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Embedding an FIR in a Body

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Abstract

We have shown that certain non-commutative integral rings can be immersed in a field by a method which is not that of the construction of the field of fractions on the right $\mathbb{R}\mathbb{R}^{*-1}$, or left $\mathbb{R}^{*-1}\mathbb{R}$, of the ring $\mathbb{R}$ under consideration.

Keywords

Rings, Bcody, Fractions, FIR, Matrix, Honest Closing

1. Introduction

In many branches of mathematics, one may have to compare two "objects" with each other by showing that one of the "objects" is a "sub-object" of the other (sometimes via injection, replacing set inclusion). In some theories, such as in differential geometry or field theory, the term embedding is completely defined, while in others it is only mentioned in intuitive contexts and is therefore not endowed with a precise meaning. Generally speaking, an embedding should be thought of as an injective morphism. We shall show, from [1] that certain non-commutative integral rings can be immersed in a field by a method that is not that of the construction of the field of fractions on the right $\mathbb{R}\mathbb{R}^{*-1}$, or on the left $\mathbb{R}^{*-1}\mathbb{R}$, of the ring $\mathbb{R}$ under consideration.

In our specific case, we will use the "honestfence" of the ring R, which if it exists and if R satisfies certain conditions, is a field. This method will apply to the case of an FIR, whose definition we recall: An R ring is called a FIR (Free Ideal Ring) on the right if it has the property of invariant rank (*i.e.* if two bases of a free R-module have the same cardinal) and if any ideal to the right of R is free ...

$$a\mathbb{R} \bigcap b\mathbb{R} \neq 0, \forall a, b \in \mathbb{R}^*$$

For a left-hand FIR, we have the analogous conclusions. Therefore, an FIR usually does not have a field of fractions on the left or right.

2. Honest $\mathbb{R}$ Closing

A) Solid Matrices – Definitions and Properties

Either $M_n(\mathbb{R})$ the ring of square matrices of order *n* with coefficients in $\mathbb{R}$.

Definition

a matrix *A* of $M_n(\mathbb{R})$ is said to be full if *A* cannot be written as the product of a matrix $n \times p$ and a matrix $p \times n$ with coefficients in $\mathbb{R}$, with $p < n$.

A matrix *A* is full if and only if the α endomorphism of $\mathbb{R}^n$ that it defines cannot be written: $\alpha = \beta\gamma$, with β homomorphism of $\mathbb{R}^p$ in $\mathbb{R}^n$, γ homomorphism of $\mathbb{R}^n$ in $\mathbb{R}^p$, and $p < n$.

Note that in the case where $\mathbb{R}$ is a subring of a ring *S*, a matrix *A* of $M_n(\mathbb{R})$ can be full without being as a matrix of $M_n(S)$. We then specify: Full on $\mathbb{R}$

Proposition 1

If $\mathbb{R}$ is a field the following properties are equivalent:

a) *A* is full

b) *A* is invertible in $M_n(\mathbb{R})$

c) *A* is a non-divisor of zero in *A*

We know the equivalence of a) and b) and the following condition:

The αendomorphism of $\mathbb{R}^n$ defined by *A* is injective. Now, if *A* is full, α is injective, because of the following factorization of α:

$$\mathbb{R}^n \to \mathbb{R}^n / Ker\ \alpha \to \mathbb{R}^n$$

Conversely, if α is injective, for any factorization of α:

$$\mathbb{R}^n \xrightarrow{\gamma} \mathbb{R}^p \xrightarrow{\beta} \mathbb{R}^n$$

We necessarily have the property: γ is injective, and consequently $p \geq n$.

In the general case, there is no implication between the properties: *A* is full, and *A* is a non-divisor of zero [2]. But we have the following proposition:

Proposition 2

If *A* is a regular element of $M_n(\mathbb{R})$ (an element with no divisor of zero to the right or left), *A* is a solid matrix.

Let *A* be a regular element of $M_n(\mathbb{R})$. Suppose: $A = BC$ with *B* matrix $n \times r$, *C* matrix $r \times n$, and $r < n$. By completing matrices *B* and *C* with zeros, we obtain matrices B' and C' of $M_n(\mathbb{R})$, which are divisors of zeros in $M_n(\mathbb{R})$ and which verify $A = B'C'$.

We also get the following propositions:

Proposition 3

Any matrix of $M_n(\mathbb{R})$, the factor of matrix full of $M_n(\mathbb{R})$ is. Full

Proposition 4

Either $A \in M_n(\mathbb{R})$, $B \in M_n(\mathbb{R})$, *C* a matrix $n \times p$. If a matrix $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$ is a matrix full of $M_{n+p}(\mathbb{R})$, *A* and *B* are full matrices.

Let us indicate, for proposition 4, that in order to prove that *A* is full, we use the relation:

$$\begin{pmatrix} PQ & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} P & C \\ 0 & B \end{pmatrix} \begin{pmatrix} Q & O \\ O & I \end{pmatrix}$$

Let us then introduce the following definition: *A* ring $\mathbb{R}$ satisfies property (*P*), if the reciprocal of proposition 4 is true:

Proposition 5

If $\mathbb{R}$ satisfies (*P*), the unit matrix of $M_n(\mathbb{R})$ is full

Corollary:

If $\mathbb{R}$ satisfies (*P*), $\mathbb{R}$ has the property of invariant rank. For $\mathbb{R}$ to have the property of invariant rank, it suffices [3], that $\mathbb{R}$ satisfy the condition: unit matrix I_n of $M_n(\mathbb{R})$ is full (this is exactly proposition 2 of [1])

B) Honest closing

Definition

A ring *S* is called an honest extension of $\mathbb{R}$ if $\mathbb{R}$ is a subring of *S* and if any solid matrix on $\mathbb{R}$ is full-on *S*. A ring *S* is called an honest $\mathbb{R}$ of fence if it satisfies the following conditions:

- *S* is an honest extension of $\mathbb{R}$
- Any solid matrix on $\mathbb{R}$ is invertible on *S*
- *S* is generated by $\mathbb{R}$ and the elements of inverse matrices are solid matrices on $\mathbb{R}$

Note $\Sigma(\mathbb{R})$ the ring obtained by adding 'freely' to $\mathbb{R}$ the elements of inverse matrices of solid matrices on $\mathbb{R}$. More precisely, let *X* be a system of generators of $\mathbb{R}$ and ϕ a system of relations defining $\mathbb{R}$. *E* is the set of elements of the solid matrices on $\mathbb{R}$, let $\sigma : E \to X'$ be a bijection and ϕ' be the set of relations: $\sum \alpha_{ik}\sigma(\alpha_{ki}) = 1$ for any matrix $A = (\alpha_{ij})$ full on $\mathbb{R}$.

$\Sigma(\mathbb{R})$ is the ring defined by the generating system $X \cup X'$ and relationships $\phi \cup \phi'$

Let f be the canonical homomorphism $\mathbb{R} \to \Sigma(\mathbb{R})$. The ring $\Sigma(\mathbb{R})$ satisfies the following property:

For any homomorphism $\mathbb{R} \to \mathbb{R}'$ such as any full matrix on $\mathbb{R}$ has an invertible image on $\mathbb{R}'$, There is a unique homomorphism $\bar{\varphi} : \Sigma(\mathbb{R}) \to \mathbb{R}'$ tel que $\varphi = \bar{\varphi} f$ [4].

We get the following theorem:

Theorem 1

A ring $\mathbb{R}$ satisfaying (*P*) to an honest closure $\mathbb{K}$ if and only if the homomorphism $\mathbb{R} \to \Sigma(\mathbb{R})$ is injective; if $\mathbb{K}$ exist, $\mathbb{K}$ is isomorphic to $\Sigma(\mathbb{R})$.

If *f* is injective, $\Sigma(\mathbb{R})$ satisfies the proprieties of an honest closure. conversely, if $\mathbb{R}$ have honest losure $\mathbb{K}$, $\mathbb{K}$ is a boby. Either the injection of $\mathbb{R}$ in $\mathbb{K}$. It exists $\bar{i} : \Sigma(\mathbb{R}) \to \mathbb{K}$ such as $i = \bar{i}f$. As a result, *f* is injective; $\Sigma(\mathbb{R})$ is a closure of $\mathbb{R}$, therefore a body. We deduce that the homomorphism

$$\bar{i} : \Sigma(\mathbb{R}) \to \mathbb{K} \quad (1)$$

is injective. Moreover, the image $\bar{i}$ generates $\mathbb{K}$ (according to the third proposition definition of a closure). From where

$$\mathbb{K} \simeq \Sigma(\mathbb{R}) \tag{2}$$

Sufficient conditions for the existence of honest closure of a satisfactory ring (*P*)

Theorem 2

A ring $\mathbb{R}$ satisfying (*P*) has an honest closure if and only if we can immerse, by a homomorphism of unit ring, the ring $M_n(\mathbb{R})$ in a satisfactory ring F_n :

a) Any matrix full if $M_n(\mathbb{R})$ has an inversible image in F_n.

La condition est évidemment nécessaire. The condition is obviously necessary. For it to be satisfactory, we will use the following lemmas:

Lemme 1

A ring F unitary has elements $e_{ij}(i,j=1,\cdots,n)$ such as:

$$\sum\nolimits_{i=1}^{n} e_{ij} = 1,\ e_{ij} = e_{hl} = \delta_{jh} e_{i\ell} \tag{3}$$

Is isomorphic to the ring $M_n(A)$, with $A = e_{11} F e_{11}$.

We verify that the application $\sigma : F \to M_n(A)$ defined by $\sigma(x) = (\alpha_{ij})$ with $\alpha_{ij} = e_{1i} e_{j1}$, is an isomorphism of unit rings.

Lemme 2

If $M_n(A)$ is immersed, by a homomorphism of unit ring, in a ring F_n satisfying, we can dive $\mathbb{R}$ in a satisfactory ring $\mathbb{R}_n$:

b) Any full matrix on $\mathbb{R}$ d'order $\leq n$ has an inversible image on $\mathbb{R}_n$.

We have an injective homomorphism: $\varphi_n : M_n(\mathbb{R}) \to F_n$ and, according to lemme 1, isomorphism $\sigma : F_n \to M_n(\mathbb{R}_n)$ with

$$\mathbb{R}_n = \varphi_n(e_{11}) F_n \varphi_n(e_{11}).$$

Either $g_n : \mathbb{R} \to \mathbb{R}_n$ the homomorphism defined by $g_n(r) = \sigma\varphi_n(e_{11})$. We can deduce

$$\overline{g}_n : M_n(\mathbb{R}) \to M_n(\mathbb{R}_n) \tag{4}$$

We checked we have: $\varphi_n = \sigma \overline{g}_n$. As a result, $\overline{g}_n$ and g_n are injective. Let us further show that $\mathbb{R}_n$ satisfies.

Either $A \in M_n(\mathbb{R})$. So that $\varphi_n(A)$ is invertible in F_n, it is necessary and sufficient that $\overline{g}_n(A)$ the so-called $M_n(\mathbb{R}_n)$, that is, the matrix $\overline{g}_n(A)$ is invertible on $\mathbb{R}_n$. Any full matrix $A \in M_n(\mathbb{R})$ has an inversible image $\overline{g}_n(A)$ ion $\mathbb{R}_n$.

Either $A \in M_p(\mathbb{R})$ (with $p \leq n$) a full matrix. As $\mathbb{R}$ satisfied (*P*), la matrice $A' = \begin{pmatrix} A & O \\ O & I \end{pmatrix}$ of order n is full. It's invertible on $\mathbb{R}_n$. It is the same for the matrix *A*.

Theorem 3

Either $\mathbb{R}$ a ring. S a part of $\mathbb{R}$ such that:

- Every element $p \in S$ is not a divisor of zero
- Every $p \in S$ $End_{\mathbb{R}}(\mathbb{R}/p\mathbb{R})$ is a body
- Every $p, q \in S$, if $p \neq q$, $Hom_{\mathbb{R}}(\mathbb{R}/p\mathbb{R}, \mathbb{R}/q\mathbb{R}) = 0$

Then the canonical homomorphism $\mathbb{R} \to \mathbb{R}_S$ is injective.

3. Ease of Use

Proposition 6

Either $\mathbb{R}$ a fir, A a matrix of $M_n(\mathbb{R})$, α endomorphism of $\mathbb{R}^n$ defined by A. The following properties are equivalent:

- A is full
- A is a regular element of $M_n(\mathbb{R})$
- There is a $\mathbb{R}$-module twist M and an exact sequence

$$(s): O \to \mathbb{R}^n \xrightarrow{\alpha} \mathbb{R}^n \to M \to O \tag{5}$$

Proposition 7

All FIR satisfied (P)

Consider full matrices $A \in M_n(\mathbb{R})$, $B \in M_n(\mathbb{R})$, the matrix $D = \begin{pmatrix} A & C \\ O & B \end{pmatrix}$.

Matrices A and B respect full define injective endomorphisms α de $\mathbb{R}^n$ and β of $\mathbb{R}^p$. We verify that the endomorphism γ of $\mathbb{R}^{n+p}$ defined by D is also injective. [5] or [6]

Proposition 8

Either $\mathbb{R}$ a FIR, so:

- All E matrices full of $M_n(\mathbb{R})$ is factorial
- If A is an automatic full matrix $M_n(\mathbb{R}) = \mathbb{R}_n$, $End_{\mathbb{R}}(\mathbb{R}_n / A\mathbb{R}_n)$ is a body.

If A and B are two non-similar automatic full matrices of $M_n(\mathbb{R})$, we have:

$$Hom_{\mathbb{R}_n}(\mathbb{R}_n / A\mathbb{R}_n, \mathbb{R}_n / \mathbb{R}_n) = 0 \tag{6}$$

Theorem 4

If $\mathbb{R}$ ais a FIR, the ring $M_n(\mathbb{R})$ perhaps immersed in a ring such as any matrix full of $M_n(\mathbb{R})$ is an invertible image.

According to the previous proposition, any full matrix admits an automatic matrix factorization. On the other hand, if two matrices A and B of $M_n(\mathbb{R})$ are similar and if A is invertible in a ring containing $M_n(\mathbb{R})$, it is the same for B.

Either S set of two-by-two non-similar automatic full matrices of $M_n(\mathbb{R})$, such as any automatic full matrix of $M_n(\mathbb{R})$ is similar to an element of S. According to proposition 8, S satisfied the conditions of the theorem 3 And the homomorphism $M_n(\mathbb{R}) \to (M_n(\mathbb{R}))_S$ is injective.

According to the choice of S, any matrix full of $M_n(\mathbb{R})$ is invertible i $(M_n(\mathbb{R}))_S$. We thus obtain, according to the theorems 1 et 2.

Theorem 5

All FIR perhaps immersed in a body.

Remark: We can weaken the hypotheses of the propositions 6, 7, 8 and therefore of theorem 5, and only assume that $\mathbb{R}$ is a $2n$-FIR (ring with invariant rank, where any ideal generated by less than $2n$ generators is free), satisfying the increasing chain condition for ideals on the right at à n generators, as well as the string condition for left ideals a n generators.

For $n = 1$, we obtain a 2-FIR atomic. As any non-zero element corresponds to a full matrix 1×1, the theorem 4 results in the following result:

The semigroup of non-zero elements of a 2-FIR atomic can be immersed in a body. Also point out in particular, as an application of the theorem 5, the following theorem:

Theorem 6

Two bodies $\mathbb{K}_1$ et $\mathbb{K}_2$ can be taken into the same body if and only if they have the same characteristics.

It is clear that the condition is necessary. To prove that it is sufficient, we use the following property: The free product of a family $\mathbb{K}_i$ of bodies on the same sub-body $\mathbb{K}$ exist [7] and is a FIR [8].

If $\mathbb{K}_1$ and $\mathbb{K}_2$ have the same characteristics, they have the same underbody $\mathbb{K}$ and can therefore be immersed in a FIR: their free product on $\mathbb{K}$, who is himself immersed in a body.

4. Conclusion

Some non-commutative integral rings can be immersed in a field by a method that is not that of the construction of the field of fractions on the right $\mathbb{R}\mathbb{R}^{*-1}$, where to the left $\mathbb{R}^{*-1}\mathbb{R}$, of the ring $\mathbb{R}$ considered.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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