Scientific Research Publishing

Advances in Pure Mathematics, 2024, 14, 556-595
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Convergence Phenomenon with Fourier Series of $\mathrm{tg}\left(\frac{x}{2}\right)$ and Alike

Alfred Wünsche

Institut für Physik, Humboldt-Universität Berlin, Berlin, Germany
Email: alfred.wuensche@physik.hu-berlin.de

How to cite this paper: Wünsche, A. (2024) Convergence Phenomenon with Fourier Series of $\mathrm{tg}\left(\frac{x}{2}\right)$ and Alike. *Advances in Pure Mathematics*, **14**, 556-595.
https://doi.org/10.4236/apm.2024.147032

Received: May 19, 2024
Accepted: July 26, 2024
Published: July 29, 2024

Abstract

The Fourier series of the 2π-periodic functions $\mathrm{tg}\left(\frac{x}{2}\right)$ and $\frac{1}{\sin(x)}$ and some of their relatives (first of their integrals) are investigated and illustrated with respect to their convergence. These functions are Generalized functions and the convergence is weak convergence in the sense of the convergence of continuous linear functionals defining them. The figures show that the approximations of the Fourier series possess oscillations around the function which they represent in a broad band embedding them. This is some analogue to the Gibbs phenomenon. A modification of Fourier series by expansion in powers $\cos^n(x)$ for the symmetric part of functions and $\sin(x)\cos^{n-1}(x)$ for the antisymmetric part (analogous to Taylor series) is discussed and illustrated by examples. The Fourier series and their convergence behavior are illustrated also for some 2π-periodic delta-function-like sequences connected with the Poisson theorem showing non-vanishing oscillations around the singularities similar to the Gibbs phenomenon in the neighborhood of discontinuities of functions.

Keywords

Gibbs Phenomenon, Generalized Functions, Weak Convergence, Chebyshev Polynomials of First and Second Kind, Even and Odd Generating Functions for Chebyshev Polynomials, Polylogarithms, Completeness Relations

1. Introduction

Periodic functions and their representation by Fourier series possess a great importance in different branches of fundamental and applied sciences, in particular, in optics. It is well known that the convergence of Fourier series of discontinuous functions show a peculiar property of oscillations in the neighborhoods of the discontinuities where they increase to higher becoming and remaining overshoots of oscillations with decreasing distances from the discontinuities about (or below) its level. This behavior is called the Gibbs phenomenon after J. W. Gibbs who described it in about 1898 [1]. The author of the short article Ul'yanov in the Mathematical Encyclopedia mentioned that the phenomenon was discovered already before Gibbs by H. Wilbraham in 1848 and gives the original citations[1]. The phenomenon is represented in detail, e.g., by Sommerfeld [2], Fichtenholz [3], Arfken [4] and surely by others. Körner [5] in his monograph also mentions Wilbraham as a precurser of the Gibbs phenomenon but without any references though with some pieces to history in the Appendices. More about Fourier series and the Gibbs phenomenon with many citations one finds in Weisstein's MathWorld [6] under the corresponding notions.

In present paper we calculate the Fourier series of the 2π-periodic function $\mathrm{tg}\left(\frac{x}{2}\right)$ and of related functions which are not convergent in usual sense but are Generalized function or linear continuous functionals over a certain space of basis function (e.g., L. Schwartz [7], Gelfand and Shilov [8], Vilyenkin [9], Vladimirov [10]) as explained in the further text and illustrated by figures in Sections 2 - 4. The Tangent and related functions show new peculiarities of convergence in their Fourier series and it seems that they are worth to be described and illustrated. Apparently this phenomenon is present if the considered periodic function possesses singularities of, at least, first or higher order (but not logarithmic singularities). Whereas the Gibbs phenomenon is connected with discontinuities of the functions the here investigated phenomenon is connected with singularities of the kind of delta functions that arise from differentiation of functions with discontinuities but in our case from differentiation of logarithmic singularities. Insofar, the Gibbs phenomenon is connected with the phenomenon described but concerns functions with poles of first order for which in addition the Fourier series is not convergent in the ordinary sense. In Sections 5 and 6 we describe and illustrate by figures this phenomenon for two other related examples, the reciprocal Sine function $\left(\sin(x)\right)^{-1}$ and the corresponding reciprocal Cosine function. In another form for identities with 2π-periodic delta functions and their Fourier series which are derived from the Poisson summation formula we investigate and illustrate the convergence phenomenon in Section 7.

In Section 8 we derive a modification of Fourier series with power-like expansions of the discussed 2π-periodic functions in (non-orthogonal) powers of Cosine and Sine functions. The considered examples of such expansions are convergent even in such cases when the Fourier series converge only in the sense of weak convergence of Generalized functions. A more general consideration of

---

[1] Wilbraham H., "Cambridge and Dublin Ma. J.", 1848, vol. 3, pp. 198-201, and Gibbs J. W., "Nature", 1898, v. 59, p. 200.

these power-like expansions with calculation of their coefficients is made in Sections 9 and 10 plus Appendices.

Another aspect is that the specially considered functions are connected with a kind of Higher Transcendental Functions called Polylogarithms (Apostol [11], Andrews, Askey, Roy [12], in last mainly Dilogarithms), where their notation $\mathrm{Li}_s(z),(s=0,1,2,\cdots)$ was introduced in more recent times (second half of last century) and which include many other Elementary and Special functions as special cases (e.g, Tangent and Logarithm function and also the Riemann zeta function $\zeta(s)$ as the special case $\mathrm{Li}_s(z=1)$ of them although the last not as a function of *z*). We take into account the connection of the here-considered functions to the function class of Polylogarithms.

2. Fourier Series of $\mathrm{tg}\left(\frac{x}{2}\right)$ and $\mathrm{ctg}\left(\frac{x}{2}\right)$

In this and in the following Sections we develop and discuss the Fourier series of the Tangent function and of related functions. Fourier series were established by Fourier in 1807 in the theory of heat and its propagation. We begin in this Section with the preliminary introduction of the Fourier series for the Tangent function and for its first integral.

We consider the 2π-periodic function $\mathrm{tg}\left(\frac{x}{2}\right)$ for which the following representations are important as starting point for further calculations

$$\mathrm{tg}\left(\frac{x}{2}\right)=\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}=\frac{1-\cos(x)}{\sin(x)}=\frac{\sin(x)}{1+\cos(x)}. \tag{2.1}$$

The function $\mathrm{ctg}\left(\frac{x}{2}\right)$ is simply related to the function $\mathrm{tg}\left(\frac{x}{2}\right)$ by

$$\mathrm{ctg}\left(\frac{x}{2}\right)=\frac{1}{\mathrm{tg}\left(\frac{x}{2}\right)}=\mathrm{tg}\left(\frac{\pi-x}{2}\right)=\sqrt{\frac{1+\cos(x)}{1-\cos(x)}}=\frac{1+\cos(x)}{\sin(x)}=\frac{\sin(x)}{1-\cos(x)}. \tag{2.2}$$

This means that in transitions from formulae for $\mathrm{tg}\left(\frac{x}{2}\right)$ to formulae for $\mathrm{ctg}\left(\frac{x}{2}\right)$ and related functions obtained by analogous differentiations or integrations one has to make the substitutions $x\to\pi-x$, in particular

$$x\to\pi-x,\quad\Leftrightarrow\quad \mathrm{tg}\left(\frac{x}{2}\right)\to\mathrm{ctg}\left(\frac{x}{2}\right):$$

$$\sin(kx)\to\sin(k(\pi-x))=(-1)^{k-1}\sin(kx),\quad \sin^n(x)\to\sin^n(x),$$

$$\cos(kx)\to\cos(k(\pi-x))=(-1)^{k}\cos(kx),\quad \cos^n(x)\to(-1)^n\cos^n(x),$$

$$\sin\left(\frac{x}{2}\right)\to\cos\left(\frac{x}{2}\right),\quad \cos\left(\frac{x}{2}\right)\to\sin\left(\frac{x}{2}\right), \tag{2.3}$$

and it is not necessary to calculate them separately.

The next problem is to obtain the Fourier series of $\mathrm{tg}\left(\frac{x}{2}\right)$. Generally, the Fourier series of a 2π-periodic function

$$f(x) = f(x+2\pi), \tag{2.4}$$

can be represented in the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos(kx) + b_k \sin(kx)\right), \tag{2.5}$$

with the well-known coefficients

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{+\pi} \mathrm{d}x\, f((x)\cos(kx), \quad (k = 0,1,2,\cdots) \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{+\pi} \mathrm{d}x\, f(x)\sin(kx), \quad (k = 1,2,3,\cdots). \end{aligned} \tag{2.6}$$

Without paying attention to convergence we may formally obtain an expression for the Fourier series of $\mathrm{tg}\left(\frac{x}{2}\right)$. In principle, it should be considered as known because in the monograph of Gel'fand and Shilov [8] and in the article of Vilenkin [9] and also in the monograph of Tolstov [13] we found the convergent Fourier series for $-\log\left(2\left|\sin\left(\frac{x}{2}\right)\right|\right)$ from which by differentiation follows the (in the usual sense) divergent Fourier series of $\mathrm{ctg}\left(\frac{x}{2}\right)$ with the result[2]

$$\mathrm{ctg}\left(\frac{x}{2}\right) = 2\sum_{k=1}^{\infty}\sin(kx). \tag{2.7}$$

Then using (2.3) we find the Fourier series of $\mathrm{tg}\left(\frac{x}{2}\right)$ with following result

$$\mathrm{tg}\left(\frac{x}{2}\right) = 2\sum_{k=1}^{\infty}(-1)^{k-1}\sin(kx). \tag{2.8}$$

Vice versa, from this follows using (2.3) the Fourier series (2.7) of $\mathrm{ctg}\left(\frac{x}{2}\right)$. In next section, we will give a more direct derivation of the Fourier series (2.8) of $\mathrm{tg}\left(\frac{x}{2}\right)$.

For further investigations, we calculate the following definite integral over the function $\mathrm{tg}\left(\frac{x}{2}\right)$ in the sense of the Principal value concerning the singularities

$$\int_{0(\pm n2\pi)}^{x(\pm n2\pi)} \mathrm{d}t\,\mathrm{tg}\left(\frac{t}{2}\right) = -2\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right), \tag{2.9}$$

---

[2] With respect to this series Gel'fand and Shilov cite the monograph of Fikhtengol'ts, vol. III [3] (pp. 469-472), see also Vilyenkin [9], p. 463, Tolstov [13] pp. 117-121.

with 2π-periodic continuation on the right-hand side[3]. Thus the integration constant of the indefinite integral is chosen here in such a way that the function on the right-hand side becomes zero for all its minima which are at once symmetry points of the function. The corresponding integral over the function $\mathrm{ctg}\left(\frac{x}{2}\right)$ is

$$\int_{\pi(\pm n2\pi)}^{\pi+x(\pm n2\pi)} \mathrm{d}t\,\mathrm{ctg}\left(\frac{t}{2}\right) = 2\log\left(\left|\sin\left(\frac{x}{2}\right)\right|\right), \tag{2.10}$$

where the integrand is not locally integrable at the singularities $x = \pm 2n\pi, (n = 1, 2, 3, \cdots)$. The mentioned functions are illustrated in **Figure 1**.

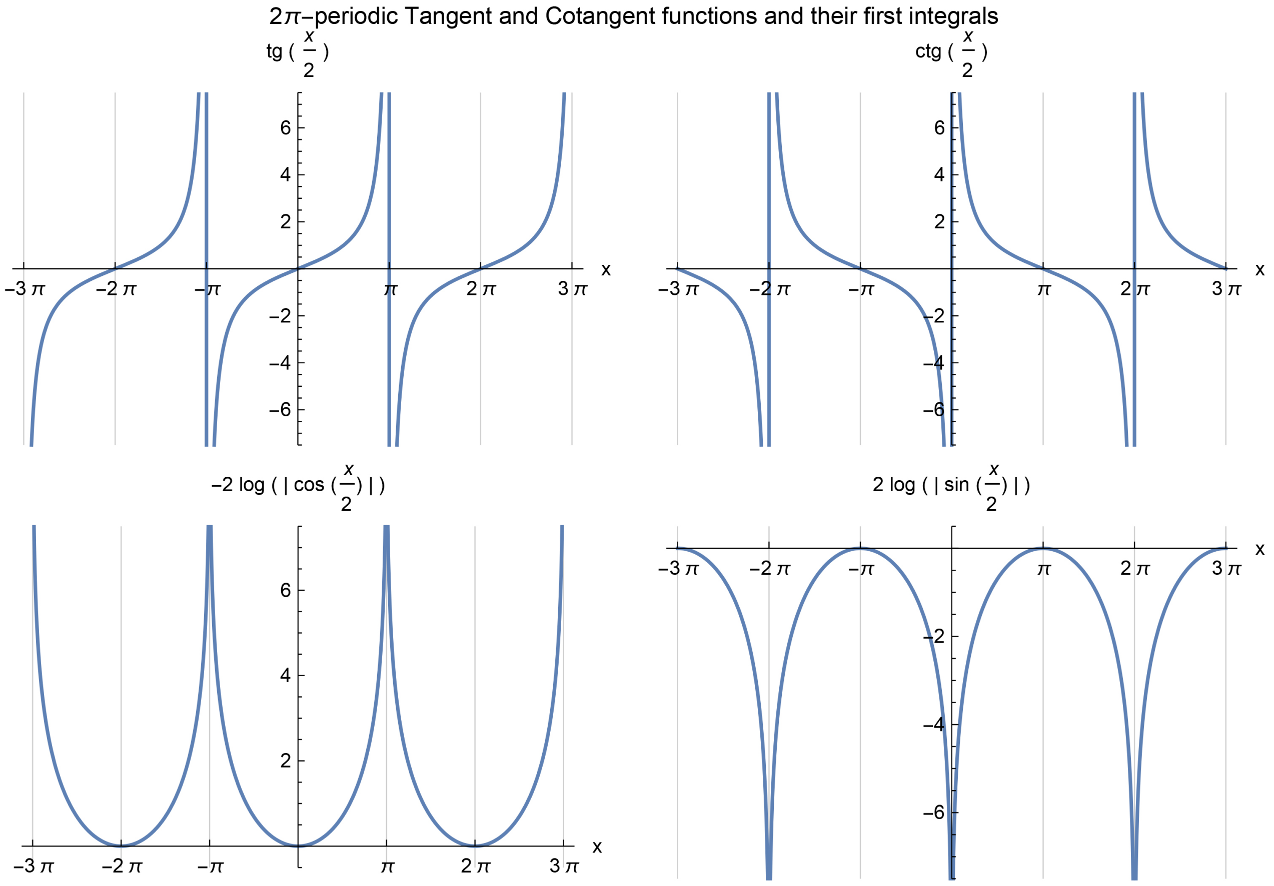


Figure 1. Tangent and Cotangent functions and their definite first integral. The Tangent and Cotangent functions with poles of first order are the derivatives of the logarithmic functions with logarithmic singularities below them.

According to (2.9) from the series (2.8) with term-by-term integration and division by factor -2 follows the convergent Fourier series with exact agreement of

---

[3] The modulus of the trigonometric function in (2.9) and similar formulae is important since the Logarithm as real-valued function of real variable x is only defined for $\left|\cos\left(\frac{x}{2}\right)\right| > 0$. This means that for Trigonometric functions without modulus in the argument of the Logarithm would arise gaps of real-valued definition everywhere where these Trigonometric functions possess negative values.

function and its approximation for argument $x = 0, \pm 2\pi, \pm 4\pi, \cdots$

$$\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\left(1 - \cos(kx)\right) = 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\sin^2\left(\frac{kx}{2}\right), \tag{2.11}$$

and for (2.10), correspondingly with the exact agreement of function and its approximation at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$

$$\log\left(\left|\sin\left(\frac{x}{2}\right)\right|\right) = \sum_{k=1}^{\infty} \frac{1}{k}\left((-1)^k - \cos(kx)\right) = 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\sin^2\left(\frac{k(x-\pi)}{2}\right). \tag{2.12}$$

The right-hand sides of the Fourier series (2.11) and (2.12) possess an uncommon form for Fourier series. Taking into account the following alternating series with conditional convergence with the well-known result in the natural order of summation of terms

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log(2), \tag{2.13}$$

and using this in (2.11) one finds

$$\log\left(\left|2\cos\left(\frac{x}{2}\right)\right|\right) = \log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right) + \log(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\cos(kx), \tag{2.14}$$

and, correspondingly, using (2.12) one finds the known Fourier series [8] [9]

$$\log\left(\left|2\sin\left(\frac{x}{2}\right)\right|\right) = \log\left(\left|\sin\left(\frac{x}{2}\right)\right|\right) + \log(2) = -\sum_{k=1}^{\infty} \frac{1}{k}\cos(kx). \tag{2.15}$$

In [3] is posed and dealt with the inverse problem to determine to a given Fourier series the corresponding function which it represents, in this case the series (2.15) with the function $\log\left(\left|2\sin\left(\frac{x}{2}\right)\right|\right)$ as the result.

3. Derivation of the Fourier Series of $\mathrm{tg}\left(\frac{x}{2}\right)$

A derivation of the Fourier series can be made if one calculates the coefficients by the integrals in (2.6) in case of $\mathrm{tg}\left(\frac{x}{2}\right)$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{+\pi} \mathrm{d}x\,\mathrm{tg}\left(\frac{x}{2}\right)\cos(kx) = 0, \quad (k = 0,1,2,\cdots), \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{+\pi} \mathrm{d}x\,\mathrm{tg}\left(\frac{x}{2}\right)\sin(kx) = 2(-1)^{k-1}, \quad (k = 1,2,\cdots). \end{aligned} \tag{3.1}$$

The integrals a_k vanish due to the antisymmetry of the integrand. The integrals for b_k can be directly calculated or can be taken from an independent calculation of the Fourier series that provides then a result for these integrals. We will present here a derivation of the Fourier series by a method using complex representation of the involved trigonometric functions which is similar to a method used in [3] (pp. 469-471). In the way of this derivation, we meet soon an apparent problem with the complexity of the solution which we try to clarify.

As preparation to the derivation we first write down the following summation formula for the sum of the Geometric series.

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \tag{3.2}$$

which is uniformly convergent for $|q| < 1$.

The representation of $\operatorname{tg}\left(\frac{x}{2}\right)$ by complex exponentials can be transformed then with $q = -\mathrm{e}^{\mathrm{i}x}$ as follows

$$\begin{aligned}
\operatorname{tg}\left(\frac{x}{2}\right) &= \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = -\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\frac{x}{2}} - \mathrm{e}^{-\mathrm{i}\frac{x}{2}}}{\mathrm{e}^{\mathrm{i}\frac{x}{2}} + \mathrm{e}^{-\mathrm{i}\frac{x}{2}}} = \mathrm{i}\frac{1-\mathrm{e}^{\mathrm{i}x}}{1+\mathrm{e}^{\mathrm{i}x}} = \mathrm{i}\left(1-\mathrm{e}^{\mathrm{i}x}\right)\sum_{k=0}^{\infty}(-1)^k \mathrm{e}^{\mathrm{i}kx} \\
&= \mathrm{i}\left(\sum_{k=0}^{\infty}(-1)^k\left(\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{-\mathrm{i}kx}\right) + \sum_{k=0}^{\infty}(-1)^k \mathrm{e}^{-\mathrm{i}kx} - \sum_{k=0}^{\infty}(-1)^k \mathrm{e}^{\mathrm{i}(k+1)x}\right) \\
&= \mathrm{i}\left(\sum_{k=0}^{\infty}(-1)^k\left(\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{-\mathrm{i}kx}\right) + \sum_{l=-\infty}^{0}(-1)^l \mathrm{e}^{\mathrm{i}lx} + \sum_{l=1}^{\infty}(-1)^l \mathrm{e}^{\mathrm{i}lx}\right) \\
&= 2\sum_{k=1}^{\infty}(-1)^{k-1}\sin(kx) + \mathrm{i}\sum_{k=-\infty}^{\infty}(-1)^k \mathrm{e}^{\mathrm{i}kx},
\end{aligned} \tag{3.3}$$

where $|q| = 1$ is only the case at the singularities $x = (2l+1)\pi$ of the function $\operatorname{tg}\left(\frac{x}{2}\right)$. Relation (3.3) is almost but not fully in agreement with (2.8) due to the presence of an additional part $\mathrm{i}\sum_{k=-\infty}^{+\infty}(-1)^k \mathrm{e}^{\mathrm{i}kx}$. The additional part is imaginary-valued but the Tangent function on the left-hand side is real-valued. However, this apparent contradiction resolves since the additional imaginary part proves to be almost zero without the delta function due an identity (see (7.3)). The notation $\operatorname{tg}\left(\frac{x}{2}\right)$ on the left-hand side corresponds then to the Principal value of the function at the singularities[4]. First we write down the mentioned identity. Using well-known formulae of the theory of Generalized functions (see Section 7, (7.1) and (7.10)) from

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} &= 1 + 2\sum_{k=1}^{\infty}\cos(kx) = 2\pi\sum_{l=-\infty}^{\infty}\delta(x - 2l\pi) \\
&= \pi\delta\left(\sin\left(\frac{x}{2}\right)\right) = \pi\delta\left(\operatorname{tg}\left(\frac{x}{2}\right)\right),
\end{aligned} \tag{3.4}$$

which rests on the Poisson summation identity transforming a sum over a lattice into a sum over the reciprocal lattice as follows (see (7.3))

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}(-1)^k \mathrm{e}^{\mathrm{i}kx} &= 1 + 2\sum_{k=1}^{\infty}(-1)^k\cos(kx) = 2\pi\sum_{l=-\infty}^{\infty}\delta\left(x - (2l+1)\pi\right) \\
&= \pi\delta\left(\cos\left(\frac{x}{2}\right)\right) = \pi\delta\left(\operatorname{ctg}\left(\frac{x}{2}\right)\right).
\end{aligned} \tag{3.5}$$

---

[4] The Principal value is often denoted by the symbols "$\mathcal{P}$" or "V. p." prepended to the function.

The singularities of the function $\mathrm{tg}\left(\frac{x}{2}\right)$ at $x=(2l+1)\pi$, $l=0,\pm1,\pm2,\cdots$ approaches the singularities of the functions $-\frac{2}{x-(2l+1)\pi}$ for which in the theory of Generalized functions is derived the well-known formula [7]-[10]

$$\frac{1}{x\pm\mathrm{i}0}=\frac{1}{x}\mp\mathrm{i}\pi\delta(x), \tag{3.6}$$

which means that one adds (or subtracts) in the argument x of the function an infinitesimal small imaginary $\mathrm{i}y=\mathrm{i}\varepsilon$ with positive $\varepsilon>0$ and lets them go $\varepsilon\to+0$. A consequence of (3.6) is

$$\begin{aligned}&-\frac{2}{x-(2l+1)\pi+\mathrm{i}0}\\&=-\frac{2}{x-(2l+1)\pi}+\mathrm{i}2\pi\delta\big(x-(2l+1)\pi\big),(l=0,\pm1,\pm2,\cdots),\end{aligned} \tag{3.7}$$

with positive imaginary amplitude of the delta function. This can be compared now with the general complex extension of $\mathrm{tg}\left(\frac{x}{2}\right)$

$$\mathrm{tg}\left(\frac{x+\mathrm{i}y}{2}\right)=\frac{\sin(x)+\mathrm{i}\,\mathrm{sh}(y)}{\cos(x)+\mathrm{ch}(y)}=\frac{\mathrm{tg}\left(\frac{x}{2}\right)}{1+\frac{\mathrm{sh}^2\left(\frac{y}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}}+\mathrm{i}\frac{\mathrm{th}\left(\frac{y}{2}\right)}{1-\frac{\sin^2\left(\frac{x}{2}\right)}{\mathrm{ch}^2\left(\frac{y}{2}\right)}}. \tag{3.8}$$

One can see that this imaginary part for $y\to0$ indeed vanishes everywhere with the exception of the points of singularities of the function $\mathrm{tg}\left(\frac{x}{2}\right)$. The Tangent function $\mathrm{tg}\left(\frac{x}{2}\right)$ and (3.3) can be written now more rigorously

$$\mathrm{tg}\left(\frac{x+\mathrm{i}0}{2}\right)=2\left(\sum_{k=1}^{\infty}(-1)^{k-1}\sin(kx)+\mathrm{i}\pi\sum_{l=-\infty}^{+\infty}\delta\big(x-(2l+1)\pi\big)\right). \tag{3.9}$$

In analogy to the above derivations for the Tangent function we obtain for the corresponding Cotangent function

$$\mathrm{ctg}\left(\frac{x+\mathrm{i}0}{2}\right)=2\left(\sum_{k=1}^{\infty}\sin(kx)-\mathrm{i}\pi\sum_{l=-\infty}^{\infty}\delta(x-2l\pi)\right). \tag{3.10}$$

The general complex extension of the function $\mathrm{ctg}\left(\frac{x}{2}\right)$ is

$$\mathrm{ctg}\left(\frac{x+\mathrm{i}y}{2}\right)=-\frac{\sin(x)-\mathrm{i}\,\mathrm{sh}(y)}{\cos(x)-\mathrm{ch}(y)}=\frac{\mathrm{ctg}\left(\frac{x}{2}\right)}{1+\frac{\mathrm{sh}^2\left(\frac{y}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}}-\mathrm{i}\frac{\mathrm{cth}\left(\frac{y}{2}\right)}{1+\frac{\sin^2\left(\frac{x}{2}\right)}{\mathrm{sh}^2\left(\frac{y}{2}\right)}}. \tag{3.11}$$

In Section 5 we will come across the function $\operatorname{tg}\left(\frac{x}{2}\right)$ displaced on the real axis by $\frac{\pi}{2}$

$$\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)=\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}=\frac{1+\sin(x)}{\cos(x)}=\frac{\cos(x)}{1-\sin(x)}. \tag{3.12}$$

Its Fourier series follows from (3.9) by the substitutions $\sin(x)\rightarrow\cos(x)$, $\cos(x)\rightarrow-\sin(x)$ and is

$$\operatorname{tg}\left(\frac{x+\frac{\pi}{2}+\mathrm{i}0}{2}\right)=2\left(\sum_{k=1}^{\infty}(-1)^{k-1}\cos(kx)+\mathrm{i}\pi\sum_{l=-\infty}^{+\infty}\delta\left(x-\frac{4l+1}{2}\pi\right)\right). \tag{3.13}$$

In this and other formulae of this kind, in particular, in (3.9) and in (3.10) one has to check the sign of the imaginary part for general values of x and for infinitesimal positive y.

4. Convergence Behavior of the Fourier Series of $\operatorname{tg}\left(\frac{x}{2}\right)$ Illustrated

The Fourier series of the Tangent function (2.8) is not convergent in usual sense but it is convergent in the sense of a continuous linear functional over a space of appropriate basis functions $\varphi=\varphi(x)$ which has to be determined that means

$$\left(\operatorname{tg}\left(\frac{x}{2}\right),\varphi\right)=2\sum_{k=1}^{\infty}(-1)^{k}\left(\sin(kx),\varphi\right). \tag{4.1}$$

In case of sufficiently well-behaved functions $f=f(x), x\in\mathbb{R}$ such as the Sine functions on the right-hand side of (4.1) the linear continuous functional is simply the integral

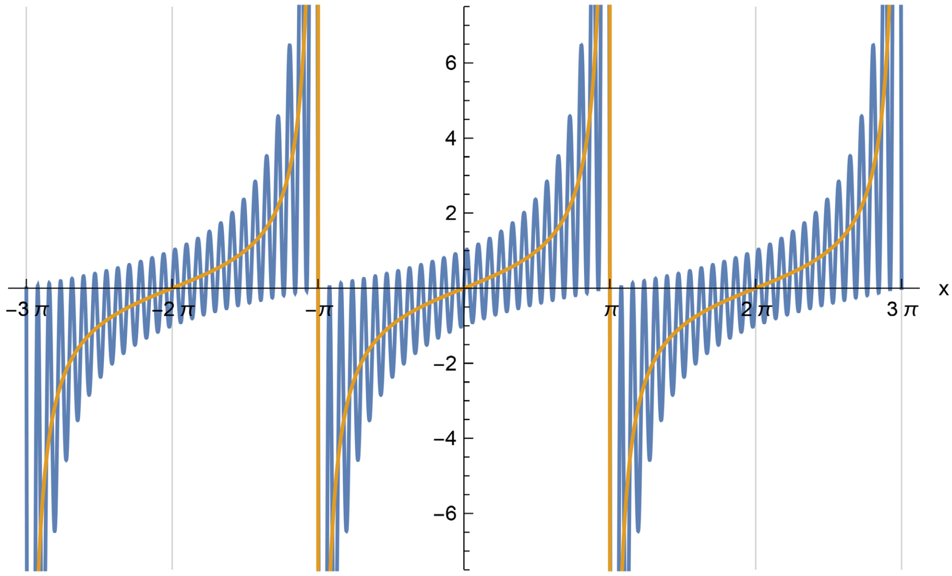
$$(f,\varphi)\equiv\int_{-\infty}^{+\infty}\mathrm{d}x\, f(x)\varphi(x). \tag{4.2}$$

In other cases such as for the delta “function” and its derivatives it has to be defined separately and it has to be proved then that it is a linear continuous functional.

The convergence behavior of Fourier series for the Tangent function (2.8) is illustrated for two different approximations of the number of first sum terms taken into account in **Figure 2**. It is seen that the approximations of the Fourier series for $\operatorname{tg}\left(\frac{x}{2}\right)$ oscillate to both sides of the curve above and below in a certain stripe. This stripe seems to be very stabile with two envelopes which do not depend on the number of sum terms in the approximation of the Fourier series but it is not very clear how to calculate these envelopes. The superposition of two different approximation in **Figure 3** shows this behavior by its homogeneous color. Furthermore, this picture shows that at variable $x = 0$ the envelope functions $\mathrm{env}(x)$, apparently, are exactly $\mathrm{env}(0) = \pm 1$ and for $x = \pm\pi$ are exactly $\mathrm{env}(\pm\pi) = 0$ (in the limit of infinite sum terms). It seems that in a neighborhood of $x = 0$ the heuristically found envelope functions

Approximations to the Fourier series of tg $\left(\frac{x}{2}\right)$

tg $\left(\frac{x}{2}\right)$ (yellow) and its Fourier series with 25 sum terms (blue)



tg $\left(\frac{x}{2}\right)$ (yellow) and its Fourier series with 50 sum terms (blue)

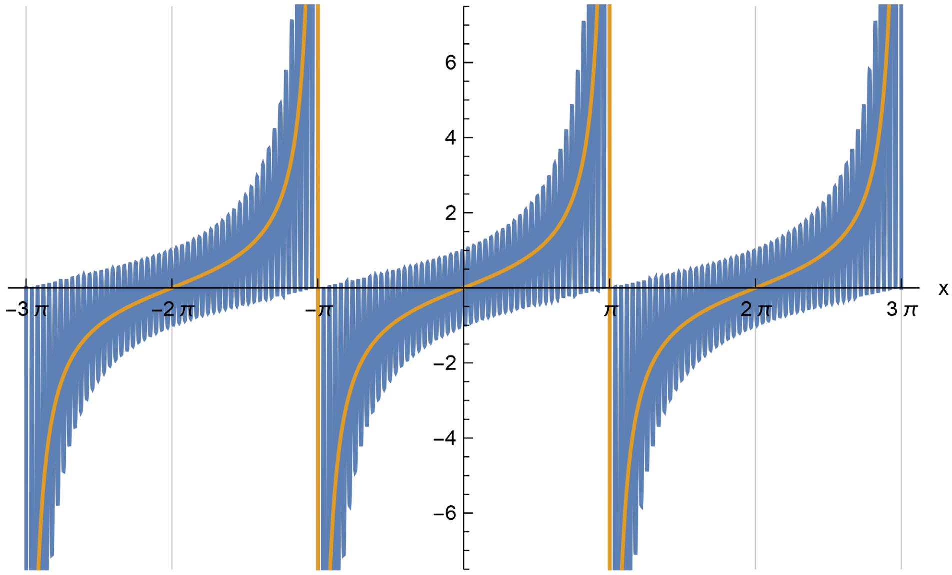


Figure 2. Fourier series of Tangent function in different approximations. In the figure above we used 25 sum terms and in the figure below 50 sum terms of the Fourier series for the approximation. The Tangent function itself is shown embedded in a broad band of oscillations of the approximations by the Fourier series.

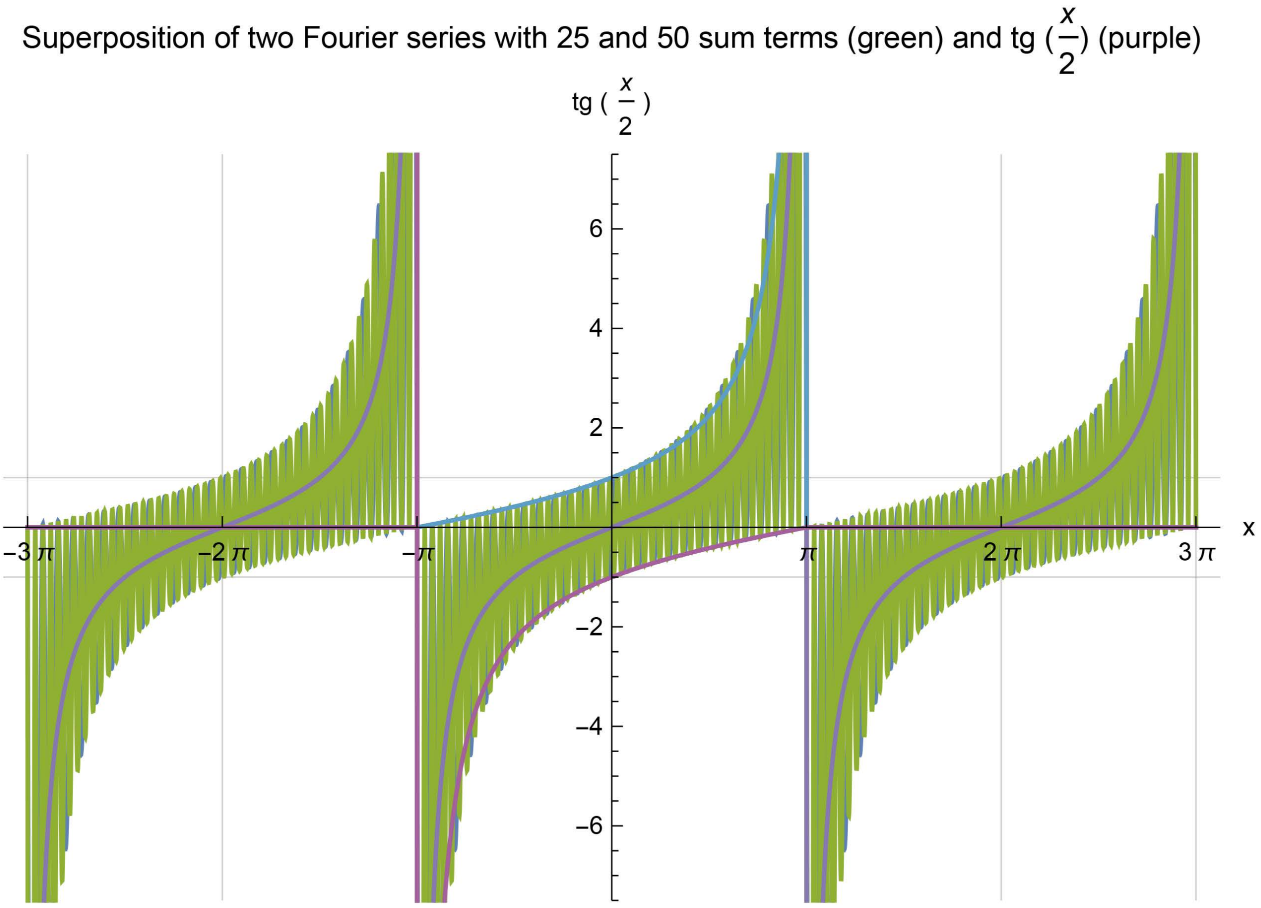


Figure 3. Superposition of two approximations with 25 and 50 sum terms in the Fourier series of $\mathrm{tg}\left(\frac{x}{2}\right)$. Within the range $-\pi \le x \le +\pi$ the conjectured approximate envelopes (4.3) are drawn. The homogeneous color shows that they seem to be very stabile and seem to be independent of the used number of sum terms in the approximation.

$$\mathrm{env}(x) = \frac{x}{2\pi} - \frac{\pi}{x \pm \pi}, \tag{4.3}$$

if periodically continued are "good" approximations for the two envelopes.

As already said the Tangent function $\mathrm{tg}\left(\frac{x}{2}\right)$ has to be considered as a Generalized function and its Fourier series converges in the sense of weak convergence to this function. As the space of basis functions can be chosen the space $\mathcal{S}$ of continuous and continuously differentiable functions which in infinity decrease faster than any power function $\frac{1}{|x|^n}, (n, 1, 2, \cdots)$ to which belong also all Gauss bell functions $\exp\left(-\frac{(x - x_0)^2}{a^2}\right)$ with arbitrary width a and displacement x_0 from the center. The more special class of finite functions of space $\mathcal{D}$ seems to be also appropriate ($\mathcal{D} \subset \mathcal{S}$) instead of $\mathcal{S}$. The Tangent function $\mathrm{tg}\left(\frac{x}{2}\right)$ can be then considered as a Generalized function of the space $\mathcal{D}' \supset \mathcal{S}'$.

When the Tangent function is combined in integrals with such functions then these integrals or linear functionals converge in ordinary sense.

The Fourier series of the first integral of the function $\operatorname{tg}\left(\frac{x}{2}\right)$ which is the function $-2\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)$ is convergent in ordinary sense. This is illustrated in **Figure 4** for two different (low) numbers of sum terms in the approximations. They do not show striking peculiarities. Fourier series within the frame of Generalized functions are discussed in all of the here cited works about Generalized functions [7]-[10] but similar to considered examples of such functions we found only in [8] [9] and additionally in [13] without support on Generalized functions.

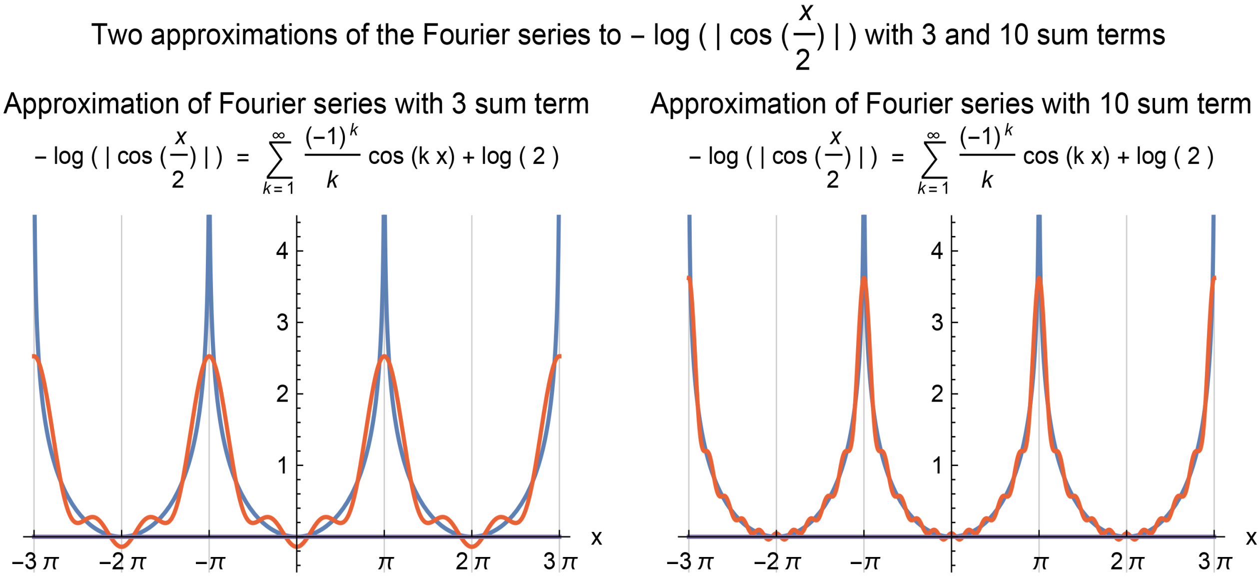


Figure 4. Approximations of the Fourier series (2.11) to function $-\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)$ with 5 and 6 terms. The Fourier series converges in ordinary sense to mentioned function. The term $\log(2)$ was taken as a whole and was not taken in the same approximation with only N sum terms of (2.13).

5. Fourier Series of $\csc(x)$ and $\sec(x)$

In analogy to Sections 2 and 3 we investigate now the Fourier series for the 2π-periodic function $\csc(x)$

$$\csc(x) \equiv \frac{1}{\sin(x)} = \frac{1}{\cos\left(x - \frac{\pi}{2}\right)}, \tag{5.1}$$

which similar to the function $\operatorname{tg}\left(\frac{x}{2}\right)$ possess poles of first order at

$x = 0, \pm 2\pi, \pm 4\pi, \cdots$ and analogously the function $\sec(x)$

$$\sec(x) \equiv \frac{1}{\cos(x)} = \frac{1}{\sin\left(x+\frac{\pi}{2}\right)}, \tag{5.2}$$

with poles of first order at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$, the same kinds of poles as $\operatorname{tg}\left(\frac{x}{2}\right)$ and $\operatorname{ctg}\left(\frac{x}{2}\right)$. The two kinds of functions $\operatorname{tg}\left(\frac{x}{2}\right)$ and $\operatorname{ctg}\left(\frac{x}{2}\right)$ on one side and $\frac{1}{\sin[x]}$ on the other side are connected by

$$\begin{aligned} \frac{1}{\sin(x)} &= \frac{1}{2\sin^2\left(\frac{x}{2}\right)} \operatorname{tg}\left(\frac{x}{2}\right) = \frac{1}{1-\cos(x)} \operatorname{tg}\left(\frac{x}{2}\right) \\ &= \frac{1}{2\cos^2\left(\frac{x}{2}\right)} \operatorname{ctg}\left(\frac{x}{2}\right) = \frac{1}{1+\cos(x)} \operatorname{ctg}\left(\frac{x}{2}\right). \end{aligned} \tag{5.3}$$

The two functions $\frac{1}{\sin(x)}$ and $\frac{1}{\cos(x)}$ are only shifted to each other by $\pm\frac{\pi}{2}$ on the real axis and therefore we make derivations only for the first of these functions.

The first integrals are for (5.1) [14]

$$\int_{\frac{\pi}{2}(\pm n2\pi)}^{\frac{\pi}{2}+x(\pm n2\pi)} \mathrm{d}t \frac{1}{\sin(t)} = \log\left(\left|\operatorname{tg}\left(\frac{x}{2}\right)\right|\right) = \log\left(\left|\sin\left(\frac{x}{2}\right)\right|\right) - \log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right), \tag{5.4}$$

and for (5.2)

$$\begin{aligned} \int_{0(\pm n2\pi)}^{x(\pm n2\pi)} \mathrm{d}t \frac{1}{\cos(t)}\left(\frac{t}{2}\right) &= \log\left(\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right) \\ &= \log\left(\left|\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right) - \log\left(\left|\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right), \end{aligned} \tag{5.5}$$

that can be checked by differentiation. The functions together with the Cosecant and Secant functions are illustrated in **Figure 5**. Since by (2.11) and (2.12) we know already the Fourier series of the two parts in (5.4) and (5.5) we have also their Fourier series and by term-wise differentiation the Fourier series of $\frac{1}{\sin(x)}$ and $\frac{1}{\cos(x)}$. The result for the Fourier series of the function (5.5)

$$\log\left(\left|\operatorname{tg}\left(\frac{x}{2}\right)\right|\right) = -2\sum_{k=0}^{\infty} \frac{1}{2k+1} \cos\left((2k+1)x\right), \tag{5.6}$$

and for the function (5.5), correspondingly, with the variable substitution $x \to x + \frac{\pi}{2}$ in (2.10)

$$\begin{aligned} \log\left(\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right) &= -2\sum_{k=0}^{\infty} \frac{1}{2k+1} \cos\left((2k+1)\left(x+\frac{\pi}{2}\right)\right) \\ &= 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \sin\left((2k+1)x\right). \end{aligned} \tag{5.7}$$

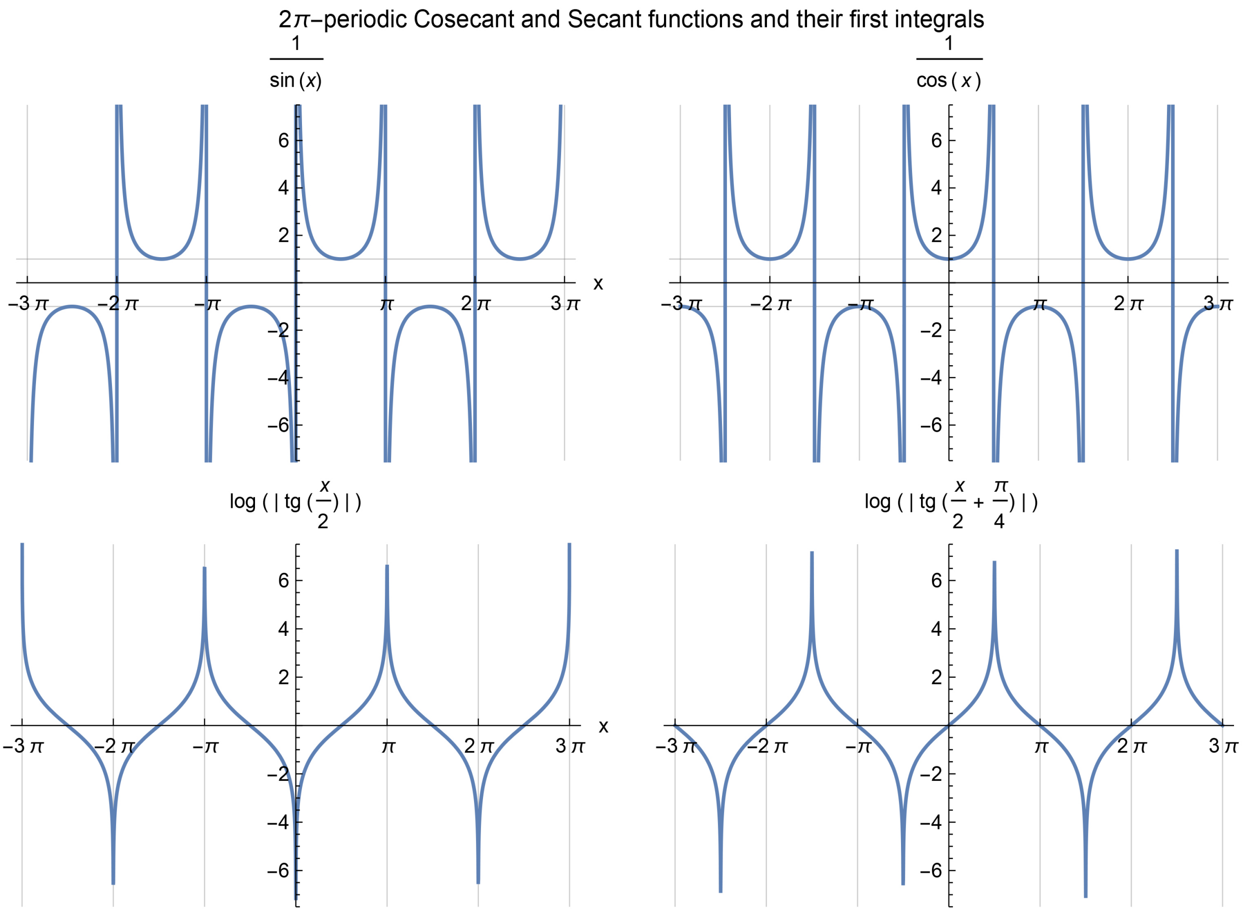


Figure 5. Cosecant and Secant functions and their definite first integral. The Cosecant and Secant functions are the derivatives of the functions below them which possess 2π-periodic logarithmic singularities.

They converge in ordinary way. By term-wise differentiation of (5.6) we find the Fourier series of $\frac{1}{\sin(x)}$

$$\frac{1}{\sin(x)} = 2\sum_{k=0}^{\infty}\sin((2k+1)x), \tag{5.8}$$

and by term-wise differentiation of (5.7) the Fourier series of $\frac{1}{\cos(x)}$

$$\frac{1}{\cos(x)} = 2\sum_{k=0}^{\infty}(-1)^k \sin((2k+1)x). \tag{5.9}$$

The last two Fourier series (5.8) and (5.9) can be only considered in the sense of Generalized functions and their convergence is weak convergence of Generalized functions.

Before we illustrate some previous results by figures in next Section we want to derive now the Fourier series (5.8) in another way in analogy to (3.3) by transition to the complex representation with application of the geometric series as follows

$$
\begin{aligned}
\frac{1}{\sin(x)} &= \mathrm{i}\frac{2}{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}} = -\mathrm{i}\frac{2\mathrm{e}^{\mathrm{i}x}}{1-\mathrm{e}^{\mathrm{i}2x}} = -\mathrm{i}2\sum_{k=0}^{\infty}\mathrm{e}^{\mathrm{i}(2k+1)x} \\
&= -\mathrm{i}2\sum_{k=0}^{\infty}\left\{\cos\left((2k+1)x\right)+\mathrm{i}\sin\left((2k+1)x\right)\right\} \\
&= 2\sum_{k=0}^{\infty}\sin\left((2k+1)x\right)-\mathrm{i}2\sum_{k=0}^{\infty}\cos\left((2k+1)x\right).
\end{aligned}
\tag{5.10}
$$

The function $\frac{1}{\sin(x)}$ is real-valued but on the right-hand side we find an imaginary-valued additional term. Due to the identity (7.9) this additional term consists of 2π-periodic delta functions with positive and negative amplitudes and is zero in most of its range. In usual notation the left-hand side of (5.10) means the principal value of the function at the singularities which is real-valued. More correct are therefore the notations

$$
\frac{1}{\sin(x+\mathrm{i}0)} = 2\sum_{k=0}^{\infty}\sin\left((2k+1)x\right)-\mathrm{i}2\sum_{k=0}^{\infty}\cos\left((2k+1)x\right),
\tag{5.11}
$$

and its principal value

$$
\frac{1}{\sin(x)} = 2\sum_{k=0}^{\infty}\sin\left((2k+1)x\right).
\tag{5.12}
$$

In this connection we give the following complex extension of the function $\frac{1}{\sin(x)}$

$$
\frac{1}{\sin(x+\mathrm{i}y)} = \frac{1}{\sin(x)}\frac{\mathrm{ch}(y)}{1+\frac{\mathrm{sh}^2(y)}{\sin^2(x)}} - \mathrm{i}\frac{1}{\mathrm{sh}(y)}\frac{\cos(x)}{1+\frac{\sin^2(x)}{\mathrm{sh}^2(y)}}.
\tag{5.13}
$$

In next Section we illustrate the convergence behavior of the Fourier series of $\frac{1}{\sin(x)}$.

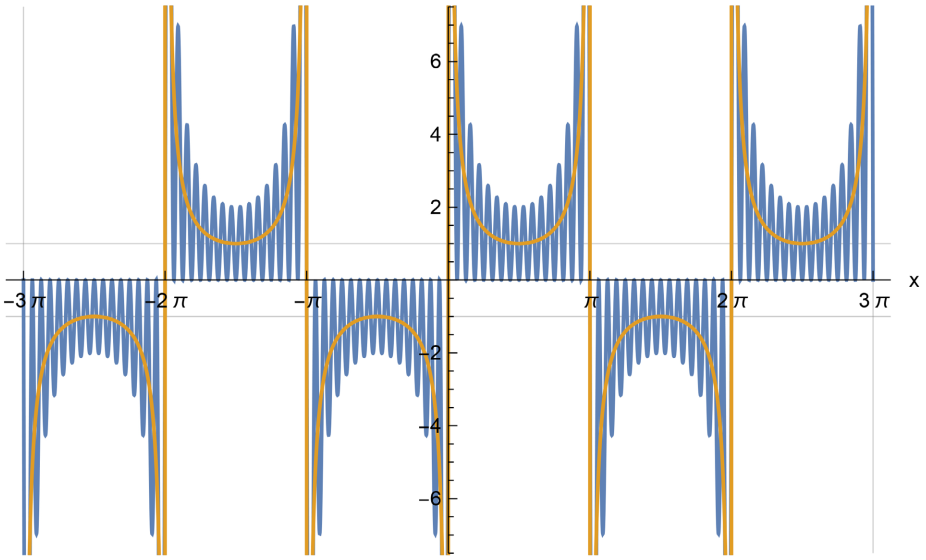
6. Convergence Behavior of the Fourier Series of $\frac{1}{\sin(x)}$ Illustrated

We illustrate now the convergence behavior of the Fourier series (5.12) for the function $\frac{1}{\sin(x)}$ (principal value) and the same for its first integral (5.6) which is the function $\log\left|\mathrm{tg}\left(\frac{x}{2}\right)\right|$.

Figure 6 shows two approximations with different numbers of taken sum terms in the Fourier series of $\frac{1}{\sin(x)}$. We see in this picture oscillations around the function which it should represent and the convergence is weak convergence in the sense of the theory of Generalized functions. This is very similar to **Figure 2** for the function $\operatorname{tg}\left(\frac{x}{2}\right)$. As in mentioned **Figure 2** one believes to see two envelopes of the oscillations which by a suited definition not available up to now could be calculated. These envelopes apparently remain very stabile if we go to higher approximations and surely to the limiting case. One of the envelopes is obviously the x-axis

Approximations to the Fourier series of $\frac{1}{\sin(x)}$

$\frac{1}{\sin(x)}$ (yellow) and its Fourier series with 15 sum terms (blue)



$\frac{1}{\sin(x)}$ (yellow) and its Fourier series with 30 sum terms (blue)

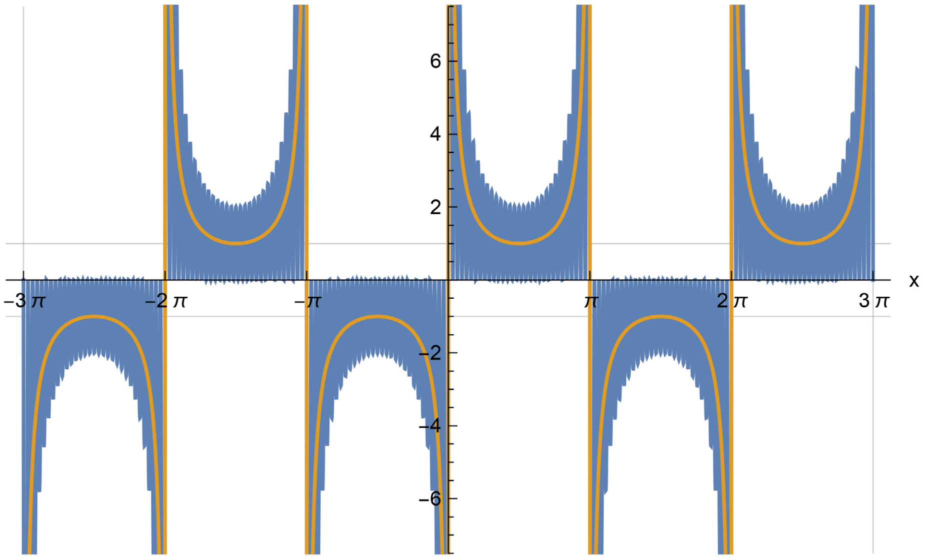


Figure 6. Fourier series of reciprocal Sine function in different approximations. In the figure above we used 15 sum terms and in the figure below 30 sum terms of their Fourier series for the approximation.

$$\mathrm{env}_1(x) = 0, \tag{6.1}$$

and for the other upper and lower envelopes we found heuristically the approximations

$$\mathrm{env}_2(x) = \pm 2\left(1 - \log\left(\left|\sin(x)\right|\right)\right). \tag{6.2}$$

In **Figure 7** is made a superposition of the two approximations of the **Figure 6** with different numbers of taken sum terms and the uniform color of the oscillations shows that for these approximations is already reached a sufficient stability of the envelopes. These envelopes are drawn in **Figure 7** in the range $-\pi \le x \le +\pi$ and they show that the approximations are very well in the neighborhood of the minima or maxima, respectively but not very well in the neighborhood of the singularities.

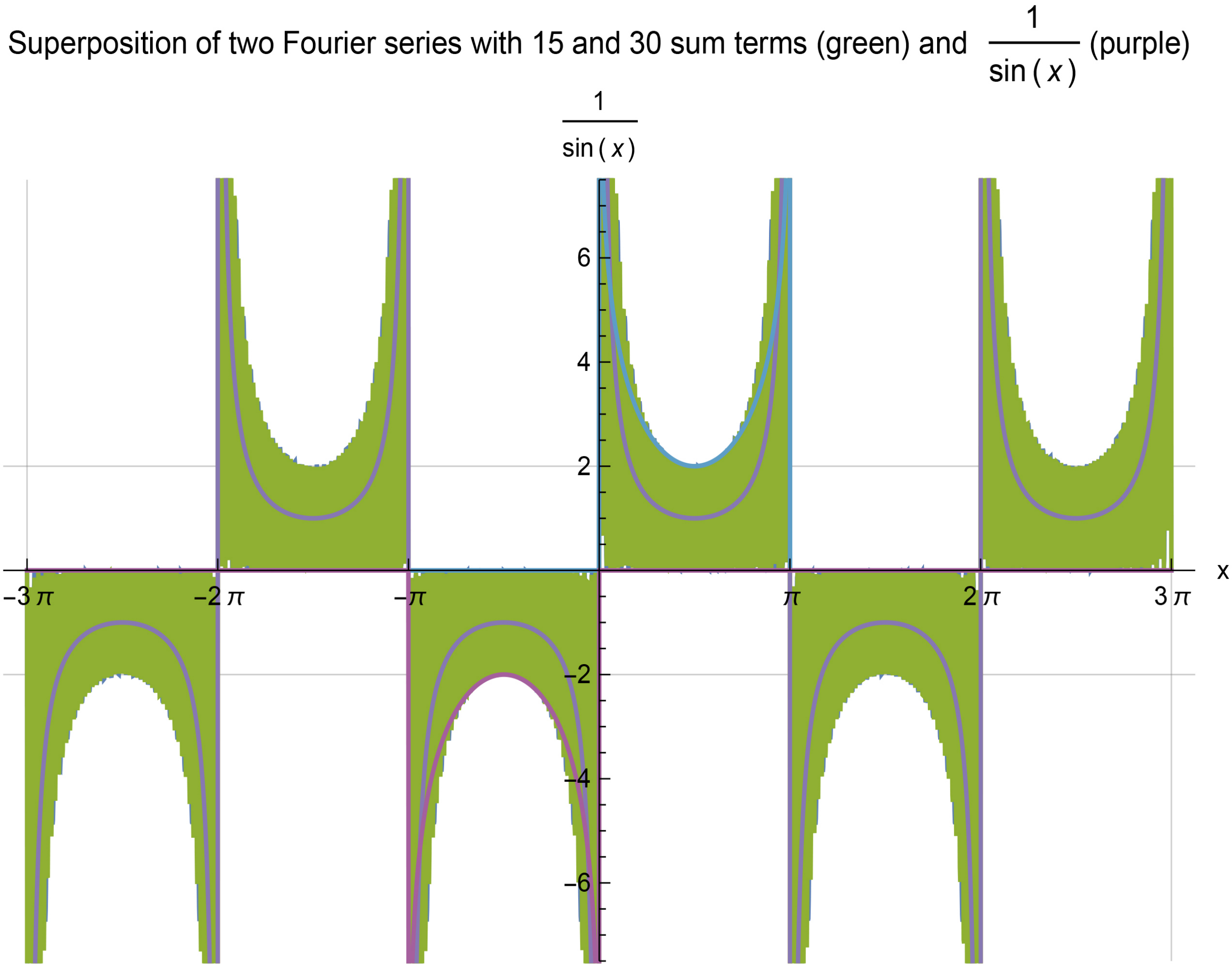


Figure 7. Superposition of two approximations with 15 and 30 sum terms in Fourier series of $\frac{1}{\sin(x)}$. This superposition shows that changing the number of taken sum terms of the Fourier series does not possess an essential influence to the envelope of the oscillations. Within the range, $-\pi < x < +\pi$ the conjectured envelope functions (6.2) are drawn. In the neighborhood of the singularities, they are not satisfying approximations.

Figure 8 shows two approximations for the first integral of the function $\frac{1}{\sin(x)}$ in analogy to **Figure 4**. The Fourier series for this function converges already in the ordinary sense of uniform convergence to the functions that they represent.

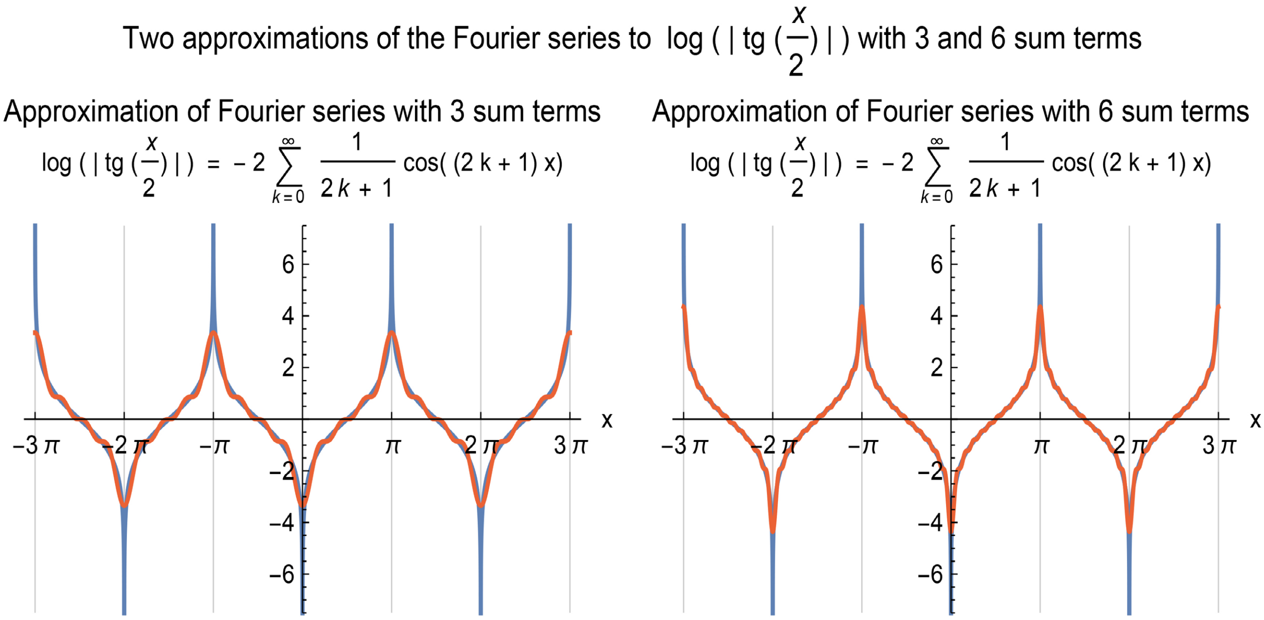


Figure 8. Approximations of the Fourier series to function $\log\left|\mathrm{tg}\left(\frac{x}{2}\right)\right|$ with 3 and 6 terms. The Fourier series converges fast in ordinary sense to the given function.

7. Derivation of Identities for 2π-Periodic Generalized Functions Resting on the Poisson Summation Formula

In the following, we derive identities for 2π-periodically distributed delta functions resting on the Poisson summation formula and investigate the convergence properties of their corresponding Fourier series. From the Poisson summation formula follows the well-known basic relation (e.g., [7]-[10])

$$\sum_{k=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}kx} = 2\pi \sum_{l=-\infty}^{\infty} \delta(x-2l\pi). \tag{7.1}$$

On the left-hand side, we have a complex-valued function and on the right-hand side a real-valued function. Therefore the imaginary part of the left-hand side has to be identically zero

$$\mathrm{i}\sum_{k=-\infty}^{\infty} \sin(kx) = 0. \tag{7.2}$$

This is also immediately seen from the antisymmetry of the Sine function $\sin(-y) = -\sin(y)$.

Using $(-1)^k \mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{\mathrm{i}k(x-\pi)}$ from (7.1) follows the identity

$$\begin{aligned}\sum_{k=-\infty}^{\infty}(-1)^k \mathrm{e}^{\mathrm{i}kx} &= \sum_{k=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(x-\pi)} = 2\pi\sum_{l=-\infty}^{\infty}\delta\big((x-\pi)-2l\pi\big)\\ &= 2\pi\sum_{l=-\infty}^{+\infty}\delta\big(x-(2l+1)\pi\big),\end{aligned} \tag{7.3}$$

with 2π-periodic delta functions on the x-axis displaced by π on the right-hand side in comparison to (7.1).

We derive now further identities from (7.1). Due to (7.2) relation (7.1) can be written

$$\sum_{k=-\infty}^{\infty}\cos(kx) = 1+2\sum_{k=1}^{\infty}\cos(kx) = 2\pi\sum_{l=-\infty}^{\infty}\delta(x-2l\pi), \tag{7.4}$$

and relation (7.3) correspondingly

$$\sum_{k=-\infty}^{\infty}(-1)^k\cos(kx) = 1+2\sum_{k=1}^{\infty}(-1)^k\cos(kx) = 2\pi\sum_{l=-\infty}^{+\infty}\delta\big(x-(2l+1)\pi\big), \tag{7.5}$$

where we used the symmetry of the Cosine function $\cos(-y)=\cos(y)$. Approximations of the Fourier series (7.4) on the left-hand side to the Generalized functions on the right-hand side are illustrated in **Figure 9** (Fourier series (7.5) is not illustrated).

From (7.4) by substitution $x \rightarrow 2x$ we find

$$\frac{1}{2}+\sum_{k=1}^{\infty}\cos(2kx) = \pi\sum_{l=-\infty}^{\infty}\delta\big(2(x-l\pi)\big) = \frac{\pi}{2}\sum_{l=-\infty}^{\infty}\delta(x-l\pi), \tag{7.6}$$

where we used a known relation for delta functions with arguments multiplied by a real factor. Furthermore, using (7.4) and (7.6)

$$\begin{aligned}\sum_{k=0}^{+\infty}\cos\big((2k+1)x\big) &= \frac{1}{2}+\sum_{k=1}^{\infty}\cos(kx)-\left(\frac{1}{2}+\sum_{k=1}^{\infty}\cos(2kx)\right)\\ &= \pi\sum_{l=-\infty}^{\infty}\delta(x-2l\pi)-\frac{\pi}{2}\sum_{l=-\infty}^{\infty}\delta(x-l\pi).\end{aligned} \tag{7.7}$$

The identity (7.6) can be alternatively represented in the form

$$\frac{1}{2}+\sum_{k=1}^{\infty}\cos(2kx) = \frac{\pi}{2}\sum_{l=-\infty}^{+\infty}\big\{\delta(x-2l\pi)+\delta\big(x-(2l+1)\pi\big)\big\}, \tag{7.8}$$

and the identity (7.7) correspondingly

$$\sum_{k=0}^{\infty}\cos\big((2k+1)x\big) = \frac{\pi}{2}\sum_{l=-\infty}^{\infty}\big\{\delta(x-2l\pi)-\delta\big(x-(2l+1)\pi\big)\big\}, \tag{7.9}$$

which are more analogous to each other than the representations (7.6) and (7.7).

We mention here in addition the well-known formula for the delta function $\delta\big(f(x)\big)$ with a function in its argument [7]

$$\delta\big(f(x)\big) = \sum_l \frac{\delta(x-x_l)}{\left|f^{(1)}(x_l)\right|},\quad \big(f(x_l)=0\big)\quad \Rightarrow\quad \delta^{(n)}\big(f(x)\big) = \sum_l \frac{\delta^{(n)}(x-x_l)}{\left|f^{(1)}(x_l)\right|}, \tag{7.10}$$

where the summation goes over all simple zeros of the function $f(x)$. Therefore

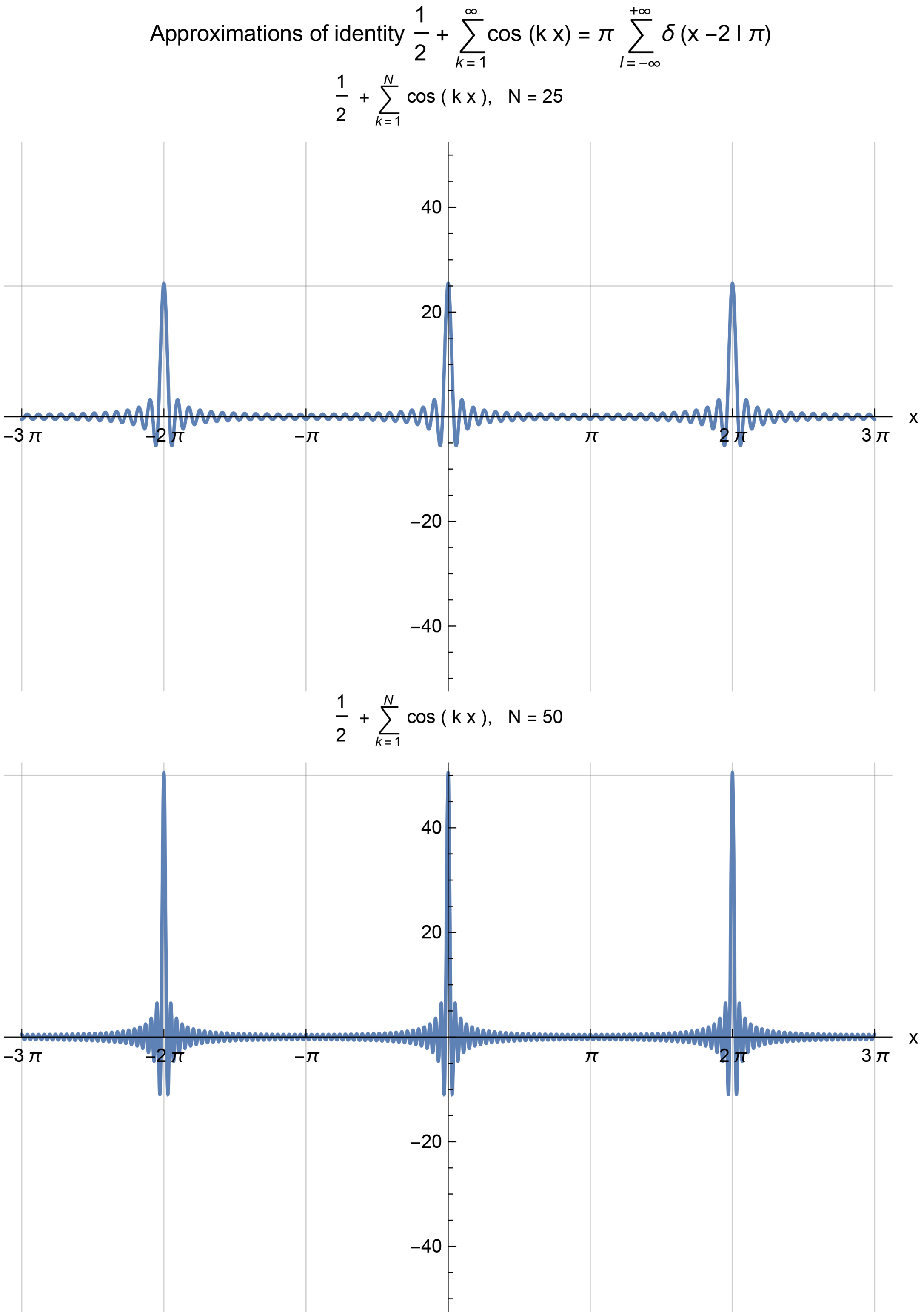


Figure 9. Approximations of the Fourier series $\frac{1}{2}+\sum_{k=1}^{\infty}\cos(kx)$ to identity (7.4). The Fourier series of the 2π-periodic delta functions show oscillations in the neighborhood of the singularities which are comparable to the oscillations in the Gibbs phenomenon in the neighborhood of the discontinuities of the functions. The envelopes to these oscillations seem to be very stabile.

$$\delta\left(\sin\left(x\right)\right)=\sum_{l=-\infty}^{+\infty}\delta\left(x-l\pi\right),$$

$$\delta\left(\cos\left(x\right)\right)=\sum_{l=-\infty}^{+\infty}\delta\left(x-\left(l+\frac{1}{2}\right)\pi\right),\tag{7.11}$$

and the right-hand sides of the above formulae and of some formulae of Section 3 can be represented with it.

Approximations of the Fourier series for the considered 2π-periodic Generalized function (7.8) are represented in **Figure 10** and for the Generalized function (7.9) in **Figure 11**. For comparison, all these Figures are represented in a frame with the same dimensions.

The delta function $\delta\left(x\right)$ becomes infinite at $x=0$ but in the sense of Generalized functions, it is not correct to speak about a function value at the singularity. In the approximations of periodic delta functions by their Fourier series, they possess finite amplitudes at all singularities which depend very much on the number of taken sum terms that is to be seen in the figures. In the neighborhood of the singularities the oscillations in the truncated Fourier series increase but independently from the number of taken sum terms they form envelopes that never go exactly to zero as one would expect from the delta function off the singularities. They even go to negative values there if we consider delta functions with positive amplitudes and to positive values if we consider delta functions with negative amplitudes. This effect is somehow similar to the Gibbs phenomenon of Fourier series at points of discontinuities of the 2π-periodic functions which they represent.

8. Representation of 2π-Periodic Function by Power-Like Series of Cosine Functions Together with a Sine Function

Using the last of the representations of $\mathrm{tg}\left(\frac{x}{2}\right)$ given in (2.1) we find that this function can be expanded in a Geometric series of powers of $\cos\left(x\right)$ (with a nonsingular multiplier function, here $\sin\left(x\right)$) as follows

$$\mathrm{tg}\left(\frac{x}{2}\right)=\sin\left(x\right)\sum_{n=0}^{\infty}\left(-1\right)^{n}\cos^{n}\left(x\right),\tag{8.1}$$

and analogously of $\mathrm{ctg}\left(\frac{x}{2}\right)$ according to (2.2)

$$\mathrm{ctg}\left(\frac{x}{2}\right)=\sin\left(x\right)\sum_{n=0}^{\infty}\cos^{n}\left(x\right).\tag{8.2}$$

The powers $\cos^{n}\left(x\right)$ of $\cos\left(x\right)$ are nonorthogonal to each other. This is similar to Taylor series $\sum_{n=0}^{\infty}\frac{f^{(n)}\left(0\right)}{n!}z^{n}$ where the power functions z^{n} of z are also nonorthogonal to each other. In the following we show that such power series as (8.1) and (8.2) possess a faster convergence behavior as the corresponding Fourier series but this must not be true for all Fourier series.

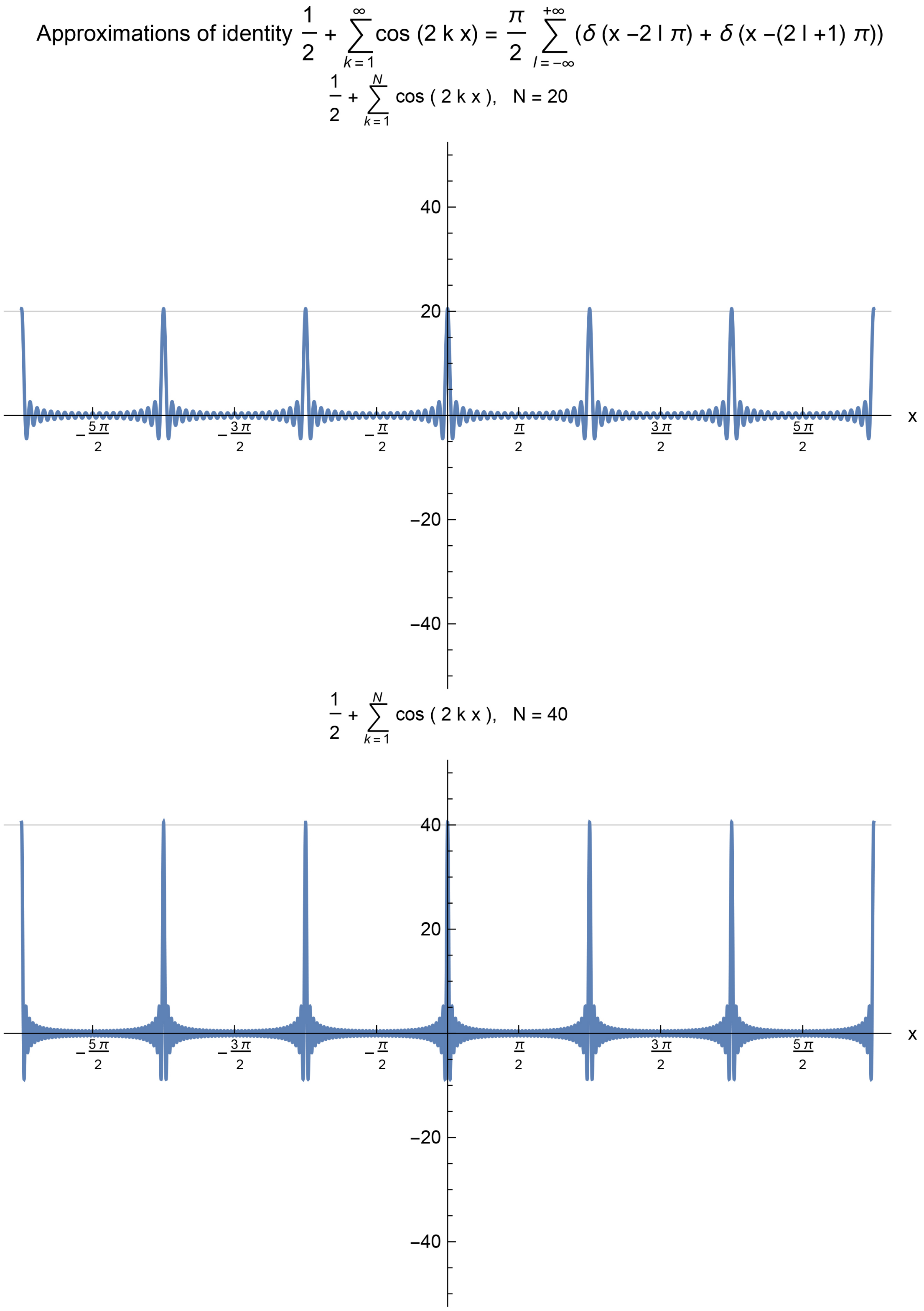


Figure 10. Approximations of Fourier series $\frac{1}{2}+\sum_{k=1}^{\infty}\cos(2kx)$ to identity (7.8). Apparently, the oscillations of the approximations in the neighborhood of the delta functions seem to be very stabile to both sides of the abscissa and do not vanish in the limiing case.

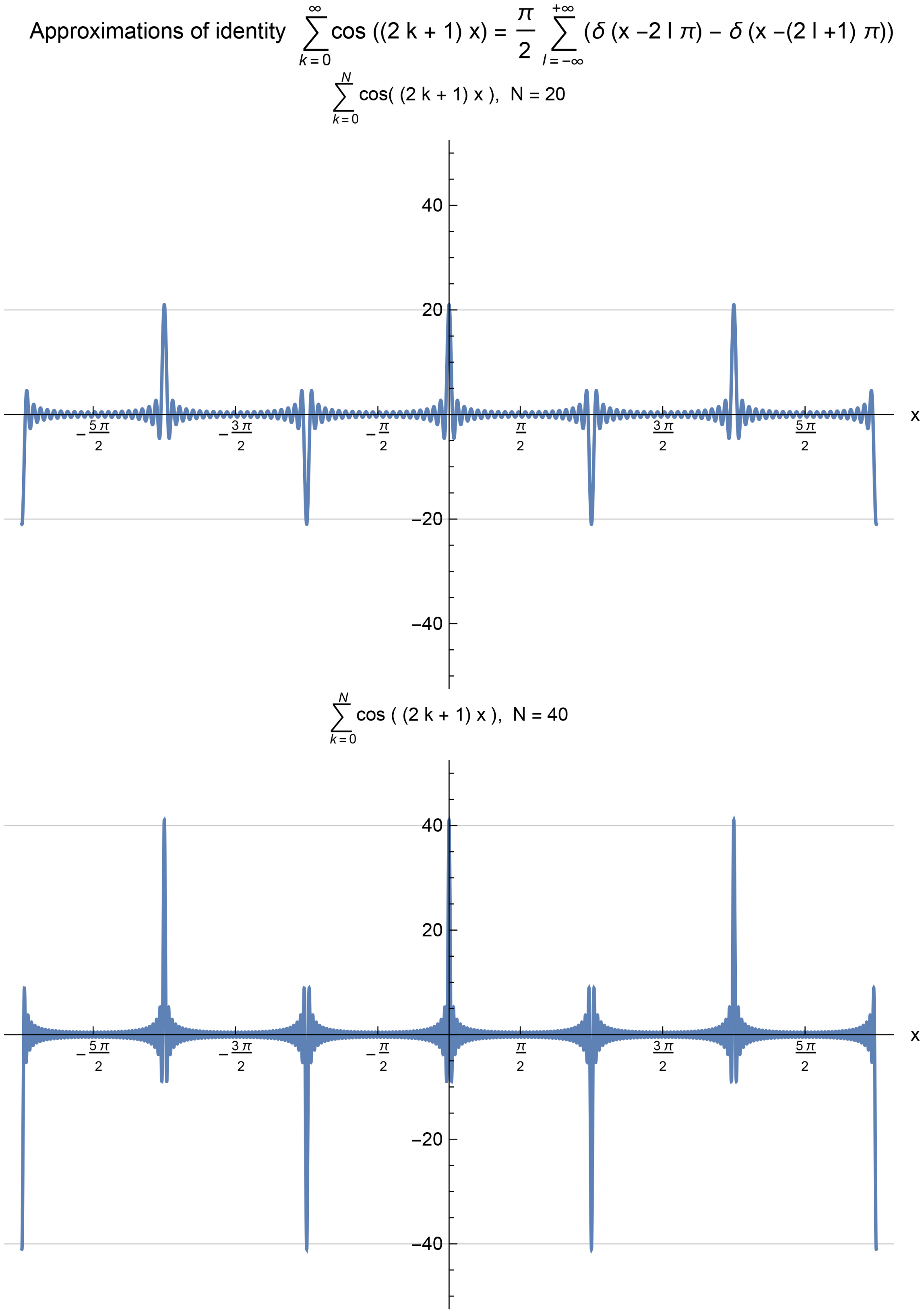


Figure 11. Approximations of Fourier series $\sum_{k=0}^{\infty} \cos((2k+1)x)$ to identity (7.9). Again, the oscillations in the neighborhood of the delta functions are very stabile in their envelope.

If we bring the factor $\sin(x)$ from the right-hand to the left-hand side and $\operatorname{tg}\left(\frac{x}{2}\right)$ or $\operatorname{ctg}\left(\frac{x}{2}\right)$ from the left-hand side to the right-hand side in (8.1) and (8.2) follows

$$\frac{1}{\sin x}=\operatorname{ctg}\left(\frac{x}{2}\right)\sum_{n=0}^{\infty}(-1)^n\cos^n(x), \tag{8.3}$$

and

$$\frac{1}{\sin x}=\operatorname{tg}\left(\frac{x}{2}\right)\sum_{n=0}^{\infty}\cos^n(x). \tag{8.4}$$

Using furthermore (5.3) we find the expansions

$$\frac{1}{2\cos^2\left(\frac{x}{2}\right)}=\frac{1}{1+\cos(x)}=\sum_{n=0}^{\infty}(-1)^n\cos^n(x), \tag{8.5}$$

and

$$\frac{1}{2\sin^2\left(\frac{x}{2}\right)}=\frac{1}{1-\cos(x)}=\sum_{n=0}^{\infty}\cos^n(x) \tag{8.6}$$

Due to the similarity to Taylor series we call such expansions power-like series of Trigonometric functions.

The convergence behavior of (8.1) and (8.2) is in contrast to the convergence behavior of its Fourier series (see **Figure 2** and **Figure 3**) and is uniform convergence on the whole real axis. This is illustrated for the function $\operatorname{tg}\left(\frac{x}{2}\right)$ in **Figure 12** for two numbers of sum terms taken into account. **Figure 13** illustrates the convergence behavior of the power-like series for the function $\frac{1}{\sin(x)}$ which is also in full contrast to the convergence behavior of its Fourier series (see **Figure 6** and **Figure 7**).

The term-wise integration of the series for $\operatorname{tg}\left(\frac{x}{2}\right)$ using (2.9) provides

$$-2\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)=\log(2)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos^n(x), \tag{8.7}$$

and from the series (2.10) for $\operatorname{tg}\left(\frac{x}{2}\right)$ using (2.9)

$$2\log\left(\left|\sin\left(\frac{x}{2}\right)\right|\right)=-\log(2)-\sum_{n=1}^{\infty}\frac{1}{n}\cos^n(x). \tag{8.8}$$

The convergence behavior of the power-like series of $-2\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)$ which is the first integral of $\operatorname{tg}\left(\frac{x}{2}\right)$ is illustrated in **Figure 14** for two numbers of sum terms taken into account. The Fourier series of this functions (see **Figure 4**) is convergent in ordinary sense and the convergence behavior of their power-like series is similar.

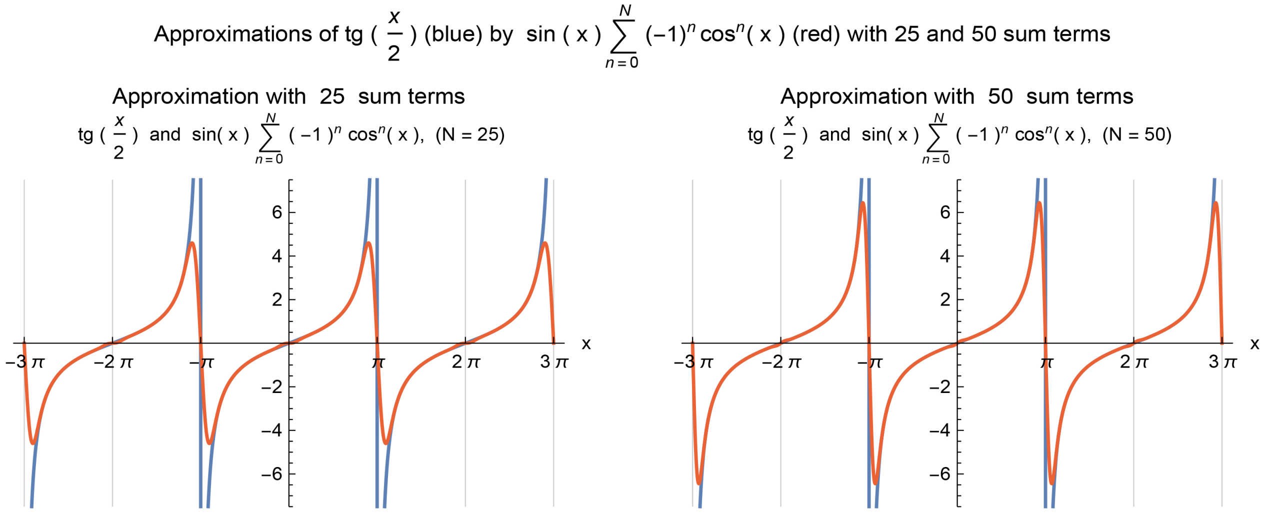


Figure 12. Two approximations of power-like series $\operatorname{tg}\left(\frac{x}{2}\right)=\sin(x)\sum_{n=0}^{\infty}(-1)^n\cos^n(x)$. These series possess another convergence behavior than the corresponding Fourier series and converge uniformly on the whole range of definition of the function. The differences between the two approximations with different numbers of sum terms are rather small.

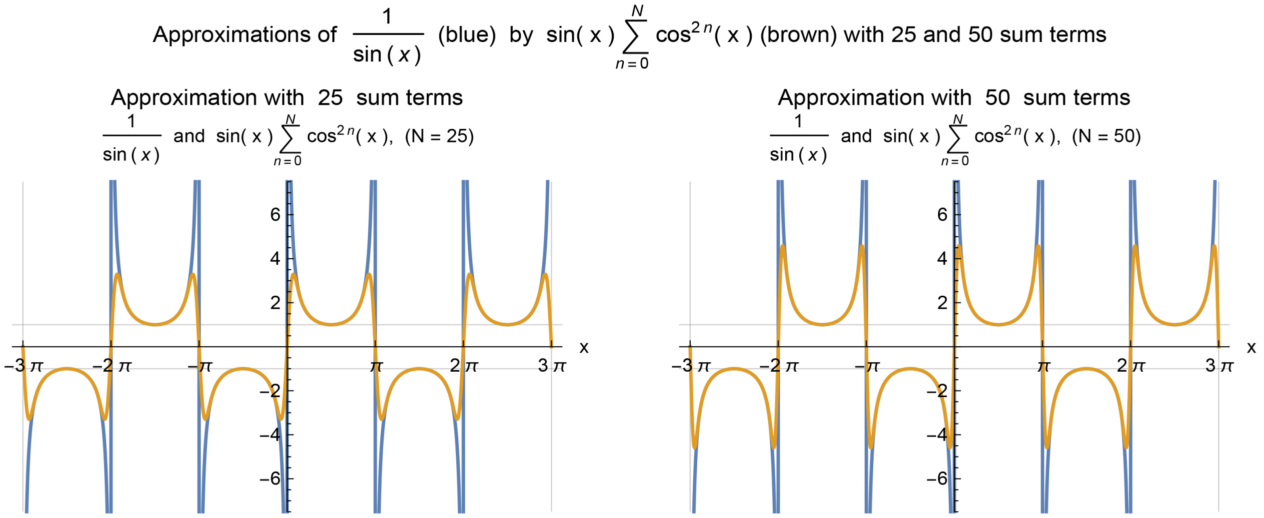


Figure 13. Two approximations of power-like series $\frac{1}{\sin(x)}=\sin(x)\sum_{n=0}^{\infty}\cos^{2n}(x)$. These series possess another convergence behavior than the corresponding Fourier series and converge uniformly on the whole range of definition of the function. The differences between the two approximations with different numbers of sum terms are rather small.

We come now to the other starlet of our investigations that is the function $\frac{1}{\sin(x)}$ and to its displaced function on the real axis $\frac{1}{\cos(x)}$ and expand them in power-like series of trigonometric functions. Since we have already its first integral in (5.4) which is closely related to the above derived expressions (8.7) and (8.8) we go the reversed way and derive the other results from its series. According to (5.4) and by applying (8.7) and (8.8) we find

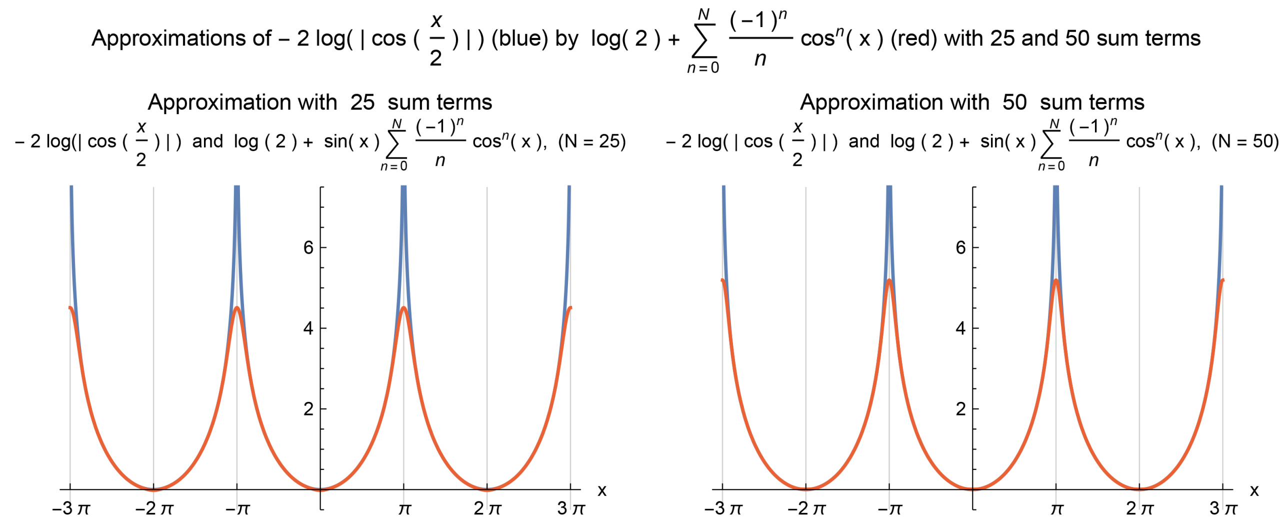


Figure 14. Two approximations of power-like series $-2\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)=\log(2)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos^n(x)$. These series possess another convergence behavior than the corresponding Fourier series and converge uniformly on the whole range of definition of the function.

$$\begin{aligned}\log\left(\left|\operatorname{tg}\left(\frac{x}{2}\right)\right|\right)&=\log\left(\left|\sin\left(\frac{x}{2}\right)\right|\right)-\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)\\&=\frac{1}{2}\left\{\left(-\log(2)-\sum_{m=1}^{\infty}\frac{1}{m}\cos^m(x)\right)+\left(\log(2)+\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\cos^m(x)\right)\right\}\\&=-\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos^{2n+1}(x).\end{aligned}\tag{8.9}$$

The convergence behavior of this power-like series is illustrated in **Figure 15** for two values of the number of sum terms taken into account in the approximation. It is widely analogous to that in **Figure 14** for the power-like series of $-2\log\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)$ and is convergence in ordinary sense.

By differentiation of the expression (8.9) one obtains together with (8.3) and (8.4) the third form of a power-like series for $\frac{1}{\sin(x)}$

$$\frac{1}{\sin(x)}=\sin(x)\sum_{n=0}^{\infty}\cos^{2n}(x).\tag{8.10}$$

The corresponding expansion power-like expansion for $\frac{1}{\cos(x)}$ is

$$\frac{1}{\cos(x)}=\sin\left(x+\frac{\pi}{2}\right)\sum_{n=0}^{\infty}\cos^{2n}\left(x+\frac{\pi}{2}\right)=\cos(x)\sum_{n=0}^{\infty}\sin^{2n}(x).\tag{8.11}$$

It is immediately to see that from (8.10) and (8.11) follow the power-like expansions

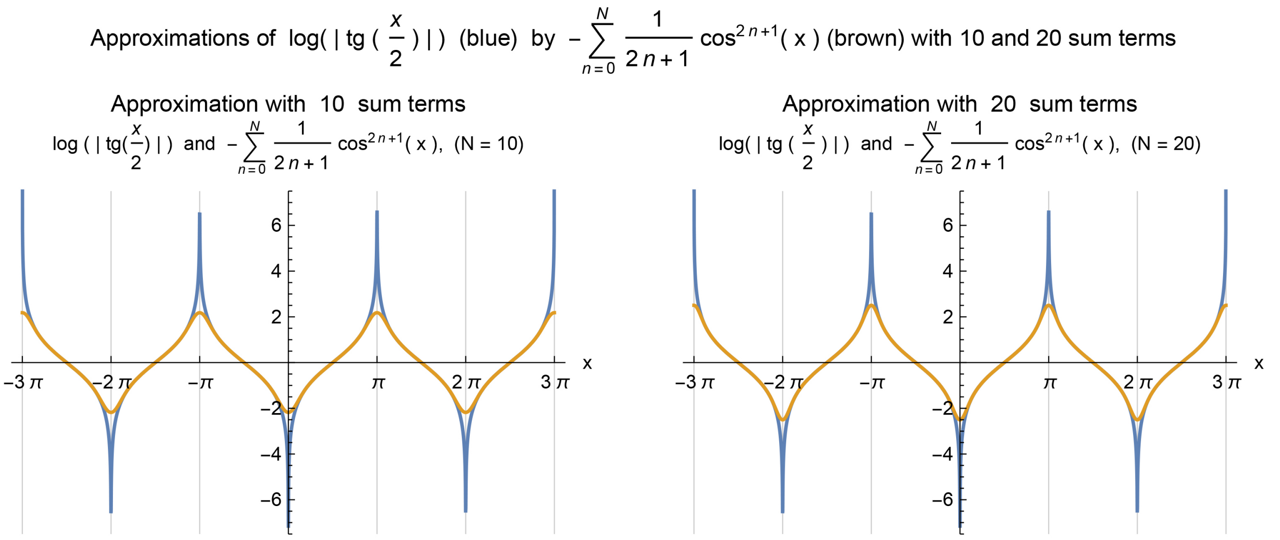


Figure 15. Two approximations of power-like series $\log\left(\left|\operatorname{tg}\left(\frac{x}{2}\right)\right|\right)=-\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos^{2n+1}(x)$. These series possess another convergence behavior than the corresponding Fourier series and converge uniformly on the whole range of definition of the function.

$$\frac{1}{\sin^2(x)}=\sum_{n=0}^{\infty}\cos^{2n}(x),\quad \frac{1}{\cos^2(x)}=\sum_{n=0}^{\infty}\sin^{2n}(x). \tag{8.12}$$

The power-like series (8.10) - (8.12) in trigonometric functions converge in ordinary sense.

In next section, we make a more general consideration of the power-like expansions in Sine and Cosine function and derive a general formula for the coefficients in these expansions.

9. Remark about Power-Like Series of Cosine and Sine Functions

In last section, we expanded some functions into power series of Sine and Cosine functions about which we make some more general remarks. Every function $f(x)$ can be separated in an even part $f_+(x)$ and an odd part $f_-(x)$ according to

$$\begin{aligned} f(x) &= f_+(x)+f_-(x),\\ f_+(x) &\equiv \frac{1}{2}\big(f(x)+f((-x)\big)=+f_{-1}(-x),\\ f_-(x) &\equiv \frac{1}{2}\big(f(x)-f(-x)\big)=-f_-(-x). \end{aligned} \tag{9.1}$$

In the Fourier series (2.5) with the coefficients (2.6) these two parts are separated by the even functions $\cos(kx)$ and the odd functions $\sin(kx)$. This is not similar for power-like series in trigonometric functions. Whereas the powers $\cos^n(x)$ are even functions for arbitrary n the powers $\sin^n(x)$ are even functions for even $n=2m$ but are odd functions for odd $n=2m+1$. Furthermore, the different powers of $\cos(x)$ and the different powers of $\sin(x)$ taken alone are, in general, no more mutually orthogonal. The last is similar to the powers in Taylor series. A complete set of symmetric and antisymmetric functions for the power-like expansion of 2π-periodic functions is formed by the two sets $\cos^n(x), \sin(x)\cos^n(x), (n = 0,1,\cdots,\infty)$. One way to get power-like expansions in Cosine and Sine functions for 2π-periodic functions is to use within the formulae for Fourier series (2.5) the following transition formulae from Cosine and Sine with multiple arguments to expansions in powers of these functions as follows

$$\cos(kx) = \frac{k}{2}\sum_{l=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^l (k-1-l)!}{l!(k-2l)!}\left(2\cos(x)\right)^{k-2l} \equiv \mathrm{T}_k\left(\cos(x)\right),$$

$$\sin\left((k+1)x\right) = \sin(x)\sum_{l=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^l (k-l)!}{l!(k-2l)!}\left(2\cos(x)\right)^{k-2l} \equiv \sin(x)\,\mathrm{U}_k\left(\cos(x)\right), \quad (9.2)$$

where $\mathrm{T}_k(z)$ and $\mathrm{U}_k(z)$ $(k = 0,1,2,\cdots)$ denotes the Chebyshev polynomials of first and second kind, correspondingly. These relations are known (e.g., Gradshteyn and Ryzhik [14]) and can be proved by complete induction. The inversion of these formulae is

$$\begin{aligned}
\cos^n(x) &= \frac{1}{2^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\cos\left((n-2k)x\right) \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mathrm{T}_{n-2k}\left(\cos(x)\right), \\
\sin^n(x) &= \frac{1}{2^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\cos\left((n-2k)\left(x-\frac{\pi}{2}\right)\right) \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mathrm{T}_{n-2k}\left(\sin(x)\right).
\end{aligned} \quad (9.3)$$

A general procedure of the transition from Fourier series to power-like series can be made using the transition relations (9.2) and the completeness relation for expansions of 2π-periodic functions into Fourier series as explained in the next Section. The final result is an expansion of the form

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos^n(x) + B_n\sin(x)\cos^{n-1}(x)\right), \quad (9.4)$$

in the two sets of nonorthogonal (symmetric) functions $\cos^n(x)$ and nonorthogonal (antisymmetric) functions $\sin(x)\cos^{n-1}(x)$.

The power-like expansions in powers of Cosine and Sine functions from (8.1) - (8.10) obtained in the previous Section are of the kind (9.4) and (8.11) is a modification of them. As the cases of expansions of $\mathrm{tg}\left(\frac{x}{2}\right)$ and of $\frac{1}{\sin(x)}$ show where their Fourier series converge only in the sense of weak convergence their power-like series converge in the usual sense of uniform convergence and one needs only a relatively "low" number of sum term to have sufficient approximations (**Figure 14** and **Figure 15**). Nevertheless, one has to keep in mind as already said that in the derivation of the coefficients A_n and B_n is made a substitution of the summation indices which causes a change of the order of summation in the double sums and which is only allowed if these series are absolutely convergent.

10. Completeness Relations to Fourier Series

The whole information contained in the three formulae for Fourier series (2.4), (2.5) and (2.6) can be compactly written in the following basic relation which from its kind is a completeness relation for the involved basic trigonometric functions in the basic interval $-\pi \le x < +\pi$ or in any equivalent interval

$$\delta(x-y) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\big(\cos(kx)\cos(ky) + \sin(kx)\sin(ky)\big), \quad (-\pi \le (x,y) < +\pi), \tag{10.1}$$

and the proof of the formulae (2.5) and (2.6) under the assumption (2.4) is equivalent to the proof of this relation. For example, the Fourier coefficients in a Fourier series of the function $f(x)$ follow by multiplication of (10.1) with $f(y)$ and integration over the basic interval

$$\begin{aligned} f(x) &= \int_{-\pi}^{+\pi} \mathrm{d}y\,\delta(x-y)\, f(y) \\ &= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}y\, f(y)}_{=\frac{a_0}{2}} \\ &\quad + \sum_{k=1}^{\infty}\left\{\cos(kx)\frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\cos(ky)\, f(y) + \sin(kx)\frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\sin(ky)\, f(y)\right\}. \end{aligned} \tag{10.2}$$

Also the orthogonality relations

$$\begin{aligned} &\frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\cos(ky)\cos(ly) = \frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\cos(ky)\cos(ly) = \delta_{k,l}, \quad (k,l = 1,2,3,\cdots), \\ &\frac{1}{2\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\cos(ky) = \delta_{k,0}, \quad (k = 0,1,2,\cdots), \quad \int_{-\pi}^{+\pi}\mathrm{d}y\cos(ky)\sin(ly) = 0, \end{aligned} \tag{10.3}$$

follow from (10.1) inserting $\cos(ly)$ and $\sin(ly)$ for $f(y)$ in (10.2) where the last in (10.3) follows from antisymmetry of the integrand.

Inserting (9.2) into the completeness relation (10.1) leads to

$$\begin{aligned} \delta(x-y) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\big\{\mathrm{T}_k(\cos(x))\mathrm{T}_k(\cos(y)) \\ &\quad + \sin(x)\mathrm{U}_{k-1}(\cos(x))\sin(y)\mathrm{U}_{k-1}(\cos(y))\big\} \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\frac{k}{2}\sum_{l=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^l(k-1-l)!}{l!(k-2l)!}(2\cos(x))^{k-2l}\cos(ky)\right. \\ &\quad \left. + \sin(x)\sum_{l=0}^{\left[\frac{k-1}{2}\right]}\frac{(-1)^l(k-1-l)!}{l!(k-1-2l)!}(2\cos(x))^{k-1-2l}\sin(ky)\right\}. \end{aligned} \tag{10.4}$$

We transform and calculate the right-hand side of this expression in detail in Appendix A. The result can be written as expansion in powers $\cos^{n}(x)$ which are symmetric functions and in functions $\sin(x)\cos^{n-1}(x)$ which are antisymmetric functions

$$\delta(x-y)=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\cos^{n}(x)A_{n}(\cos(y))+\sin(x)\cos^{n-1}(x)\sin(y)B_{n}(\cos(y))\right\}. \tag{10.5}$$

with the abbreviations

$$\begin{aligned}A_{n}(\cos(y))&\equiv\frac{2^{n}}{n!}\sum_{l=0}^{\infty}\frac{(-1)^{l}(n-1+l)!}{l!}\frac{n+2l}{2}\mathrm{T}_{n+2l}(\cos(y)),\\B_{n}(\cos(y))&\equiv\frac{2^{n-1}}{(n-1)!}\sum_{l=0}^{\infty}\frac{(-1)^{l}(n-1+l)!}{l!}\mathrm{U}_{n-1+2l}(\cos(y)).\end{aligned} \tag{10.6}$$

In Section 8 we obtained expansions of the kind (9.4) in special cases more directly. The coefficients of these expansions (10.6) are related to the Generating functions of the Chebyshev polynomials with even and odd indices as the following representations show

$$\begin{aligned}A_{n}(\cos(y))&=\frac{2^{n-1}}{n!}\frac{\partial}{\partial y}\left\{\frac{\partial^{n-1}}{\partial t^{n-1}}\sum_{l=0}^{\infty}t^{n-1+l}(-1)^{l}\sin((n+2l)y)\right\}_{t=1}\\&=\frac{2^{n-1}}{n!}\frac{\partial}{\partial y}\left\{\frac{\partial^{n-1}}{\partial t^{n-1}}\sum_{l=0}^{\infty}t^{n-1+l}(-1)^{l}\sin(y)\mathrm{U}_{n-1+2l}(\cos(y))\right\}_{t=1},\\B_{n}(\cos(y))&=\frac{2^{n-1}}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial t^{n-1}}\sum_{l=0}^{\infty}t^{n-1+l}(-1)^{l}\mathrm{U}_{n-1+2l}(\cos(y))\right\}_{t=1}.\end{aligned} \tag{10.7}$$

In Appendix B we derive basic Generating functions for Chebyshev polynomials with even and odd indices which show the kind of functions with which we, principally, have to do it.

11. Representations of Power-Like Arithmetic Expansions by Polylogarithms

In Section 8 we found in explicit form expansions in functions $f_{n}(x)=\cos^{n}(x)$ and $g_{n}(x)=\sin(x)\cos^{n-1}(x)$. These expansions can be expressed by Special Higher Transcendental functions which are called Polylogarithms [11] [12].

Polylogarithms with the notation $\mathrm{Li}_{n}(z)$ are defined by the series

$$\mathrm{Li}_{n}(z)\equiv\sum_{k=1}^{\infty}\frac{z^{k}}{k^{n}},\quad(z\leq 1). \tag{11.1}$$

The special case $n=0$ is related to the Geometric series with absent first sum term

$$\mathrm{Li}_{0}(z)\equiv\sum_{k=1}^{\infty}z^{k}=\frac{z}{1-z}. \tag{11.2}$$

The next special case $n=1$ is the logarithmic function

$$\mathrm{Li}_1(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z). \tag{11.3}$$

The special case $n=2$ is called Dilogarithm and it is the following definite integral of the logarithmic function (11.3)

$$\mathrm{Li}_2(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \mathrm{d}t \frac{\log(1-t)}{t}. \tag{11.4}$$

It is already not representable in a finite way using only the basic operations of addition, multiplication and division of elementary functions.

The Polylogarithms can be differentiated in a simple way according to

$$\mathrm{Li}_n^{(1)}(z) \equiv \frac{\partial}{\partial z} \mathrm{Li}_n(z) = \frac{1}{z} \mathrm{Li}_{n-1}(z), \tag{11.5}$$

but their integration is not very simple.

In the following we provide the obtained representations of power-like expansions by Polylogarithm functions. On the right-hand side we add in addition the Fourier series of the functions. From (8.1) and (8.2) follow

$$\mathrm{tg}\left(\frac{x}{2}\right) = \sin(x)\left(\mathrm{Li}_0(-\cos(x))+1\right) = 2\sum_{k=1}^{\infty}(-1)^{k-1}\sin(kx)$$

$$\mathrm{ctg}\left(\frac{x}{2}\right) = \sin(x)\left(\mathrm{Li}_0(\cos(x))+1\right) = 2\sum_{k=1}^{\infty}\sin(kx). \tag{11.6}$$

For the power-like expansions of the function $\frac{1}{\sin(x)}$ we found in Section 8 the three different power-like representations (8.1), (8.2) and (8.10) depending on the factors in front of the relations on the right-hand side

$$\frac{1}{\sin(x)} = \mathrm{ctg}\left(\frac{x}{2}\right)\left(\mathrm{Li}_0(-\cos(x))+1\right) = 2\sum_{k=0}^{\infty}\sin((2k+1)x),$$

$$\frac{1}{\sin(x)} = \mathrm{tg}\left(\frac{x}{2}\right)\left(\mathrm{Li}_0(\cos(x))+1\right),$$

$$\frac{1}{\sin(x)} = \sin(x)\left(\mathrm{Li}_0\left(\cos^2(x)\right)-1\right). \tag{11.7}$$

They converge in an ordinary sense that is illustrated in comparison to each other in **Figure 16**. The last partial picture of **Figure 16** represents the convergence behavior of the power-like series of the function $\frac{1}{1+\cos(x)}$. Together with $\frac{1}{1-\cos(x)}$ which is displaced by π relatively to $\frac{1}{1+\cos[x]}$ one finds

$$\frac{1}{1+\cos(x)} = \frac{1}{2\cos^2\left(\frac{x}{2}\right)}\mathrm{Li}_0(-\cos(x))+1 = \sum_{k=0}^{\infty}(-1)^k\cos^k(x)$$

$$= 2\sum_{k=0}^{\infty}(-1)^k(k+1)\cos((k+1)x) \quad \left(= \frac{\partial}{\partial x}\mathrm{tg}\left(\frac{x}{2}\right)\right),$$

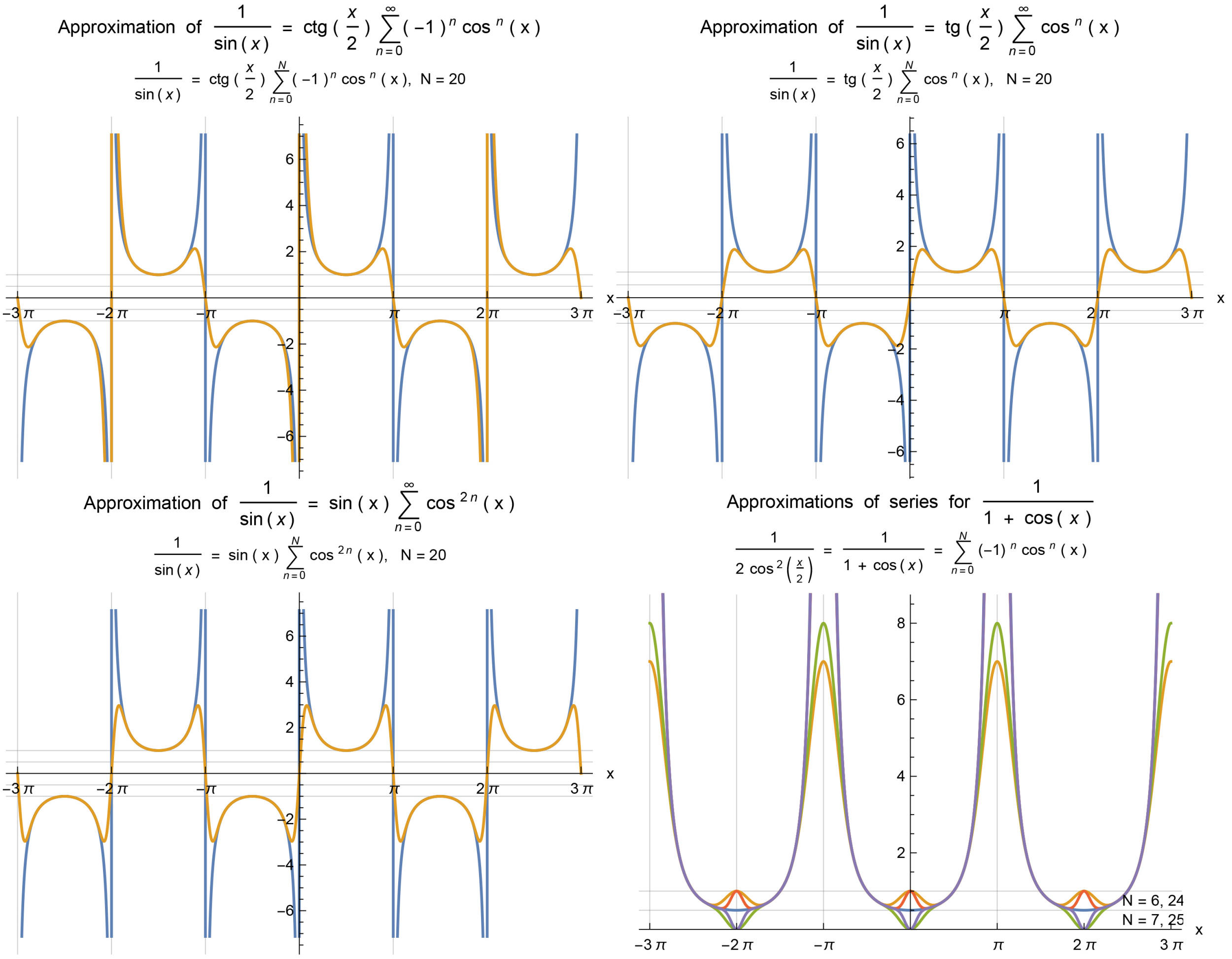


Figure 16. Three different power-like series for $\frac{1}{\sin(x)}$ and for $\frac{1}{1+\cos(x)}$ the last in lower row with approximations by N sum terms. In the last power-like series we see a noticeable difference for approximations with even and odd sum terms. The even approximations remain on the concave side and the odd approximations on the convex side of the curves with stabile height of the hump and only their widths decrease with higher approximations.

$$\begin{aligned}\frac{1}{1-\cos(x)} &= \frac{1}{2\sin^2\left(\frac{x}{2}\right)}\mathrm{Li}_0\left(\cos(x)\right)+1=\sum_{k=0}^{\infty}\cos^k(x)\\ &= -2\sum_{k=0}^{\infty}(k+1)\cos\left((k+1)x\right)\quad\left(=-\frac{\partial}{\partial x}\mathrm{ctg}\left(\frac{x}{2}\right)\right).\end{aligned}\tag{11.8}$$

An interesting phenomenon of the power-like representations which converge in ordinary sense is here that the approximations with even and odd sum terms show a different behavior. For even numbers of sum terms taken they remain at x = 0 on the concave side of the curves and for odd numbers of sum terms taken they are fully on the convex side of the curves. Only the widths of the humps change when the even or odd number of sum terms taken is changed and go to zero in the limit. The last given representations are the Fourier series which converge in the sense of the theory of Generalized functions similar to what is

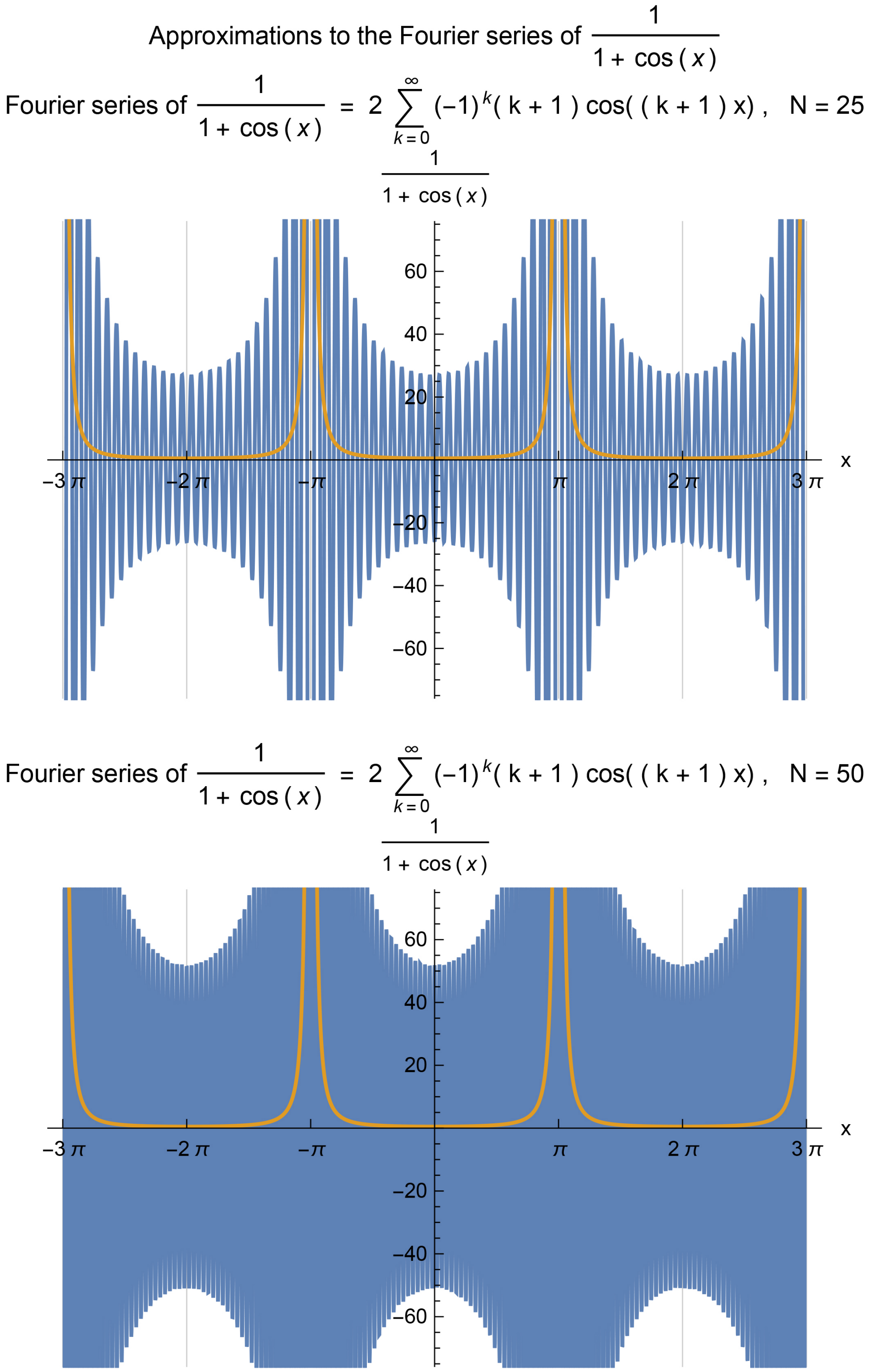


Figure 17. Fourier series of $\frac{1}{1+\cos(x)}$ for two approximations by N sum terms. It converges in the sense of weak convergence of Generalized functions and shows that the envelope of the oscillations is here not stabile changing the number of taken sum terms. The last is likely caused because we have here in addition a linear term in "k" in its Fourier series.

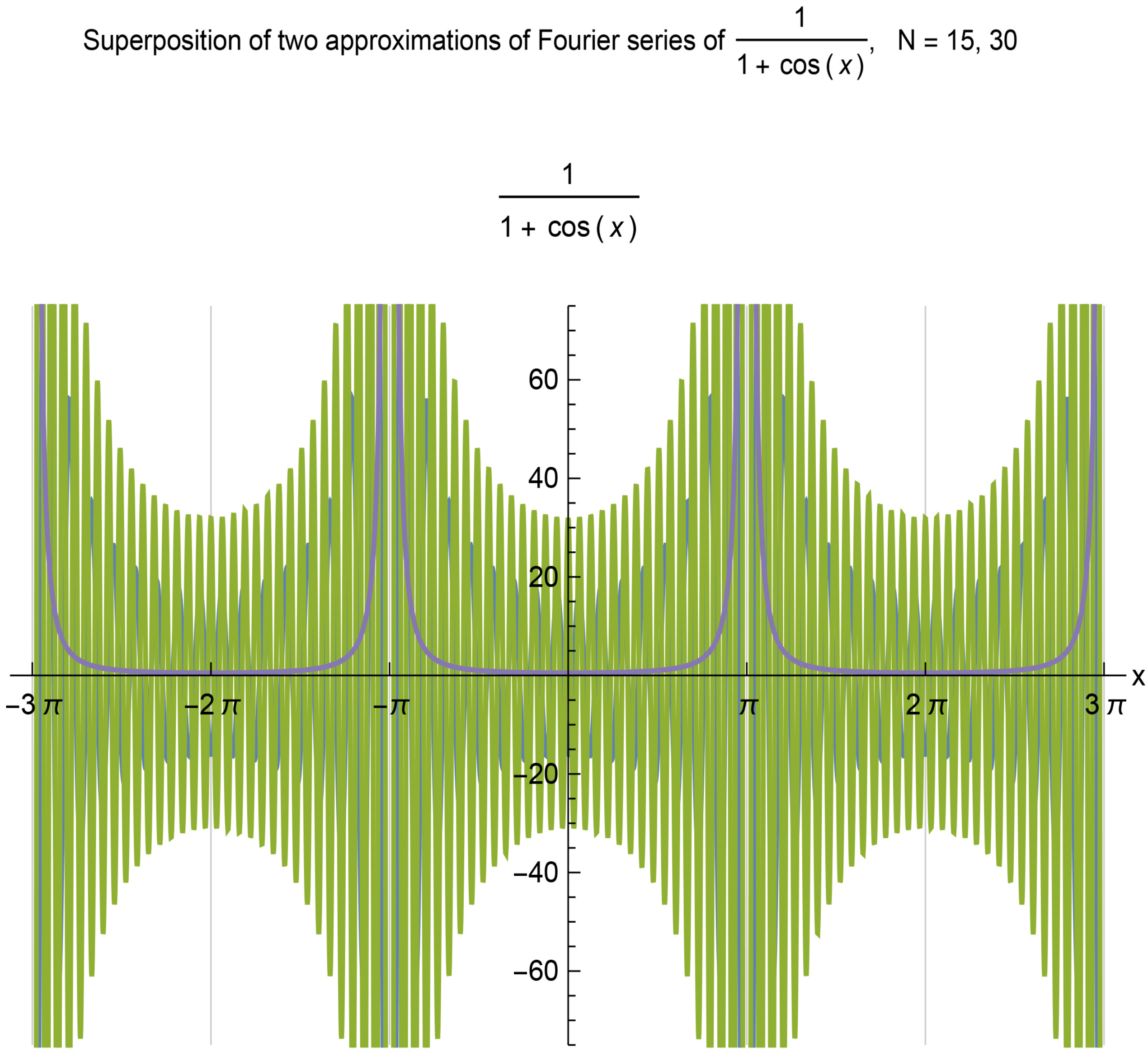


Figure 18. Superposition of two approximation to Fourier series of $\frac{1}{1+\cos(x)}$ with $N = 15$ and $N = 30$ sum terms. It shows that the envelopes for this function depend on the sum terms in the approximation.

shown in Figure 2 and Figure 3 and Figure 6 and Figure 7. They are obtained by term-wise differentiation of the Fourier series for $\mathrm{tg}\left(\frac{x}{2}\right)$ and $-\mathrm{ctg}\left(\frac{x}{2}\right)$, correspondingly. This is represented for $\frac{1}{1+\cos[x]}$ for two numbers of taken sum terms of the Fourier series in Figure 17. In Figure 18 is represented the superposition of two approximations with different numbers of taken sum terms in the Fourier series of $\frac{1}{1+\cos(x)}$ where the inhomogeneous color shows the dependence of the envelopes from its number.

One of the analogous representations to (11.7) for $\frac{1}{1+\cos(x)}$ is

$$\frac{1}{\cos(x)} = \cos(x)\left(\mathrm{Li}_0\left(\sin^2(x)\right)+1\right) = 2\sum_{k=0}^{\infty}(-1)^k \sin((2k+1)x). \qquad (11.9)$$

With the representation of the Fourier series and of the power-like series by Polylogarithms we connected the hope that we can also determine the higher than first-order integrals of $\mathrm{tg}\left(\frac{x}{2}\right)$ and of $\frac{1}{\sin(x)}$. Indeed, the program "Mathematica 10.4" provides also a result for the second-order integrals but it is so complicated and lengthy and with complex arguments of the Polylogarithms that it is hardly to use and we do not give it.

12. Conclusion

The convergence behavior of Fourier series of 2π-periodic functions which converge in the sense of weak convergence of Generalized functions was investigated and represented in figures that seems to be new. In particular, this concerns the basic functions $\mathrm{tg}\left(\frac{x}{2}\right), \frac{1}{\sin(x)}, \frac{1}{1+\cos(x)}$ from which some further related functions such as their first integrals may be derived. It was found that the oscillations of the approximations are distributed in broad bands to both sides of the functions. The Fourier series of functions with weak convergence can be applied in connection with the multiplication by "sufficiently well-behaved" functions which rapidly decrease in infinity such as the Gaussian bell functions and integration over such products. In the course of calculations we came across to expansions which were called power-like expansions and which converge in ordinary sense also in cases where the Fourier series are only weakly convergent. These are expansions in powers $\cos^n(x)$ for even parts of functions and $\sin(x)\cos^n(x)$ for odd parts of functions. The derivation of general formulae for the coefficients of the terms in such expansion was not fully successful up to now. The obtained examples were found by application of the Geometric series. Furthermore, we investigated distributed 2π-periodic delta function with origin from the Poisson identity. In the neighborhood of the singularities (the delta functions) on the *x*-axis they show oscillations which also in the limiting case of their Fourier series do not vanish. This is very similar to the Gibbs phenomenon of Fourier series in the neighborhood of discontinuities of the functions. It is represented in some of the figures.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Ul'yanov, P.L. (1977) Matematicheskaya entsyklopedia, Tom 1 (from 5 volumes). "Sovetskaya Encyclopedia" Publishing House, 958-959.

[2] Sommerfeld, A. (1958) Partielle Differentialgleichungen der Physik, 4. Sauter, Akademische Verlagsgesellschaft Geest und Portig.

[3] Fikhtengol'ts, G.M. (1960) Kurs differentialnovo i integral'novo is'chisleniya, tom III. Fizmatgiz.

[4] Arfken, G. (1985) Mathematical Methods for Physicists. Third Edition, Academic Press.

[5] Körner, T.W. (1988) Fourier Analysis. Cambridge University Press. https://doi.org/10.1017/CBO9781107049949

[6] Weisstein, E.W. (1995) Fourier Series. https://mathworld.wolfram.com/FourierSeries.html

[7] Schwartz, L. (1961) Méthodes Mathématiques pour les Sciences Physiques. Hermann.

[8] Gel'fand, I.M. and Shilov, G.E. (1959) Obobstchonnyje funktsii, vol. 1. Fizmatgiz.

[9] Vilyenkin, N.Y. (1972) Obobshtchennyje funktzii. In: Krein, S.G., Ed., *Funktsional'nyj analiz*, 2nd Edition, Naúka, 455-544.

[10] Vladimirov, V.S. (1976) Obobstshonnyje funktsii v matematicheskoy fizike. Nauka.

[11] Apostol, T.M. (2010) Zeta and Related Functions. In: Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W., Eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, 602-616.

[12] Andrews, G.E., Askey, R. and Roy, R. (1999) Special Functions. Cambridge University Press. https://doi.org/10.1017/CBO9781107325937

[13] Tolstov, G.P. (1980) Ryady Fur'ye. Naúka.

[14] Gradshteyn, I.S. and Ryzhik, I.M. (1963) Tablitsy integralov, sum, ryadov i proizvedyenij. 4th edition, Fizmatgiz.

Appendix A

Calculation of the completeness relation for the power-like expansion in trigonometric functions

Starting from the completeness relation for Fourier series of 2π-periodic functions (10.1) we insert the transition relations (9.2) that leads to

$$\begin{aligned}\delta(x-y) &= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\cos(kx)\cos(ky)+\sin(kx)\sin(ky)\right\}\\ &= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\left(\frac{k}{2}\sum_{l=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^l(k-1-l)!}{l!(k-2l)!}2^{k-2l}\cos^{k-2l}(x)\right)\cos(ky)\right. \quad (\mathrm{A}.1)\\ &\quad \left.+\sin(x)\left(\sum_{l=0}^{\left[\frac{k-1}{2}\right]}\frac{(-1)^l(k-1-l)!}{l!(k-1-2l)!}2^{k-1-2l}\cos^{k-1-2l}(x)\right)\sin(ky)\right\}.\end{aligned}$$

In this relation, we make a reordering of the sum terms of the double sum by the substitution $n\equiv k-2l\rightarrow k-n+2l$ using the formula

$$\sum_{k=m}^{\infty}\sum_{l=0}^{\left[\frac{k}{2}\right]}f_{k,l}=\sum_{n=m}^{\infty}\sum_{l=0}^{\infty}f_{n+2l,l},\quad (m=0,1,2,\cdots), \tag{A.2}$$

with $m=1$ in our case and with the result

$$\begin{aligned}\delta(x-y) &= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\cos^n(x)\frac{2^{n-1}}{n!}\sum_{l=0}^{\infty}\frac{(-1)^l(n-1+l)!}{l!}(n+2l)\cos((n+2l)y)\right.\\ &\quad \left.+\sin(x)\cos^{n-1}(x)\frac{2^{n-1}}{(n-1)!}\sum_{l=0}^{\infty}\frac{(-1)^l(n-1+l)!}{l!}\sin((n+2l)y)\right\}.\end{aligned} \tag{A.3}$$

If we express in this formula $\cos((n+2l)y)$ and $\sin((n+2l)y)$ by the Chebyshev polynomials of first and second kind (see (9.2)) we find

$$\begin{aligned}\delta(x-y) &= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\cos^n(x)\frac{2^{n-1}}{n!}\sum_{l=0}^{\infty}\frac{(-1)^l(n-1+l)!}{l!}(n+2l)\mathrm{T}_{n+2l}(\cos(y))\right.\\ &\quad \left.+\sin(x)\cos^{n-1}(x)\frac{2^{n-1}}{(n-1)!}\sum_{l=0}^{\infty}\frac{(-1)^l(n-1+l)!}{l!}\sin(y)\mathrm{U}_{n-1+2l}(\cos(y))\right\}.\end{aligned} \tag{A.4}$$

This representation by the Chebyshev polynomials shows that we may introduce two functions which depend only on $\cos(y)$ as follows

$$\begin{aligned}\delta(x-y) &\equiv \frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\cos^n(x)A_n(\cos(y))\right.\\ &\quad \left.+\sin(x)\cos^{n-1}(x)\sin(y)B_n(\cos(y))\right\},\end{aligned} \tag{A.5}$$

with the abbreviations

$$A_n\left(\cos\left(y\right)\right) \equiv \frac{2^{n-1}}{n!}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l\left(n-1+l\right)!}{l!}\left(n+2l\right)\mathrm{T}_{n+2l}\left(\cos\left(y\right)\right),\quad \left(n=1,2,\cdots\right),$$

$$B_n\left(\cos\left(y\right)\right) \equiv \frac{2^{n-1}}{\left(n-1\right)!}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l\left(n-1+l\right)!}{l!}\mathrm{U}_{n-1+2l}\left(\cos\left(y\right)\right),\quad \left(n=1,2,\cdots\right). \quad \text{(A.6)}$$

Expression (A.5) is the new completeness relation for power-like expansions of 2π-periodic function in the sets of functions $\cos^n\left(x\right), \sin\left(x\right)\cos^n\left(x\right), \left(n=0,1,2,\cdots\right)$. It is asymmetric with respect to functions of variables x and y (some duality of these functions). Formula (10.6) does not uniquely determine $A_0\left(\cos\left(y\right)\right)$ which has to be set $A_0\left(\cos\left(y\right)\right) \equiv 1$. If we multiply (A.5) by an arbitrary function $f\left(y\right)$ and integrate over a basis range of length 2π one finds

$$f\left(x\right) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left\{\cos^2\left(x\right)A_n + \sin\left(x\right)\cos^{n-1}\left(x\right)B_n\right\}, \quad \text{(A.7)}$$

with the coefficients

$$A_n = \frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\, f\left(y\right)A_n\left(\cos\left(y\right)\right),\quad A_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\, f\left(y\right),$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{+\pi}\mathrm{d}y\, f\left(y\right)B_n\left(\cos\left(y\right)\right),\quad \left(n=1,2,\cdots\right). \quad \text{(A.8)}$$

Altogether, the general formulae for the determination of the coefficients in power-like expansions are very difficult. All power-like expansions of 2π-periodic functions explicitly calculated in present paper were simpler determined with the help of the Geometric series than by the here-discussed general procedure.

Appendix B

Generating functions for Chebyshev polynomials with even or odd indices

The further treatment of relations (10.6) is connected with Generating functions for the Chebyshev polynomials with summation over even or odd indices, respectively to which we make here first steps. From the basic well-known Generating functions for Chebyshev polynomials

$$\sum_{n=0}^{\infty} t^n \mathrm{T}_n\left(\cos\left(x\right)\right) = \frac{1-t\cos\left(x\right)}{1-2t\cos\left(x\right)+t^2},$$

$$\sum_{n=0}^{\infty} t^n \mathrm{U}_n\left(\cos\left(x\right)\right) = \frac{1}{1-2t\cos\left(x\right)+t^2}, \quad \text{(B.1)}$$

we derive now Generating functions for Chebyshev polynomials with even or odd indices and Cosine functions in the argument.

For Chebyshev polynomials of first kind $\mathrm{T}_n\left(z\right)$ with even or odd indices only one finds

$$\begin{aligned}
&\sum_{m=0}^{\infty} t^{2m}\mathrm{T}_{2m}\left(\cos(x)\right) = \sum_{m=0}^{\infty} t^{2m}\mathrm{T}_{m}\left(\cos(2x)\right)\\
&\qquad = \frac{1-t^2\cos(2x)}{1-2t^2\cos(2x)+t^4},\\
&\sum_{m=0}^{\infty} t^{2m+1}\mathrm{T}_{2m+1}\left(\cos(x)\right)\\
&= \sum_{n=0}^{\infty} t^{n}\mathrm{T}_{n}\left(\cos(x)\right) - \sum_{m=0}^{\infty} t^{2m}\mathrm{T}_{2m}\left(\cos(x)\right)\\
&= \frac{1-t\cos(x)}{1-2t\cos(x)+t^2} - \frac{1-t^2\cos(2x)}{1-2t^2\cos(2x)+t^4}\\
&= \frac{t\left(1-t^2\right)\cos(x)}{1-2t^2\cos(2x)+t^4}.
\end{aligned} \tag{B.2}$$

Analogously, for Chebyshev polynomials of second kind $\mathrm{U}_n(z)$ with even or odd indices only one finds

$$\begin{aligned}
&\sum_{m=0}^{\infty} t^{2m+1}\mathrm{U}_{2m+1}\left(\cos(x)\right)\\
&= 2\cos(x)\sum_{m=0}^{\infty} t^{2m+1}\mathrm{U}_{m}\left(\cos(2x)\right) - \frac{2t\cos(x)}{1-2t^2\cos(2x)+t^4},\\
&\sum_{m=0}^{\infty} t^{2m}\mathrm{U}_{2m}\left(\cos(x)\right)\\
&= \sum_{n=0}^{n} t^{n}\mathrm{U}_{n}\left(\cos(x)\right) - \sum_{m=0}^{\infty} t^{2m+1}\mathrm{U}_{2m+1}\left(\cos(x)\right)\\
&= \frac{1}{1-2t\cos(x)+t^2} - \frac{2t\cos(x)}{1-2t^2\cos(2x)+t^4}\\
&= \frac{1+t^2}{1-2t^2\cos(2x)+t^4}.
\end{aligned} \tag{B.3}$$

One may simplify these results by the substitutions $t^2 \to t$ and finds finally from (B.2)

$$\begin{aligned}
\sum_{m=0}^{\infty} t^{m}\mathrm{T}_{2m}\left(\cos(x)\right) &= \frac{1-t\cos(2x)}{1-2t\cos(2x)+t^2},\\
\sum_{m=0}^{\infty} t^{m}\mathrm{T}_{2m+1}\left(\cos(x)\right) &= \frac{(1-t)\cos(x)}{1-2t\cos(2x)+t^2}.
\end{aligned} \tag{B.4}$$

and from (B.3)

$$\begin{aligned}
\sum_{m=0}^{\infty} t^{m}\mathrm{U}_{2m+1}\left(\cos(x)\right) &= \frac{2\cos(x)}{1-2t\cos(2x)+t^2},\\
\sum_{m=0}^{\infty} t^{m}\mathrm{U}_{2m}\left(\cos(x)\right) &= \frac{1+t}{1-2t\cos(2x)+t^2}.
\end{aligned} \tag{B.5}$$

It is easy to write these Generating functions by variables $\left(t, z \equiv \cos(x)\right)$ but it is not so easy to transform them to indices with substitution $2m \to n+2m$ or

$2m+1 \rightarrow n+2m+1$ with arbitrary integers n where a certain number of initial sum terms has to be excluded and which would fully finish the explicit calculation of (A.6).