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How to Prove Riemann Conjecture by Riemann's Four Theorems

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Abstract

Riemann (1859) had proved four theorems: analytic continuation $\zeta(s)$, functional equation $\xi(z)=G(s)\zeta(s)$ ($s=1/2+iz$, $z=t-i(\sigma-1/2)$), product expression $\xi_1(z)$ and Riemann-Siegel formula $Z(z)$, and proposed Riemann conjecture (RC): All roots of $\xi(z)$ are real. We have calculated ξ and ζ, and found that $\xi(z)$ is alternative oscillation, which intuitively implies RC, and the property of $\zeta(s)$ is not good. Therefore Riemann's direction is correct, but he used the same notation $\xi(t)=\xi_1(t)$ to confuse two concepts. So the product expression only can be used in contraction. We find that if ξ has complex roots, then its structure is destroyed, so RC holds. In our proof, using Riemann's four theorems is sufficient, needn't cite other results. Hilbert (1900) proposed Riemann hypothesis (RH): The non-trivial roots of ζ have real part 1/2. Of course, RH also holds, but can not be proved directly by $\zeta(s)$.

Keywords

Riemann Conjecture, Zeta-Function, Xi-Function, Functional Equation, Product Expression, Contradiction

1. Introduction

D. Hilbert (1900) proposed 23 problems and stated in the eighth problem that [1].

Riemann Hypothesis (RH). *The nontrivial zeros of* $\zeta(s)$ *have real part* 1/2.

Since then it has been accepted as a classical formulation. S. Smale [2] (1998) proposed 18 problems and listed RH as the first. In 2000, Clay Mathematics Institute opened seven Millennium Problems, including RH, see official reviews E. Bombieri [3] (2000) and P. Sarnak [4] (2005). In the 20th century, extremely large-scale computations of ζ confirm that RH holds (up to $t = 10^9 : 10^{13}$) [5]-[7], which have enhanced our belief. But we don't know how to prove it.

A century has passed, J. Conrey [8] (2003) pointed out that: "*In my belief, RH is a genuinely arithmetic problem, likely don't succumb to the method of analysis*". We have to consider another direction of the research.

We recall Riemann's paper ([9], p. 300), after analytically continuing $\zeta(s)$ by contour integral, he gave an important explanation:

"$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)$ *remains unchanged when* s *is replaced by* $1-s$. *This property of the function motivated me to consider the integral* $\Gamma(s/2)$ *instead of the integral* $\Gamma(s)$ *in the general term of* $\sum n^{-s}$, *which leads to a very convenient expression of the function* $\zeta(s)$".

Clearly, his aim is to reconstruct a new analytic function $\zeta(s)$ symmetric with respect to $s = 1/2$, and then define the entire function $\xi(z) = G(s)\zeta(s)$ and product expression. He proposed.

Riemann Conjecture (RC). *All the roots of* $\xi(z)$ *are real.*

where the transform $s = 1/2 + iz$, $z = t - i\beta$, $\beta = \sigma - 1/2$ is used.

Facing to this extremely difficult problem, our only starting method is to detect unknown by calculation (in §5). We find that $\xi(z) = u + iv$ is alternative oscillation, which intuitively implies RC true. But the property of $\zeta(s)$ is not good. Therefore Riemann's direction is correct. To study RC, we concentrate our attention on the product expression $\xi_1(z)$, and a hidden fault of Riemann is found (in §3). The function $\xi(z)$ is a sharp expression produced by $\zeta(s)$, up to now no complex roots are found. Whereas the product expression $\xi_1(z)$ bases on another principle, whose presupposition is that all roots z_j are given. Actually, there are two possibilities. If $\xi(z)$ has no complex roots, then RC is assumed (needn't discuss). If $\xi(z)$ has complex roots, then RC is denied, we must prove impossible. But Riemann had used the same notation $\xi(z) = \xi_1(z)$ to confuse two different concepts. So the product expression must include complex roots, which only can be used in contradiction. Finally, we find that if $\xi(z)$ has complex roots, its structure is destroyed, then RC holds (in §6). In our proof, the product expression $\xi_1(z)$ is the most powerful tool, and using four theorems of Riemann is sufficient, needn't cite other results.

Therefore RH also holds, but it can not be proved directly by $\zeta(s)$, because studying the infinite series $\zeta(s) \neq 0$ has surpassed the ability of existing analysis.

We clarify three notations used in this paper:

1) Euler $\zeta(s)$-function is analytic in the whole complex plane with a pole $s = 1$.

2) Call Riemann function $\xi(z) = G(s)\zeta(s)$ (not $\xi(s)$ used in literatures).

3) Construct the product expression $\xi_1(z)$ by all roots of $\xi(z)$.

c2. Follow Riemann's Thinking (Three Theorems and RC)

In Riemann's paper, only two pages focused on RC, see [9], pp. 300-302.

2.1. Analytic Continuation $\zeta(s)$

Riemann took the product formula of primes of Euler as a start,

$$\zeta(\sigma)=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma}}=\prod_{p\in primes}\left(1-\frac{1}{p^{\sigma}}\right)^{-1},\quad \sigma>1. \tag{1}$$

Taking $s=\sigma+it,\sigma>1$ and $y=n^2\pi x$ in gamma integral

$$\Gamma\left(\frac{s}{2}\right)=\int_0^{\infty}y^{s/2-1}\mathrm{e}^{-y}\mathrm{d}y=n^s\pi^{s/2}\int_0^{\infty}x^{s/2-1}\mathrm{e}^{-n^2\pi x}\mathrm{d}x,$$

and summing over *n*, Riemann had

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}=\pi^{s/2}\Gamma^{-1}\left(\frac{s}{2}\right)\int_0^{\infty}x^{s/2-1}\psi(x)\mathrm{d}x,\quad \psi(x)=\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x},$$

where Jacobi function $\psi(x)$ satisfies $2\psi(x)+1=x^{-1/2}\left(2\psi\left(\frac{1}{x}\right)+1\right)$. By $z=1/x$, there is

$$\int_0^1 z^{s/2-1}\psi(z)\mathrm{d}z=\frac{1}{s(s-1)}+\int_1^{\infty}x^{-s/2-1/2}\psi(x)\mathrm{d}x.$$

The singularity $x=0$ is eliminated. Riemann got

Theorem 1. *There is an integral representation*

$$\zeta(s)=\pi^{s/2}\Gamma^{-1}\left(\frac{s}{2}\right)\left\{\frac{1}{s(s-1)}+\int_1^{\infty}\left(x^{s/2-1}+x^{-(s+1)/2}\right)\psi(x)\mathrm{d}x\right\}, \tag{2}$$

which is analytically continued over the whole complex plane except a pole $s=1$.

Here $\Gamma^{-1}(s/2)$ has zeros $s=-2,-4,\cdots$, called trivial zeros of $\zeta(s)$, no interest for us.

2.2. Functional Equation $\xi(z)$ and RC

Multiplying (2.2) by $G(s)$, Riemann directly took $s=1/2+it$ and defined

$$\xi(t)=G(s)\zeta(s),\quad G(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right),\quad |G(s)|\approx Ct^{7/4}\mathrm{e}^{-t\pi/4}. \tag{3}$$

(Many scholars have accepted another notation $\xi(s)=G(s)\zeta(s)$, [9] p. 17, but Riemann's notation $\xi(t)$ is more concise in research, see (2.4), (2.5) and (2.6)). Inserting ζ into (2.3) and applying integration by parts twice, one has [9] p. 17,

$$\begin{aligned}\xi(t)&=\frac{1}{2}+\frac{s(s-1)}{2}\int_1^{\infty}\left(x^{s/2-1}+x^{-s/2-1/2}\right)\psi(x)\mathrm{d}x,\\&=r_1+\int_1^{\infty}\left(x^{s/2-1}+x^{-s/2-1/2}\right)g(x)\mathrm{d}x,\quad g(x)=2x^2\psi''+3x\psi',\end{aligned}$$

where $r_1 = \frac{1}{2} + \psi(1) + 4\psi'(1) = 0$. Riemann got a real function [9] pp. 301-302,

$$\xi(t) = 2\int_1^\infty \cos\left(\frac{t}{2}\ln x\right) x^{-3/4} g(x)\,\mathrm{d}x, \tag{4}$$

and considered t as complex variable. In fact, Riemann used translation $\beta = \sigma - 1/2$ and rotation $s = 1/2 + iz$, $z = t - i\beta$, see Figure 1, and got an even entire function $\xi(z)$ by (2.4). We state

Theorem 2. *The entire function* $\xi(z)$ *satisfies functional equation*

$$\xi(z) = G(s)\zeta(s), \quad s = \sigma + it = 1/2 + iz, \quad z = t - i\beta, \quad \beta = \sigma - 1/2, \tag{5}$$

which has symmetry $\xi(z) = \xi(-z)$ *and conjugate* $\xi(\bar{z}) = \overline{\xi(z)}$.

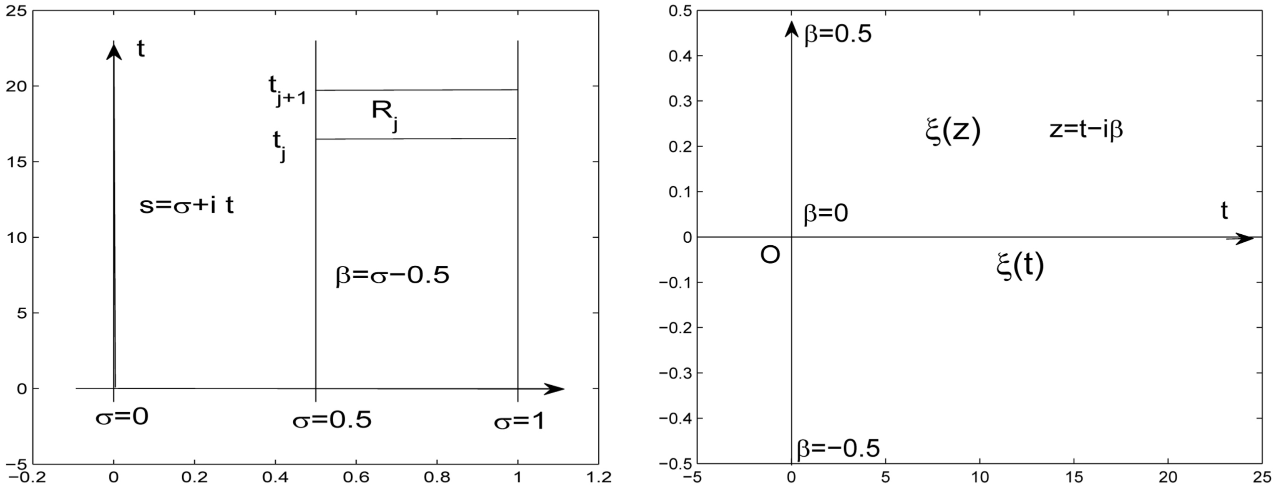

Figure 1. Translation $\beta = \sigma - 1/2$ and rotation $z = t - i\beta$.

Riemann continued:

"… *the function* $\xi(t)$ *can vanish only when the imaginary part of t lies between* $\frac{1}{2}i$ *and* $-\frac{1}{2}i$. *The number of roots of* $\xi(t) = 0$ *whose real parts lie between* 0 *and T is about*

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}$$

… One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the research for such a proof after some fleeting vain attempts, …"

He proposed an important statement in critical strip $\Omega = \{z = t - i\beta : |\beta| \le 1/2, 0 \le t < \infty\}$,

Riemann conjecture (RC). *All the roots of function* $\xi(z)$ *are real.*

2.3. Product Expression $\xi_1(z)$

Riemann finally pointed out that

"*If one denotes by α the roots of the equation* $\xi(\alpha)=0$, *then one can express* $\log\xi(t)$ *as*

$$\sum\log\left(1-\frac{t^2}{\alpha^2}\right)+\log\xi(0),$$

because, since the density of roots of size t grows only like $\log(t/2\pi)$ *as t grows, this expression converges and for infinite t is only infinite like* $t\log t$; *Thus it differs from* $\log\xi(t)$ *by a function of* t^2 *which is continuous and finite for finite t and which, when divided by* t^2, *is infinitely small for infinite t. This difference is therefore a constant, the value of which can be determined by setting* $t=0$."

We have seen that Riemann proved the following

Theorem 3. *If* $\{z_j\}$ *are all roots of* $\xi(z)$, *there is a product expression*

$$\xi_1(z)=\xi(0)\prod_{j=1}^{\infty}\left(1-\frac{z^2}{z_j^2}\right),\quad z=t-i\beta,\quad \xi(0)=0.497120778\cdots. \tag{6}$$

Here Riemann didn't give the explanation more. Are the *α* real or complex? This just is a key in the whole problem.

3. A Hidden Fault of Riemann and Our New Thinking

We have a very difficult process to recognize the role of the functional equation. The function $\xi(z)=G(s)\zeta(s)$ is a sharp expression produced by $\zeta(s)$, one can calculate all real roots, up to now no complex roots are found. We wanted to prove RC by $\xi(z)$, several attempts fail, for example, use geometric analysis [10] [11], although attained some progressions, but RC can not be completely proved. Finally, we think the lack of some condition, only $\xi(z)$ is not sufficient, and we have to consider the product expression. So a hidden fault of Riemann is found [12] [13].

The product expression $\xi_1(z)$ has a presupposition, *i.e.*, its all roots are given in advance. Riemann said, "*If one denotes by α the roots of the equation* $\xi(\alpha)=0$", where *α* are real or complex? Very ambiguous. Actually, there are two possibilities:

1). If ξ has only real roots, then RC is assumed, $\xi(t)=\xi_1(t)=w(t)$ is defined by all real roots, needn't discuss.

2). If $\xi(z)$ has complex roots, then RC is denied, $\xi_1(t)=w(t)Q(t)$, in which $Q(t)$ is defined by all complex roots. We must prove impossible.

But Riemann had used the same notation $\xi=\xi_1$ to confuse two different concepts. This is a hidden fault, and also is the main reason to fail.

Because the product expression must include all roots, it can be used only in contradiction. We find that if $\xi(z)$ has complex roots, then its structure in symmetric line will be destroyed, so RC holds.

If using Riemann's notation $\xi(z)=G(s)\zeta(s)$ and product expression, A. Hinkkanen [14] (1997) and J. Lagarias [15] (1999) have proved the following *Equivalence.* "*The positivity* $Re(\xi_\beta/\xi)>0$, $\beta\in(0,1/2]$, *is a sufficient and* *necessary condition for RC true*". This is an important property of $\xi(z)$, which also supports us to study RC by the product expression.

E. Bombieri [3] pointed out that "*We do not have algebraic and geometric models to guide our thinking, and entirely new ideas may be needed to study these intriguing objects*". This is a valuable advice. We find that the product expression (*i.e.*, an algebraic model, called multiplicative group) is the most powerful tool to study RC.

4. Riemann-Siegel Formula Is Valid for the Complex Variable

Using asymptotic expansion of gamma function

$$\Gamma(s)=\sqrt{2\pi}s^{s-1/2}\mathrm{e}^{-s}\left(1+\frac{1}{12s}+\frac{1}{288s^2}-\frac{139}{51740s^3}+O\left(s^{-4}\right)\right), \tag{7}$$

where $|\arg(s)|<\pi$, we have in $s=1/2+it$

$$\begin{cases} G(s)=\dfrac{1}{2}s(s-1)\pi^{s/2}\Gamma(s/2)=M(t)\mathrm{e}^{i\phi(t)}, \\ M(t)=\left(\dfrac{\pi}{2}\right)^{1/4}t^{7/4}\mathrm{e}^{-t\pi/4}\left(1+O\left(t^{-2}\right)\right), \\ \phi(t)=\dfrac{t}{2}\ln\dfrac{t}{2\pi}-\dfrac{t}{2}+\dfrac{7\pi}{8}+\dfrac{1}{48t}+\dfrac{7}{5760t^3}+\cdots \end{cases} \tag{8}$$

where $M(t)$ is exponent decay, a troublesome factor. Delete $M(t)$ in $\xi(t)$, we get

$$Z(t)=\frac{\xi(t)}{M(t)}=\mathrm{e}^{i\phi(t)}\zeta(1/2+it),\quad |Z(t)|=|\zeta(1/2+it)|. \tag{9}$$

where $\mathrm{e}^{i\phi(t)}$ is an important symmetrizer.

C. Siegel [16] (1932) found a formula in Riemann manuscript unpublished.

Theorem 4 (Riemann-Siegel formula). *For all* $t\in(1,\infty)$, *there is*

$$\begin{cases} Z(t)=2\displaystyle\sum_{n=1}^{N}\frac{\cos(\phi(t)-t\ln n)}{n^{1/2}}+R_N(t), \\ \phi(t)=\dfrac{t}{2}\left(\ln\dfrac{t}{2\pi}-1\right)+\dfrac{7\pi}{8}+\dfrac{1}{48t}+\dfrac{7}{5760t^3}+\cdots \\ R_N(t)=(-1)^N\left(\dfrac{t}{2\pi}\right)^{-1/4}\left\{C_0+C_1t^{-1/2}+C_2t^{-1}+C_3t^{-3/2}+\cdots\right\} \end{cases} \tag{10}$$

where $N=[X]$ *is the integer part of* $X=(t/2\pi)^{1/2}$ *and the coefficients*

$$\begin{cases} C_0=\psi(q)=\dfrac{\cos\left(q^2+3\pi/8\right)}{\cos\left(\sqrt{2\pi}q\right)},\quad q=\sqrt{t}-(N+1/2)\sqrt{2\pi}, \\ C_1=-\dfrac{1}{2^3\cdot 3}\psi'''(q),\ C_2=\dfrac{1}{2^4}\psi''(q)+\dfrac{1}{2^7\cdot 3^2}\psi^{(6)}(q),\cdots \end{cases} \tag{11}$$

Should point out that in [16], $\psi(q)$ is a smooth function and $0.3827<\sin(\pi/8)\le\psi(q)\le\cos(\pi/8)=0.9239$. $\psi(q)=\dfrac{0}{0}$ is removable singularity at $q = \pm\sqrt{\pi/2}/2$. In [9] adopts $p = X - N$ and $\psi(p) = \dfrac{\cos\left(2\pi\left(p^2 - p - 1/16\right)\right)}{\cos(2\pi p)}$, which is not original statement. Besides, Siegel deleted a factor $s(s-1) = -\left(t^2 + 1/4\right)$, $\phi(t)$ contains $-\pi/8$ and R_N contains factor $(-1)^{N-1}$, so $Z(t)$ and $\xi(t)$ have opposite signs. Now in (10), $Z(t)$ and $\xi(t)$ have same sign.

Today $Z(t)$ is the most efficient tool in computing real roots, only requires $N = \left[(t/2\pi)^{1/2}\right]$ terms. Riemann at least calculated the first several real roots, e.g., $t_1 = 14.135$, $t_2 = 21.022$, $t_3 = 25.011$. Riemann said, "*One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real*". We believe that Riemann had accepted such a research method, *i.e.* combine theoretical analysis (symmetry and conjugate) with finite calculations.

Recall that the function $\xi(t)$ is valid for complex *z*. We wake up the following.

Corollary 1. *R-S formula* $Z(z)$ *is valid for the complex variable.*

This is obvious for Riemann. He said, "*after some fleeting vain attempts,…*", we guess that he likely studied RC by the finite series $Z(z)$, but failed. Siegel said, "In a letter to Weierstrass (1859) Riemann mentioned a new formula of $\zeta(s)$ which, however, he had not yet simplified enough to be able to include in his published paper." The proof of R-S formula has 20 pages, which is improved by Edwards [9] and simplified by us to 12 pages. Using the contour integral of $\zeta(s)$, Riemann introduced $\xi(t) = G(1/2+it)\zeta(1/2+it)$ and $Z(t) = \mathrm{e}^{i\phi(t)}\zeta(1/2+it)$, here the main term $2\sum_{n=1}^{N} n^{-1/2}\cos(\phi(t) - t\ln n)$ is obtained by residue theorem and the remainder R_N is estimated by a wonderful saddle method. Therefore the correctness of $Z(t) = \xi(t)/M(t)$ is verified. Besides, although Siegel [16] (p. 290 in English) suggested to take $t = -i(s-1/2)$ and the integer $N = \left[\left(|t|/2\pi\right)^{1/2}\right]$, but he didn't implement. For this we have derived a new $Z_1(z)$ by taking $z = t - i\beta$ and an integer $N = \left[(t/2\pi)^{1/2}\right]$, and found that $Z(z) = Z_1(z)$. Therefore the Corollary 1 is valid. We shall compute $Z(z)$ and $\zeta(s) = \mathrm{e}^{-i\phi(z)}Z(z)$ in §5.

By R-S formula it is easy to derive (see [9], p. 200)

$$\begin{aligned}|Z(t)| &\le 2\left(1 + \sum_{n=2}^{N} n^{-1/2}\right) + |R_N| \le 2\left(1 + \int_1^N x^{-1/2}\right)\mathrm{d}x + 2 \\ &\le 4\left(\frac{t}{2\pi}\right)^{1/4} - 2 + 2 \le 4t^{1/4},\quad t \ge 10.\end{aligned}$$

and get

Lemma 1. *For* $t \ge 10$, *there is* $|Z(t)| = |\zeta(1/2+it)| \le 4t^{1/4}$.

The roots of $Z(t)$ are irregular distribution. These roots $t_n \approx 2\pi n/\ln n$ have averaging spacing $\Delta_n \approx 2\pi/\ln n$. As the sign of $R_N(t)$ is unchanged,

$U(t)=2\sum_{n=1}^{N}n^{-1/2}\cos\left(\theta(t)-t\ln n\right)$ and $Z(t)=U(t)+R_N$ all are high-frequency oscillation, we can define an infinite subset

$$E_N=\begin{cases}E_N^+=\left\{t:Z(t)\ge R_N>0,\text{ if } N \text{ is even and } U(t)\ge 0\right\}\\ E_N^-=\left\{t:Z(t)\le R_N<0,\text{ if } N \text{ is odd and } U(t)\le 0\right\}\end{cases}$$

So $|Z(t)|\ge|R_N|\ge 0.5t^{-1/4}$ in E_N. To avoid all roots of $\zeta(1/2+it)=0$, we have

Lemma 2. *Define the infinite point-set*

$$L(\omega)=\left\{t:t\ge 20\text{ and }\left|\zeta(1/2+it)\right|=\omega(t)\right\},\quad \omega(t)=0.5t^{-1/4}. \tag{12}$$

Their averaging spacing $O(1/\ln t)$ *gets more and more small.*

Calculate the curve $\left|\zeta(1/2+it)\right|$ with $N=12$ in Figure 2, which has 90 roots in $(900,1000)$. The $L(\omega)$ contains many points. Taking $\omega=1$ makes the figure clear.

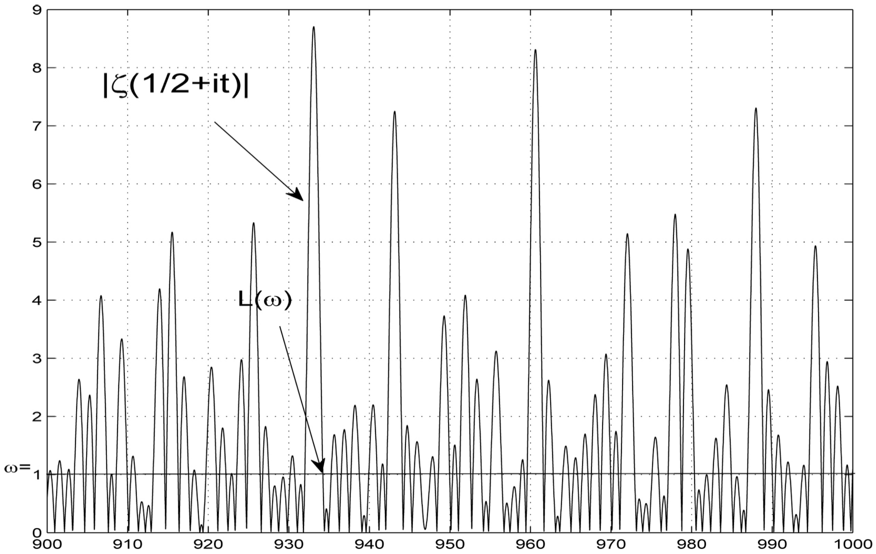

Figure 2. $\left|\zeta(1/2+it)\right|=\omega$ contains many points.

5. Computing Can Detect the Properties of ξ and ζ

5.1. Liuhui Proposed Three Kinds of Thinking Method, Rather than Two Kinds

Descartes pointed out that "*thinking has two methods: intuition and deduction*". Poincare emphasized, "*Logic is tool of proof, intuition is tool of discovery*". But the failure of studying RC indicates that "deduction and intuition" are invalid. We have to turn to Chinese mathematics, which emphasizes algorithms and calculations. We find that Liuhui (a.d.225-295) pointed out in preface of "Nine Chapters Mathematics" (a.d.263) [17]:

"*Analyze the reason by logic, explain the essence by figures*".

"*Computing can distinguish tiny, detect unknown.*"

He very early proposed "Logic and intuition", and emphasized third thinking "computing can detect unknown". We state it as:

Liuhui thinking. *Computing detects unknown and reinforce geometric intuition are correct and reliable research method, which is always considered together with discovery and proof.*

We shall adopt this method to detect the clue of proving RC.

5.2. $Z(z)=u+iv$ Is Alternative Oscillation

The complex R-S formula has opened a new way to study RC.

Taking $N=2$ and $t\in J_2=[25,55]$, we calculate the curves (u,v) for $\beta=0,0.1,0.3,0.5$ in **Figure 3**. As $v(t,0)=0$, the real curve $Z(t,0)=u(t,0)$ has 9 zeros in J_2. For different $\beta>0$, $u(t,\beta)$ is even function in β and changes less, which also has 9 zeros. While $v(t,\beta)$ is odd (almost linear) function in β, its zero falls between two zeros of $u(t,\beta)=0$. Then these curves (u,v) are alternative oscillation without common zeros, which intuitively implies that RC holds.

5.3. $\{|u|,|v|\}$ Has Peak-Valley Structure

We studied $\xi(z)$ by the asymptotic analysis, several attempts fail, then have to turn to geometric analysis. Denote each root-interval $I_n=[t_n,t_{n+1}]$ of $u(t,\beta)$. If $u(t,\beta)$ has only one peak in I_n, called single peak, else called multiple peaks. We assume that $u(t,0)$ is single peak.

Using Cauchy-Riemann equations $u_\beta=v_t, v_\beta=-u_t$ (as $z=t-i\beta$), we have

$$v(t,\beta)=v(t,0)+\int_0^\beta v_\beta(t,y)\,\mathrm{d}y=0-\int_0^\beta u_t(t,y)\,\mathrm{d}y. \tag{13}$$

$$\begin{aligned} u(t,\beta)-u(t,0)&=\int_0^\beta u_\beta(t,y)\,\mathrm{d}(y-\beta),\ \text{as}\ u_\beta(t,0)=0,\\ &=0-\int_0^\beta u_{tt}(t,y)(\beta-y)\,\mathrm{d}y. \end{aligned} \tag{14}$$

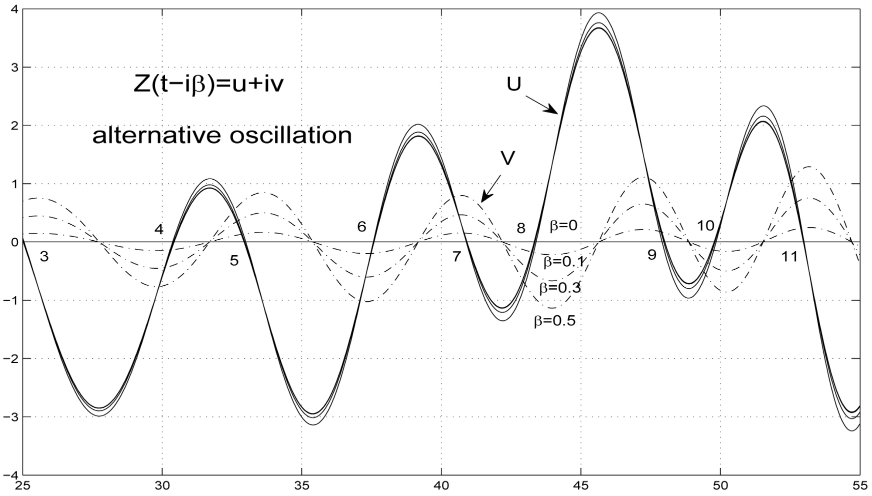

Figure 3. (u,v) for $\beta>0$ are alternative oscillation.

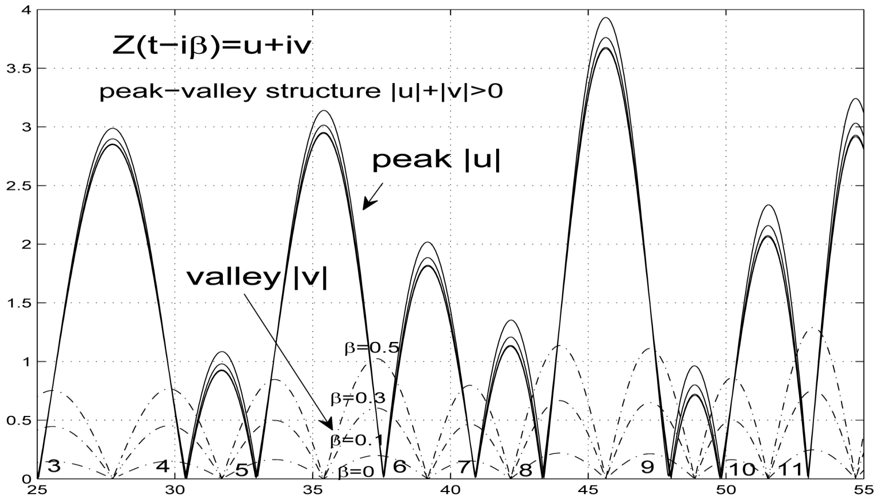

Figure 4. $\{|u|,|v|\}$ for $\beta>0$ have peak-valley structure.

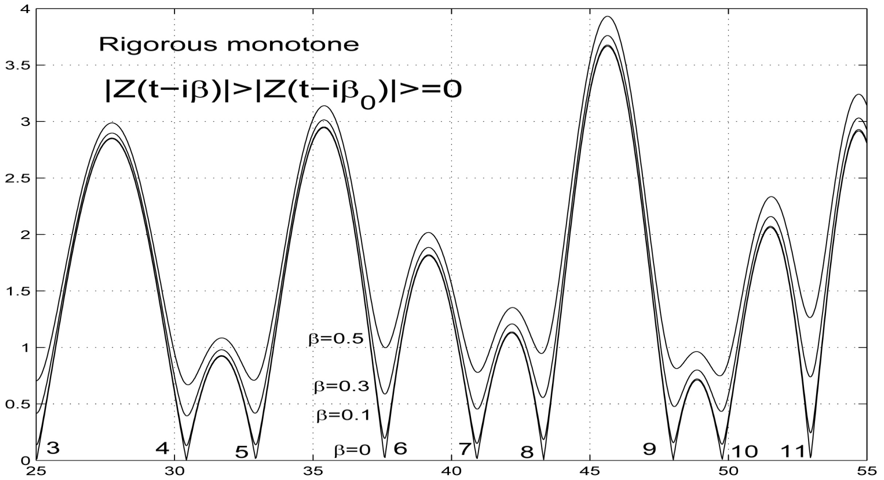

Figure 5. Monotone $|Z(t-i\beta)|>|Z(t-i\beta_0)|$ for $\beta>\beta_0\geq 0$.

So when $\beta>0$, $u(t,\beta)$ expands toward its convex direction, and then $|u(t,\beta)|>0$ inside I_j. Besides, as $v(t,\beta)$ and $u_t(t,\beta)$ at two end-points of I_j have opposite signs, there surely exists an inner point t_j^* of I_j so that $v(t^*,\beta)=0$. So $\{|u|,|v|\}$ forms a peak-valley structure for $\beta>0$, *i.e.* $|u|+|v|>0$ or $\sqrt{u^2+v^2}>0$ in each I_j, see **Figure 4** and **Figure 5**.

On the other hand, if assuming RC and using theorem 3, we can prove that $u(t,0)$ is single peak [10] [11], but which can not be proved only by theorem 2 (or Corollary 1). We feel the lack of a condition, have to consider theorem 3 and then a hidden fault of Riemann is found. This is a turning point in our research.

5.4. Euler Curves $\zeta(s)=U+iV$ Are Not Good

We calculate $\zeta(s)=\mathrm{e}^{-\phi(z)}Z(z)=U+iV$ in **Figure 6**, its property is bad.

1). U and V in critical line $\sigma=1/2$ have not symmetry and alternative oscillation, even which are almost tangent in some points. So proving RH near critical line is impossible.

2). For $\sigma = 0.6, 0.8, 1.0$, the imaginary part V has many zeros, whereas real part U gradually goes away from *t*-axes so that $|U| > 0$, RH holds. But estimating the series $\zeta(s) \neq 0$ has surpassed the ability of existing analysis.

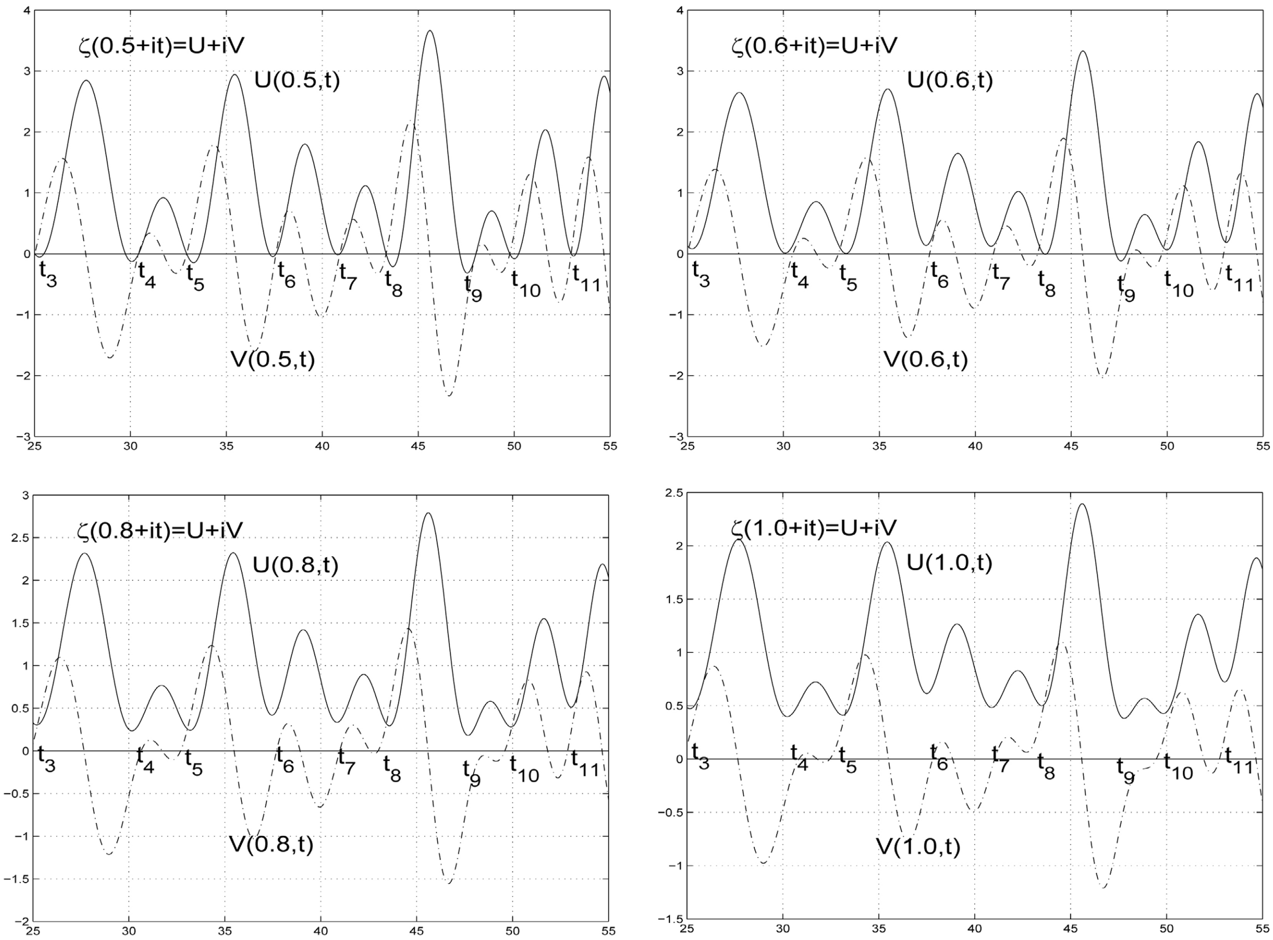

Figure 6. For different *σ*, {*U*, *V*} oscillate.

6. Proof of Riemann Conjecture

The Lemma 2 can be stated as

Lemma 3. *In any cases there is*

$$|Z(t)| = |\zeta(1/2+it)| = \omega(t) > 0, \ \text{for } t \in L(\omega). \tag{15}$$

The Lemma 3 is obtained directly by the functional equation $\xi(t)$, but we don't know if $\xi(z)$ has complex roots. By now Riemann had finished 99% of the whole work, only 1% remains, *i.e.* use the product expression and its contradiction to prove RC, which will be completed below. We point out that the whole research can be completed in the symmetry line $s_0 = 1/2 + it$, which accords with original thinking of Riemann.

Reconsider the product expression. If ξ has complex roots $z_j = t'_j + i\alpha_j$,

$0<\left|\alpha_j\right|\le 1/2$, its modulus $R_j=\left|z_j\right|\ge 20$. By the symmetry and conjugate, there are 4 conjugate complex roots $z_j=\pm\left(t_j'\pm i\alpha_j\right)$ and quantic factor

$$q_j(z)=\left(1-\frac{z^2}{\left(t_j'\pm i\alpha_j\right)^2}\right)\left(1-\frac{z^2}{\left(t_j'-i\alpha_j\right)^2}\right)$$

$$=1-2z^2\frac{R_j^2-2\alpha_j^2}{R_j^4}+\frac{z^4}{R_j^4}=\left(1-\frac{z^2}{R_j^2}\right)^2+\frac{4z^2\alpha_j^2}{R_j^4},\quad z=t-i\beta,$$

which is a positive function in the symmetric line $z=t$

$$q_j(t)=\left(1-\frac{t^2}{R_j^2}\right)^2+\mu_j>0,\quad \mu_j=\left(\frac{t}{R_j}\right)^2\left(\frac{2\alpha_j}{R_j}\right)^2>0. \tag{16}$$

Denote all real roots t_k and complex roots z_j of $\xi(z)$, by theorem 3 we have

$$\xi_1(t)=w(t)Q(t),\ w(t)=\xi(0)\prod_{k=1}^{\infty}\left(1-\frac{t^2}{t_k^2}\right),\ Q(t)=\prod_{j=1}^{\infty}q_j(t)>0. \tag{17}$$

So if ξ has no complex roots, then $\xi(t)=w(t)$, $Q(t)=1$. As ξ has complex roots, then $\xi(t)=\xi_1(t)=w(t)Q(t)$. We have the following key result.

Lemma 4. *In any cases there is*

$$\frac{\left|w(t)\right|}{M(t)}=\omega(t),\ \text{for}\ \ t\in L(\omega), \tag{18}$$

Proof. As the roots $\left\{t_j\right\}$ have irregular distribution, directly estimating $w(t)$ is impossible, we have to use an indirect method. First assume no complex roots, then $\left|\xi(t)\right|=M(t)\left|\zeta(s_0)\right|=\left|w(t)\right|$, using Lemma 3 leads (18). Once (18) is gotten, we find that $w(t),M(t)$ all are independent of complex roots, therefore it holds in any cases. The Lemma is proved.

Contradiction. If $\xi(z)$ has complex roots z_j. Denote the modulus $R_j=\left|z_j\right|\ge 20$, where $R_1\le R_2\le R_3\le\cdots$ is a nondecreasing sequence. Noting $0<\left(2\alpha_j\right)^2\le 1$, we have

$$Q(t)=\prod_{j=1}^{\infty}q_j(t),\quad q_j(t)=\left(1-\frac{t^2}{R_j^2}\right)^2+\frac{t^2}{R_j^2}\left(\frac{2\alpha_j}{R_j}\right)^2>0. \tag{19}$$

Obviously $q_j(0)=1$. For $t\in I_j=\left[0,R_j\right]$ we consider the derivative

$$q_j'(t)=4\frac{t}{R_j^2}\left\{-1+\frac{t^2+2\alpha_j^2}{R_j^2}\right\}<0,\quad \text{if } t^2+2\alpha_j^2<R_j^2,$$

then $q_j(t)$ decreases. It takes the minimum at $\hat{t}^2=R_j^2-2\alpha_j^2$

$$q_j(\hat{t})=\left(\frac{2\alpha_j^2}{R_j^2}\right)^2+\left(1-\frac{2\alpha_j^2}{R_j^2}\right)\left(\frac{2\alpha_j}{R_j}\right)^2=\frac{4\alpha_j^2}{R_j^2}-\frac{4\alpha_j^4}{R_j^4}>0.$$

When $t^2>R_j^2-2\alpha_j^2$, $q_j(t)$ will slowly increase and attain $q_j\left(R_j\right)=\left(\frac{2\alpha_j}{R_j}\right)^2\ll 1$. So $q_j(t)\le 1$ in I_j. In particular, $Q(t)\le q_1(t)\le 1$ for

$t \in I_1 = [0, R_1]$, see Figure 7.

Whatever the number and distribution of the complex roots, we consider only the first modulus $R_1 \ge 20$, in $t \in J = [R_1/2, R_1]$

$$q_1(t) = \left(1 - \left(\frac{t}{R_1}\right)^2\right)^2 + \frac{t^2}{R_1^2}\left(\frac{2\alpha_1}{R_1}\right)^2 \le \frac{9}{16} + 20^{-2} < \frac{10}{16} < 1, \tag{20}$$

As the length $R_1/2$ of *J* is large, there surely exist many points $t \in L(\omega)$.

Assume that $\xi(z)$ has complex roots. So $\xi(t) = G(s_0)\zeta(s_0) = w(t)Q(t)$ and

$$|\zeta(1/2 + it)| = \frac{|w(t)|}{M(t)} Q(t). \tag{21}$$

By Lemma 3, Lemma 4 and (20), we derive a contradiction

$$\omega(t) = \omega(t)Q(t) < \omega(t), \quad \text{for } t \in L(\omega). \tag{22}$$

Therefore the assumption above is wrong. Riemann conjecture is proved.

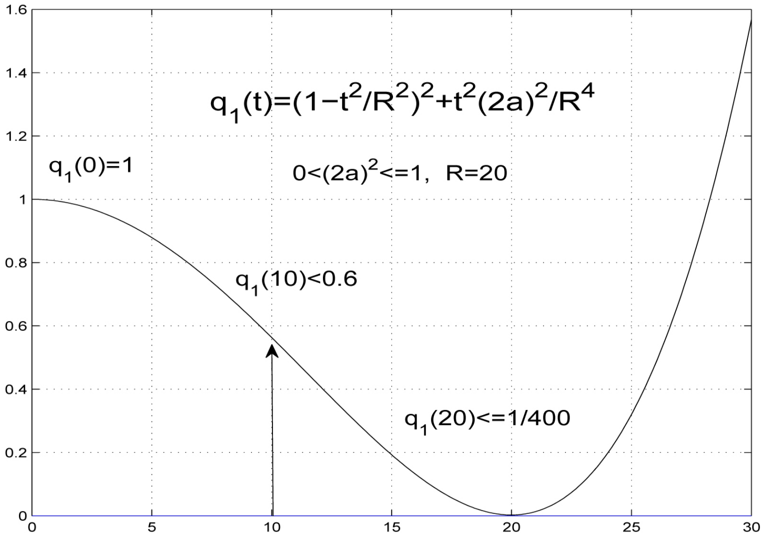

Figure 7. Take $R = 20$, the curve $q_1(t) < 0.6$ in $[10, 20]$.

Remark 1. The Lemma 2 can be proved by other method, we [13] have used the unboundedness of $|\zeta(s)|$ derived by Lindelöf theorem (1908), see [9], p. 184. Why the complicated R-S formula is used in this paper? Because we want to confirm such a fact, if Riemann could find his mistaken, likely he had already proved RC by his four theorems.

Remark 2. P. Sarnak [4] discussed the analytic continuation $L(s, \chi)$, the even entire function $\Lambda(s, \chi)$ and the corresponding Grand-RH (related to Goldbach conjecture). Likely our method is useful.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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