Scientific Research Publishing

Advances in Pure Mathematics, 2024, 14, 515-522
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Solving Invariant Problem of Cauchy Means Based on Wronskian Determinant

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How to cite this paper: Ni, Y.J. and Wang, F. (2024) Solving Invariant Problem of Cauchy Means Based on Wronskian Determinant. *Advances in Pure Mathematics*, **14**, 515-522.
https://doi.org/10.4236/apm.2024.147029

Received: July 2, 2024
Accepted: July 20, 2024
Published: July 23, 2024

Abstract

This paper studied the invariance of the Cauchy mean with respect to the arithmetic mean when the denominator functions satisfy certain conditions. The partial derivatives of Cauchy's mean on the diagonal are obtained by using the method of Wronskian determinant in the process of solving. Then the invariant equation is solved by using the obtained partial derivatives. Finally, the solutions of invariant equations when the denominator functions satisfy the same simple harmonic oscillator equation or the denominator functions are power functions that have been obtained.

Keywords

Cauchy Mean, Wronskian Determinant, Arithmetic Mean, Invariant Equation

1. Introduction

Throughout this paper, let $I \subseteq \mathbb{R}$ be a nonempty open interval. In the sequel, the classes of continuous strictly monotone and continuous positive real-valued functions defined on I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively.

Let $I \subset \mathbb{R}$ be an open interval. A two-variable function $M: I^2 \to I$ is called a mean on the interval I if

$$\min(x, y) \le M(x, y) \le \max(x, y), \quad x, y \in I,$$

holds. If for all $x, y \in I$, $x \ne y$, these inequalities are strict, M is called strict ([1]). Obviously, if M is a mean, then M is reflexive, *i.e.*, $M(x, x) = x$ for all $x \in I$.

Let $M, N: I^2 \to I$ be means. A mean $K: I^2 \to I$ is called invariant with respect to the mean-type mappings (M, N), shortly, (M, N)-invariant ([2]), if

$$K\left(M\left(x,y\right),N\left(x,y\right)\right)=K\left(x,y\right),\ \ x,y\in I.$$

The simplest example of when the invariance equation holds is the well-known identity

$$G\left(\mathcal{A}\left(x,y\right),H\left(x,y\right)\right)=G\left(x,y\right),\ \ x,y>0,$$

where

$$A\left(x,y\right)=\frac{x+y}{2},\ G\left(x,y\right)=\sqrt{xy},\ H\left(x,y\right)=\frac{2xy}{x+y},$$

are arithmetic mean, geometric mean and harmonic mean respectively.

The *Cauchy mean* $C_{f,g}:I^2\to I$ is defined by

$$C_{f,g}\left(x,y\right):=\left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f\left(tx+\left(1-t\right)y\right)\mathrm{d}t}{\int_0^1 g\left(tx+\left(1-t\right)y\right)\mathrm{d}t}\right),\ \ x,y\in I,$$

$f,g:I\to\mathbb{R}$ are two continuous functions such that $g\in\mathcal{CP}\left(I\right)$ and $\frac{f}{g}$ is bijective.

In 1984, Leach and Sholander [3] proposed the Cauchy mean theorem for difference quotients and the definition of Cauchy mean. Losonczi solved the comparison and equality problems of Cauchy means for more than two variables and the equality problem for the two-variable Cauchy means under the assumption of seven times differentiability in [4]-[6]. Then Matkowski reduced the regularity condition for the equality problem for the two-variable Cauchy means to first-order differentiability [7]. Other definitions for Cauchy have been in [8] [9]. The invariant equations of some special Cauchy mean-type mappings with geometric mean or arithmetic mean have been studied in [10] [11]. Other invariance of more generalized means generated by measure integral have been considered in [12] [13]. Recently, Zhang [14] gave a survey of results dealing with the problem of means.

In this paper, we will study the invariant equation of Cauchy mean with respect to the arithmetic mean, that is to solve

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f\left(tx+\left(1-t\right)y\right)\mathrm{d}t}{\int_0^1 g\left(tx+\left(1-t\right)y\right)\mathrm{d}t}\right)+\left(\frac{h}{k}\right)^{-1}\left(\frac{\int_0^1 h\left(tx+\left(1-t\right)y\right)\mathrm{d}t}{\int_0^1 k\left(tx+\left(1-t\right)y\right)\mathrm{d}t}\right), \tag{1.1}$$

where $x,y\in I$, $f,g,h,k:I\to\mathbb{R}$ are continuous functions such that $g,k\in\mathcal{CP}\left(I\right)$ and $\frac{f}{g},\frac{h}{k}$ are injective functions.

2. Auxiliary Results

In order to describe the regularity conditions related to the two unknown functions f,g generating the mean $C_{f,g}$, we introduce some notations. The class $\mathcal{C}_0\left(I\right)$ consists of all those pairs $\left(f,g\right)$ of continuous functions $f,g:I\to\mathbb{R}$ such that $g\in\mathcal{CP}\left(I\right)$ and $f/g\in\mathcal{CM}\left(I\right)$. For $n\in\mathbb{N}$, we say that the pair (f,g) is in the class $\mathcal{C}_n(I)$ if (f,g) are *n*-times continuously differentiable functions such that $g \in \mathcal{CP}(I)$ and the function $f'g - fg'$ does not vanish anywhere on *I*. Obviously, the latter condition implies that f/g is strictly monotone, *i.e.*, $f/g \in \mathcal{CM}(I)$.

For $(f,g) \in \mathcal{C}_2(I)$, we also introduce the notation

$$\Phi_{f,g} := \frac{W_{f,g}^{2,0}}{W_{f,g}^{1,0}}, \quad \Psi_{f,g} := -\frac{W_{f,g}^{2,1}}{W_{f,g}^{1,0}},$$

where the (i,j)-order Wronskian operator $W_{i,j}$ is defined in terms of *i*th and *j*th derivatives by

$$W_{f,g}^{i,j} := \begin{vmatrix} f^{(i)} & f^{(j)} \\ g^{(i)} & f^{(j)} \end{vmatrix}.$$

Lemma 1 (*Lemma 5 in* [15]) *Let* $(f,g) \in \mathcal{C}_0(I)$. *Then* f,g *are solutions of the second-order differential equation*

$$y'' = \Phi_{f,g} y' + \Psi_{f,g} y.$$

In what follows, we will give some formulae for the high-order directional derivatives of $C_{f,g}$ at the diagonal points of the Cartesian product $I \times I$. Given a pair $(f,g) \in \mathcal{C}_0(I)$ and a fixed element $x \in I$, define the function $m_x := m_{x;f,g}$ in a neighborhood of origin by

$$m_x(u) = m_{x;f,g}(u) := C_{f,g}\left(x + \frac{1}{2}u, x - \frac{1}{2}u\right). \tag{2.1}$$

Using Lemma 2.2 from [16], we get the following

Lemma 2 *Let* $n \in \mathbb{N}$, $(f,g) \in \mathcal{C}_n(I)$. *Then, for fixed* $x \in I$, *the function* m_x *defined by* (*2.1*) *is n-times continuously differentiable at the origin and*

$$\sum_{i=0}^{n} \binom{n}{i} \begin{vmatrix} f^{(i)}(x) & (f \circ m_x) f^{(n-i)}(0) \\ g^{(i)}(x) & (g \circ m_x) g^{(n-i)}(0) \end{vmatrix} = 0.$$

Furthermore, $m_x^{(2k)}(0) = x$ *for* $k \in \mathbb{N}$ *and for the cases* $n = 2,4$, *we have*

$$m''_{x;f,g}(0) = \frac{1}{12}\Phi_{f,g}(x),$$

$$m_{x;f,g}^{(4)}(0) = -\frac{1}{48}\left(\Phi_{f,g}^3 + 2\Phi_{f,g}\Psi_{f,g}\right)(x) + \frac{1}{80}\left(\Phi''_{f,g} + 3\Phi'_{f,g}\Phi_{f,g} + \Psi_{f,g}^3 \right.$$
$$\left. + 2\Phi_{f,g}\Psi_{f,g} + 2\Psi'_{f,g}\right)(x).$$

Lemma 3 *If* $(f,g),(h,k) \in \mathcal{C}_1(I)$, *and Equation* (*1.1*) *holds, then*

$$\Phi_{f,g} = -\Phi_{h,k} := \Phi. \tag{2.2}$$

Proof. By Equation (2.1), the invariant Equation (1.1) can be rewritten by

$$m_{x;f,g}(u) + m_{x;h,k}(u) = 0. \tag{2.3}$$

By differentiating the above equation twice with respect to variable *u*, it can be obtained that

$$m''_{x;f,g}(u) + m''_{x;h,k}(u) = 0,$$

letting $u=0$

$$m''_{x;f,g}(0)+m''_{x;h,k}(0)=0.$$

And because

$$m''_{x;f,g}(0)=\frac{1}{12}\Phi_{f,g}(x),\ m''_{x;h,k}(0)=\frac{1}{12}\Phi_{h,k}(x),$$

then

$$m''_{x;f,g}(0)+m''_{x;h,k}(0)=\frac{1}{12}\left(\Phi_{h,k}+\Phi_{h,k}\right)(x)=0,$$

that is

$$\Phi_{f,g}=-\Phi_{h,k}=\Phi.$$

□

Lemma 4 *If* $(f,g),(h,k)\in\mathcal{C}_4(I)$, *and Equation* (*1.1*) *holds, then*

$$\frac{3}{80}\Phi\Phi'-\frac{1}{120}\Phi\left(\Psi_{f,g}-\Psi_{f,g}\right)+\frac{1}{80}\left(\Psi'_{f,g}+\Psi'_{h,k}\right)=0. \tag{2.4}$$

Proof. By differentiating Equation (2.3) fourth with respect to variable *u*, it can be obtained that

$$m^{(4)}_{x;f,g}(u)+m^{(4)}_{x;h,k}(u)=0,$$

letting $u=0$, we have

$$m^{(4)}_{x;f,g}(0)+m^{(4)}_{x;h,k}(0)=0.$$

Since

$$\begin{aligned}m^{(4)}_{x;f,g}(0)=&-\frac{1}{48}\left(\Phi^3_{f,g}+2\Phi_{f,g}\Psi_{f,g}\right)(x)\\&+\frac{1}{80}\left(\Phi''_{f,g}+3\Phi'_{f,g}\Phi_{f,g}+\Psi^3_{f,g}+2\Phi_{f,g}\Psi_{f,g}+2\Psi'_{f,g}\right)(x),\end{aligned}$$

$$\begin{aligned}m^{(4)}_{x;h,k}(0)=&-\frac{1}{48}\left(\Phi^3_{h,k}+2\Phi_{h,k}\Psi_{h,k}\right)(x)\\&+\frac{1}{80}\left(\Phi''_{h,k}+3\Phi'_{h,k}\Phi_{h,k}+\Psi^3_{h,k}+2\Phi_{h,k}\Psi_{h,k}+2\Psi'_{h,k}\right)(x),\end{aligned}$$

then we get

$$\frac{3}{80}\Phi\Phi'-\frac{1}{120}\Phi\left(\Psi_{f,g}-\Psi_{f,g}\right)+\frac{1}{80}\left(\Psi'_{f,g}+\Psi'_{h,k}\right)=0.$$

□

3. Solutions to Invariant Equations under Special Conditions

Although it is difficult to directly solve the Cauchy mean invariant equation with respect to the arithmetic mean, some types of invariant equations can also be solved with certain prerequisites. Next, we will solve the Cauchy mean invariant equation with respect to the arithmetic mean when the denominator functions g,k are equal and satisfy the harmonic oscillator equation

$$y'' = py. \tag{3.1}$$

for some $p \in \mathbb{R}$. Then, we introduce the sine and cosine type functions $S_p, C_p : \mathbb{R} \to \mathbb{R}$ by

$$S_p(x) := \begin{cases} \sin\sqrt{-p}x, & p < 0, \\ x, & p = 0, \\ \sinh\left(\sqrt{p}x\right), & p > 0, \end{cases} \quad C_p(x) := \begin{cases} \cos\sqrt{-p}x, & p < 0, \\ x, & p = 0, \\ \cosh\left(\sqrt{p}x\right), & p > 0. \end{cases}$$

Due to basic results on the second-order linear differential equations, the functions S_p and C_p given above form a fundamental system of solutions for the differential Equation (3.1).

Theorem 1 *If* $(f,g),(h,k) \in \mathcal{C}_4(I)$. *Assume* $k = g$ *satisfying Equation* (*3.1*) *and Equation* (*1.1*) *holds, then exists* $c \in \mathbb{R}$ *such that*

$$f = g\int \frac{1}{g^2}\exp\left(\int cg^{\frac{4}{9}}\right), h = g\int \frac{1}{g^2}\exp\left(-\int cg^{\frac{4}{9}}\right). \tag{3.2}$$

Proof. Since Equation (3.1) holds, by the definitions of $\Phi_{f,g}$ and $\Psi_{f,g}$ we get

$$\Psi_{f,g} = -\frac{\begin{vmatrix} f'' & f' \\ g'' & g' \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}} = -\frac{\begin{vmatrix} f'' & f' \\ pg & g' \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}} = -\frac{g'}{g}\Phi_{f,g} + p.$$

Similarly, we have

$$\Psi_{h,k} = -\frac{k'}{k}\Phi_{h,k} + p.$$

Using (2.2) and the above two equalities, we have

$$\Psi_{f,g} + \Psi_{h,k} = 2p, \quad \Psi_{f,g} - \Psi_{h,k} = -2\frac{g'}{g}\Phi,$$

and thus $\Psi'_{f,g} + \Psi'_{h,k} = 0$. Consequently, (2.4) becomes

$$\frac{3}{80}\Phi\Phi' + \frac{g'}{60g}\Phi^2 = 0,$$

which implies that there exists some $c \in \mathbb{R}$ such that

$$\Phi = cg^{\frac{4}{9}}.$$

Hence, we obtain

$$\Phi_{f,g} = \frac{\left(W_{f,g}^{1,0}\right)'}{W_{f,g}^{1,0}} = cg^{\frac{4}{9}}, \quad \Phi_{h,k} = \frac{\left(W_{h,k}^{1,0}\right)'}{W_{h,k}^{1,0}} = -cg^{\frac{4}{9}},$$

Since $W_{f,g}^{1,0} = \left(\frac{f}{g}\right)g^2$ and $W_{h,k}^{1,0} = \left(\frac{h}{k}\right)k^2$, integrating the above equations, we get (3.2). ☐

In what follows, we will restrict to the basic solutions to the equation of harmonic oscillator, *i.e.* consider the functions

$(H1)$ $g(x)=x$ for $x\in I\subseteq(0,+\infty)$;
$(H2)$ $g(x)=\mathrm{e}^{x}$ for $x\in I\subseteq\mathbb{R}$;
$(H3)$ $g(x)=\sin x$ for $x\in I\subseteq\left(0,\frac{\pi}{2}\right)$.

Theorem 2 *If* $(f,g),(h,k)\in\mathcal{C}_4(I)$. *Assume* $k=g$ *satisfying one of* the *conditions* $(H1)$ - $(H3)$ *and Equation* (*1.1*) *holds, then*

1) For case $(H1)$, there exists $b_1,b_2\in\mathbb{R}_{+},c\in\mathbb{R}$,

$$f(x)=b_1\int\frac{1}{x^2}\exp\left(cx^{\frac{13}{9}}\right),h(x)=b_2\int\frac{1}{x^2}\exp\left(cx^{\frac{13}{9}}\right),\quad x\in I.$$

2) For case $(H2)$, there exists $b_1,b_2\in\mathbb{R}_{+},c\in\mathbb{R}$,

$$f(x)=b_1\mathrm{e}^{x}\int\mathrm{e}^{-2x}\exp\left(c\mathrm{e}^{\frac{4x}{9}}\right),h(x)=b_2b_1\mathrm{e}^{x}\int\mathrm{e}^{-2x}\exp\left(-c\mathrm{e}^{\frac{4x}{9}}\right),\quad x\in I.$$

3) For case $(H3)$, there exists $c\in\mathbb{R}$,

$$f(x)=\cos x\int\frac{1}{\cos^2 x}\exp\left(\int c\cos^{\frac{4}{9}}x\right),$$

$$h(x)=\cos x\int\frac{1}{\cos^2 x}\exp\left(\int -c\cos^{\frac{4}{9}}x\right),\quad x\in I.$$

Next, we will consider the solution of the invariant equation for (1.1) when the denominator functions are power functions, that is $g(x)=x^{p}$, $k(x)=x^{r}$.

Theorem 3 *Let* $g(x)=x^{p},k(x)=x^{r}$, $p=r$ *and the invariant Equation* (*1.1*) *holds, then there exists* $c_1,c_2\in\mathbb{R}$ *such that*

$$f=g\int\frac{1}{g^2}\exp\left(\pm\int\sqrt{c_1\beta+c_2}\right),h=g\int\frac{1}{g^2}\exp\left(\pm\int\sqrt{c_1\beta+c_2}\right),$$

where $\beta=\frac{p^3(1-p)}{x^2}$.

Proof. When $g=k=x^{p}$ we have $g'(x)=k'(x)=px^{p-1}$ and $g''(x)=k''(x)=p(p-1)x^{p-2}$, the relationship between $\Phi_{f,g}$ and $\Psi_{f,g}$ is

$$\Psi_{f,g}(x)=-\frac{p}{x}\Phi_{f,g}(x)+\frac{p(p-1)}{x^2}.$$

Similarly

$$\Psi_{h,k}(x)=-\frac{p}{x}\Phi_{h,k}(x)+\frac{p(p-1)}{x^2}.$$

Using (2.2) and the above two equalities, we have

$$\Psi_{f,g}(x)+\Psi_{h,k}(x)=\frac{2p(p-1)}{x^2},\quad \Psi_{f,g}(x)-\Psi_{h,k}(x)=-\frac{2p\Phi}{x},$$

and thus

$$\Psi'_{f,g}(x)+\Psi'_{h,k}(x)=-\frac{4p(p-1)}{x^3}.$$

Consequently, (2.4) becomes

$$\frac{3}{80}\Phi(x)\Phi'(x)+\frac{p}{60x}\Phi^2(x)-\frac{p(p-1)}{20x^3}=0,$$

which implies that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\Phi^2(x)=c_1\frac{p^3(1-p)}{x^2}+c_2.$$

Hence, we obtain

$$\Phi_{f,g}=\frac{\left(W_{f,g}^{1,0}\right)'}{W_{f,g}^{1,0}}=cg^{\frac{4}{9}},\quad \Phi_{h,k}=\frac{\left(W_{h,k}^{1,0}\right)'}{W_{h,k}^{1,0}}=-cg^{\frac{4}{9}}.$$

By $W_{f,g}^{1,0}=\left(\frac{f}{g}\right)g^2$ and $W_{h,k}^{1,0}=\left(\frac{h}{k}\right)k^2$, integrating the above equations, we get the result. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Cauchy, A.L. (1821) Cours d'analyse de l'École Royale Polytechnique, Ire partie: Analyse Algébrique. Imprimérie Royale.

[2] Daróczy, Z. and Páles, Z. (2002) Gauss-Composition of Means and the Solution of the Matkowski—Sutô Problem. *Publicationes Mathematicae Debrecen*, **61**, 157-218. https://doi.org/10.5486/pmd.2002.2713

[3] Leach, E.B. and Sholander, M.C. (1984) Multi-Variable Extended Mean Values. *Journal of Mathematical Analysis and Applications*, **104**, 390-407. https://doi.org/10.1016/0022-247x(84)90003-9

[4] Losonczi, L. (2000) Equality of Cauchy Mean Values. *Publicationes Mathematicae Debrecen*, **57**, 217-230. https://doi.org/10.5486/pmd.2000.2338

[5] Losonczi, L. (2002) On the Comparison of Cauchy Mean Values. *Journal of Inequalities and Applications*, **2002**, Article ID: 876016. https://doi.org/10.1155/s1025583402000024

[6] Losonczi, L. (2003) Equality of Two Variable Cauchy Mean Values. *Aequationes Mathematicae*, **65**, 61-81. https://doi.org/10.1007/s000100300004

[7] Matkowski, J. (2004) Solution of a Regularity Problem in Equality of Cauchy Means. *Publicationes Mathematicae Debrecen*, **64**, 391-400. https://doi.org/10.5486/pmd.2004.2939

[8] Berrone, L.R. and Moro, J. (2000) On Means Generated through the Cauchy Mean Value Theorem. *Aequationes Mathematicae*, **60**, 1-14. https://doi.org/10.1007/s000100050131

[9] Berrone, L.R. (2015) Generalized Cauchy Means. *Aequationes Mathematicae*, **90**, 307-328. https://doi.org/10.1007/s00010-015-0341-7

[10] Głazowska, D. (2011) Some Cauchy Mean-Type Mappings for Which the Geometric Mean Is Invariant. *Journal of Mathematical Analysis and Applications*, **375**, 418-430. https://doi.org/10.1016/j.jmaa.2010.09.036

[11] Zhang, Q., Xu, B. and Han, M. (2020) Some Cauchy Mean-Type Mappings for

Which the Arithmetic Mean Is Invariant. *Aequationes Mathematicae*, **95**, 13-34. https://doi.org/10.1007/s00010-020-00743-0

[12] Zhang, Q. and Xu, B. (2011) An Invariance of Geometric Mean with Respect to Generalized Quasi-Arithmetic Means. *Journal of Mathematical Analysis and Applications*, **379**, 65-74. https://doi.org/10.1016/j.jmaa.2010.12.025

[13] Zhang, Q. and Xu, B. (2017) On Some Invariance of the Quotient Mean with Respect to Makó-Páles Means. *Aequationes Mathematicae*, **91**, 1147-1156. https://doi.org/10.1007/s00010-017-0502-y

[14] Zhang, Q. (2024) Means and the Equations of Means. *Advances in Mathematics* (*China*), **53**, 468-498.

[15] Páles, Z. and Zakaria, A. (2018) On the Invariance Equation for Two-Variable Weighted Nonsymmetric Bajraktarević Means. *Aequationes Mathematicae*, **93**, 37-57. https://doi.org/10.1007/s00010-018-0560-9

[16] Zhang, Q. and Li, L. (2023) On the Invariance of Generalized Quasiarithmetic Means. *Journal of Applied Analysis & Computation*, **13**, 1581-1596. https://doi.org/10.11948/20220380