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Some Inequalities Involving the Exponential Transform

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Abstract

The exponential transform or modified Fourier transform is an integral transform where the Kernel function is $K_a(\xi,t)=a^{-i\xi t}$, and $a\in\left]1,+\infty\right[$. This is a general case of the Fourier transform where the Kernel function is of the form $K_e(\xi,t)=\exp(-i\xi t)$. Joseph Fourier, the famous French Mathematician and Engineer, was the pioneer and he studied the properties in his seminal works. This important tool created an avenue of research later and it is very important in tackling problems in strong differential form and studying spectral properties of various ordinary and partial differential operators. In our article, we will obtain explicit bounds for the modified Fourier transform and derive some corollaries using in the Kernel function a multiple of the Euler-Mascheroni constant γ. The bounds obtained involve the Riemann zeta function for positive integers at the right-hand side.

Keywords

Modified Fourier Transform, Inequalities, Riemann Zeta Function, Euler-Mascheroni Constant

1. Introduction

Integral transforms are elegant and powerful tools for solving differential equations by transforming the differential problems into simpler ones (see the literature [1] [2]). With integral transforms, not only we can solve differential equations, but also, we can study in detail the spectral properties of various types of ordinary and partial differential operators. This area of study is called Spectral analysis of differential operators. One of the most popular and ubiquitous integral transforms in the mathematical world is the Fourier transform where the Kernel function is the $K_e(\xi,t)=\exp(-i\xi t)$. The Fourier transform is a useful and powerful tool for tackling differential problems arising from engineering, and it is also very critical to fields such as signal processing, wavelet theory, telecommunications, etc. There are various works in the literature with regards to the integral Fourier transform ([3]-[11]). In our article, we focus on the modified Fourier transform or exponential transform where the Kernel function is $K_a(\xi,t)=a^{-i\xi t}$, where $a\in\left]1,+\infty\right[$. The work is motivated by articles [12]-[14]. More explicitly, we obtain various bounds in modulus of modified Fourier transforms for particularly defined complex Fourier numbers and have some assumptions on the function $u(t)$ in terms of its integrability properties.

Similar work has been done and submitted [15], when $K_e(\xi,t)=\mathrm{e}^{-i\xi t}$. Other available literature is mentioned here ([16]-[18]). Based on the obtained inequalities (Theorem 1 - Theorem 15), we proceed to obtain corollaries assuming that the kernel function is $K_{4\gamma}(\xi,t)=(4\gamma)^{-i\xi t}$, where γ is the so-called Euler-Mascheroni constant. The estimates we obtain, reveal that we can bound the modulus of the Fourier transformed quantities by the 2-norm of the function $u(t)$ times some other quantities where at the right hand side, the Riemann zeta function appears for positive integers.

2. Preliminaries

Before proceeding to obtain the various inequalities, we make some assumptions for the function $u=u(t)$. The assumptions are the following:

$u=u(t):\ \Delta_{t_+}\to U(\Delta_{t_+})\subset\mathbb{R}$, where $\Delta_{t_+}=\left]0,+\infty\right[$. Also, the function $u(t)$ belongs to the class of continuous functions and it has continuous derivative. The space $C^0(\Delta_{t_+})$ denotes the continuity of the function $u(t)$ and $C^1(\Delta_{t_+})$ is the space for the continuity of the function $u'(t)$. Another assumption for the function $u(t)$ is that it belongs to the space of square integrable functions with finite $L_2(\Delta_{t_+})$ norm. More precisely, the $L_2(\Delta_{t_+})$ function space is defined as:

$$L_2(\Delta_{t_+})=\left\{u:\Delta_{t_+}\to U(\Delta_{t_+})\,|\left(\int_0^{+\infty}|u(t)|^2\,\mathrm{d}t\right)<+\infty\right\}.$$

The complex exponential transform, is an integral transform defined as

$$\widetilde{u(\xi)}=\int_0^{+\infty}u(t)K_a(\xi,t)\mathrm{d}t$$

where $K_a(\xi,t)=a^{-i\xi t}$ is the Kernel function, with $\xi\in\mathbb{C}$.

With regards to the novelty of the work, the following should be considered: In many cases the function $u(t)$ could be a function such that when trying to calculate the integral transform explicitly in closed form, then it becomes a difficult task. Instead, estimates are obtained in modulus, and these estimates can be used to define the bounds of the spectrum. Examples will be provided.

Theorem 1. For $0<\mathrm{Im}(\xi)<+\infty$, $0<\mathrm{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u(\xi^*)}\right|\le\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\mathrm{Im}(\xi)\ln(a)\right)^{-\frac{1}{2}} \tag{1}$$

where $\xi = \mathrm{Re}(\xi) - i\,\mathrm{Im}(\xi)$.

Proof.

$$
\begin{aligned}
\left|\widetilde{u(\xi^*)}\right| &= \left|\int_0^{+\infty} u(t) K_a(\xi^*, t)\,\mathrm{d}t\right| = \left|\int_0^{+\infty} u(t) a^{-i\xi^* t}\,\mathrm{d}t\right| \\
&\le \int_0^{+\infty} |u(t)| \left|a^{-i\xi^* t}\right| \mathrm{d}t = \int_0^{+\infty} |u(t)| \left|\mathrm{e}^{-i\xi^* t \ln(a)}\right| \mathrm{d}t \\
&\le \int_0^{+\infty} |u(t)| \left|\mathrm{e}^{-\mathrm{Im}(\xi) t \ln(a)}\right| \mathrm{d}t \\
&\le \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\mathrm{Im}(\xi)\ln(a)\right)^{-\frac{1}{2}}
\end{aligned}
$$

by using the Schwarz-Cauchy inequality.

Theorem 2. For $0 < \mathrm{Im}(\xi) < +\infty$, $0 < \mathrm{Re}(\xi) < +\infty$, the following estimate holds

$$
\left|\widetilde{u(w^*)}\right| \le \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\mathrm{Re}(\xi)\ln(a)\right)^{-\frac{1}{2}} \tag{2}
$$

where $w^* = \mathrm{Im}(\xi) - i\,\mathrm{Re}(\xi)$.

Proof.

$$
\begin{aligned}
\left|\widetilde{u(w^*)}\right| &= \left|\int_0^{+\infty} u(t) K_a(w^*, t)\,\mathrm{d}t\right| \\
&\le \int_0^{+\infty} |u(t)| \left|K_a(w^*, t)\right| \mathrm{d}t \\
&\le \left(\int_0^{+\infty} |u(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^{+\infty} \left|K_a(w^*, t)\right|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \left(\int_0^{+\infty} |u(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^{+\infty} \mathrm{e}^{-2\mathrm{Re}(\xi) t \ln(a)}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\mathrm{Re}(\xi)\ln(a)\right)^{-\frac{1}{2}},
\end{aligned}
$$

by employing the Schwarz-Cauchy inequality.

Theorem 3. For $0 < \mathrm{Im}(\xi) < +\infty$, $0 < \mathrm{Re}(\xi) < +\infty$, the following estimate holds

$$
\left|\widetilde{u(\xi^*)} + \widetilde{u(w^*)}\right| \le \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\ln(a)\right)^{-\frac{1}{2}} \left(\mathrm{Im}(\xi)^{-\frac{1}{2}} + \mathrm{Re}(\xi)^{-\frac{1}{2}}\right) \tag{3}
$$

Proof.

$$
\begin{aligned}
\left|\widetilde{u(\xi^*)} + \widetilde{u(w^*)}\right| &\le \left|\widetilde{u(\xi^*)}\right| + \left|\widetilde{u(w^*)}\right| \\
&\le \left|\int_0^{+\infty} u(t) K_a(\xi^*, t)\,\mathrm{d}t\right| + \left|\int_0^{+\infty} u(t) K_a(w^*, t)\,\mathrm{d}t\right| \\
&\le \int_0^{+\infty} |u(t)| \left|K_a(\xi^*, t)\right| \mathrm{d}t + \int_0^{+\infty} |u(t)| \left|K_a(w^*, t)\right| \mathrm{d}t \\
&\le \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\mathrm{Im}(\xi)\ln(a)\right)^{-\frac{1}{2}} + \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\mathrm{Re}(\xi)\ln(a)\right)^{-\frac{1}{2}} \\
&\le \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \left(\ln(a)\right)^{-\frac{1}{2}} \left(\mathrm{Im}(\xi)^{-\frac{1}{2}} + \mathrm{Re}(\xi)^{-\frac{1}{2}}\right),
\end{aligned}
$$

by employing the triangle inequality, and exploiting the estimates (1) and (2).

Theorem 4. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u(\xi^*)}\widetilde{u(w^*)}\right|\le\frac{1}{2}\|u\|_{L_2(\Delta_{t_+})}^2(\ln(a))^{-1}(\operatorname{Re}(\xi)\operatorname{Im}(\xi))^{-\frac{1}{2}}\qquad(4)$$

where $w^*=\operatorname{Im}(\xi)-i\operatorname{Re}(\xi)$, $\xi^*=\operatorname{Re}(\xi)-i\operatorname{Im}(\xi)$.

Proof.

$$\begin{aligned}\left|\widetilde{u(\xi^*)}\widetilde{u(w^*)}\right|&=\left|\widetilde{u(\xi^*)}\right|\left|\widetilde{u(w^*)}\right|\\&=\left|\int_0^{+\infty}u(t)K_a(\xi^*,t)\mathrm{d}t\right|\left|\int_0^{+\infty}u(t)K_a(w^*,t)\mathrm{d}t\right|\\&\le\left(\int_0^{+\infty}|u(t)||K_a(\xi^*,t)|\mathrm{d}t\right)\left(\int_0^{+\infty}|u(t)||K_a(w^*,t)|\mathrm{d}t\right)\\&\le\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}(\operatorname{Im}(\xi)\ln(a))^{-\frac{1}{2}}\times\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}(\operatorname{Re}(\xi)\ln(a))^{-\frac{1}{2}}\\&\le\frac{1}{2}\|u\|_{L_2(\Delta_{t_+})}^2(\ln(a))^{-1}(\operatorname{Re}(\xi)\operatorname{Im}(\xi))^{-\frac{1}{2}}\end{aligned}$$

by employing the inequalities (1) and (2).

Theorem 5. The following estimate holds

$$\left|\widetilde{u((\xi^*)^n)}\right|\le\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}|\xi^*|^{-\frac{n}{2}}(\ln(a))^{-\frac{1}{2}}\left(\sqrt{\sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)}\right)^{-1}\qquad(5)$$

under the restrictions

$$\begin{aligned}&\operatorname{Im}(\xi)>0,\operatorname{Re}(\xi)>0,\sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)>0,\\&\cos\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)>0,n\in\mathbb{Z}^+.\end{aligned}$$

Proof.

$$\begin{aligned}\left|\widetilde{u((\xi^*)^n)}\right|&=\left|\int_0^{+\infty}u(t)K_a((\xi^*)^n,t)\mathrm{d}t\right|\\&\le\int_0^{+\infty}|u(t)||K_a((\xi^*)^n,t)|\mathrm{d}t\\&\le\left(\int_0^{+\infty}|u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^{+\infty}|K_a((\xi^*)^n,t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\\&\le\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}|\xi^*|^{-\frac{n}{2}}(\ln(a))^{-\frac{1}{2}}\left(\sqrt{\sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)}\right)^{-1}\end{aligned}$$

by employing the Schwarz-Cauchy inequality and De Moivre's theorem.

Theorem 6. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, $\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)^2>1$, the following bound holds

$$\left|\widetilde{u(\xi^*w)}\right|\le\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}(\ln(a))^{-\frac{1}{2}}\left(\operatorname{Im}(\xi)^2-\operatorname{Re}(\xi)^2\right)^{-\frac{1}{2}}\qquad(6)$$

Proof.

$$\begin{aligned}\left|\widetilde{u\left(\xi^{*}w\right)}\right| &= \left|\int_0^{+\infty} u(t)K_a\left(\xi^{*}w,t\right)\mathrm{d}t\right| \\ &\le \left(\int_0^{+\infty}\left|u(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^{+\infty}\left|K_a\left(\xi^{*}w,t\right)\right|^2\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\ln(a)\right)^{-\frac{1}{2}}\left(\mathrm{Im}(\xi)^2-\mathrm{Re}(\xi)^2\right)^{-\frac{1}{2}}\end{aligned}$$

using the Schwarz-Cauchy inequality.

Theorem 7. For $0<\mathrm{Im}(\xi)<+\infty$, $0<\mathrm{Re}(\xi)<+\infty$, $\left(\frac{\mathrm{Re}(\xi)}{\mathrm{Im}(\xi)}\right)^2>1$, the following bound holds

$$\left|\widetilde{u\left(\xi w^{*}\right)}\right| \le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\ln(a)\right)^{-\frac{1}{2}}\left(\mathrm{Re}(\xi)^2-\mathrm{Im}(\xi)^2\right)^{-\frac{1}{2}} \tag{7}$$

Proof.

$$\begin{aligned}\left|\widetilde{u\left(\xi w^{*}\right)}\right| &= \left|\int_0^{+\infty} u(t)K_a\left(\xi w^{*},t\right)\mathrm{d}t\right| \\ &\le \int_0^{+\infty}\left|u(t)\right|\left|K_a\left(\xi w^{*},t\right)\right|\mathrm{d}t \\ &\le \left(\int_0^{+\infty}\left|u(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^{+\infty}\left|K_a\left(\xi w^{*},t\right)\right|^2\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\ln(a)\right)^{-\frac{1}{2}}\left(\mathrm{Re}(\xi)^2-\mathrm{Im}(\xi)^2\right)^{-\frac{1}{2}}\end{aligned}$$

by employing Schwarz-Cauchy inequality.

Theorem 8. For $0<\mathrm{Im}(\xi)<+\infty$, $0<\mathrm{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u'\left(\xi^{*}\right)}\right| \le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\mathrm{Im}(\xi)\right)^{-\frac{1}{2}}\sqrt{\mathrm{Re}(\xi)^2+\mathrm{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right| \tag{8}$$

Proof.

$$\begin{aligned}\left|\widetilde{u'\left(\xi^{*}\right)}\right| &= \left|\int_0^{+\infty} u'(t)K_a\left(\xi^{*},t\right)\mathrm{d}t\right| \\ &= \left|\ln(a)i\xi^{*}\int_0^{+\infty} u(t)K_a\left(\xi^{*},t\right)\mathrm{d}t-u(0)\right| \\ &\le \ln(a)\left|\xi\right|\left|\int_0^{+\infty} u(t)K_a\left(\xi^{*},t\right)\mathrm{d}t\right|+\left|u(0)\right| \\ &\le \ln(a)\left|\xi\right|\int_0^{+\infty}\left|u(t)\right|\left|K_a\left(\xi^{*},t\right)\right|\mathrm{d}t+\left|u(0)\right| \\ &\le \ln(a)\left|\xi\right|\|u\|_{L_2\left(\Delta_{t_+}\right)}\left\|K_a\left(\xi^{*},t\right)\right\|_{L_2\left(\Delta_{t_+}\right)}+\left|u(0)\right| \\ &\le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\mathrm{Im}(\xi)\right)^{-\frac{1}{2}}\sqrt{\mathrm{Re}(\xi)^2+\mathrm{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right|\end{aligned}$$

using integration by parts, the triangle inequality, the Schwarz-Cauchy inequality and the bound (1).

Theorem 9. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u'\left(w^*\right)}\right| \le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Re}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right| \tag{9}$$

Proof.

$$\begin{aligned}
\left|\widetilde{u'\left(w^*\right)}\right| &= \left|\int_0^{+\infty} u'(t)K_a\left(w^*,t\right)\mathrm{d}t\right| \\
&= \left|\ln(a)i\xi^*\int_0^{+\infty} u(t)K_a\left(w^*,t\right)\mathrm{d}t-u(0)\right| \\
&\le \ln(a)|\xi|\left|\int_0^{+\infty} u(t)K_a\left(w^*,t\right)\mathrm{d}t\right|+\left|u(0)\right| \\
&\le \ln(a)|\xi|\int_0^{+\infty}\left|u(t)\right|\left|K_a\left(w^*,t\right)\right|\mathrm{d}t+\left|u(0)\right| \\
&\le \ln(a)|\xi|\|u\|_{L_2\left(\Delta_{t_+}\right)}\left\|K_a\left(w^*,t\right)\right\|_{L_2\left(\Delta_{t_+}\right)}+\left|u(0)\right| \\
&\le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Re}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right|
\end{aligned}$$

using integration by parts, the triangle inequality, the Schwarz-Cauchy inequality and the bound (2).

Theorem 10. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following inequality holds

$$\begin{aligned}
\left|\widetilde{u'\left(\xi^*\right)}+\widetilde{u'\left(w^*\right)}\right| \le{} & \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}} \\
& \times\left(\operatorname{Im}(\xi)^{-\frac{1}{2}}+\operatorname{Re}(\xi)^{-\frac{1}{2}}\right)+2\left|u(0)\right|
\end{aligned} \tag{10}$$

Proof.

$$\begin{aligned}
&\left|\widetilde{u'\left(\xi^*\right)}+\widetilde{u'\left(w^*\right)}\right| \le \left|\widetilde{u'\left(\xi^*\right)}\right|+\left|\widetilde{u'\left(w^*\right)}\right| \\
&\le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Im}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right| \\
&+\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Re}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right| \\
&\le \frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}\left(\operatorname{Im}(\xi)^{-\frac{1}{2}}+\operatorname{Re}(\xi)^{-\frac{1}{2}}\right)+2\left|u(0)\right|
\end{aligned}$$

by employing the triangle inequality and the estimates (8), (9).

Theorem 11. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following inequality holds

$$\begin{aligned}
&\left|\widetilde{u'\left(\xi^*\right)}\widetilde{u'\left(w^*\right)}\right| \\
&\le\left(\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Im}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right|\right) \\
&\times\left(\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Re}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln(a)\right)^{\frac{1}{2}}+\left|u(0)\right|\right)
\end{aligned} \tag{11}$$

Proof.

$$\begin{aligned}
&\left|\widetilde{u'(\xi^*)}\widetilde{u'(w^*)}\right| = \left|\widetilde{u'(\xi^*)}\right|\left|\widetilde{u'(w^*)}\right| \\
&= \left|\int_0^{+\infty} u'(t) K_a(\xi^*, t)\,\mathrm{d}t\right|\left|\int_0^{+\infty} u'(t) K_a(w^*, t)\,\mathrm{d}t\right| \\
&\le \left(\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}(\mathrm{Im}(\xi))^{-\frac{1}{2}}\sqrt{\mathrm{Re}(\xi)^2+\mathrm{Im}(\xi)^2}(\ln(a))^{\frac{1}{2}}+|u(0)|\right) \\
&\quad\times\left(\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}(\mathrm{Re}(\xi))^{-\frac{1}{2}}\sqrt{\mathrm{Re}(\xi)^2+\mathrm{Im}(\xi)^2}(\ln(a))^{\frac{1}{2}}+|u(0)|\right)
\end{aligned}$$

by doing integration by parts, applying the triangle inequality and making use of the estimates (8) and (9).

Theorem 12. The following estimate holds

$$\left|\widetilde{u'\left(\left(\xi^*\right)^n\right)}\right| \le \frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left|\xi^*\right|^{\frac{n}{2}}(\ln(a))^{\frac{1}{2}}\left(\sqrt{\sin\left(n\tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)\right)}\right)^{-1}+|u(0)| \quad (12)$$

under the restrictions

$$\begin{aligned}
&\mathrm{Im}(\xi)>0, \mathrm{Re}(\xi)>0, \sin\left(n\tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)\right)>0, \\
&\cos\left(n\tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)\right)>0, n\in\mathbb{Z}^+.
\end{aligned}$$

Proof.

$$\begin{aligned}
\left|\widetilde{u'\left(\left(\xi^*\right)^n\right)}\right| &= \left|\int_0^{+\infty} u'(t) K_a\left(\left(\xi^*\right)^n, t\right)\mathrm{d}t\right| \\
&= \left|\ln(a) i\left(\xi^*\right)^n \int_0^{+\infty} u(t) K_a\left(\left(\xi^*\right)^n, t\right)\mathrm{d}t - u(0)\right| \\
&\le \ln(a)\left|\left(\xi^*\right)^n\right|\int_0^{+\infty}|u(t)|\left|K_a\left(\left(\xi^*\right)^n, t\right)\right|\mathrm{d}t + |u(0)| \\
&\le \frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left|\xi^*\right|^{\frac{n}{2}}(\ln(a))^{\frac{1}{2}}\left(\sqrt{\sin\left(n\tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)\right)}\right)^{-1}+|u(0)|
\end{aligned}$$

Integrating by parts, using De Moivre's theorem, applying the triangle inequality and employing the estimate (5).

Theorem 13. For $0<\mathrm{Im}(\xi)<+\infty$, $0<\mathrm{Re}(\xi)<+\infty$, $\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)^2>1$, the following bound holds

$$\begin{aligned}
\left|\widetilde{u'(\xi^* w)}\right| \le{}& \frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}(\ln(a))^{\frac{1}{2}}\left(\mathrm{Im}(\xi)^2-\mathrm{Re}(\xi)^2\right)^{-\frac{1}{2}} \\
&\times\left(\mathrm{Im}(\xi)^2+\mathrm{Re}(\xi)^2\right)+|u(0)|
\end{aligned} \quad (13)$$

Proof.

$$
\begin{aligned}
&\left|\widetilde{u'\left(\xi^{*}w\right)}\right| = \left|\int_{0}^{+\infty} u'(t)\, K_{a}\left(\xi^{*}w,t\right)\mathrm{d}t\right| \\
&= \left|\ln(a)\, i\xi^{*} w \int_{0}^{+\infty} u(t)\, K_{a}\left(\xi^{*}w,t\right)\mathrm{d}t - u(0)\right| \\
&\le \ln(a)\left|\xi^{*}\right|\left|w\right|\left|\int_{0}^{+\infty} u(t)\, K_{a}\left(\xi^{*}w,t\right)\mathrm{d}t\right| + \left|u(0)\right| \\
&\le \ln(a)\left|\xi^{*}\right|\left|w\right|\int_{0}^{+\infty} \left|u(t)\right|\left|K_{a}\left(\xi^{*}w,t\right)\right|\mathrm{d}t + \left|u(0)\right| \\
&\le \frac{\sqrt{2}}{2}\left\|u\right\|_{L_2\left(\Delta_{t_+}\right)}\left(\ln(a)\right)^{\frac{1}{2}}\left(\mathrm{Im}(\xi)^{2} - \mathrm{Re}(\xi)^{2}\right)^{-\frac{1}{2}} \times \left(\mathrm{Im}(\xi)^{2} + \mathrm{Re}(\xi)^{2}\right) + \left|u(0)\right|
\end{aligned}
$$

by integration by parts, using the triangle inequality and employing the inequality (6).

Theorem 14. For $0 < \mathrm{Im}(\xi) < +\infty$, $0 < \mathrm{Re}(\xi) < +\infty$, $\left(\frac{\mathrm{Re}(\xi)}{\mathrm{Im}(\xi)}\right)^{2} > 1$, the following bound holds

$$
\begin{aligned}
\left|\widetilde{u'\left(\xi w^{*}\right)}\right| \le{}& \left(\mathrm{Re}(\xi)^{2} + \mathrm{Im}(\xi)^{2}\right)\frac{\sqrt{2}}{2}\left\|u\right\|_{L_2\left(\Delta_{t_+}\right)}\left(\mathrm{Re}(\xi)^{2} - \mathrm{Im}(\xi)^{2}\right)^{-\frac{1}{2}} \\
&\times\left(\ln(a)\right)^{\frac{1}{2}} + \left|u(0)\right|
\end{aligned}
\tag{14}
$$

Proof.

$$
\begin{aligned}
&\left|\widetilde{u'\left(\xi w^{*}\right)}\right| = \left|\int_{0}^{+\infty} u'(t)\, K_{a}\left(\xi w^{*},t\right)\mathrm{d}t\right| \\
&\le \left(\mathrm{Re}(\xi)^{2} + \mathrm{Im}(\xi)^{2}\right)\frac{\sqrt{2}}{2}\left\|u\right\|_{L_2\left(\Delta_{t_+}\right)}\left(\mathrm{Re}(\xi)^{2} - \mathrm{Im}(\xi)^{2}\right)^{-\frac{1}{2}} \times \left(\ln(a)\right)^{\frac{1}{2}} + \left|u(0)\right|
\end{aligned}
$$

by integration by parts, using the triangle inequality and employing the inequality (7).

Theorem 15. Let $g(t) = t^{\frac{m}{2}} u(t)$. For $0 < \mathrm{Im}(\xi) < +\infty$, $0 < \mathrm{Re}(\xi) < +\infty$, $m \in \mathbb{Z}^{+}$, the following inequality holds

$$
\left|\widetilde{g\left(\xi^{*}\right)}\right| \le \left\|u\right\|_{L_2\left(\Delta_{t_+}\right)}\sqrt{\Gamma(m+1)}\, 2^{-\frac{m+1}{2}}\left(\ln(a)\right)^{-\frac{m+1}{2}}\left(\mathrm{Im}(\xi)\right)^{-\frac{m+1}{2}} \tag{15}
$$

Proof.

$$
\begin{aligned}
\left|\widetilde{g\left(\xi^{*}\right)}\right| &= \left|\int_{0}^{+\infty} g(t)\, K_{a}\left(\xi^{*},t\right)\mathrm{d}t\right| \le \int_{0}^{+\infty}\left|t^{\frac{m}{2}} u(t)\, K_{a}\left(\xi^{*},t\right)\right|\mathrm{d}t \\
&\le \left(\int_{0}^{+\infty}\left|u(t)\right|^{2}\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{0}^{+\infty}\left|t^{\frac{m}{2}}\right|^{2}\left|K_{a}\left(\xi^{*},t\right)\right|^{2}\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \left\|u\right\|_{L_2\left(\Delta_{t_+}\right)}\left(\frac{\Gamma(m+1)}{\left(2\ln(a)\,\mathrm{Im}(\xi)\right)^{m+1}}\right)^{\frac{1}{2}} \\
&\le \left\|u\right\|_{L_2\left(\Delta_{t_+}\right)}\sqrt{\Gamma(m+1)}\, 2^{-\frac{m+1}{2}}\left(\ln(a)\right)^{-\frac{m+1}{2}}\left(\mathrm{Im}(\xi)\right)^{-\frac{m+1}{2}}
\end{aligned}
$$

using integration by parts, the triangle inequality, and the properties of gamma function.

3. Corollaries

In the following section, we provide corollaries stemming from the Theorems 1 - 15. To derive the bounds, we inject $a=4\gamma$, where γ is the Euler Mascheroni constant and we use the mathematical relationship $\gamma=\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}$, (Gourdon and Sebah 2003, p. 3) where ζ is the Riemann-zeta function. The kernel function in this case is $K_{4\gamma}(\xi,t)=(4\gamma)^{-i\xi t}$.

Corollary 1. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u(\xi^*)}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\operatorname{Im}(\xi)\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{-\frac{1}{2}}.$$

Corollary 2. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u(w^*)}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\operatorname{Re}(\xi)\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{-\frac{1}{2}}.$$

Corollary 3. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\begin{aligned}&\left|\widetilde{u(\xi^*)}+\widetilde{u(w^*)}\right|\\&\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{-\frac{1}{2}}\left(\operatorname{Im}(\xi)^{-\frac{1}{2}}+\operatorname{Re}(\xi)^{-\frac{1}{2}}\right).\end{aligned}$$

Corollary 4. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u(\xi^*)}\widetilde{u(w^*)}\right|\leq\frac{1}{2}\|u\|^2_{L_2(\Delta_{t_+})}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{-1}\left(\operatorname{Re}(\xi)\operatorname{Im}(\xi)\right)^{-\frac{1}{2}}.$$

Corollary 5. The following estimate holds

$$\begin{aligned}&\left|\widetilde{u\left((\xi^*)^n\right)}\right|\\&\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left|\xi^*\right|^{-\frac{n}{2}}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{-\frac{1}{2}}\left(\sqrt{\sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)}\right)^{-1}\end{aligned}$$

under the restrictions

$$\operatorname{Im}(\xi)>0, \operatorname{Re}(\xi)>0, \sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)>0,$$

$$\cos\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)>0, n\in\mathbb{Z}^{+}.$$

Corollary 6. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, $\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)^{2}>1$, the following bound holds

$$\left|\widetilde{u(\xi^{*}w)}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^{l}\zeta(l)}{l}\right)\right)^{-\frac{1}{2}}\left(\operatorname{Im}(\xi)^{2}-\operatorname{Re}(\xi)^{2}\right)^{-\frac{1}{2}}.$$

Corollary 7. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, $\left(\frac{\operatorname{Re}(\xi)}{\operatorname{Im}(\xi)}\right)^{2}>1$, the following bound holds

$$\left|\widetilde{u(\xi w^{*})}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^{l}\zeta(l)}{l}\right)\right)^{-\frac{1}{2}}\left(\operatorname{Re}(\xi)^{2}-\operatorname{Im}(\xi)^{2}\right)^{-\frac{1}{2}}.$$

Corollary 8. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u'(\xi^{*})}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\operatorname{Im}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^{2}+\operatorname{Im}(\xi)^{2}}$$
$$\times\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^{l}\zeta(l)}{l}\right)\right)^{\frac{1}{2}}+|u(0)|.$$

Corollary 9. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following estimate holds

$$\left|\widetilde{u'(w^{*})}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\left(\operatorname{Re}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^{2}+\operatorname{Im}(\xi)^{2}}$$
$$\times\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^{l}\zeta(l)}{l}\right)\right)^{\frac{1}{2}}+|u(0)|.$$

Corollary 10. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following inequality holds

$$\left|\widetilde{u'(\xi^{*})}+\widetilde{u'(w^{*})}\right|\leq\frac{\sqrt{2}}{2}\|u\|_{L_2(\Delta_{t_+})}\sqrt{\operatorname{Re}(\xi)^{2}+\operatorname{Im}(\xi)^{2}}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^{l}\zeta(l)}{l}\right)\right)^{\frac{1}{2}}$$
$$\times\left(\operatorname{Im}(\xi)^{-\frac{1}{2}}+\operatorname{Re}(\xi)^{-\frac{1}{2}}\right)+2|u(0)|.$$

Corollary 11. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, the following inequality holds

$$
\begin{aligned}
&\left|\overline{u'\left(\xi^*\right)}\,\overline{u'\left(w^*\right)}\right| \\
&\leq\left(\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Im}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{\frac{1}{2}}+|u(0)|\right) \\
&\times\left(\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Re}(\xi)\right)^{-\frac{1}{2}}\sqrt{\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{\frac{1}{2}}+|u(0)|\right).
\end{aligned}
$$

Corollary 12. The following estimate holds

$$
\begin{aligned}
\left|\overline{u'\left(\left(\xi^*\right)^n\right)}\right| &\leq\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left|\xi^*\right|^{\frac{n}{2}}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{\frac{1}{2}} \\
&\times\left(\sqrt{\sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)}\right)^{-1}+|u(0)|
\end{aligned}
$$

under the restrictions

$$
\begin{aligned}
&\operatorname{Im}(\xi)>0,\operatorname{Re}(\xi)>0,\sin\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)>0, \\
&\cos\left(n\tan^{-1}\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)\right)>0,n\in\mathbb{Z}^+.
\end{aligned}
$$

Corollary 13. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, $\left(\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right)^2>1$, the following bound holds

$$
\begin{aligned}
\left|\overline{u'\left(\xi^* w\right)}\right| &\leq\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{\frac{1}{2}}\left(\operatorname{Im}(\xi)^2-\operatorname{Re}(\xi)^2\right)^{-\frac{1}{2}} \\
&\times\left(\operatorname{Im}(\xi)^2+\operatorname{Re}(\xi)^2\right)+|u(0)|.
\end{aligned}
$$

Corollary 14. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, $\left(\frac{\operatorname{Re}(\xi)}{\operatorname{Im}(\xi)}\right)^2>1$, the following bound holds

$$
\begin{aligned}
\left|\overline{u'\left(\xi w^*\right)}\right| &\leq\left(\operatorname{Re}(\xi)^2+\operatorname{Im}(\xi)^2\right)\frac{\sqrt{2}}{2}\|u\|_{L_2\left(\Delta_{t_+}\right)}\left(\operatorname{Re}(\xi)^2-\operatorname{Im}(\xi)^2\right)^{-\frac{1}{2}} \\
&\times\left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l\zeta(l)}{l}\right)\right)^{\frac{1}{2}}+|u(0)|.
\end{aligned}
$$

Corollary 15. Let $g(t)=t^{\frac{m}{2}}u(t)$. For $0<\operatorname{Im}(\xi)<+\infty$, $0<\operatorname{Re}(\xi)<+\infty$, $m\in\mathbb{Z}^+$, the following inequality holds

$$\left|\widetilde{g\left(\xi^*\right)}\right| \le \|u\|_{L_2\left(\Delta_{t_+}\right)} \sqrt{\Gamma(m+1)}\, 2^{-\frac{m+1}{2}} \left(\ln\left(4\sum_{l=2}^{+\infty}\frac{(-1)^l \zeta(l)}{l}\right)\right)^{-\frac{m+1}{2}} \left(\mathrm{Im}(\xi)\right)^{-\frac{m+1}{2}}.$$

4. Conclusion

In this article, we derived estimates for the modulus of the modified Fourier transform of $u(t)$ for specifically defined Fourier numbers, and estimates for the function $u'(t)$. Additionally, we have obtained estimates when the kernel function is of the form $K_{4\gamma}(\xi,t) = (4\gamma)^{-i\xi t}$ where γ is the Euler-Mascheroni constant. We have observed then that the bounds obtained have at the right-hand side, terms of the Riemann zeta function, using the formula $\gamma = \sum_{l=2}^{+\infty}\frac{(-1)^l \zeta(l)}{l}$.

5. Examples and Illustrations

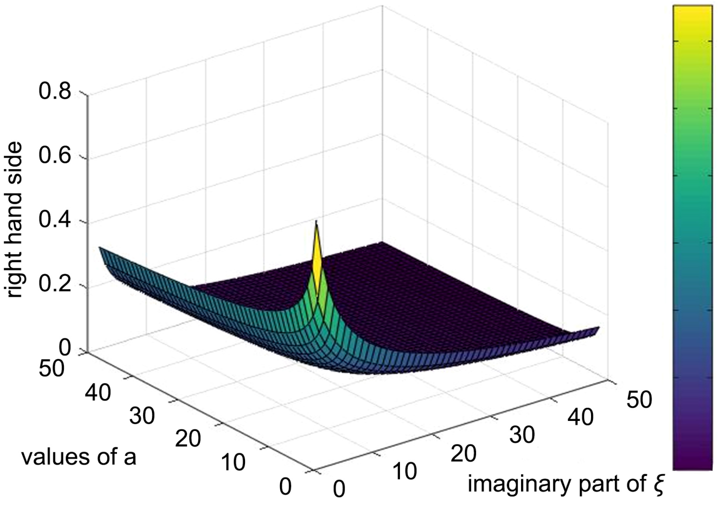

Figure 1. The right hand side of estimate (1).

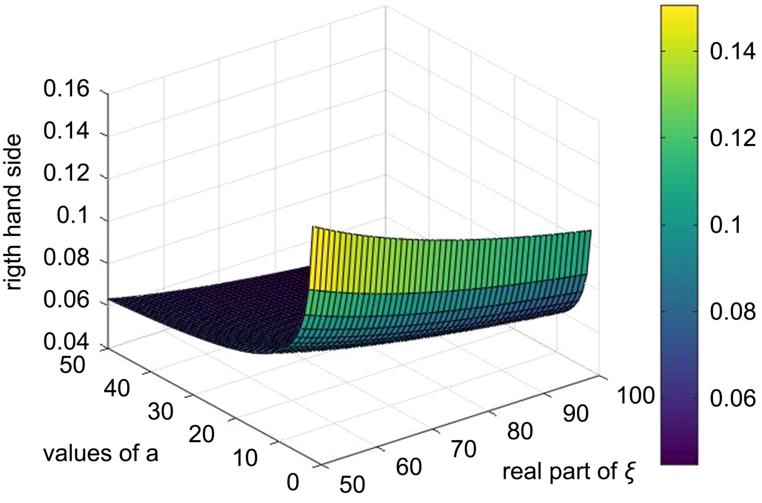

Figure 2. The right hand side of estimate (2).

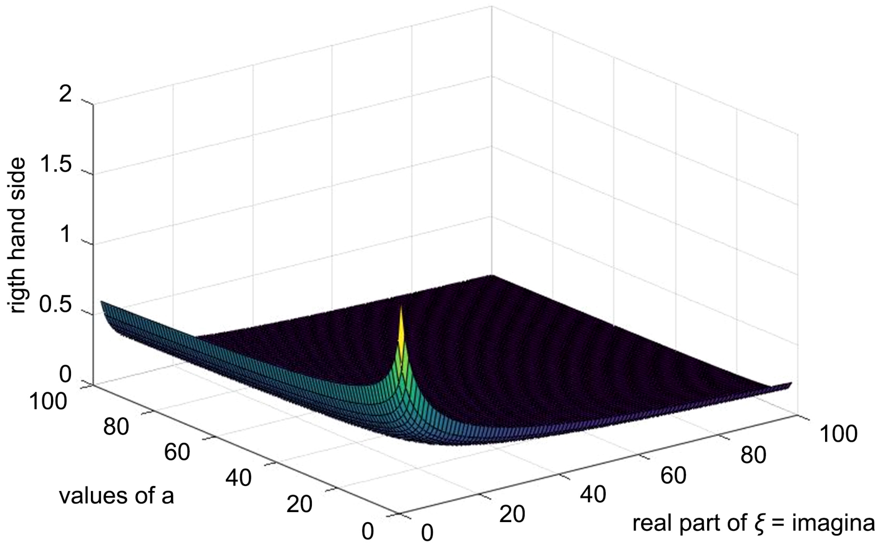

Figure 3. The right hand side of estimate (3) taking the imaginary and real part of Fourier variable to be equal.

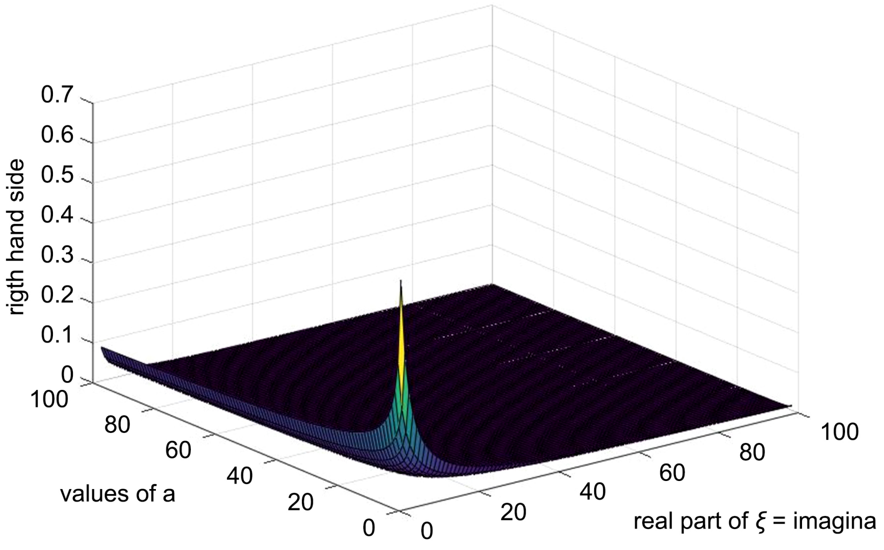

Figure 4. The right hand side of estimate (4) taking the imaginary and real part of Fourier variable to be equal.

In this section, we provide some examples with illustrations for a few theorems. The function that we use for testing is $u(t)=\frac{1}{\sqrt{t^2+1}} \quad \forall t\in\Delta_{t_+}$. All the graphs that follow are the right hand sides of estimates (1) up to (4) as functions of the parameter a that appears in the kernel function and the real-imaginary parts of the Fourier variable, see **Figures 1-4**. For the last two figures (**Figure 3** and **Figure 4**) we take that the imaginary and real parts of the Fourier variable are identical. All the plots have been derived using MatLab software package. Similar graphical depiction can be done for all the theorems, however, we exhibit a selected number of cases.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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