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Connected Components in Bipolar Fuzzy Digital Plane

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Abstract

The concepts of connectedness play a critical role in digital picture segmentation and analyses. However, the crisp nature of set theory imposes hard boundaries that restrict the extension of the underlying topological notions and results. Whilst fuzzy set theory was introduced to address this inherent drawback, most human processes are not just fuzzy but also double-sided. Most phenomena will exhibit both a positive side and a negative side. Therefore, it is not enough to have a theory that addresses imprecision, uncertainty and ambiguity; rather, the theory must also be able to model polarity. Hence the study of bipolar fuzzy theory is of potential significance in an attempt to model real-life phenomena. This paper extends some concepts of fuzzy digital topology to bipolar fuzzy subsets including some important basic properties such as connectedness and surroundedness.

Keywords

Fuzzy, Bipolar Fuzzy, Digital Topology

1. Introduction

The study of digital topology was initiated by Azriel Rosenfeld in the late 1960s. Despite its name, the theory developed out of the utilization of graph-theoretic methods rather than topological methods. It was not until the late 1980s that a topological approach, which later became known as the axiomatic approach, was developed. The search for a convenient and plausible theory for image analysis has given birth to several interest, among them, the extension of the digital plane to fuzzy environments.

Fuzzy digital topology was introduced by [1] in an attempt to generalize the topological relationships, including connectedness and surroundness on parts of a digital picture, to fuzzy subsets. His justification was born out of the fact that segmentation of a picture into subsets represents a very strong commitment which can be overcome by extracting fuzzy subsets, rather than ordinary subsets from the picture. He therefore developed some of the basic properties of these generalized concepts.

Human experiences are bipolar [2]. In 1994 [3] Zhang introduced the notion of bipolar fuzzy set. Many researchers including [4], [5] and Jun and [6], proceeded to further develop the theory via BCK/BCI-algebras. The notion of a bipolar fuzzy topological space was introduced by and [7], who defined a bipolar fuzzy point and extensively introduced the notions of fuzzy topology into bipolar fuzzy sets.

In this paper, we develop extensions of the properties of digital pictures by weakening the commitment imposed by fuzzy subsets. Bipolar subsets allow us to study both vagueness and duality in the properties of digital objects. We will then study some properties of these generalized concepts.

2. Literature Review

Digital topology has been developed to address problems in image processing and analysis—An area of computer science that deals with the analysis and manipulation of pictures by computer [8]. The results from digital topology help provide a sound mathematical basis for image processing operations such as object counting, boundary detection, data compression and thinning. There are two main approaches to the study of digital topology, namely; the Graph-theoretic approach and the Axiomatic approach. Below are definitions of some important concepts in digital topology.

2.1. Digital *n*-Space $\mathbb{Z}^n$

A digital n-space $\mathbb{Z}^n$ is the n-tuple $(x_1, x_2, \cdots, x_n)$ of the Euclidean n-space having integer coordinates. A point with integer coordinates is called a digital point. In computer grapics, the most commonly used representations are the 2- or 3-space, $\mathbb{Z}^2$ and $\mathbb{Z}^3$ respectively. [1]

2.2. Adjacency Relation

An adjacency relation π is a binary operation on the digital space that describes the connectedness behavior of digital points (or lack of it). Such a relation plays a very vital role in the grouping process and must therefore be as close as possible to the idea of the nearness of points in an intuitive sense. Two distinct points $x, y \in \mathbb{Z}^n$ are called adjacent if $(x, y) \in \pi$ [9].

The adjacency relation extends the notion of topological connectedness onto π-connectedness. A digital space is π-connected if $\forall x, y \in \mathbb{Z}^n$, $(x, y) \in \pi$. A very initial problem in digital topology was to determine whether these two notions, topological connectedness and π-connectedness, were equivalent. It was shown that the two notions are related but not equivalent (*i.e.* there exists digital π-connected that are not topological).

The digital space $(\mathbb{Z}^n, \pi)$ if for any two points x and y there exists a finite sequence $\{x_0, x_1, \cdots, x_n\}$ of points in $\mathbb{Z}^n$ such that $x = x_0$, $y = x_n$ and $(x_j, x_{j+1}) \in \pi \quad \forall 0 \le j \le n-1$. This definition well elaborated in [9].

2.3. Graph-Theoretic Digital Topology

According to [1] the graph-based approach, and any approach for that matter, should be in agreement with classical topology, most especially with respect to connectedness and validity of the Jordan Curve Theorem. However, [10], noted that neither the 4-adjacency nor the 8-adjacency as introduced by Rosenfeld, allows an analogue of the Jordan Curve Theorem (JCT). It is important that the JCT be definable in the digital plane since it guarantees a mathematical interpretation of the boundary properties of a space.

2.4. *k*-Adjacency

To study *nD* digital images, we say that two distinct points $p, q \in \mathbb{Z}^n$ are *k*-(or *k*(*t*, *n*)-)adjacent if for $t \in \mathbb{N}$ s.t $1 \le t \le n$ at most *t* of their coordinates differ by ±1 and all the others coincide [10].

We can obtain the *k*-adjacencies of $\mathbb{Z}^n$ as follows

$$k = \sum_{i=n-t}^{n-1} 2^{n-i} c_i^n, \quad \text{where } c_i^n = \frac{n!}{(n-i)!i!}$$

The configuration of the digital *k*-connectivity $\mathbb{Z}^n$, $n = \{1, 2, 3\}$ are represented in **Figure 1** below.

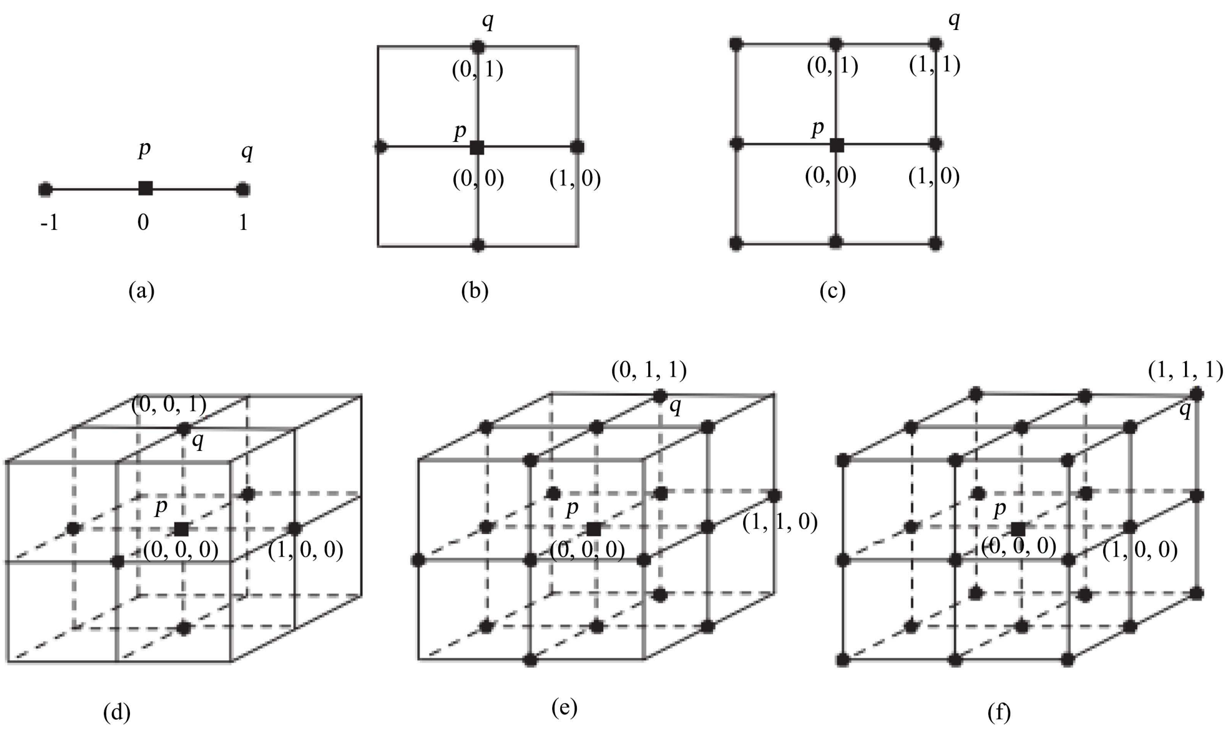

Figure 1. Configuration of the digital *k*-connectivity of $\mathbb{Z}^n$, $n \in \{1, 2, 3\}$. (a) 2-adjacency, (b) 4-adjacency, (c) 8-adjacency, (d) 6-adjacency, (e) 18-adjacency, (f) 26-adjacency.

The set $X \subset \mathbb{Z}^n$ with *k*-adjacency is called a digital image, denoted by (X,k). [1]

2.5. Digital *k*-Neighborhood

A digital *k*-neighborhood of a point $p \in \mathbb{Z}^n$ is the set $N_k(p) = \{q : p$ is *k*-adjacent to $q\}$.

Both the *k*-adjacency relation of $\mathbb{Z}^n$ [11].

2.6. Digital *k*-Interval

For $a,b \in \mathbb{Z}^n$ with $a < b$, the set $[a,b] = \{n \in \mathbb{Z} : a \le n \le b\}$ with 2-adjacencies is called a digital interval [11].

Two subsets $(A,k),(B,k) \subset (X,k)$ are *k*-adjacent to each other if $A \cap B = \varnothing$ and there exists $a \in A, b \in B$ such that a,b are *k*-adjacent to each other. $X \subset \mathbb{Z}^n$ is *k*-connected if there exists $A,B \in X$ such that $A \cup B = X$ and $A \cap B = \varnothing$.

For a digital image (X,k), the *k*-component of $x \in X$ is the largest *k*-connected subset of (X,k) containing *x* [12].

2.7. Path and Connectedness

The following definitions have been extracted from:

A simple *k*-path with $l+1$ elements in $\mathbb{Z}^n$ is an injective sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are *k*-adjacent iff $|i - j = 1|$. If $x_0 = x$ and $x_i = y$ then the length of the simple *k*-path, denoted by $l_k(x,y)$ is l [13].

A *k*-path is called a *k*-arc if it has the additional property that for any two points p_i and p_j which are not endpoints $p_i \in N(p_j)$ implies that $|i - j| \le 1$, that is, and arc is a path that does not intersect or touch itself with the possible exception of its endpoints [14].

Remark

k-connected is an equivalence relation and hence this relation partitions *X* into equivalence classes, which are maximal.

Suppose *X* and *Y* are disjoint subsets of $\mathbb{Z}^n$. We say that *X* surrounds *Y* if any path from *Y* to the border of $\mathbb{Z}^n$ must meet *X*, where the border points of $\mathbb{Z}^n$ are elements of the complement of *X*.

Lemma 2.7.1

Let *P* be a path with two endpoints. Then there exists an arc P_o which is completely contained in *P* and has the same endpoints [14].

2.8. Bipolar Fuzzy Relation

The following definitions are provided in [15]. Let $X, Y \ne \varnothing$ and δ and μ be bipolar fuzzy subsets of *X* and *Y* respectively. A bipolar fuzzy subset $\tilde{R} = (\tilde{R}^+, \tilde{R}^-) \subset X \times Y$ is called a bipolar fuzzy relation from *X* to *Y* if

$$\tilde{R}^{+}(x,y) \le \min\{\delta^{+}(x), \mu^{+}(y)\}$$
$$\tilde{R}^{-}(x,y) \le \max\{\delta^{-}(x), \mu^{-}(y)\} \quad \forall x \in \delta, y \in \mu$$

2.9. Bipolar Fuzzy Graph

A bipolar fuzzy graph with X as the underlying set is defined in [15] a pair $G = (\mu, \tilde{R})$ such that $\mu : X \to [-1,1]$ is a bipolar fuzzy subset of X and $\tilde{R} : X \times X \to [-1,1]$ is a bipolar fuzzy relation on A *i.e.* $\forall x, y \in X$

$$\tilde{R}^{+}(x,y) \le \min\{\mu^{+}(x), \mu^{+}(y)\}$$
$$\tilde{R}^{-}(x,y) \le \max\{\mu^{-}(x), \mu^{-}(y)\}$$

μ is called the bipolar fuzzy vertex set of G and $\tilde{R}$ the bipolar fuzzy edge set of G. For the definition to make sense, we assume the underlying set X is finite [16].

2.10. Path and Connectedness

In [17], a path ρ in a fuzzy graph (A, R) is a sequence of distinct vertices $x_0, x_1, \cdots, x_n$ such that $R(x_{i-1}, x_i) > 0 \quad \forall i = 1, 2, \cdots, n$. $n \ge 1$ is called the length of the path.

3. Results

3.1. Path Strength

A path $\rho : x = x_0, x_1, \cdots, x_n = y$ from x to y in a bipolar fuzzy graph $(\mu, \tilde{R})$ is a sequence of distinct vertices $x_0, x_1, \cdots, x_n$ such that

$$\tilde{R}^{+}(x_{i-1}, x_i) > 0$$
$$\tilde{R}^{-}(x_{i-1}, x_i) > -1 \quad \forall i = 1, 2, \cdots, n$$

To define the strength of a path in the bipolar fuzzy subset, we have to consider both the strength of the path in the direction representing the satisfaction degree from x to y and the direction representing the satisfaction of x and y to some implicit counter-property.

Definition 3.1.1

The μ^{+}-strength $S_{\mu^{+}}(\rho)$ and μ^{-}-strength $S_{\mu^{-}}(\rho)$ of the path $\rho : x = x_0, x_1, \cdots, x_n = y$ is the weight of the weakest edge of the path, *i.e.*

$$S_{\mu^{+}}(\rho) = \min\{\tilde{R}(x_{i-1}, x_i)\}$$
$$S_{\mu^{-}}(\rho) = \max\{\tilde{R}^{-}(x_{i-1}, x_i)\} \quad i = 1, 2, \cdots, n$$

The length $\ell(\rho)$ of a path from x to y is the number, $n \ge 0$ of vertices or nodes between the points. If the path has length 0, it is convenient to define its strength to be;

$$S_{\mu^{+}}(\rho) = \mu^{+}(x_0)$$
$$S_{\mu^{-}}(\rho) = \mu^{-}(x_0)$$

Two vertices joined by a path are said to be connected.

3.2. Degree of Connectedness

The degree of connectedness of x and y is the weight of the strongest path from x to y. Similarly, we have both the degree of connectedness with respect to the positive membership degree of the vertices in the path ρ from x to y and the degree of connectedness with respect to the negative membership degree of the vertices in the path ρ from x to y, *i.e.* $\begin{matrix} C_{\mu^+}(x,y)=\max\left(S_{\mu^+}(\rho)\right) \\ C_{\mu^-}(x,y)=\min\left(S_{\mu^-}(\rho)\right) \end{matrix}$.

Note: The max and min are taken over all possible paths from x to y.

Theorem 3.2.1

Let $x,y\in A$. Then

1) $C_{\mu^+}(x,x)=\mu^+(x)$ (resp. $C_{\mu^-}(x,x)=\mu^-(x)$) and

2) $C_{\mu^+}(x,y)=C_{\mu^+}(y,x)$ (resp. $C_{\mu^-}(x,y)=C_{\mu^-}(y,x)$)

Proof:

1) Any path ρ x to x from passes through x. Therefore $S_{\mu^+}(\rho)=\min\left\{\tilde{R}(x_{i-1},x_i)\right\}\le\mu^+(x)$. On the other hand, x is itself a path of length 0, for which $S_{\mu^+}(\rho)=\mu^+(x)$. Hence $C_{\mu^+}(x,x)=\max\left(S_{\mu^+}(\rho)\right)=\mu^+(x)$. The same argument shows $C_{\mu^-}(x,x)=\mu^-(x)$

2) Path reversal preserves path strength. Therefore $C_{\mu^+}(x,y)=C_{\mu^+}(y,x)$ (resp. $C_{\mu^-}(x,y)=C_{\mu^-}(y,x)$)

Theorem 3.2.2

$\forall x,y\in A$, $C_{\mu^+}(x,y)\le\min\left(\mu^+(x),\mu^+(y)\right)$ and $C_{\mu^-}(x,y)\le\max\left(\mu^-(x),\mu^-(y)\right)$.

Proof:

Suppose $\rho: x=x_0,x_1,\cdots,x_n=y$ is a path from x to y. Then

$$S_{\mu^+}(\rho)=\min\left\{\tilde{R}(x_{i-1},x_i)\right\}\le\min\left(\mu^+(x),\mu^+(y)\right), i=1,2,\cdots,n$$

But $C_{\mu^+}(x,y)=\max\left(S_{\mu^+}(\rho)\right)\le\min\left(\mu^+(x),\mu^+(y)\right), i=1,2,\cdots,n$

Similarly,

$$S_{\mu^-}(\rho)=\min\left\{\tilde{R}^-(x_{i-1},x_i)\right\}\le\max\left(\mu^-(x),\mu^-(y)\right), i=1,2,\cdots,n$$

But $C_{\mu^-}(x,y)=\min\left(S_{\mu^+}(\rho)\right)\le\max\left(\mu^-(x),\mu^-(y)\right), i=1,2,\cdots,n$

Proposition 3.2.1

The degree of connectedness $C_{\mu^+}(x,y)$ (resp. $C_{\mu^-}(x,y)$) is a reflexive and symmetric relation but not necessarily transitive

Proof:

From proposition 1 above;

Reflexive: $C_{\mu^+}(x,x)=\mu^+(x)$ (resp. $C_{\mu^-}(x,x)=\mu^-(x)$) and

Symmetry: $C_{\mu^+}(x,y)=C_{\mu^+}(y,x)$ (resp. $C_{\mu^-}(x,y)=C_{\mu^-}(y,x)$)

Transitive: Suppose $x,y,z\in A$ and $\mu^+(x)=\mu^+(z)=1$ and $\mu^+(y)<1$ then (x,y) are connected since $S_{\mu^+}(\rho)=\min\left(\mu^+(x),\mu^+(y)\right)=\mu^+(y)$. Similarly (y,z) are connected since $S_{\mu^+}(\rho)=\min(\mu^+(y),\mu^+(z))=\mu^+(y)$. However (x,z) are not connected since $S_{\mu^+}(\rho)\neq\min(\mu^+(x),\mu^+(z))$.

$C_{\mu^+}(x,y)$ (resp. $C_{\mu^-}(x,y)$) is not an equivalence relation.

Just like in fuzzy connectedness, this concept of bipolar fuzzy connectedness is not an equivalence relation. Nevertheless, it remains a useful relation since the analogous notion of "connected components" may be defined in the bipolar fuzzy setting. In the next section, this definition is explored and the accompanying properties are discussed.

3.3. Connected Components in Bipolar Fuzzy Graphs

To define the connected components in a bipolar fuzzy graph, both the positive and negative edges while traversing the graph are considered. The traversal process will involve following positive edges in one direction and negative edges in the opposite direction to ensure a comprehensive understanding of the relationships. Consequently, working with bipolar fuzzy graphs will involve the inclusion of both fuzzy memberships and bipolar weights.

In this section, the concepts of fuzzy components are extended by defining their analogous versions in bipolar fuzzy graphs. These concepts are; Plateaus, Tops and Bottoms

Definition 3.3.1

A μ^+-plateau (resp. μ^--plateau) in a bipolar fuzzy subset μ is a maximal μ^+-connected (resp. μ^--connected) connected of subset A on which μ^+ (resp. μ^-) has a constant value. That is, $A\subseteq X$ is a plateau if

1) A is μ^+-connected (resp. μ^--connected),

2) $\forall x,y\in A$, $\mu^+(x)=\mu^+(y)$ (resp. $\mu^-(x)=\mu^-(y)$),

3) For all pairs of adjacent vertices such that $x\in A$, and $y\notin A$, $\mu^+(x)\neq\mu^+(y)$ (resp. $\mu^-(x)\neq\mu^-(y)$).

Note: 1) If A is both a μ^+-plateau and a μ^--plateau then we say that A is a μ-plateau.

2) Any $x\in A$ can only belong to one and only one μ^+-plateau (resp. μ^--plateau).

Definition 3.3.2 μ^+-Top (resp. μ^--Top)

A μ^+-plateau (resp. μ^--plateau) is called a μ^+-Top (resp. μ^--Top) if whenever $x\in A$, and $y\notin A$, $\mu^+(x)>\mu^+(y)$ (resp. $\mu^-(x)<\mu^-(y)$. If A is both a μ^+-Top and a μ^--Top then we say that A is a μ-Top.

A μ^+-plateau (resp. μ^--plateau) is called a μ^+-Bottom (resp. μ^--Bottom) if whenever $x\in A$, and $y\notin A$, $\mu^+(x)<\mu^+(y)$ (resp. $\mu^-(x)>\mu^-(y)$). If A is both a μ^+-Bottom and a μ^--Bottom then we say that A is a μ-Bottom.

Proposition 3.3.1

Let A be a bipolar fuzzy subset

1) A is a μ-plateau *iff* it is a $(1-\mu)$-plateau.

2) A is a μ-Bottom *iff* it is a $(1-\mu)$-Top (resp. A is a μ-Top *iff* it is a $(1-\mu)$-Bottom).

Proof:

The proofs are intuitive hence omitted.

Remark [1]: In the crisp sense, the plateaus will represent connected components of the underlying set X and of its complement X^c.

Definition 3.3.3

Suppose A is a μ-Top, then we may associate to A the following sets;

1) $\Omega_{A^+} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^+(x_{i-1}) \le \mu^+(x_i)\}$
$\Omega_{A^-} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^-(x_{i-1}) \ge \mu^+(x_i)\}$

2) $\Pi_{A^+} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^+(x) \le \mu^+(x_i) \le \mu^+(y)\}$
$\Pi_{A^-} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^-(x) \ge \mu^-(x_i) \ge \mu^-(y)\}$

3) $\Sigma_{A^+} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^+(x) \le \mu^+(x_i)\}$
$\Sigma_{A^-} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^-(x) \ge \mu^-(x_i)\}$

Theorem 3.3.1

Let A be a μ-Top, then

$$A^+ \subseteq \Omega_{A^+} \subseteq \Pi_{A^+} \subseteq \Sigma_{A^+}$$
$$A^- \subseteq \Omega_{A^-} \subseteq \Pi_{A^-} \subseteq \Sigma_{A^-}$$

From the definitions above, a point x belongs in Ω_{A^+} (resp Ω_{A^-}) if there exists a monotonically nondecreasing bipolar fuzzy path from x to A. Consequently, it is not possible to have a peak higher than the *Top A*. By the same argument, if a point x belongs in Π_{A^+} (resp Π_{A^-}) or x belongs in Σ_{A^+} (resp Σ_{A^-}) then there cannot exist a peak higher than the *Top A* between x and A.

Corollary 3.3.1

Let A and B be two μ-Top. Then A and B cannot be adjacent to each other.

Proof:

If they have the same height, then A and B are a single μ-Top. If A and B have different heights, then the shorter one cannot be a μ-Top.

Proposition 3.3.2

Let A^+ be a μ^+-Top and $\Sigma_{A^+} = \{x \in A \mid \exists\rho : x = x_0, x_1, \cdots, x_n = y \text{ with } \mu^+(x) \le \mu^+(x_i)\}$. Then Σ_{A^+} is essentially the set of all points of X that are connected to points of A^+.

Proof

Let X, a nonempty set of integer coordinate points endowed with a k-adjacency relation, be connected to $y \in A^+$. Then there exists a path ρ from x to y such that for all points x_i on the path ρ, $\mu^+(x_i) \ge \min(\mu^+(x), \mu^+(y))$.

If $\mu^+(x) > \mu^+(y)$ then $x \notin A^+$ and $\mu^+(x) \ge \mu^+(y)$ $\forall x_i$ on the path ρ. But form the proposition above this is not possible since the path ρ must pass through a point y' adjacent to A^+ but not in A^+. Therefore, it is mandatory that $\mu^+(y') < \mu^+(A)$ hence $\mu^+(x) < \mu^+(y)$ and $\mu^+(x_i) \ge \mu^+(x)$ $\forall x_i$ on the path ρ.

Conversely, if $x \in \Sigma_{A^+}$ then $\mu^+(x) \le \mu^+(A)$ by above proposition. Therefore, there exists a path ρ from x to a point y of A^{+} such that $\forall x_i$ on the path ρ, then $\mu^{+}(x_i) \geq \mu^{+}(x) = \min(\mu^{+}(x), \mu^{+}(y))$, implying that x is connected to y.

3.4. Bipolar Fuzzy Surroundness

The extent to which a digital structure is surrounded is related to how much the path must change direction in order to reach the boundary without intersections.

Definition 3.4.1

Let $A = (A^{+}, A^{-}), B = (B^{+}, B^{-}), C = (C^{+}, C^{-})$ be bipolar fuzzy subsets of X. B is said to separate A from C if $\forall x \in X$ and all paths ρ from x to y there exists a point y on the path ρ such that

$$B^{+}(y) \geq \min(A^{+}(x), C^{+}(z)) \text{ and } B^{-}(y) \leq \max(A^{-}(x), C^{-}(z))$$

In this formulation, B surrounds A if it separates A from the border of X.

Theorem 3.4.1

The relation B surrounds A is a weak partial order. *i.e.* the relation is reflexive, antisymmetric and Transitive.

Proof

Let $A = (A^{+}, A^{-}), B = (B^{+}, B^{-}), C = (C^{+}, C^{-})$ be bipolar fuzzy subsets of X. Then,

1) Reflexivity: A surrounds A is intuitive.

2) Transitivity: Let $x \in X$, and ρ be any path from x to the border of B. If B surrounds C, then there exists a point $y \in X$ on the path ρ such that $B^{+}(y) \geq C^{+}(x)$ and if A surrounds B, then similarly there exists a point $y \in X$ on the path ρ such that $A^{+}(y) \geq B^{+}(x)$ and $A^{-}(y) \leq B^{-}(x)$. Therefore $A^{+}(y) \geq C^{+}(x)$ and $A^{-}(y) \leq C^{-}(x)$ hence A surrounds C.

3) Antisymmetry: suppose A surrounds B and B surrounds A. Then to proof that the relation is antisymmetric, it is enough to show that $A \wedge B$ surrounds both A and B. Now let ρ be any path from x to the border of B and y be the last point on the path such that $B^{+}(y) \geq A^{+}(x)$ and $B^{-}(y) \leq A^{-}(x)$. Since A surrounds B then there exists a point y' on the path ρ beyond y (or possibly y itself) such that $A^{+}(y') \geq B^{+}(y)$ and $A^{-}(y') \leq B^{-}(y)$. Similarly, since B surrounds A, then there exists a point y'' on the path ρ beyond y' (or possibly y' itself) such that $B^{+}(y'') \geq A^{+}(y') \geq A^{+}(x)$ and $B^{-}(y'') \leq A^{-}(y') \leq A^{-}(x)$. From the choice of, then $y = y' = y''$ so that $A^{+}(y) \wedge B^{+}(y) \geq A^{+}(x)$ and $A^{-}(y) \wedge B^{-}(y) \leq A^{-}(x)$. Since x is arbitrary, then $A \wedge B$ surrounds A and similarly surrounds B.

The relation is a weak partial order.

4. Conclusion

This paper has extended some very fundamental concepts of fuzzy digital topology into the realm of bipolar fuzzy digital topology. The utility of these concepts in the bipolar fuzzy environment is critical in the accurate modelling of real-life phenomena since both the positive and negative membership degrees of belonging have been accommodated. The consistency of connected components in bipolar fuzzy sets has been confirmed, implying that these concepts may be applied in solving problems in fields such as decision-making and pattern recognition.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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