\cdot 10^k = \sum_{k=1}^{2} 9\cdot 10^k = 9\cdot 10^1 + 9\cdot 10^2 = 990$. Then by construction, $l(m) = l(a)-2$ and all de digits of *c* are all equal to $10-1$, except the last digit which is equal to 0. Moreover, considering *d* the number obtained by adding 0 to *m* twice and just before the first digit of *m*, we have $l(a) = l(d)$ and $a+m = a+d$. The digits of *d* are all equal to $b-1$, except the first, the second and the last which are equal to 0. We suppose that $a_{l(a)-1} \neq 0$. Then setting $x = a+d$, $l(x) = l(a)$ if and only if $a_1 \neq b-1$ and $a_2 \neq b-1$. Hence, we have $\sigma_{l(a)}(a,d) = 0$ by definition of σ, and $\sigma_{l(a)-1}(a,d) = 0$ because $a_{l(a)} + 0 \le b-1$. Also, $\sigma_1(a,d) = 0$ because $a_2 \neq b-1$, and for all $i \in \{2,\cdots,l(a)-2\}$, $\sigma_i(a,d) = 1$ since $a_{i+1} + 10 - 1 + \sigma_{i+1}(a,d) \ge 10$ (we remind that the the $\sigma_i(a,d)$ are determined in the decreasing order). From the $\sigma_i(a,d)$, the digits of *x* are now given by $x_1 = \mathcal{U}(a_1 + \sigma_1(a,d)) = a_1$ since $a_1 \neq b-1$ and $\sigma_1(a,d) = 0$. $x_2 = \mathcal{U}(a_2 + \sigma_2(a,d)) = \mathcal{U}(a_2+1) = a_2+1$ since $a_2 \neq b-1$. Moreover, for all $i \in \{3,\cdots,l(a)-2\}$, $x_i = \mathcal{U}(a_i + b - 1 + \sigma_i(a,d)) = \mathcal{U}(a_i + b - 1 + 1) = \mathcal{U}(a_i + b) = \mathcal{U}(a_i + 10) = a_i$ (by lemma 1). Also,

$$x_{l(a)-1} = \mathcal{U}\left(a_{l(a)-1} + b - 1 + \sigma_{l(a)-1}(a,d)\right) = \mathcal{U}\left(a_{l(a)-1} + b - 1\right) = \mathcal{U}\left(a_{l(a)-1} + 10 - 1\right) = a_{l(a)-1} - 1$$

(also by lemma 1) and $x_{l(a)} = \mathcal{U}\left(a_{l(a)} + 0 + \sigma_{l(a)}(a,d)\right) = \mathcal{U}\left(a_{l(a)} + 0 + 0\right) = a_{l(a)}$. Hence, whether $l(\phi(x)) = l(\phi(a)) = l(a)$ or $l(\phi(x)) = l(\phi(a)) = l(a)+1$, $\phi(x)_i = \phi(a)_i$ for all $i \in \{1,\cdots,l(\phi(a))\}$.

□

Corollary 1. Let $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)} \in N$ be a Lychrel number in base *b* such that $3 \le b \le 10$ and $l(a) \ge 2$.

i) If $1 \le a_1 \le b-2$ and $a_{l(a)} > 0$, then $a + 10^{l(a)-1} - 1$ is a Lychrel number;

ii) If $l(a)$ is even, $1 \le a_{\frac{l(a)}{2}} \le b-2$ and $a_{\frac{l(a)}{2}+1} > 0$, then $a + (b-1)\cdot 10^{\frac{l(a)}{2}-1}$ is a Lychrel number;

iii) If $l(a)$ is even, $(a_1, a_{l(a)}) \neq (b-1, b-1)$, $a_1 + a_{l(a)} \ge 10$, $a_{\frac{l(a)}{2}-1} + a_{\frac{l(a)}{2}+2} \ge 10$, $1 \le a_{\frac{l(a)}{2}} \le b-2$ and $a_{\frac{l(a)}{2}+1} = 0$, then $a + (b-1)\cdot 10^{\frac{l(a)}{2}-1}$ is a Lychrel number;

iv) If $l(a) \ge 5$, $l(a)$ is odd, $a_1 \neq b-1$, $a_2 \neq b-1$ and $a_{l(a)-1} \neq 0$, then $a + \sum_{k=1}^{l(a)-3}(b-1)\cdot 10^k$ is a Lychrel number.

Proof. Let $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)} \in N$ be such Lychrel number. i) Since $a_1 \le b-2$ and $a_{l(a)} > 0$, then $\phi(a) = \phi\left(a + 10^{l(a)-1} - 1\right)$, from i) of proposition 5. Hence, for all $n \in N$, $\phi_n(a) = \phi_n\left(a + 10^{l(a)-1} - 1\right)$. Since *a* is a Lychrel number, for all $n \in N$, $\phi_{n+1}(a) \neq 2\phi_n(a)$, which implies that $\phi_{n+1}\left(a + 10^{l(a)-1} - 1\right) \neq 2\phi_n\left(a + 10^{l(a)-1} - 1\right)$ for all $n \in N$. Hence, $a + 10^{l(a)-1} - 1$ is also a Lychrel number.

ii), iii) and iv) are proven the same way like i). □

4.2. An Application to the Classification of Lychrel Numbers

In base 10, when we take the first Lychrel number **196** and iteratively apply 1) of proposition 5 on it, we obtain the other elements of the first column of the table bellow, from 295 to 790 and we conclude from i) of corollary 1 that they too are Lychrel numbers. Also, considering the first iteration of 196 which is 887 and iteratively applying the same result, we obtain the numbers 689, 788 and 986 and conclude from the same corollary that they too are Lychrel numbers. The same situation occurs for 879 and we obtain 978. Hence, with 196, $\phi(196)=887$ and 879, we obtain all the other elements of Table 3 below.

Table 3. Equivalence classes of 196, 879 and 887.

196	689
295	788
394	**887**
493	986
592	**879**
691	978
790	

Now we take the second iteration of 196, which is 1675, and we iteratively apply 1) of proposition 5 on it to obtain the elements of the third column of Table 4 below, which are 2674, 3673, 4672, 5671 and 6670. We then conclude from i) of corollary 1 that they are also Lychrel numbers. On each element of this third column, we iteratively apply 2) of proposition 5 to obtain from each of them all the other elements of the corresponding line. We then conclude from ii) of corollary 1 that they too are Lychrel numbers. Hence, all the 35 other elements of the table below are obtained just from a single element which is 1675 and belong to an equivalence class of $\mathcal{R}_4$.

Table 4. Equivalence class of 1675.

1495	1585	**1675**	1765	1855	1945
2494	2584	2674	2764	2854	2944
3493	3583	3673	3763	3853	3943
4492	4582	4672	4762	4852	4942
5491	5581	5671	5761	5851	5941
6490	6580	6670	6760	6850	6940

The same process is applied when we consider the Lychrel number 1857 which is the first iteration of 879, and 1997 as we can see in the following Table

5 and Table 6.

Table 5. Equivalence class of 1857.

1497	1587	1677	1767	**1857**	1947
2496	2586	2676	2766	2856	2946
3495	3585	3675	3765	3855	3945
4494	4584	4674	4764	4854	4944
5493	5583	5673	5763	5853	5943
6492	6582	6672	6762	6852	6942
7491	7581	7671	7761	7851	7941
8490	8580	8670	8760	8850	8940

Table 6. Equivalence class of 1997.

1997	2996	3995	4994	5993	6992	7991	8990

The elements of Table 6 above are iteratively obtained from 1997 by applying 1) of proposition 5. In this table, it can be seen that one of the obtained element, **4994** is a Palindrome. We called it **Palindrome-Lychrel number**.

Considering the third iteration of 196 which is 7436, when we iteratively apply 1) of proposition 5 to it we obtain the elements of the fifth column of Table 7 below. The elements of this column are then Lychrel numbers, from i) of corollary 1. Moreover when we iteratively apply 2) of proposition 5 to each element of this column we obtain the elements from column 1 to column 8 (except column 5). The 48 elements of the first 8 columns of the table belong to the same equivalence class of the relation $\mathcal{R}_4$ having 7436 as the representative, and are Lychrel numbers from ii) of corollary 1. Now we consider column 8 of the table and iteratively apply 3) of proposition 5 to each element of this column. We then obtain column 9 of the table and again, we iteratively apply 2) of proposition 5 to each element of this column to obtain the elements in columns 10 and 11 of the table. All the 18 elements in columns 9 to 11 of the table also belong to the same equivalence class which is different from the one containing the elements in columns 1 to 8 and are Lychrel numbers from iii) of corollary 1. Hence, all these 66 elements of the table are Lychrel numbers.

Table 7. Equivalence classes of 7436 and 4799.

4079	4169	4259	4349	4439	4529	4619	4709	4799	4889	4979
5078	5168	5258	5348	5438	5528	5618	5708	5798	5888	5978
6077	6167	6257	6347	6437	6527	6617	6707	9797	6887	6977
7076	7166	7256	7346	**7436**	7526	7616	7706	7796	7886	7976
8075	8165	8255	8345	8435	8525	8615	8705	8795	8885	8975
9074	9164	9254	9344	9434	9524	9614	9704	9794	9884	9974

The same situation occurs with **Table 8** below where starting with the Lychrel number 9438 which is the second iteration of 879, and following the same steps as in the previous case, we obain all the 22 elements of the table. The first 8 columns of the table also constitute an equivalence class of the relation $\mathcal{R}_4$. 3) of proposition 5 is also iteratively applied to each element of column 8 to obtain the elements of column 9, and 2) of proposition 5 is iteratively applied to each element of this column to obtain the other elements in columns 10 and 11 of the table. Also, all the 22 elements are Lychrel numbers.

Table 8. Equivalence classes of 9438 and 8799.

8079	8169	8259	8349	8439	8529	8619	8709	8799	8889	8979
9078	9168	9258	9348	**9438**	9528	9618	9708	9798	9888	9978

The following **Table 9** is obtained in the same way. We start with the Lychrel number 9988 which is the first iteration of 1997 and iteratively apply 1) of proposition 5 to it to obtain the element of its column. After that, 2) of proposition 5 is applied to obtain the column 10 of the table. To end, 2) of proposition 5 enables the deduction of column 9 from column 10 and again, 2) of proposition 5 is applied to the elements of column 9 to obtain the other elements of the table.

Table 9. Equivalence classes of 9988 and 8089.

8089	8179	8269	8359	8449	8539	8629	8719	8809	8899	8989
9088	9178	9268	9358	9448	9538	9628	9718	9808	9898	**9988**

With **Table 10** below, the smallest element is the Lychrel number 7059. Hence, 1) of proposition 5 is applied to 7059 to obtain the elements of its column, which are also Lychrels from i) of corollary 1. After, 2) of proposition 5 is applied to each element of this column to obtain the elements in columns 2 to 6 of the table, which are Lychrel numbers too, by ii) of corollary 1. Also, 3) of proposition 5 is applied to each element of column 6 to obtain the elements of column 7 and at the end, 2) of proposition 5 is iteratively applied to each element of column 7 to obtain the elements in columns 8 to 11 of the table. Note that columns 1 to 6 and columns 7 to 11 are different equivalent classes of $\mathcal{R}_4$. All the elements of the table are Lychrel numbers.

Table 10. Equivalence classes of 7059 and 7599.

7059	7149	7239	7329	7419	7509	**7599**	7689	7779	7869	7959
8058	8148	8238	8328	8418	8508	8598	8688	8778	8968	8958
9057	9147	9237	9327	9417	9507	9597	9687	9777	9867	9957

Hence, it can then be observed that with only four Lychrel numbers 196, 879, 1997 and 7059, one can obtain all the 248 Lychrel numbers base 10 less than 10000, by using the iteration function on them and by iteratively applying proposition 5.

Proposition 5 also enables us to describe Lychrel numbers above 10,000 as it is shown in Table 11 below in which from 13,783 which is the fifth iteration of 196, and applying 1) of proposition 5, the numbers in the second column of the table are obtained and are Lychrel numbers too, by i) of corollary 1. After that, we apply 4) of proposition 5 on each element of this column to obtain all the elements of the other columns which are also Lychrel numbers. Note that this table is also an equivalence class of the relation $\mathcal{R}_5$.

Table 11. Equivalence class of 13,783.

127,93	**13,783**	14,773	15,763	16,753	17,743	18,733	19,723
22,792	23,782	24,772	25,762	26,752	27,742	28,732	29,722
32,791	33,781	34,771	35,761	36,751	37,741	38,731	39,721
42,790	43,780	44,770	45,760	46,750	47,740	48,730	49,720

Also, considering the Lychrel numbers **17,787** and **18,887** which are respectively the third iteration of 879 and the second iteration of 1997, and applying 1) of proposition 5 and 4) of proposition 5 with the same process as in the previous case, we obtain the following Table 12 and Table 13 of new Lychrel numbers.

Table 12. Equivalence class of 17,787.

16,797	**17,787**	18,777	19,767
26,796	27,786	28,776	29,766
36,795	37,785	38,775	39,765
46,794	47,784	48,774	49,764
56,793	57,783	58,773	59,763
66,792	67,782	68,772	69,762
76,791	77,781	78,771	79,761
86,790	87,780	88,770	89,760

Table 13. Equivalence class of 18,887.

17,897	**18,887**	19,877
27,896	28,886	29,876
37,895	38,885	39,875
47,894	48,884	49,874
59,893	58,883	59,873
69,892	68,882	69,872
79,891	78,881	79,871
89,790	88,880	89,870

One can observe that the number **18,887** yields another Palindrome-Lychrel number in the last table above, which is **48,884**. Note that the two tables above are equivalent classes of $\mathcal{R}_5$. It is also the case with **Table 14** from the first iteration of 7059, which is **16,566**.

Table 14. Equivalence class of 16,566.

13,596	14,856	15,576	**16,566**	17,556	18,546	19,536
23,595	24,585	25,575	26,565	27,555	28,545	29,535
33,594	34,584	35,574	36,564	37,554	38,544	39,534
43,593	44,583	45,573	46,563	47,553	48,543	49,533
53,592	54,582	55,572	56,562	57,552	58,542	59,532
63,591	64,581	65,571	66,561	67,551	68,541	69,531
73,590	74,580	75,570	76,560	77,550	78,540	79,530

Having shown that 196, 879, 1997 and 7059 are Lychrel numbers, and presented the list of all the Lychrel numbers related to them and which are below 10,000, we now focus on some numbers above 10,000 in the last four previous tables. The choices are 19,723, 89,760, 58,883 and 43,593, because they belong to different equivalence classes of $\mathcal{R}_5$ and for each table, there is no need testing all the elements. We also include 4994 and 8778 because they are palindromes, and also belong to different equivalence classes of $\mathcal{R}_4$.

The results of the tests of the six chosen numbers using the algorithm presented in the previous section are presented in **Table 15** below:

Table 15. Lychrel number test on some chosen natural numbers superior to 10,000.

a	4994	8778	19,723	89,760	43,593	58,883
$l(a)$	4	4	5	5	5	5
$\omega_a(n)$	0.41464	0.41429	0.41466	0.41495	0.41468	0.41495
$\lvert\omega_a(n)-0.414\rvert$	0.00064	0.00029	0.00066	0.00095	0.00068	0.00095
$p(A_{n+1})$	0	0	0	0	0	0

The 0s in the last line of the above table represent the values of $p(A_{n+1})$ which are less than $\left(\frac{1}{10}\right)^{330}$. Hence, for each of the chosen numbers, $p(A_{n+1})$ is less than $\left(\frac{1}{10}\right)^{330}$. Therefore, the probability $P(a)$ of reaching the first palindrome is such that $P(a) < P_{n+1}(a) < P_u(a) < p(A_{n+1}) < \left(\frac{1}{10}\right)^{330}$. We then conclude that 19,723, 89,760, 58,883 and 43,593 are effectively Lychrel numbers.

Also, 4884 and 8778 are palindrome-Lychrel numbers according to the probabilistic definition and to theorem 4.

5. Palindrome-Lychrel Numbers

The present section addresses the notion of Palindrome-Lychrel numbers and explains how to obtain them.

Let consider a Lychrel number $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)}$. If a Palindrome-Lychrel number $k(a)$ can be obtained from *a*, then $k(a)$ is both a lychrel number and a palindrome. But $k(a)$ is obtained from *a* by the iterative application of 1) of proposition 5 to *a*. By so doing, $a_2 \cdots a_{l(a)-1}$ always remains the same for all the iterative numbers obtained, including $k(a)$. Hence, $k(a)_2 \cdots k(a)_{l(a)-1} = a_2 \cdots a_{l(a)-1}$ and since $k(a)$ is a palindrome, $k(a)_2 \cdots k(a)_{l(a)-1}$ is also a palindrome and then $a_2 \cdots a_{l(a)-1}$ is a palindrome too. Moreover, when 1) of proposition 5 is applied to a number *a* to obtain a number *m*, one always has $a_1 + a_{l(a)} = m_1 + m_{l(m)}$. We then have $a_1 + a_{l(a)} = k(a)_1 + k(a)_{l(k(a))}$ and since $k(a)$ is a palindrome, $k(a)_1 = k(a)_{l(k(a))}$ and then $k(a)_1 + k(a)_{l(k(a))}$ is even. It then comes that $\{a_1, a_{l(a)}\}$ is a couple of digits with the same parity.

Conversely, we suppose that $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)}$ is a Lychrel number with $a_2 \cdots a_{l(a)-1}$ been a palindrome and $\{a_1, a_{l(a)}\}$ a couple of digits with the same parity. We set $n = \max(a_1, a_{l(a)}) - \frac{a_1 + a_{l(a)}}{2}$. Then after iteratively applying 1) of proposition 5 to *a* *n* times, the number $k(a) = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)} + \frac{a_{l(a)} - a_1}{|a_{l(a)} - a_1|} \cdot n \cdot (10^{l(a)-1} - 1)$ is obtained and is a palindrome. Since $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)}$ is a Lychrel number, we conclude that $k(a)$ is a Palindrome-Lychrel number. We can now present the following result:

Proposition 6. Let $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)}$ be a Lychrel number. Then the following assertions are equivalent:

1) A Palindrome-Lychrel number can be obtained from *a*;

2) $a_2 \cdots a_{l(a)-1}$ is a palindrome and $\{a_1, a_{l(a)}\}$ a couple of digits with the same parity.

The above result explains why Lychrel numbers 1997, 7779 or 18,887 generate Palindrome-Lychrel numbers in their respective equivalence classes.

However, even if $a = a_1 a_2 \cdots a_{l(a)-1} a_{l(a)}$ is not a Lychrel number, by iteratively applying 1) proposition 5 on *a*, one can still obtain a palindrome if $a_2 \cdots a_{l(a)-1}$ is a palindrome and $\{a_1, a_{l(a)}\}$ a couple of digits with the same parity. For example, when we consider the natural number 26,568, we can see that the digits 2 and 8 have the same parity and 656 is a palindrome.

Hence setting $n = \max(2,8) - \frac{2+8}{2} = 3$, we can conclude that after iteratively applying 1) of proposition 5 to 26,568 3 times, we will reach a palindrome as follows: $26568 + 9999 = 36567$, $36567 + 9999 = 46566$ and $46566 + 9999 = 56565$

which is a palindrome.

The summary of the work is presented in de flowchart in Figure 2.

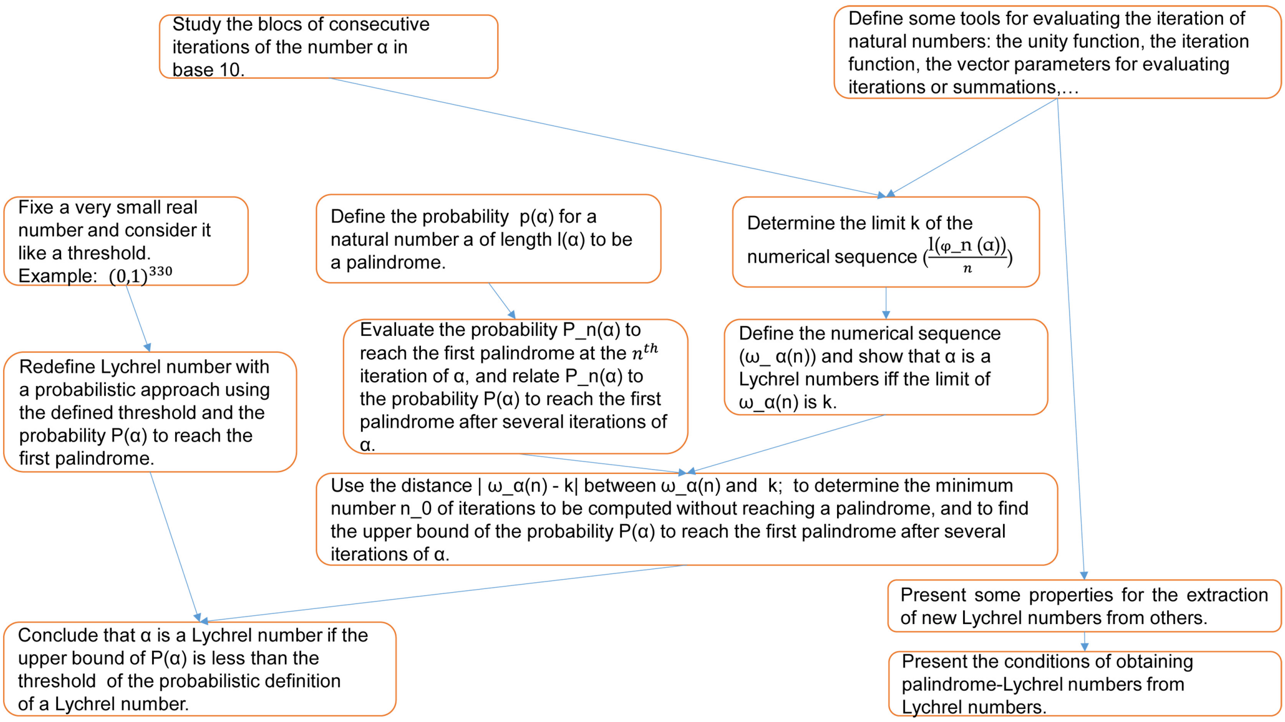

Figure 2. The Flowchart explaining the methodology of the work.

6. Conclusion

In this paper, we have defined some tools that permit the obtention of an iteration of a natural number and the proposal of a probabilistic characterization of Lychrel numbers. Moreover, we have provided some properties for the mathematical construction of new Lychrel numbers from others, without the use of a computer programme. These properties led us to the discovery of Lychrel numbers greater than 10,000 and to the discovery of Palindrome-Lychrel numbers. The characterization proposed in the paper can be extended to other bases b ($3 \le b \le 9$), provided the blocks of consecutive iterations of natural numbers in these bases are accurately studied. However one of the limitations of the work is that the characterization of Lychrel numbers here remains probabilistic. Our future work is then to find an algebraic Characterization of Lychrel numbers and Palindrome-Lychrel numbers, in order to make their handling more easy.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Notation

N: the set of natural numbers;

a, m, d, l, i, j, k : natural numbers;

$Rev(a)$: the reverse number of a;

$l(a)$: the length of *a*, meaning the number of digits of a;

N_l : the set of natural numbers of length *l*;

a_i : the i^{th} digit of a $(1 \le i \le l(a))$;

$\phi_i(a)$: the i^{th} iteration of a;

$\epsilon^i(a)$: the vector parameter used for the calculation of $\phi_i(a)$;

$\epsilon^i_j(a)$: the j^{th} element of the vector parameter used for the calculation of $\phi_i(a)$;

$\phi(a)_i$: the i^{th} digit of $\phi(a)$;

$\sigma(a, m)$: the vector parameter used for the calculation of $a + m$;

$\sigma_j(a, m)$: the j^{th} element of the vector parameter used for the calculation of $a + m$;

C^k_m : the number of *k*-combinations in a set containing *m* elements;

A_i : the event which makes $\phi_i(a)$ a palindrome;

B_i : the event for which $\phi_i(a)$ is not a palindrome;

$p(a)$: the probability for a to be a palindrome;

$p(A)$: the probability of the event *A*;

$P(G)(m)$: the probability for *m* to be a starting point of blocs of iterations from *G*;

$P(a)$: the probability of reaching the first palindrome after several iterations of a;

$P_n(a)$ the probability to reach the first palindrome at the n^{th} iteration of a.