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Asymptotic Behaviors of Hankelians, Whose Entries Involve Regularly- or Rapidly-Varying Functions, as the Variable Tends to +∞. Part I

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Abstract

The rich literature concerning "asymptotic behavior of Hankel determinants" concerns the behavior, as the order n tends to ∞, of Hankel determinants whose entries are numbers, e.g., with a combinatorial interest or arising as values of special classes of functions. Such determinants are numbers depending on n, playing roles in number theory, combinatorics, random matrices and the like; and mathematicians in the involved fields have been interested in their asymptotic behaviors as n goes to ∞, as previously mentioned, with no single exception to the author's knowledge. The study carried on in the present paper treats an altogether different situation as suggested by the specification in the title "*as the variable tends to* $+\infty$". We deal with those types of Hankel determinants (purposely called Hankelians) which are special cases of Wronskians and, continuing our work on the asymptotics of Wronskians, we study the asymptotic behaviors of n-order Hankelians, whose entries involve either regularly- or rapidly-varying functions, when the variable tends to $+\infty$. As in the study of Wronskians, the treatment of this case also needs the whole apparatus of the theory of higher-order types of asymptotic variation, but the most demanding results are not automatic corollaries of the general theory. In fact, in the study of generic Wronskians (study motivated by applications to asymptotic expansions), the entries were required to belong to one of the classes of "higher-order regular or rapid variation"; on the contrary, in the case of Hankelians, we are confronted with functions whose logarithms are either "regularly- or rapidly-varying functions", roughly classifiable as "ultrarapidly-varying functions", and the study requires both special devices and a number of preliminary lemmas about products and linear combinations of functions in the mentioned classes.

Keywords

Asymptotic Behaviors of Hankelians, Asymptotic Expansions in the Real Domain, Regularly-, Rapidly- and Exponentially-Varying Functions of Higher Order

1. Introduction

1.1. Presentation of the Problem

In a previous work of ours in two parts, [1] [2], we studied in a detailed manner the asymptotic behaviors of Wronskians, and the key to a fruitful approach was the theory of higher-order types of asymptotic variation separately developed in [3]-[6]. The present paper is a direct continuation of [1] [2] and here we study the asymptotic behaviors of those special Wronskians, which are Hankel determinants whose entries involve either regularly- or rapidly-varying functions; the results are then applied to the theory of asymptotic expansions in the real domain.

The object of our study is Wronskians of type:

$$H_n\left[\phi(x)\right] := W\left(\phi(x), \phi'(x), \cdots, \phi^{(n-1)}(x)\right)$$
$$\equiv \begin{vmatrix} \phi(x) & \phi'(x) & \cdots & \phi^{(n-1)}(x) \\ \phi'(x) & \phi''(x) & \cdots & \phi^{(n)}(x) \\ \phi''(x) & \phi'''(x) & \cdots & \phi^{(n+1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{(n-1)}(x) & \phi^{(n)}(x) & \cdots & \phi^{(2n-2)}(x) \end{vmatrix}, \tag{1.1}$$

which are a special type of Hankel determinants intentionally called Hankelians in this paper to distinguish from the more widespread types of Hankel determinants. Obviously $H_1\left[\phi(x)\right] := \phi(x)$. It was quite natural in [1] [2] to subject a Wronskian $W\left(\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\right)$ to the a-priori restriction that the n-tuple $\left(\phi_1, \phi_2, \cdots, \phi_n\right)$ formed an asymptotic scale at $+\infty$, *i.e.*

$$\phi_1(x) \gg \phi_2(x) \gg \cdots \gg \phi_n(x), \quad x \to +\infty,$$

a restriction motivated by both the problem that originated the study (namely, asymptotic expansions in the real domain) and the goal of finding out the precise principal parts of the involved Wronskians and not mere O- or o-estimates for their growth-orders. All the results in [1] [2] specifying the principal parts concern Wronskians of asymptotic scales. Hence, regarding $H_n\left[\phi(x)\right]$ we are interested in results under the restriction of either

$$\phi(x) \gg \phi'(x) \gg \cdots \gg \phi^{(n-1)}(x), \quad x \to +\infty, \tag{1.2}$$

or the converse

$$\phi^{(n-1)}(x) \gg \phi^{(n-2)}(x) \gg \cdots \gg \phi'(x) \gg \phi(x), \quad x \to +\infty. \tag{1.3}$$

When the following restrictions on the signs are added:

$$\begin{cases} \phi, \phi', \cdots, \phi^{(n-2)} \text{ ultimately non-zero in the scale } (1.2), \\ \phi \text{ ultimately non-zero in the scale } (1.3), \end{cases}$$

these two scales characterize two classes of functions respectively denominated "hypo-exponentially (≡ subexponentially) varying of order $n-1$" or "hyper-exponentially (≡ superexponentially) varying of order $n-1$" at $+\infty$ (in the strong sense): ([4], Def. 8.1, pp. 832-833, and Def. 8.2, pp. 839-840). Together with the third class of "exponentially-varying functions" they generalize the asymptotic behaviors of the elementary functions $\exp(cx^\gamma)$, $c \neq 0$, in the three cases: $0 < \gamma < 1$, $\gamma = 1$, $\gamma > 1$.

- Typical *hypoexponentially-varying functions* are all the regularly-varying functions and also:

$$\begin{cases} R(x) \cdot \exp\left(c_1 x^{\gamma_1} + \cdots + c_n x^{\gamma_n}\right) \\ \text{where } R \text{ is regularyly varying, } c_i \neq 0, 0 < \gamma_i < 1. \end{cases} \tag{1.4}$$

- Typical *exponentially-varying functions* are:

$$H(x) \cdot \exp(cx) \text{ where } H \text{ is hypoexponentially varying, } c \neq 0. \tag{1.5}$$

- Typical *hyperexponentially-varying functions* are:

$$\begin{cases} E(x) \cdot \exp\left(c_1 x^{\gamma_1} + \cdots + c_n x^{\gamma_n}\right) \\ \text{where } E \text{ is exponentially varying, } c_i \neq 0, \gamma_i > 1. \end{cases} \tag{1.6}$$

As for the behavior of $H_n[\phi(x)]$ in these three cases it seems natural to expect different results in the sense of asymptotic formulas with quite different structures. But for the elementary case of the above-mentioned exponentials we have:

$$\begin{cases} \phi(x) := \exp(cx^\gamma), \, c \neq 0, \, \gamma > 0; \;\; \phi'(x) = c\gamma x^{\gamma-1}\phi(x); \\ \phi''(x) = \left[(c\gamma)^2 x^{2\gamma-2} + c\gamma(\gamma-1)x^{\gamma-2}\right]\phi(x) \\ \qquad \sim (c\gamma)^2 x^{2\gamma-2}\phi(x), \, x \to +\infty; \\ H_2[\phi(x)] \equiv \phi(x)\phi''(x) - (\phi'(x))^2 = c\gamma(\gamma-1)x^{\gamma-2}(\phi(x))^2, \end{cases} \tag{1.7}$$

and we see that "$H_2[\phi(x)] \equiv (\phi(x))^2 l(x)$" where $l(x)$ denotes the term with the lowest (!) growth-order in the expression of ϕ''/ϕ whatever $\gamma > 0$; hence the behavior of $H_2[\phi(x)]$ does not depend on the type of exponential variation of ϕ whereas the asymptotic scales (1.2) - (1.3) do. A hint on guessing the right approach comes from the factorization

$$H_2[\phi(x)] \equiv (\phi(x))^2 \cdot (\phi'(x)/\phi(x))', \tag{1.8}$$

which shows that, apart from the factor $(\phi(x))^2$, the principal part of H_2 may depend on the type of asymptotic variation of the logarithmic derivative rather than of ϕ itself. To give a glimpse of the theory to be developed consider the function

$$\begin{cases}\phi(x) := \exp(R(x)); \\ H_2[\phi(x)] = \exp(2R(x)) \cdot R''(x).\end{cases} \tag{1.9}$$

Assuming that R is regularly varying at $+\infty$ with a finite index γ and the same for R' with a suitable index β, a result in the pertinent theory grants the asymptotic relations:

$$\begin{cases}R'(x) \sim \gamma x^{-1} R(x),\ x \to +\infty, \\ R''(x) \sim \gamma(\gamma-1) x^{-2} R(x),\ x \to +\infty,\ (\text{if } \gamma \neq 0,1),\end{cases} \tag{1.10a}$$

whence the meaningful result for the function in (1.9):

$$H_2[\phi(x)] \sim \gamma(\gamma-1) x^{-2} R(x) \exp(2R(x)),\ x \to +\infty,\ (\gamma \neq 0,1), \tag{1.10b}$$

of which the identity in the fourth line of (1.7) is a very special case. Assuming instead that R and R' are rapidly varying at $+\infty$ this means the validity of the relation

$$\begin{cases}R'(x)/R(x) \sim R''(x)/R'(x),\ x \to +\infty, \text{ which implies} \\ R''(x) \sim (R'(x))^2 / R(x),\ x \to +\infty,\end{cases} \tag{1.11a}$$

whence

$$H_2[\phi(x)] \sim (R(x))^{-1} (R'(x))^2 \exp(2R(x)),\ x \to +\infty, \tag{1.11b}$$

whose structure differs from (1.10b). An instance is

$$H_2\left[\exp\left(c_1 e^{c_2 x}\right)\right] \sim c_1 (c_2)^2 \exp(c_2 x) \exp\left(2c_1 e^{c_2 x}\right),\ x \to +\infty. \tag{1.11c}$$

We shall develop this approach in a two-part paper obtaining meaningful results based on the properties of regularly- or rapidly-varying functions of higher order for which we refer the reader to [3] [4] or to the summary below in this section. In the present Part I we treat the simple case wherein the function ϕ in (1.9) is regularly-varying of a certain order, and the demanding case wherein ϕ is the exponential of such a function, whereas in Part II we shall treat the still more complicated case wherein ϕ is the exponential of a rapidly-varying function. The last two cases involve functions whose logarithms are either "regularly- or rapidly- varying functions". Such functions, which may roughly be classifiable as "ultrarapidly-varying functions", do not require an ex-novo theory but the calculations in this paper require special devices and a number of preliminary lemmas about products and linear combinations of functions in the mentioned classes, results reported in §2 from a previous paper.

We wish to clarify the position of the present research in the context of the pertinent literature. First of all, the reader is warned not to think of any link whatsoever between the results in the present paper and other known results labelled as "asymptotic behavior of Hankel determinants" which, as outlined in the abstract, refer to the behavior, as n tends to ∞, of those n-order Hankel determinants much familiar in combinatorics, orthogonal polynomials and so on. Second, the context of this work is the following. The theory of higher-order types of asymptotic variation has been painstakingly systematized (but not discovered) in [3]-[6] by the author, spurred by possible applications to asymptotic expansions in the real domain, which is his field of interest. And some results of the type have been obtained in [1] [2] after finding out the asymptotic behaviors (the exact principal part and not rough estimates!) of suitable Wronskians. Now, specializing those results to the special Wronskians here called Hankelians seemed at first mechanical exercises to the author, but this was not the case. Apart from the case that the function ϕ in (1.1) is regularly varying, (treated in §3 below), the involved calculations required a great number of preliminary lemmas collected in §2 below. So, what innocently looked an exercise caused a lot of hard work. The results in this papers have a relevance in the framework of "the theory of higher-order types of asymptotic variation and its applications to asymptotic expansions in the real domain" and they will be a chapter in this theory the author is currently building up. This is, to the author's knowledge, the first published paper collecting results on asymptotic behaviors of Hankelians as the variable goes to $+\infty$ and perhaps, completed by the second part, it may well be the last! For these reasons, the reader may forgive the author for the seemingly narcissistic references, which include only his own papers, with one exception. See also the "Conclusions" in §6 below.

1.2. Notations

After listing a few notations we shall briefly recall only those definitions and relations needed in the present paper.

- $\mathbb{N} := \{1, 2, \cdots\}$; $\mathbb{R} :=$ the real line.
- $\overline{\mathbb{R}} :=$ the extended real line $\equiv \mathbb{R} \cup \{\pm\infty\}$.
- Factorial powers:

$$\alpha^{\underline{0}} := 1;\ \alpha^{\underline{1}} := \alpha;\ \alpha^{\underline{k}} := \alpha(\alpha-1)\cdots(\alpha-k+1);\ \alpha \in \mathbb{C},\ k \in \mathbb{N}; \tag{1.12}$$

where $\alpha^{\underline{k}}$ is termed the "*k-th falling* ($\equiv$ *decreasing*) *factorial power of* α".

Notice: $0^{\underline{0}} := 1$.

- $f \in AC^0(I) \equiv AC(I) \Leftrightarrow f$ is absolutely continuous on each compact subinterval of *I*.
- $f \in AC^k(I) \Leftrightarrow f^{(k)} \in AC(I)$.
- For $f \in AC^k(I)$ we write

$$\lim_{x \to x_0} f^{(k+1)}(x) \begin{cases} \text{meaning that } x \text{ runs through the points} \\ \text{wherein } f^{(k+1)} \text{ exists as a finite number.} \end{cases} \tag{1.13}$$

- $Df(x) \equiv \dfrac{\mathrm{d}}{\mathrm{d}x} f(x) := f'(x)$; $D^k f(x) := f^{(k)}(x)$.
- $D_\ell f(x) := f'(x)/f(x)$.
- Some asymptotic notations:

$$\begin{cases} f(x) \preceq g(x),\ x \to x_0, & \Leftrightarrow g(x) \succeq f(x),\ x \to x_0, \\ & \overset{\text{def}}{\Leftrightarrow} f(x) = O(g(x)),\ x \to x_0; \\ f(x) \ll g(x),\ x \to x_0, & \Leftrightarrow g(x) \gg f(x),\ x \to x_0, \\ & \overset{\text{def}}{\Leftrightarrow} f(x) = o(g(x)),\ x \to x_0; \\ f(x) \sim g(x),\ x \to x_0, & \overset{\text{def}}{\Leftrightarrow} \begin{cases} f(x) = \ell(x) g(x) \\ \text{with } \lim_{x \to x_0} \ell(x) = 1. \end{cases} \end{cases} \tag{1.14}$$

$$f(x) = \begin{cases} +\infty(g(x)), \\ -\infty(g(x)), \end{cases} x \to x_0, \overset{\text{def}}{\Leftrightarrow} \begin{cases} f(x) = v(x) g(x) \text{ where} \\ \lim_{x \to x_0} v(x) = \begin{cases} +\infty, \\ -\infty; \end{cases} \end{cases} \tag{1.15a}$$

whence, as $x \to x_0$:

$$f(x) \gg 1 \Leftrightarrow |f(x)| = +\infty(1) \Leftrightarrow \lim_{x \to x_0} |f(x)| = +\infty. \tag{1.15b}$$

- Notations for iterated natural logarithms and exponentials:

$$\begin{cases} \ell_k(x) := \underbrace{\log(\log(\cdots(\log}_{k} x)\cdots)),\ k \geq 1; \\ \ell_0(x) := x;\ e_0(x) := 1;\ (\text{these two are just agreements}); \\ e_k(x) \equiv \exp_k(x) := \underbrace{\exp(\exp(\cdots(\exp}_{k} x)\cdots)),\ k \geq 1; \end{cases} \tag{1.16}$$

where ℓ_k is defined for x large enough.

- $V(\alpha_1, \cdots, \alpha_n)$ denotes the Vandermonde determinant (≡Vandermondian) of the n numbers $\alpha_1, \cdots, \alpha_n$.

1.3. Summary of Higher-Order Variation

Definition 1.1. (Higher-Order Regular Variation). (I) *A function, which is absolutely continuous and strictly positive on some neighborhood of* $+\infty$, *is either regularly varying or rapidly varying at* $+\infty$ (*in the* "*strong*" *sense*) *if the following limit, meant as in* (1.13), *exists in the extended real line*:

$$\lim_{x \to +\infty} x f'(x) / f(x) \equiv \alpha,\ \alpha \in \overline{\mathbb{R}}, \begin{cases} \text{slow variation if } \alpha = 0, \\ \text{regular variation if } \alpha \in \mathbb{R}, \\ \text{rapid variation if } \alpha = \pm\infty. \end{cases} \tag{1.17}$$

Here α is the "*index of* (*asymptotic*) *variation*" *and a possible notation for the corresponding class is* $\mathcal{R}_\alpha(+\infty)$. (This is the standard definition with the explicit restriction on the sign: "$f(x) > 0$" for x large enough.)

(II) *A function* $f \in AC^{n-1}[T, +\infty)$, $n \geq 1$, *is termed* "*regularly varying at* $+\infty$ *of order n* (*in the strong sense*)" *if each of the functions* $|f|, |f'|, \cdots, |f^{(n-1)}|$ *never vanishes on a neighborhood of* $+\infty$ *and is regularly varying at* $+\infty$ *with its own index of variation provided that the limit* $\lim_{x \to +\infty} x f^{(n)}(x) / f^{(n-1)}(x)$ *is meant as in* (1.13). *If this is the case we use the notation*:

$$\begin{cases} f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}, \\ \alpha := \text{the index of regular variation of } f,\ \alpha \in \mathbb{R}. \end{cases} \tag{1.18}$$

Whenever needed we denote the indexes of the derivatives as follows:

$$\left|f^{(k)}\right| \in \mathcal{R}_{\alpha_k}(+\infty),\ 0 \le k \le n-1;\ \alpha_0 \equiv \alpha. \tag{1.19}$$

Explicitly notice that, for an $f \in AC^0[T,+\infty)$ the concept in part (I) with $\alpha \in \mathbb{R}$ corresponds to "regular variation of order 1" with the stated traditional restriction on the sign, hence consistency between the definitions in parts (I) - (II) requires the use of absolute values in (1.19). Regular variation of order *n* involves derivatives up to order *n*.

Proposition 1.1. (Basic Properties of Higher-Order Regular Variation).

(I) *If a function* $f \in AC^{n-1}[T,+\infty)$, $n \ge 1$, *is regularly varying at* $+\infty$ *of order n then the following relations hold true between the indexes* α_k *defined in* (1.19):

$$\alpha \notin \{0,1,\cdots,n-2\},\ n \ge 2 \Rightarrow \alpha_k = \alpha - k,\ 1 \le k \le n-1; \tag{1.20}$$

$$\begin{aligned} &\alpha \equiv k_0 \in \{0,1,\cdots,n-2\},\ n \ge 2 \\ &\Rightarrow \begin{cases} \alpha_k = k_0 - k & \text{if } 0 \le k \le k_0; \\ \alpha_{k_0+1} = \beta, & \text{for some } \beta \le -1; \\ \alpha_k = \beta - (k - k_0 - 1) & \text{if } k_0 + 1 \le k \le n-1. \end{cases} \end{aligned} \tag{1.21}$$

If $f \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}$, $n \ge 2$, *we have:*

$$\alpha_{n-1} < \alpha_{n-2} < \cdots < \alpha_1 \le -1. \tag{1.22}$$

(II) (Principal Parts of Higher Derivatives). *If* $f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}$, $n \ge 1$, *then the relations*

$$\begin{aligned} f^{(k)}(x)/f(x) &= \alpha(\alpha-1)\cdots(\alpha-k+1)x^{-k} + o(x^{-k}) \\ &\equiv \alpha^{\underline{k}} x^{-k} + o(x^{-k}),\ x \to +\infty,\ 1 \le k \le n, \end{aligned} \tag{1.23}$$

hold true whichever $\alpha \in \mathbb{R}$ *may be.*

Definition 1.2. (Higher-Order Smooth Variation). *A function* $f \in AC^{n-1}[T,+\infty)$, $n \ge 1$, *is termed* "*smoothly varying at* $+\infty$ *of order n and index* $\alpha \in \mathbb{R}$" *if* $f(x) \ne 0$ $\forall x$ *large enough and relations in* (1.23) *hold true. We denote this class of functions by* {$\mathcal{SR}_\alpha(+\infty)$ *of order n*}.

Useful characterizations for this class of functions are reported in ([3], Prop. 3.2, p. 801), but we report only a basic one here.

Proposition 1.2. *The following relation holds true*:

$$\begin{cases} \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\} = \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\} \\ \text{if } n = 1 \text{ or } \{n \ge 2; \alpha \ne 0,1,\cdots,n-2\}; \end{cases} \tag{1.24}$$

whereas in the cases not included in (1.24) *we have the inclusion*:

$$\{\mathcal{R}_\alpha(+\infty) \text{ of order } n\} \subsetneqq \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\}. \tag{1.25}$$

The reason for the inclusion being strict is that some derivatives of a smoothly-varying function may vanish (*even identically*) *or change sign infinitely often. For* *instance, a constant function* $f(x) \equiv c \neq 0$ *belongs to the class* $\{\mathcal{R}_0(+\infty) \text{ of order } 1\}$ *and of no greater order as* $f'(x) \equiv 0$, *but it belongs to the class* $\{\mathcal{SR}_0(+\infty) \text{ of any order } n \in \mathbb{N}\}$ *Analogously, each integer power* $x^p, p \in \mathbb{N}$, *belongs to the class* $\{\mathcal{SR}_p(+\infty) \text{ of any order } n \in \mathbb{N}\}$: (*read the two lines after formula* (1.35) *below*).

Relations in (1.24) *and* (1.20) *imply that, under the stated restrictions on the index* α, *the asymptotic relations in* (1.23) *for* $n \geq 2$ *hold true if and only if* "$\left|f^{(k)}\right| \in \{\mathcal{R}_{\alpha-k}(+\infty)\}$, $0 \leq k \leq n-1$". *For further reference explicitly notice the following partial inferences*:

$$\begin{cases} f \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\},\ n \geq 1, (\alpha \in \mathbb{R}), \Rightarrow |f| \in \mathcal{R}_\alpha(+\infty); \\ f \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\},\ n \geq 2, (\alpha \neq 0), \Rightarrow |f'| \in \{\mathcal{R}_{\alpha-1}(+\infty)\}; \\ f \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\},\ n \geq 3, (\alpha \neq 0,1), \Rightarrow \begin{cases} |f'| \in \{\mathcal{R}_{\alpha-1}(+\infty)\}, \\ |f''| \in \{\mathcal{R}_{\alpha-2}(+\infty)\}; \end{cases} \\ \cdots \\ f \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\},\ \alpha \neq 0,1,\cdots,k \text{ and } n \geq k+2 \\ \Rightarrow \left|f^{(h)}\right| \in \{\mathcal{R}_{\alpha-h}(+\infty)\},\ 1 \leq h \leq k+1. \end{cases} \tag{1.26}$$

Definition 1.3. (Higher-Order Rapid Variation).

(I) (First order). *A function* $f \in AC^1[T,+\infty)$, *is called* "*rapidly varying at* $+\infty$ *of order* 1 (*in the strong restricted sense*)" *if*:

$$\begin{cases} f(x), f'(x) \neq 0 & \forall x \text{ large enough;} \\ f(x)/f'(x) = o(x), & x \to +\infty; \\ \left(f(x)/f'(x)\right)' = o(1), & x \to +\infty. \end{cases} \tag{1.27}$$

(II) (Higher order). *A function* $f \in AC^n[T,+\infty)$, *is called* "*rapidly varying at* $+\infty$ *of order* $n \geq 2$ (*in the strong restricted sense*)" *if all the functions* $f, f', \cdots, f^{(n-1)}$ *are rapidly varying at* $+\infty$ *in the above sense, and this amounts to say that the following conditions hold true as* $x \to +\infty$:

$$f^{(k)}(x) \neq 0 \quad \forall x \text{ large enough and } 0 \leq k \leq n; \tag{1.28}$$

$$\begin{cases} f(x)/f'(x) = o(x);\ f'(x)/f''(x) = o(x);\cdots \\ \cdots; f^{(n-1)}(x)/f^{(n)}(x) = o(x); \end{cases} \tag{1.29}$$

$$\begin{cases} \left(f(x)/f'(x)\right)' = o(1);\ \left(f'(x)/f''(x)\right)' = o(1);\cdots \\ \cdots; \left(f^{(n-1)}(x)/f^{(n)}(x)\right)' = o(1). \end{cases} \tag{1.30}$$

Explicitly notice that rapid variation of order n, in our present definition, involves derivatives up to order $n+1$ *unlike regular variation of order n.*

Proposition 1.3. (Basic Properties of Rapid Variation and Two Separate Classes).

(I) *If* f *is rapidly varying at* $+\infty$ *of order* $n \geq 2$ *in the just-defined sense then it can be proved that all the functions* $|f|, |f'|, \cdots, \left|f^{(n-1)}\right|$ *belong to the same* *class, either* $\mathcal{R}_{-\infty}(+\infty)$ *or* $\mathcal{R}_{+\infty}(+\infty)$, *hence we shall use notation* $f \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$ *to denote that* f *enjoys the properties in* (1.28) - (1.29) - (1.30) *plus the appropriate value* "$\pm\infty$" *of the limit in* (1.17).

(Relations in (1.29), which immediately follow from those in (1.30), are written down explicitly just for convenience.)

(II) (Principal Parts of Higher Derivatives). *Let* $f \in AC^n[T,+\infty)$ *and conditions in* (1.28) *be satisfied. Then* $f \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\} \cup \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\}$, *i.e. conditions in* (1.30) *hold true, if and only if the following four equivalent sets of conditions are satisfied*:

$$\begin{cases} f'(x)/f(x) \sim f''(x)/f'(x) \sim \cdots \sim f^{(n)}(x)/f^{(n-1)}(x) \sim f^{(n+1)}(x)/f^{(n)}(x), \\ \text{i.e. } D_\ell\left(f^{(k)}(x)\right) \sim D_\ell(f(x)),\ x \to +\infty,\ 1 \le k \le n; \end{cases} \tag{1.31}$$

$$f^{(k+2)}(x) \sim \left(f^{(k+1)}(x)\right)^2 / f^{(k)}(x),\ x \to +\infty,\ 0 \le k \le n-1; \tag{1.32}$$

$$f^{(k+2)}(x)/f(x) \sim \left(f'(x)/f(x)\right)^{k+2},\ x \to +\infty,\ 0 \le k \le n-1; \tag{1.33}$$

$$\begin{gathered} f(x) \equiv f'(x)/D_\ell(f(x)) \sim f''(x)/\left(D_\ell(f(x))\right)^2 \sim \cdots \\ \cdots \sim f^{(n+1)}(x)/\left(D_\ell(f(x))\right)^{n+1},\ x \to +\infty. \end{gathered} \tag{1.34}$$

It follows that even $f^{(n+1)}(x) \ne 0$ *for almost all x large enough.*

1.4. Various Remarks

1) Notation "$f \in \mathcal{R}_{\pm\infty}(+\infty)$" implies "$f$ ultimately > 0" in accord with the standard agreement, whereas "$f \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } 1\}$" simply implies "$f, f'$ ultimately strictly one-signed". More precisely, for an f ultimately > 0 we have:

$$\begin{cases} f \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\} \Rightarrow f^{(k)} > 0 \text{ ultimately for } 0 \le k \le n+1; \\ f \in \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\} \Rightarrow (-1)^k f^{(k)} > 0 \text{ ultimately for } 0 \le k \le n+1. \end{cases}$$

2) A list of typical functions

- Typical functions in the class $\{\mathcal{R}_\alpha(+\infty) \text{ of any order } n \in \mathbb{N}\}$, $\alpha \in \mathbb{R}$, are:

$$\begin{cases} x^\alpha \cdot \left[\prod_{k=1}^{p_1} \left(\ell_k(x)\right)^{\beta_k}\right] \cdot \left[\prod_{k=1}^{p_2} \exp\left(c_k (\log x)^{\gamma_k}\right)\right] \cdot \left[\prod_{k=2}^{p_3} \exp\left(d_k \left(\ell_k(x)\right)^{\delta_k}\right)\right], \\ \alpha, \beta_k, c_k, d_k \in \mathbb{R};\ 0 < \gamma_k < 1;\ 0 < \delta_k; \end{cases} \tag{1.35}$$

provided they do not reduce to an integer power $x^p, p \in \mathbb{N}$, which belongs to the class $\{\mathcal{R}_p(+\infty) \text{ of exact order } p+1\}$.

- Typical functions in the classes $\{\mathcal{R}_{\pm\infty}(+\infty) \text{ of any order } n \in \mathbb{N}\}$, are:

$$\begin{cases} R_\alpha(x) \cdot \left[\prod_{k=1}^{p_2} \exp\left(c_k (\log x)^{\gamma_k}\right)\right] \cdot \left[\prod_{k=1}^{p_3} \exp\left(d_k x^{\delta_k}\right)\right], \\ R_\alpha \in \{\mathcal{R}_\alpha(+\infty) \text{ of any order } n \in \mathbb{N}\}; \\ c_k, d_k \in \mathbb{R};\ \gamma_k > 1;\ 0 < \delta_k; \end{cases} \tag{1.36}$$

provided they do not reduce to the single function R_α and where the pertinent class, either $\{\mathcal{R}_{-\infty}(+\infty)\}$ or $\{\mathcal{R}_{+\infty}(+\infty)\}$, is determined by the behavior of the absolute value of the function, as $x \to +\infty$, according as it converges to zero or diverges to $+\infty$.

- The following functions

$$f_1(x) := x^p + \mathrm{e}^{-x};\ f_2(x) := x^p + \mathrm{e}^{-x}\sin x\ \ (p \in \mathbb{R}); \tag{1.37}$$

belong to the class "$\{\mathcal{SR}_p(+\infty)\text{ of any order } n \in \mathbb{N}\}$" for any $p \in \mathbb{R}$. For $p \in \mathbb{N} \cup \{0\}$ they belong also to the class $\{\mathcal{R}_p(+\infty)\text{ of exact order } p+1\}$ but for different reasons:

$$\begin{cases} f_1^{(p+1)}(x) := (-1)^{p+1}\mathrm{e}^{-x} \in \{\mathcal{R}_{-\infty}(+\infty)\text{ of any order } n \in \mathbb{N}\}; \\ f_2^{(p+1)}\text{ does not belong to any of the previous classes due to its changing sign}. \end{cases}$$

3) As for the Wronskians we shall use only the two identities:

$$W(v(x)u_1(x),\cdots,v(x)u_n(x)) = (v(x))^n \cdot W(u_1(x),\cdots,u_n(x)); \tag{1.38}$$

$$W(u_1(g(x)),\cdots,u_n(g(x))) = (g'(x))^{n(n-1)/2} \cdot [W(u_1(y),\cdots,u_n(y))]_{y=g(x)}. \tag{1.39}$$

And for the reader's convenience we also report Faà Di Bruno's formula for derivatives of a composition, taken from ([4], p. 818):

$$\begin{aligned} (f(g(x)))^{(k)} &\equiv \frac{\mathrm{d}^k}{\mathrm{d}x^k}(f(g(x))) \\ &= \sum_{0 \le i_j \le k}^{i_1+2i_2+\cdots+ki_k=k} \frac{k!}{i_1!\cdots i_k!(1!)^{i_1}(2!)^{i_2}\cdots(k!)^{i_k}} \\ &\quad \times f^{(i_1+\cdots+i_k)}(g(x))\cdot(g'(x))^{i_1}\cdot(g''(x))^{i_2}\cdots\left(g^{(k)}(x)\right)^{i_k}, \end{aligned} \tag{1.40}$$

where the summation is taken over all possible ordered k-tuples of non-negative integers i_j such that

$$i_1 + 2i_2 + \cdots + ki_k = k\ \ (\text{hence } 1 \le i_1 + i_2 + \cdots + i_k \le k). \tag{1.41}$$

* * *

Some basic properties of Hankel determinants of the type we are studying may be found in ([7], Ch. 2, §7; pp. 70-77) but we shall need only a few of them. When referring to Theorem 10 in ([1], pp. 19-20), the reader is warned to take account of some remarks and corrections reported in ([2], pp. 39-40).

4) An important remark on the indexes of regular variation

Some meaningful examples illustrating the results in this Part I concern regularly-varying functions of index α and, according to the results about products stated in ([4], §7.2), the values $\alpha \in \mathbb{N}$ might be excluded as those requiring unnatural restrictions in most cases. In [6], we filled this gap by giving precise results on the indexes of the higher derivatives in these exceptional cases, results relevant for the functions in (1.35) as well. Moreover, certain calculations in the sequel require the evaluation of the index of variation for a linear combination, which is possible only under some restrictions and in [6] we also pointed out a simple complement to known results enabling to bypass those restrictions. In §2 we report the statements of the results needed in the sequel.

5) A remark on regularity

As for the regularity of the function ϕ in (1.1) we simply assume "$\phi \in C^{2n-2}[T,+\infty)$" recalling that all the results are unchanged under the weaker condition "$\phi \in C^{2n-3}[T,+\infty)$, and $\phi^{(2n-3)}$ absolutely continuous" provided that the asymptotic relations involving the highest-order derivatve $\phi^{(2n-2)}$ are read in the way specified in (1.13).

The choice of simplifying the assumptions will hopefully avoid visual complications in reading the various statements, which are complicated enough in themselves. Anyway the reader needs to bother in no way with checking regularity assumptions in the many examples because these only involve infinitely-differentiable functions.

2. Some Frequently-Used Results on Operations with Higher-Order Types of Asymptotic Variation

For the reader's convenience, we report in this section, with few or without comments, some frequently-used facts concerning products and linear combinations of functions with various types of asymptotic variation: facts proved in [4] or [6]. The reader may omit reading this section referring to a result therein only when needed.

2.1. Some Facts on Factorizations

In the inference

$$\mathcal{L} \in \{\mathcal{R}_0(+\infty) \text{ of order } n\} \Rightarrow \begin{cases} x^\alpha \mathcal{L}(x) \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\} \\ \text{if } \alpha \neq 0,1,\cdots,n-2, \end{cases} \tag{2.1}$$

which follows from both (1.24) and a relation in ([4], formula (4.3),p. 820), the restrictions on α can be suppressed for an important class of higher-order slowly-varying functions including the iterated logarithms.

Lemma 2.1. ([6], Lem. 2.1, p. 694).

If $\left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty), 0 \le k \le n-1, n \ge 2$, *then the following relations hold true*:

$$D^k\left(x^\alpha \mathcal{L}(x)\right) \begin{cases} \sim \alpha^{\underline{k}} x^{\alpha-k} \mathcal{L}(x),\ x \to +\infty, \left(\alpha \notin \mathbb{N} \cup \{0\}, 1 \le k \le n\right); \\ \sim \alpha^{\underline{k}} x^{\alpha-k} \mathcal{L}(x),\ x \to +\infty, \left(\alpha \in \mathbb{N}, 1 \le k \le \alpha\right); \end{cases} \tag{2.2}$$

$$D^{\alpha+1}\left(x^\alpha \mathcal{L}(x)\right) \sim \alpha!\mathcal{L}'(x) = o\left(x^{-1}\mathcal{L}(x)\right),\ x \to +\infty, \text{ if } \alpha \in \mathbb{N}; \tag{2.3}$$

$$\begin{cases} D^{\alpha+m}\left(x^\alpha \mathcal{L}(x)\right) \sim (-1)^{m-1} \alpha!(m-1)!x^{1-m}\mathcal{L}'(x),\ x \to +\infty, \\ \text{if } \alpha \in \mathbb{N},\ m > 1, (\text{and obviously})\ \alpha + m \le n. \end{cases} \tag{2.4}$$

For the special case of $\mathcal{L}(x) := x^p \log x,\ p \in \mathbb{N}$, *we have the formulas*:

$$\begin{cases} D\left(x^p \log x\right) = px^{p-1}\log x + x^{p-1}; \\ D^2\left(x^p \log x\right) = p(p-1)x^{p-2}\log x + px^{p-2} + (p-1)x^{p-2}; \\ D^k\left(x^p \log x\right) = p^{\underline{k}}x^{p-k}\log x + (\text{constant})\cdot x^{p-k},\ 1 \le k \le p-1; \\ D^p\left(x^p \log x\right) = p!\log x + (\text{constant}); \\ D^{p+m}\left(x^p \log x\right) = p!D^m(\log x) \sim (-1)^{m-1}p!(m-1)!x^{-m},\ m \ge 1, \end{cases} \tag{2.5}$$

the exact values of the constants being not presently needed.

Proposition 2.2. ([6], Prop. 2.2, p. 694). *The inference holds true*:

$$\begin{cases} \left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty),\ 0 \le k \le n-1; \\ f(x) := x^\alpha \mathcal{L}(x),\ \alpha \in \mathbb{R}\setminus\{0\}; \end{cases} \Rightarrow \begin{cases} f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}; \\ \left|f^{(k)}\right| \in \mathcal{R}_{\alpha-k}(+\infty), \\ 0 \le k \le n-1; \end{cases} \tag{2.6}$$

noticing that the simpler inference

$$\begin{cases} \mathcal{L} \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}, \\ f(x) := x^\alpha \mathcal{L}(x),\ \alpha \in \mathbb{R}\setminus\{0\}, \end{cases} \Rightarrow f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\},$$

without specifying the indexes of variation for the derivatives of $\mathcal{L}$, *is in general false as shown by the function* $g(x) = 1 + x^{-p}$, $p \in \mathbb{N}$, *which belongs to the class* $\{\mathcal{R}_0(+\infty) \text{ of any order } n\}$ *whereas* $x^p g(x)$ *belongs to the class* $\{\mathcal{R}_p(+\infty)$ *of exact order* $p+1\}$.

2.2. Results on Products

Proposition 2.3. (A special result on product of higher-order regularly-varying functions: ([6], Prop. 2.4, p. 697)).

(I) (The slow variation case). *The inference*

$$\begin{cases} \left\{\left|f^{(k)}\right|, \left|g^{(k)}\right| \in \mathcal{R}_{-k}(+\infty), 0 \le k \le n-1, n \ge 2\right\} \\ \Rightarrow \begin{cases} f \cdot g \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}, \\ \left|(f\cdot g)^{(k)}\right| \in \mathcal{R}_{-k}(+\infty), 0 \le k \le n-1, \end{cases} \end{cases} \tag{2.7}$$

holds true under any of the two following additional conditions:

$$\begin{cases} \text{either } f'(x)g(x) \gg f(x)g'(x),\ x \to +\infty, \text{ or} \\ \operatorname{sign}\left(f'(x)g(x)\right) = \operatorname{sign}\left(f(x)g'(x)\right) \text{ ultimately.} \end{cases} \tag{2.8}$$

(II) (The regular variation case). *Let* f, g *satisfy the assumptions in* (2.7) *and anyone of the two conditions in* (2.8) *and let*

$$f_1(x) := x^\alpha f(x),\ f_2(x) := x^\beta g(x),\ (\alpha, \beta \in \mathbb{R}). \tag{2.9}$$

Then:

$$\begin{cases} f_1 \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}, \\ f_2 \in \{\mathcal{R}_\beta(+\infty) \text{ of order } n\}, \\ f_1 \cdot f_2 \in \{\mathcal{R}_{\alpha+\beta}(+\infty) \text{ of order } n\}, \\ \left|(f_1 \cdot f_2)^{(k)}\right| \in \mathcal{R}_{\alpha+\beta-k}(+\infty),\ 0 \le k \le n-1. \end{cases} \tag{2.10}$$

The case $f \cdot g$, *with* $g = 1/f$, *is a good counterexample if both conditions in* (2.8) *are lacking as* $f' \cdot g = -f \cdot g'$.

Part II of the above proposition follows at once from Proposition 2.2 and Proposition 2.3-(I).

Proposition 2.4. (General results on products of higher-order regularly- or rapidly-varying functions: ([4], Prop.7.3)).

(I) If

$$\begin{cases} f_i \in \{\mathcal{R}_{\alpha_i}(+\infty) \text{ of order } n\},\ n \ge 2,\ (i = 1, \cdots, p); \\ \alpha_1 + \cdots + \alpha_p \ne 0, 1, \cdots, n-2, \end{cases} \tag{2.11}$$

then

$$\begin{cases} \prod_{i=1}^{p} f_i \in \{\mathcal{R}_\beta(+\infty) \text{ of order } n\} \text{ with } \beta = \alpha_1 + \cdots + \alpha_p; \\ \left|\left(\prod_{i=1}^{p} f_i\right)^{(k)}\right| \in \mathcal{R}_{\beta-k}(+\infty),\ 1 \le k \le n. \end{cases} \tag{2.12}$$

If the assumption in (2.11) *is replaced by the weaker condition* " $f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n\}$ " *then the first claim in* (2.12) *is replaced by* $\prod_{i=1}^{p} f_i \in \{\mathcal{SR}_\beta(+\infty) \text{ of order } n\}$ *without requiring any restriction on the* α_i*'s as follows from* ([4], Prop. 7.1). *The restrictions on the* α_i*'s imply the complete claim in* (2.12) *by the inferences in* (1.26). *In particular*:

$$\begin{cases} f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\},\ n \ge 2,\ \alpha \in \mathbb{R}, \\ m \in \mathbb{N},\ m\alpha \ne 0, 1, \cdots, n-2, \end{cases}$$
$$\Rightarrow \begin{cases} f^m \in \{\mathcal{R}_{m\alpha}(+\infty) \text{ of order } n\}, \\ D^k f^m \in \{\mathcal{R}_{m\alpha-k}(+\infty)\},\ 1 \le k \le n. \end{cases}$$

(II)

$$f_i \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n \ge 1\}\ \forall i \Rightarrow \prod_{i=1}^{p} f_i \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\}; \tag{2.13}$$

$$f_i \in \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n \ge 1\}\ \forall i \Rightarrow \prod_{i=1}^{p} f_i \in \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\}. \tag{2.14}$$

(III) *For any* $\alpha \in \mathbb{R}$:

$$\left.\begin{cases} f \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n \ge 1\}, \\ g \in \{\mathcal{SR}_{\alpha}(+\infty) \text{ of order } n \ge 1\}, \end{cases}\right\} \tag{2.15}$$
$$\Rightarrow f \cdot g \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\},$$

noticing that the assumption on g is milder than $g \in \{\mathcal{R}_{\alpha}(+\infty) \text{ of order } n\}$.

2.3. Index of Higher-Order Variation for a Linear Combination

Lemma 2.5. (The case of two smoothly-varying functions ([4], relations in (7.4), p. 820)).

If $f \in \{\mathcal{SR}_{\alpha}(+\infty) \text{ of order } n\}$ *and* $g \in \{\mathcal{SR}_{\beta}(+\infty) \text{ of order } n\}$ *then*:

$$\begin{cases} c_1 f + c_2 g \in \{\mathcal{SR}_{\max(\alpha,\beta)}(+\infty) \text{ of order } n\} \\ \forall \alpha, \beta, c_1, c_2 \in \mathbb{R},\ \alpha \ne \beta,\ c_i \ne 0; \end{cases} \tag{2.16a}$$

$$\begin{cases} c_1 f + c_2 g \in \{\mathcal{SR}_{\alpha}(+\infty) \text{ of order } n\} \\ \text{if } \alpha = \beta \text{ and under the restrictions either} \\ \{c_i > 0;\ f, g > 0\} \text{ or } \{c_i \ne 0;\ f(x) \gg g(x),\ x \to +\infty\}. \end{cases} \tag{2.16b}$$

And the following is the extension to more than two functions.

Proposition 2.6. (Linear combinations of higher-order smoothly-varying functions: ([6], Prop. 3.2, p. 699)).

Let

$$f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n\},\ 1 \le i \le p. \tag{2.17}$$

(I) *If* " $c_i > 0,\ f_i(x) > 0$ *ultimately*", *and* " $\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \cdots \ge \alpha_p$," *then*:

$$\sum_{i=1}^{p} c_i f_i \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}. \tag{2.18}$$

(II) *If* " $f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p$," *a condition granted by the restriction* " $\alpha_1 > \alpha_2 \ge \cdots \ge \alpha_p$ ", *then*:

$$\sum_{i=1}^{p} c_i f_i \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}\ \ \forall c_i = \text{constant} \ne 0. \tag{2.19}$$

(III) *In particular, in either of the two previous cases and for* $n \ge 2$:

$$\alpha_1 \ne 0, 1, \cdots, n-2,\ \Rightarrow \sum_{i=1}^{p} c_i f_i \in \{\mathcal{R}_{\alpha_1}(+\infty) \text{ of order } n\}. \tag{2.20}$$

The import of the statement in part (II) is that there is *one* function, namely f_1, with the maximal growth-order and, even if we cannot be sure that the linear combination of $f_2, \cdots, f_p$ is smoothly varying of order n (whatever the index may be) we have the desired conclusion.

Some caution is needed whenever some rapidly-varying function is involved because our adopted concept of "*n*th-order rapid variation" does not simply means the validity of the limits:

$$\lim_{x\to+\infty} x f^{(k+1)}(x) / f^{(k)}(x) = \pm\infty,\ 0 \le k \le n,$$

with the suitable signs (the same for all k's), but requires the additional conditions in (1.30) or the equivalent formulations in (1.31) - (1.34). The following is an easy result.

Proposition 2.7. (Positive linear combinations involving rapid variation: ([6], Prop. 3.3, p. 701)). *If*

$$f_i(x) > 0 \text{ ultimately}, \; c_i > 0 \text{ for all the involved indexes } i, \tag{2.21}$$

then:

$$\begin{cases} \left|f_i^{(k)}\right| \in \mathcal{R}_{-\infty}(+\infty), \; 1 \le i \le p, \; 0 \le k \le n, \\ \Rightarrow \sum_{i=1}^{p} c_i \left|f_i^{(k)}\right| \in \mathcal{R}_{-\infty}(+\infty), \; 0 \le k \le n; \end{cases} \tag{2.22}$$

$$\begin{cases} f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), \; 1 \le i \le p, \; 0 \le k \le n, \\ \Rightarrow \sum_{i=1}^{p} c_i f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), \; 0 \le k \le n; \end{cases} \tag{2.23}$$

$$\left\{\begin{matrix} f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), \; 1 \le i \le p, \\ f_i^{(k)} \in \mathcal{R}_{\alpha_i}(+\infty), \; p+1 \le i \le q, \\ 0 \le k \le n, \; \alpha_i \in \mathbb{R}, \end{matrix}\right\} \Rightarrow \sum_{i=1}^{q} c_i f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), \; 0 \le k \le n. \tag{2.24}$$

(The reader may notice the lack of absolute values in (2.23) and in (2.24) as, in fact, they would be redundant: in (2.23) by the remark after (1.34), and in (2.24) by the further reason that $\sum_{i=1}^{q} c_i f_i^{(k)} \sim c_1 f_1^{(k)}$.)

Looking at the claims in Proposition 2.7 it must be pointed out that the analogous inferences wherein one of the properties "$\mathcal{R}_{\pm\infty}(+\infty), 0 \le k \le n$," is replaced by the corresponding property "$\{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$" both in the hypotheses and in the theses, are not automatic facts. The following are non-obvious results which may be completed by a nontrivial counterexample, ([6], pp. 704-705), not reported here.

Proposition 2.8. (Arbitrary linear combinations of various types of asymptotic variations: ([6], Prop. 3.4, p. 702)).

Warning. The notation "$f_i \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$" in the next statements means that each f_i belongs to its own class, not necessarily the same for all of the f_i's.

(I) *Let*

$$\begin{cases} f_i \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}, \; 1 \le i \le p; \\ f_1(x) \gg f_i(x), \; x \to +\infty, \; 2 \le i \le p; \\ g(x) := \sum_{i=1}^{p} c_i f_i(x), \; c_i \in \mathbb{R} \setminus \{0\}. \end{cases} \tag{2.25}$$

If anyone of the following additional conditions is satisfied, either

$$f_1^{(k)}(x) \gg f_i^{(k)}(x), \; x \to +\infty, \; 2 \le i \le p, \; 1 \le k \le n+1, \tag{2.26}$$

or

$$f_i'(x)/f_i(x) = O\left(f_1'(x)/f_1(x)\right),\ 2 \le i \le p, \tag{2.27}$$

then:

$$g^{(k)}(x)/g(x) \sim \left(f_1'(x)/f_1(x)\right)^k,\ 2 \le k \le n+1, \tag{2.28}$$

which implies that "*g belongs to the same class of* f_1".

(II) *If*

$$\begin{cases} f_i \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\},\ 1 \le i \le p; \\ f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n+1\},\ p+1 \le i \le q,\ \alpha_i \in \mathbb{R}; \\ f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p; \\ \text{one of the conditions in } (2.26)\text{-}(2.27) \\ \text{for all the indexes } i: 2 \le i \le p; \\ f_{p+1}(x) \gg f_i(x),\ x \to +\infty,\ p+2 \le i \le q; \\ h(x) := \sum_{i=1}^q c_i f_i(x),\ c_i \in \mathbb{R} \setminus \{0\}; \end{cases} \tag{2.29}$$

then

$$h^{(k)}(x)/h(x) \sim \left(f_1'(x)/f_1(x)\right)^k,\ 2 \le k \le n+1, \tag{2.30}$$

which implies that "$h \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\}$".

(III) *If*

$$\begin{cases} f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n\},\ 1 \le i \le p;\ \alpha_i \in \mathbb{R}; \\ f_i \in \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\},\ p+1 \le i \le q; \\ f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p; \\ f_{p+1}(x) \gg f_i(x),\ x \to +\infty,\ p+2 \le i \le q; \\ \text{one of the conditions in } (2.26)\text{-}(2.27) \text{ referred} \\ \text{to the indexes } i: p+2 \le i \le q; \\ h(x) := \sum_{i=1}^q c_i f_i(x),\ c_i \in \mathbb{R} \setminus \{0\}; \end{cases} \tag{2.31}$$

then

$$h^{(k)}(x) = \alpha_1^k x^{-k} + o\left(x^{-k}\right),\ 1 \le k \le n, \tag{2.32}$$

which means, by definition, that "$h \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}$".

(IV) *If*

$$\begin{cases} \left|f^{(k)}\right| \in \mathcal{R}_{\alpha_k}(+\infty),\ 0 \le k \le n,\ \alpha_k \in \mathbb{R}, \\ \left|g^{(k)}\right| \in \mathcal{R}_{-\infty}(+\infty),\ 0 \le k \le n, \end{cases}$$

then:

$$\left|(f+g)^{(k)}\right| \in \mathcal{R}_{\alpha_k}(+\infty),\ 0 \le k \le n.$$

The proof of the claim in part (IV), not reported elsewhere, is quite immediate. In fact, the elementary asymptotic relations in ([3], (2.19), (2.20), (2.41)), namely:

$$\begin{cases} x^{\alpha_k-\epsilon} \ll f^{(k)}(x) \ll x^{\alpha_k+\epsilon} & \forall \epsilon > 0, \\ g^{(k)}(x) \ll x^{-m} & \forall m > 0, \end{cases} \quad x \to +\infty,$$

at once imply $g^{(k)}(x) \ll f^{(k)}(x)$ whence:

$$\frac{f^{(k)}(x)+g^{(k)}(x)}{f^{(k-1)}(x)+g^{(k-1)}(x)} \sim \frac{f^{(k)}(x)}{f^{(k-1)}(x)}.$$

3. Results for Regularly-Varying Functions

The case of Hankelians of regularly-varying functions is simple to treat and also serves for the general case in the next section.

3.1. The Main Results

First of all, notice the following identities for the Hankelians of powers (special cases of more general identities to be taken into consideration in Part II of this paper):

$$H_n\left[x^\alpha\right] \begin{cases} \equiv 0 \quad \forall x \in \mathbb{R} \text{ if } \alpha \in \{0,1,\cdots,n-2\}, n \geq 2, \\ \text{with the agreement } x^0 := 1 \quad \forall x \in \mathbb{R}; \\ = (-1)^{n(n-1)/2} \left(\prod_{i=1}^{n-1} i!\right) \cdot \left(\prod_{i=1}^{n-1} \alpha^{\underline{i}}\right) \cdot x^{n(\alpha-n+1)} \text{ for all the} \\ \text{other real values of } \alpha \text{ and all the admissible values of } x; \end{cases} \tag{3.1}$$

where the admissible values of x are those allowed by the exponent α and the structure of $H_n\left[x^\alpha\right]$; for instance: (i) $x \in \mathbb{R}$ with $\alpha \in \mathbb{N}$; (ii) $x > 0$ with $\alpha < 0$ or $0 < \alpha < 2n-2$. Formulas in (3.1) may be simply proved by writing:

$$H_n\left[x^\alpha\right] = W\left(x^\alpha, \alpha^{\underline{1}} x^{\alpha-1}, \alpha^{\underline{2}} x^{\alpha-2}, \cdots, \alpha^{\underline{n-1}} x^{\alpha-n+1}\right);$$

then factoring the various constants $\alpha^{\underline{k}}$ out of the Wronskian and using the results in ([1], formula (68), p. 10, with $g(x) = x$) and in ([1], formula (60), p. 9, with $k = -1$). Replacing x^α by a regularly-varying function ϕ we can obtain the principal part of $H_n\left[\phi(x)\right]$ directly from a claim proved in ([1], Th. 9, p. 18) which we rewrite here in a shortened form as:

Lemma 3.1. *Let the functions* ϕ_i *satisfy:*

$$\begin{cases} \phi_i \in C^{n-1}\left[T, +\infty\right), \ \phi_i(x) \neq 0 \text{ for } x \text{ large enough}, \ 1 \leq i \leq n; \\ \phi_i^{(k)}(x)/\phi_i(x) = x^{-k}\left[(\alpha_i)^{\underline{k}} + o(1)\right], \ x \to +\infty, \\ (1 \leq k \leq n-1; 1 \leq i \leq n), \ \alpha_1 > \alpha_2 > \cdots > \alpha_n; \end{cases} \tag{3.2}$$

hence they are smoothly varying at $+\infty$ of order $n-1$ and respective indexes $\alpha_1, \cdots, \alpha_n$. Then $(\phi_1, \cdots, \phi_n)$ is an asymptotic scale at $+\infty$ and, as $x \to +\infty$:

$$W\left(\phi_1(x), \cdots, \phi_n(x)\right) \sim V\left(\alpha_1, \cdots, \alpha_n\right) \cdot \left(\prod_{i=1}^{n} \phi_i(x)\right) \cdot x^{-n(n-1)/2}. \tag{3.3}$$

In this statement the essential condition is that the α_i*'s are distinct numbers otherwise the Vandermondian vanishes.*

Theorem 3.2. (Principal part of $H_n[\phi(x)]$ for a regularly-varying function).

(I) *Let* $\phi \in C^{2n-2}[T,+\infty)$ *be regularly varying at* $+\infty$ *of order* $2n-2$ *which implies that each derivative* $\phi^{(i)}, 0 \le i \le n-1$, *is regularly varying at* $+\infty$ *of order* $2n-2-i$ *with its own index* α_i *where, by* (1.21), *the indexes satisfy* "$\alpha_0 > \alpha_1 > \cdots > \alpha_{n-1}$". *Hence the n-tuple* $(\phi, \phi', \cdots, \phi^{(n-1)})$ *satisfies* (3.2) *and for* $n \ge 2$:

$$H_n[\phi(x)] \sim V(\alpha_0, \cdots, \alpha_{n-1}) \cdot \left(\prod_{i=0}^{n-1} \phi^{(i)}(x) \right) \cdot x^{-n(n-1)/2}, \; x \to +\infty. \tag{3.4}$$

(II) *With the restriction*

$$\alpha_0 \notin \{0, 1, \cdots, n-2\}, \tag{3.5}$$

we have $\alpha_i = \alpha_0 - i \; \forall i$ *by* (1.20), *and the above Vandermondian is, by* ([1], *formula* (60), p. 9):

$$V(\alpha_0, \alpha_0 - 1, \cdots, \alpha_0 - n + 1) = (-1)^{n(n-1)/2} \prod_{i=1}^{n-1} i!. \tag{3.6}$$

In this case, by (1.23) *the principal parts of* $\phi', \cdots, \phi^{(n-1)}$ *may be expressed in terms of* ϕ *and* (3.4) *takes on the simple explicit form for each* $n \ge 2$ *as* $x \to +\infty$:

$$H_n[\phi(x)] \sim (-1)^{n(n-1)/2} \left(\prod_{i=1}^{n-1} i! \right) \cdot \left(\prod_{i=1}^{n-1} \alpha_0^{i} \right) \cdot (\phi(x))^n \cdot x^{-n(n-1)}. \tag{3.7}$$

(III) (The excluded cases). *For* $n = 2$ *and* $\alpha_0 = 0$ *the assumptions are*:

$$\phi'(x) = o(x^{-1}\phi(x)); \; \phi''(x) = x^{-1}\phi'(x)[\alpha_1 + o(1)], \; (\alpha_1 \le -1); \tag{3.8}$$

whence:

$$\begin{aligned} H_2[\phi(x)] &\equiv \phi(x)\phi''(x) - (\phi'(x))^2 \\ &= x^{-1}\phi(x)\phi'(x)[\alpha_1 + o(1)] - (\phi'(x))^2 \\ &\sim \alpha_1 x^{-1}\phi(x)\phi'(x), \; x \to +\infty. \end{aligned} \tag{3.9}$$

If $\alpha_0 \in \{0, 1, \cdots, n-2\}, n \ge 3$, *then* $\alpha_k = 0$ *for one value of* $k \le n-2$, *which, by* (1.21), *means the validity of the following relations as* $x \to +\infty$:

$$\begin{cases} \phi'(x) \sim \alpha_0 x^{-1}\phi(x); \; \phi''(x) \sim (\alpha_0 - 1)x^{-1}\phi'(x); \\ \cdots \\ \phi^{(k)}(x) \sim (\alpha_0 - k + 1)x^{-1}\phi^{(k-1)}(x); \\ \phi^{(k+1)}(x) = o\left(x^{-1}\phi^{(k)}(x)\right); \\ \phi^{(k+2)}(x) \sim \alpha_{k+1}x^{-1}\phi^{(k+1)}(x), \text{ with } \alpha_{k+1} \le -1; \\ \cdots \\ \phi^{(n-1)}(x) \sim (\alpha_{k+1} - n + k + 3)x^{-1}\phi^{(n-2)}(x); \end{cases} \tag{3.10}$$

wherein the relation for ϕ' *is missing if* $k = 0$, *and the other relations involve derivatives up to order* $n-1$. *In order to simplify the final formula as much as* *possible, we may express the derivatives up to order k in terms of* ϕ' *, and those of order higher than* $k+1$ *in terms of* $\phi^{(k+1)}$ *leaving* ϕ' *or* $\phi^{(k+1)}$ *unchanged. We get:*

$$\begin{cases} \prod_{i=0}^{n-1} \phi^{(i)}(x) \equiv \phi(x)\phi'(x)\prod_{i=2}^{n-1}\phi^{(i)}(x) \\ \qquad \sim \left(\prod_{i=1}^{n-2}(\alpha_1)^{\underline{i}}\right)\cdot x^{-(n-1)(n-2)/2}\cdot\phi(x)\cdot(\phi'(x))^{n-1}, \\ x\to+\infty,\ n\ge 3;\ (\text{if } \alpha_0 = 0 = k); \end{cases} \tag{3.11}$$

$$\begin{cases} \prod_{i=0}^{n-1}\phi^{(i)}(x) = \left(\prod_{i=0}^{k}\phi^{(i)}(x)\right)\cdot\phi^{(k+1)}(x)\cdot\left(\prod_{i=k+2}^{n-1}\phi^{(i)}(x)\right) \\ \qquad \sim \left(\prod_{i=1}^{k}(\alpha_0)^{\underline{i}}\right)\cdot(\phi(x))^{k+1}\cdot\phi^{(k+1)}(x) \\ \qquad \times\left(\prod_{i=1}^{n-k-2}(\alpha_{k+1})^{\underline{i}}\right)\cdot\left(\phi^{(k+1)}(x)\right)^{n-k-2} x^{-(n-1)(n-2)/2}, \\ x\to+\infty,\ n\ge 3,\ (\text{if } \alpha\in\{1,\cdots,n-2\}); \end{cases}$$

whence:

$$\begin{aligned} \prod_{i=0}^{n-1}\phi^{(i)}(x) &\sim \left(\prod_{i=1}^{k}(\alpha_0)^{\underline{i}}\right)\left(\prod_{i=1}^{n-k-2}(\alpha_{k+1})^{\underline{i}}\right)\cdot x^{-(n-1)(n-2)/2} \\ &\quad\times(\phi(x))^{k+1}\cdot\left(\phi^{(k+1)}(x)\right)^{n-k-1},\ x\to+\infty,\ (n\ge 3); \end{aligned} \tag{3.12}$$

wherein the product with index i running from 1 *to* $n-k-2$ *is void for* $k=n-2$ *, hence equals* 1 *by an usual agreement. The principal parts in* (3.11)-(3.12) *may be used into* (3.4).

3.2. A Special Important Case

In the context of the foregoing theorem the principal part of $H_n(\phi)$ can always be expressed in terms of at most two of the involved functions: ϕ itself and, possibly, one of its derivatives. The following corollary emphasizes a most important case wherein, thanks to Lemma 2.1, all relations involve only ϕ,ϕ'.

Corollary 3.3. *Let* $\mathcal{L}$ *be slowly varying at* $+\infty$ *and such that* $\left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty)$ $\forall k\in\mathbb{N}$ *as, for instance, any function in* (1.35) *after suppressing the power* x^α*. This is the case of* (3.11) *with* $k=0$ *and* $\alpha_1=-1$*; the constants in* (3.4) *and* (3.11) *are respectively:*

$$V(0,-1,\cdots,-(n-1)) = (-1)^{n(n-1)/2}\left(\prod_{i=1}^{n-1}i!\right);\ \prod_{i=1}^{n-2}(-1)^{\underline{i}} = (-1)^n\left(\prod_{i=1}^{n-2}i!\right);$$

and the final relation for $n\ge 3$ *is:*

$$\begin{aligned} H_n[\mathcal{L}(x)] &\sim (-1)^{n(n+1)/2}(n-1)\cdot\left(\prod_{i=1}^{n-2}i!\right)^2 \\ &\quad\times x^{-(2n-1)(n-1)/2}\cdot\mathcal{L}(x)\cdot(\mathcal{L}'(x))^{n-1},\ x\to+\infty. \end{aligned} \tag{3.13}$$

Multiplying $\mathcal{L}(x)$ *by* x^α*, with no restriction on* $\alpha\ne 0$*, Proposition* 2.2

grants the relations "$\left|f^{(k)}\right| \in \mathcal{R}_{\alpha-k}(+\infty) \quad \forall k \in \mathbb{N}$," *i.e.* "$\alpha_i = \alpha - i \quad \forall i$", *and formula* (3.6) *holds true with* $\alpha_0 = \alpha$. *With the further restriction in* (3.5) *we may use formula* (3.7) *that now reads*:

$$\begin{cases} H_n\left[x^{\alpha}\mathcal{L}(x)\right] \sim (-1)^{n(n-1)/2}\left(\prod_{i=1}^{n-1} i!\right)\cdot\left(\prod_{i=1}^{n-1}\alpha^{\underline{i}}\right)\cdot x^{-n(n+\alpha-1)}\cdot(\mathcal{L}(x))^n, \\ x \to +\infty, \left(n \ge 2; \alpha \notin \{0,1,\cdots,n-2\}\right). \end{cases} \tag{3.14}$$

For the remaining values of the exponent we must use (3.12) *from whence, we change the notation to better distinguish the two cases*

$$p := \alpha_0,\ \alpha_p = 0,\ \alpha_{p+1} = -1,$$

and get relation:

$$\begin{cases} H_n\left[x^{p}\mathcal{L}(x)\right] \sim (-1)^{[n(n+1)-2p]/2}\left(\prod_{i=1}^{n-1} i!\right)\cdot\left(\prod_{i=1}^{n-p-2} i!\right) \\ \qquad \times x^{-(2n-1)(n-1)/2}\cdot(\mathcal{L}(x))^{p+1}\cdot\left(\mathcal{L}^{(p+1)}(x)\right)^{n-p-1}; \\ x \to +\infty, \left(n \ge 3, p \in \{1,\cdots,n-2\}\right). \end{cases} \tag{3.15a}$$

Using (2.3) *for* $\mathcal{L}^{(p+1)}$ *we get the final relation*:

$$\begin{cases} H_n\left[x^{p}\mathcal{L}(x)\right] \sim (-1)^{[n(n+1)-2p]/2}\left(\prod_{i=1}^{n-1} i!\right)\cdot\left(\prod_{i=1}^{n-p-2} i!\right) \\ \qquad \times (p!)^{n-p-1}\cdot x^{-(2n-1)(n-1)/2}\cdot(\mathcal{L}(x))^{p+1}\cdot\left(\mathcal{L}'(x)\right)^{n-p-1}, \\ x \to +\infty, \left(n \ge 3, p \in \{1,\cdots,n-2\}\right). \end{cases} \tag{3.15b}$$

Examples 3.1. As particular cases of relations in (3.13) we have:

$$\begin{cases} H_n\left[\log(x)\right] \sim (-1)^{n(n+1)/2}(n-1)\cdot\left(\prod_{i=1}^{n-1} i!\right)^2 \\ \qquad \times x^{-(2n+1)(n-1)/4}\cdot\log x,\ x \to +\infty, (n \ge 3); \end{cases} \tag{3.16a}$$

$$\begin{cases} H_n\left[c+\log(x)\right] \sim (-1)^{n(n+1)/2}(n-1)\cdot c\cdot\left(\prod_{i=1}^{n-1} i!\right)^2 \\ \qquad \times x^{-(2n+1)(n-1)/4},\ x \to +\infty, (n \ge 3; c \ne 0); \end{cases} \tag{3.16b}$$

$$\begin{cases} H_n\left[\ell_k(x)\right] \sim (-1)^{n(n+1)/2}(n-1)\cdot\left(\prod_{i=1}^{n-1} i!\right)^2\cdot x^{-(2n+1)(n-1)/4} \\ \qquad \times \ell_k(x)\left[\ell_{k-1}(x)\cdots\ell_1(x)\right]^{1-n},\ x \to +\infty, (n \ge 3; k \ge 2); \end{cases} \tag{3.17}$$

$$\begin{cases} H_n\left[\exp\left(c(\log x)^{\beta}\right)\right] \sim (-1)^{n(n+1)/2}(n-1)\cdot\left(\prod_{i=1}^{n-1} i!\right)^2(c\beta)^{n-1} \\ \qquad \times x^{-(2n+1)(n-1)/4}\cdot(\log x)^{(\beta-1)(n-1)}\cdot\exp\left(nc(\log x)^{\beta}\right), \\ x \to +\infty, (n \ge 3; c \ne 0; 0 < \beta < 1). \end{cases} \tag{3.18}$$

Examples 3.2. As particular cases of relations in (3.14) - (3.15) we have:

$$\begin{cases} H_n\left[x^\alpha \log(x)\right] \sim (-1)^{n(n-1)/2}\left(\prod_{i=1}^{n-1} i!\right)\cdot\left(\prod_{i=1}^{n-1}\alpha^{\underline{i}}\right)\times x^{-n(n+\alpha-1)}\cdot(\log(x))^n, \\ x\to+\infty, \left(n\ge 2; \alpha\notin\{0,1,\cdots,n-2\}\right); \end{cases} \quad (3.19)$$

$$\begin{cases} H_n\left[x^p \log(x)\right] \sim (-1)^{[n(n+1)+2p(n-p)]/2}\left(\prod_{i=1}^{n-1} i!\right)\cdot\left(\prod_{i=1}^{n-p-2} i!\right) \\ \qquad\times(p!)^{n-p-1}\cdot x^{-[(2n-1)(n-1)+2(p+1)(n-p-1)]/2}(\log(x))^{p+1}, \\ x\to+\infty, \left(n\ge 3; p\in\{1,\cdots,n-2\}\right). \end{cases} \quad (3.20)$$

3.3. An Application to Asymptotic Expansions

The last result in this section is a corollary of the general theory developed in [8]-[10] but for a proper understanding it is enough to read the brief summary given in ([1], §6, pp. 26-27) which we do not report here.

Theorem 3.4. *Let* $\phi^{(k)}\in\mathcal{R}_{\alpha-k}(+\infty)\ \ \forall k\in\{0,1,\cdots,2n-2\}$; *and let* L_n *be the unique linear ordinary differential operator of type*:

$$\begin{cases} L_n u := u^{(n)} + \alpha_{n-1}(x)u^{(n-1)} + \cdots + \alpha_0(x)u, \\ \alpha_i \in L^1_{loc}[T,+\infty[,\ 0\le i\le n-1, \end{cases} \quad (3.21)$$

acting on the space $AC^{n-1}[T,+\infty[$ *and such that*: $\ker L_n = \operatorname{span}(\phi_1,\cdots,\phi_n)$. *Then a function* $f\in AC^{n-1}[T,+\infty[$ *admits of an asymptotic expansion of type*:

$$f(x) = a_0\phi(x) + a_1\phi'(x) + \cdots + a_{n-1}\phi^{(n-1)}(x) + o\left(\phi^{(n-1)}(x)\right),\ x\to+\infty, \quad (3.22)$$

formally differentiable $n-1$ *times in the sense of* ([1], §6), *provided that*:

$$\int^{+\infty} t^{n-1}\left|(\phi(t))^{-1}\cdot L_n\left[f(t)\right]\right| \mathrm{d}t < +\infty. \quad (3.23)$$

Formal differentiability in the present context refers to the validity of the following n expansions as $x\to+\infty$:

$$\begin{cases} f = a_0\phi + a_1\phi' + \cdots + \phi^{(n-1)}\cdot\left[a_{n-1}+o(1)\right]; \\ (f/\phi)' = a_1(\phi'/\phi)' + \cdots + \left(\phi^{(n-1)}/\phi\right)'\cdot\left[a_{n-1}+o(1)\right]; \\ \left((f/\phi)'/(\phi'/\phi)'\right)' = a_2\left((\phi''/\phi)'/(\phi'/\phi)'\right)' + \cdots \\ \qquad + \left(\left(\phi^{(n-1)}/\phi\right)'/(\phi'/\phi)'\right)'\cdot\left[a_{n-1}+o(1)\right]; \end{cases} \quad (3.24)$$

and so on, dividing both sides of each expansion by the first term (*constant apart*) *in the right-hand side and then differentiating both sides to obtain the next expansion until differentiating* $n-1$ *times.*

For the proof just notice that the hypotheses imply conditions in (3.2) hence relations in (3.3), so that the assumptions in ([1], p. 26) are satisfied, and the ratio of Wronskians, ([1], formula (198), p. 26), in the present context is $H_{n-1}\left[\phi(x)\right]/H_n\left[\phi(x)\right]$.

4. The Approach and Preliminary Examples for Rapidly-Varying Functions

4.1. The Right Approach

Corresponding general results for a rapidly-varying function ϕ of higher order cannot be directly inferred from Theorem 10 in ([1], p. 19). In fact, Proposition 3 in ([1], p. 8) states that the logarithmic derivatives of the functions $\phi,\phi',\cdots,\phi^{(n-1)}$ are asymptotically equivalent to each other, hence we are in the situation of Theorem 10-(IV) in ([1], p. 20) with $c_i = 1 \quad \forall i$, which only grants the "o"-asymptotic estimates:

$$\begin{cases} H_2\left[\phi(x)\right] = o\left(\left(\phi'(x)\right)^2\right), \; x \to +\infty; \\ H_n\left[\phi(x)\right] = o\left(\left(\dfrac{\phi'(x)}{\phi(x)}\right)^{n(n-1)/2} \cdot \displaystyle\prod_{i=0}^{n-1} \phi^{(i)}(x)\right), \; x \to +\infty. \end{cases} \tag{4.1}$$

Instead, referring to formula (1.8) we shall try to use the factorized expression

$$\begin{aligned} H_n\left[\phi(x)\right] &= \left(\phi(x)\right)^n \cdot W\left(\left(\frac{\phi'(x)}{\phi(x)}\right)', \left(\frac{\phi''(x)}{\phi(x)}\right)', \cdots, \left(\frac{\phi^{(n-1)}(x)}{\phi(x)}\right)'\right) \\ &\equiv \left(\phi(x)\right)^n \cdot W\left(\psi_1'(x),\cdots,\psi_{n-1}'(x)\right), \; n \ge 2, \left(\psi_k(x) := \phi^{(k)}(x)/\phi(x)\right), \end{aligned} \tag{4.2}$$

looking for results concerning rapidly-varying functions of type

$$\phi(x) := \exp\left(R(x)\right) \text{ with } R(+\infty) = \pm\infty; \tag{4.3}$$

a notation that simplifies calculations. For such a function we have

$$\phi'(x)/\phi(x) = R'(x); \; H_2\left[\exp\left(R(x)\right)\right] = \exp\left(2R(x)\right) \cdot R''(x); \tag{4.4}$$

whereas, for $n \ge 3$, we resort to Faà Di Bruno's formula (1.40) which now takes on the following form for $k \ge 2$:

$$\frac{\phi^{(k)}(x)}{\phi(x)} = \left(R'(x)\right)^k + \sum_{0 \le i_j \le k;\, i_1 \le k-1}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1,\cdots,i_k} \left(R'(x)\right)^{i_1} \cdot \left(R''(x)\right)^{i_2} \cdots \left(R^{(k)}(x)\right)^{i_k}, \tag{4.5a}$$

where $a_{i_1,\cdots,i_k}$ are suitable positive coefficients and the summation is taken over all possible ordered k-tuples of non-negative integers i_j such that "$i_1 + 2i_2 + \cdots + ki_k = k$". Having isolated the term corresponding to the k-tuple "$i_1 = k$, $i_2 = \cdots = i_k = 0$", it is essential to notice that:

$$\begin{cases} \text{the exponents of the factors into the summation symbol satisfy:} \\ 1 \le i_1 + \cdots + i_k \le k-1. \end{cases} \tag{4.5b}$$

4.2. Worked-Out Examples

To highlight the procedure, we work out four preliminary examples encompassing all the special cases wherein the function $R(x)$ in (4.3) is $cx^\alpha (\log x)^\beta$. *In each example there is a part of text written in italics for a reason explained at the* *end of this section.*

Example 4.1. For the function

$$\phi_1(x) := \exp(cx^\alpha),\ c \neq 0,\ 0 < \alpha \neq 1, \tag{4.6}$$

we have:

$$\begin{cases} R_1(x) := cx^\alpha;\ R_1'(x) = c\alpha x^{\alpha-1}; \\ R_1^{(i)} : \text{either} \equiv 0 \text{ or } = c\alpha^{\underline{i}} x^{\alpha-i},\ \forall i \geq 2; \end{cases} \tag{4.7}$$

and in formula (4.5a) each term into the summation symbol is either $\equiv 0$ or a "non-zero constant" times the pertinent power:

$$x^{i_1(\alpha-1)+i_2(\alpha-2)+\cdots+i_k(\alpha-k)} = x^{\alpha(i_1+\cdots+i_k)-(i_1+2i_2+\cdots+ki_k)}$$
$$= x^{\alpha(i_1+\cdots+i_k)-k} = o\left(x^{k(\alpha-1)}\right),\ x \to +\infty,$$

wherein the last estimate depends on the restrictions in (4.5b) and $\alpha > 0$. By suitably reordering we get an expression of the following type and the consequent asymptotic relation:

$$\begin{aligned} \psi_k(x) &:= \frac{\phi_1^{(k)}(x)}{\phi_1(x)} = (c\alpha)^k x^{k(\alpha-1)} + \sum_{j=1}^{k-1} a_j x^{\alpha j-k} \\ &\sim (c\alpha)^k x^{k(\alpha-1)},\ x \to +\infty,\ (k \geq 1), \end{aligned} \tag{4.8}$$

no matter what the coefficients a_j may be. Hence:

ψ_k is a linear combination of real powers; it is $\not\equiv 0$ and, by Proposition 2.6-(II), *smoothly varying at $+\infty$ of index $k(\alpha-1)$ and any order $n \in \mathbb{N}$. Moreover, by the second line in* (1.26), *condition $k(\alpha-1) \neq 0$ implies that ψ_k' is smoothly varying at $+\infty$ of index "$k(\alpha-1)-1$" and any order $n \in \mathbb{N}$. Obviously the sequence $\{k(\alpha-1)-1\}_k$ is strictly decreasing or increasing according as $\alpha < 1$ or $\alpha > 1$, and applying Lemma* 3.1 *to the Wronskian in* (4.2) *we get the partial result*:

$$\begin{cases} H_n\left[\exp(cx^\alpha)\right] \sim V(\alpha-2, 2\alpha-3, \cdots, (n-1)\alpha-n) \\ \times x^{-(n-1)(n-2)/2} \cdot \left(\exp(ncx^\alpha)\right) \cdot \prod_{k=1}^{n-1} \psi_k'(x),\ x \to +\infty,\ (\alpha \neq 1), \end{cases} \tag{4.9}$$

noticing that for the excluded value $\alpha = 1$ this relation becomes trivial as the Hankelian is ≡0: (which will be highlighted in Part II of this work).

Now, condition "ψ_k' smoothly varying of index $k(\alpha-1)-1$" and (4.7) imply as $x \to +\infty$:

$$\begin{cases} \psi_k'(x) \sim k(\alpha-1)(c\alpha)^k \cdot x^{k(\alpha-1)-1},\ x \to +\infty; \\ \prod_{k=1}^{n-1} \psi_k'(x) \sim (n-1)!(\alpha-1)^{n-1} (c\alpha)^{n(n-1)/2} \cdot x^{1-n+(\alpha-1)n(n-1)/2}; \end{cases} \tag{4.10}$$

whence the final relation:

$$\begin{cases} H_n\left[\exp(cx^\alpha)\right] \sim V(\alpha-2, 2\alpha-3, \cdots, (n-1)\alpha-n) \cdot (n-1)! \\ \qquad \times (\alpha-1)^{n-1} \cdot (c\alpha)^{n(n-1)/2} x^{(\alpha-2)n(n-1)/2} \cdot \exp(ncx^\alpha), \\ x \to +\infty,\ (n \geq 2;\ c \neq 0;\ 0 < \alpha \neq 1); \end{cases} \tag{4.11}$$

and in particular for $n \ge 2$:

$$H_n\left[\exp\left(cx^2\right)\right] \sim \left(\prod_{i=1}^{n-1} i!\right) \cdot (2c)^{n(n-1)/2} \cdot \exp\left(ncx^2\right),\ x \to +\infty. \tag{4.12}$$

Example 4.2. For the function

$$\phi_2(x) := \exp\left(c(\log x)^\beta\right),\ c \ne 0,\ \beta > 1, \tag{4.13}$$

we have:

$$\begin{cases} R_2(x) := c(\log x)^\beta;\ R_2'(x) = c\beta x^{-1}(\log x)^{\beta-1}; \\ R_2^{(i)}(x) = x^{-i}\sum_{1\le j\le i} c_j (\log x)^{\beta-j},\ i \ge 1, \end{cases} \tag{4.14}$$

with suitable coefficients c_j (as trivially checked by induction) and $c_1 = (-1)^{i-1} c\beta$. Hence:

$$R_2^{(i)}(x) \sim (-1)^{i-1} c\beta x^{-i} (\log x)^{\beta-1},\ x \to +\infty, (i \ge 1); \tag{4.15}$$

and each product $\prod_j \left(R_2^{(j)}(x)\right)^{i_j}$ into the summation symbol in (4.5a) is asymptotically equivalent to a "non-zero constant" times the function:

$$x^{-i_1-2i_2-\cdots-ki_k}(\log x)^{(\beta-1)(i_1+\cdots+i_k)} = x^{-k}(\log x)^{(\beta-1)(i_1+\cdots+i_k)},$$

hence it is $o\left(x^{-k}(\log x)^{(\beta-1)k}\right)$, as $i_1 + \cdots + i_k < k$ and $\beta - 1 > 0$. This estimate implies that the leading term in the right-hand side in (4.5a) turns out to be the first one so that:

$$\phi_2^{(k)}(x)/\phi_2(x) \sim \left(R_2'(x)\right)^k = (c\beta)^k x^{-k} (\log x)^{k(\beta-1)}. \tag{4.16}$$

Now, the right-hand side in (4.5a) *is a linear combination of functions of type* $x^{-k}(\log x)^{a_i}$ *so that, after properly grouping and rearranging, it looks like a sum of type*

$$\sum_{i=1}^{m} c_i x^{-k}(\log x)^{a_i},\ \text{where } a_1 > a_2 > \cdots > a_m,\ a_1 = k(\beta-1).$$

and because each term in the sum belongs to the class $\{\mathcal{R}_{-k}(+\infty) \text{ of any order } n\}$ *we infer from both Proposition* 2.6-(III) *and the second line in* (1.26) *that*

$$\begin{cases} \phi_2^{(k)}/\phi_2 \in \{\mathcal{R}_{-k}(+\infty) \text{ of any order } n\}, \\ \left(\phi_2^{(k)}/\phi_2\right)' \in \{\mathcal{R}_{-k-1}(+\infty) \text{ of any order } n\}, \end{cases} \quad \forall k \ge 1, \tag{4.17}$$

which implies:

$$\begin{cases} \left(\phi_2^{(k)}(x)/\phi_2(x)\right)' \sim -kx^{-1}\phi_2^{(k)}(x)/\phi_2(x) \\ \qquad \sim -k(c\beta)^k x^{-k-1}(\log x)^{k(\beta-1)}; \\ \prod_{k=1}^{n-1}\left(\dfrac{\phi_2^{(k)}(x)}{\phi_2(x)}\right)' \sim (-1)^{n-1}(n-1)!(c\beta)^{n(n-1)/2} \\ \qquad \times x^{-(n+2)(n-1)/2} \cdot (\log x)^{(\beta-1)n(n-1)/2}. \end{cases} \tag{4.18}$$

Referring to (3.3), with n replaced by $n-1$, we see that, in this case, $\alpha_k = -k-1$, which is the index of $\left(\phi_2^{(k)}/\phi_2\right)'$, and the Vandermondian is $V(-2,\cdots,-n)$ *i.e.*, by ([1], formula (60), p. 9):

$$\begin{aligned} V(-2,\cdots,-n) &= V(-2,-2-1,\cdots,-2-(n-2)) \\ &= (-1)^{(n-1)(n-2)/2}\left(\prod_{i=0}^{n-2} i!\right), n \geq 2. \end{aligned}$$

By (3.3), (4.2) and (4.18) the exponent of x turns out to be

$$[-(n+2)(n-1)/2]-[(n-1)(n-2)/2] = -n(n-1),$$

and the final relation is:

$$\begin{cases} H_n\left[\exp\left(c(\log x)^\beta\right)\right] \sim (-1)^{n(n-1)/2}\left(\prod_{i=1}^{n-1} i!\right)(c\beta)^{n(n-1)/2} \\ \qquad \times x^{-n(n-1)} \cdot (\log x)^{(\beta-1)n(n-1)/2} \cdot \exp\left[nc(\log x)^\beta\right], \\ x \to +\infty, (n \geq 2; c \neq 0; \beta > 1); \end{cases} \tag{4.19}$$

which is checked at once for $n=2$. The reader will notice that for $0<\beta<1$ and $n \geq 3$ the principal part, given by relation (3.18), has a different expression. As a matter of fact the function ϕ_2 is slowly varying for $0<\beta<1$ and rapidly varying for $\beta>1$.

Example 4.3. For the function

$$\phi_3(x) := \exp\left(cx^\alpha(\log x)^\beta\right), c \neq 0, \beta \neq 0, 0 < \alpha \neq 1, \tag{4.20}$$

we have:

$$\begin{cases} R_3(x) := cx^\alpha(\log x)^\beta; R_3'(x) = c\alpha x^{\alpha-1}(\log x)^\beta + c\beta x^{\alpha-1}(\log x)^{\beta-1}; \\ R_3^{(i)}(x) = x^{\alpha-i}\sum_{0\leq j\leq i} c_j(\log x)^{\beta-j}, i \geq 1, \end{cases} \tag{4.21}$$

with suitable coefficients c_j and $c_0 = c\alpha^{\underline{i}} \neq 0$. Hence:

$$R_3^{(i)}(x) \sim c\alpha^{\underline{i}} x^{\alpha-i}(\log x)^\beta, x \to +\infty, (i \geq 1); \tag{4.22}$$

and each product $\prod_j\left(R_3^{(j)}(x)\right)^{i_j}$ into the summation symbol in (4.5a) is asymptotically equivalent to a "non-zero constant" times the quantity

$$x^{(\alpha-1)i_1+(\alpha-2)i_2+\cdots+(\alpha-k)i_k}(\log x)^{\beta(i_1+\cdots+i_k)} = x^{\alpha(i_1+\cdots+i_k)-k}(\log x)^{\beta(i_1+\cdots+i_k)}. \tag{4.23}$$

By (4.5b) and taking account that β may also be negative whereas $\alpha>0$, we estimate the last quantity as follows:

$$\begin{aligned} &x^{\alpha(i_1+\cdots+i_k)-k}(\log x)^{\beta(i_1+\cdots+i_k)} \ll x^{\alpha(k-1)-k}(\log x)^{\beta(i_1+\cdots+i_k)} \\ &= x^{k(\alpha-1)}\left[x^{-\alpha}(\log x)^{\beta(i_1+\cdots+i_k)}\right] \\ &\ll x^{k(\alpha-1)}(\log x)^{k\beta}, x \to +\infty, \text{ whatever } \beta \neq 0, \end{aligned} \tag{4.24}$$

and, as in Examples 4.1-4.2, we conclude that the leading term in the right-hand side in (4.5a) is the first one:

$$\phi_3^{(k)}(x)/\phi_3(x) \sim (R_3'(x))^k \sim (c\alpha)^k x^{k(\alpha-1)} (\log x)^{k\beta},\ x \to +\infty. \tag{4.25}$$

Now the same reasoning as in the preceding example, using Proposition 2.6-(III) *and the second line in* (1.26), *shows that, under the restriction* "$k(\alpha-1) \neq 0$":

$$\begin{cases} \phi_3^{(k)}/\phi_3 \in \{\mathcal{R}_{k(\alpha-1)}(+\infty) \text{ of any order } n\} \\ (\phi_3^{(k)}/\phi_3)' \in \{\mathcal{R}_{k(\alpha-1)-1}(+\infty) \text{ of any order } n\} \end{cases},\quad \forall k \geq 1. \tag{4.26}$$

But in this special case, the algebraic structure of the terms in the right-hand side in (4.5a) *makes applicable Proposition* 2.2 *so that relations in* (4.26) *hold for each* α *as in* (4.20). *Hence, with no additional restriction on* α *we get*:

$$\begin{cases} \left(\phi_3^{(k)}(x)/\phi_3(x)\right)' \sim k(\alpha-1)\cdot x^{-1}\phi_3^{(k)}(x)/\phi_3(x) \sim k(\alpha-1)(c\alpha)^k \cdot x^{k(\alpha-1)-1}\cdot(\log x)^{k\beta}; \\ \prod_{k=1}^{n-1}\left(\dfrac{\phi_3^{(k)}(x)}{\phi_3(x)}\right)' \sim (\alpha-1)^{n-1}(n-1)!(c\alpha)^{n(n-1)/2}\cdot x^{(n-1)[(\alpha-1)n-1]/2}\cdot(\log x)^{\beta n(n-1)/2}. \end{cases} \tag{4.27}$$

The Vandermondian is

$$\begin{aligned} &V((\alpha-1)-1, 2(\alpha-1)-1, \cdots, (n-1)(\alpha-1)-1) \\ &= V(\alpha-2, 2\alpha-3, \cdots, (n-1)\alpha-n) \equiv V_\alpha; \end{aligned} \tag{4.28}$$

the exponent of x is

$$[(n-1)[(\alpha-1)n-1]-(n-1)(n-2)]/2 = (n-1)[(\alpha-2)n+1]/2;$$

and the final relation is:

$$\begin{cases} H_n\left[\exp\left(cx^\alpha(\log x)^\beta\right)\right] \sim V_\alpha\cdot(n-1)!(\alpha-1)^{n-1}(c\alpha)^{n(n-1)/2} \\ \times x^{(n-1)[(\alpha-2)n+1]/2}\cdot(\log x)^{\beta n(n-1)/2}\cdot\exp\left(ncx^\alpha(\log x)^\beta\right),\ x\to+\infty, \\ (n\geq 2; c\neq 0; \beta\neq 0; 0<\alpha\neq 1);\ \text{with } V_\alpha \text{ defined in } (4.28). \end{cases} \tag{4.29}$$

4.3. A Last More Complicated Example

Example 4.4. The exceptional case $\alpha = 1$ in (4.20). For the function

$$\phi_4(x) := \exp(cx(\log x))^\beta,\ c\neq 0,\ \beta\neq 0, \tag{4.30}$$

we infer from (4.14) the formulas:

$$\begin{cases} R_4(x) := cx(\log x)^\beta;\ R_4'(x) = c(\log x)^\beta + c\beta(\log x)^{\beta-1} \sim c(\log x)^\beta; \\ R_4^{(i)}(x) = x^{1-i}\sum_{1\leq j\leq i} c_j(\log x)^{\beta-j},\ i\geq 2, \end{cases} \tag{4.31}$$

with suitable coefficients c_j and, by Lemma 2.1:

$$R_4^{(i)}(x) \sim (-1)^i(i-2)!c\beta x^{1-i}(\log x)^{\beta-1},\ x\to+\infty; (i\geq 2). \tag{4.32}$$

Each product $\prod_j \left(R_4^{(j)}(x)\right)^{i_j}$ into the summation symbol in (4.5a) is asymptotically equivalent to a "non-zero constant" times the quantity

$$x^{-i_2-2i_3-\cdots-(k-1)i_k}\left(\log x\right)^{\beta i_1+(\beta-1)(i_2+\cdots+i_k)}=x^{i_1+\cdots+i_k-k}\left(\log x\right)^{\beta i_1+(\beta-1)(i_2+\cdots+i_k)}, \quad (4.33)$$

and, as "$i_1+\cdots+i_k-k\le -1$" by (4.5b), we conclude as in (4.25) that:

$$\phi_4^{(k)}(x)/\phi_4(x)\sim\left(R_4'(x)\right)^k\sim c^k\left(\log x\right)^{k\beta},\ x\to+\infty. \quad (4.34)$$

Moreover, after proper grouping and rearranging as in the Example 4.2, *the right-hand side in* (4.5b) *becomes a sum of type*

$$\left(R_4'(x)\right)^k+\sum_{\cdots}x^{i_1+\cdots+i_k-k}\sum_i c_i\left(\log x\right)^{a_i},$$

wherein, by its very algebraic structure, the first term belongs to the class $\left\{\mathcal{R}_0(+\infty)\text{ of any order }n\right\}$ *with its i-th derivative in the class* $\left\{\mathcal{R}_{-i}(+\infty)\text{ of any order }n\right\}$ *whereas each other single term is in the class* $\left\{\mathcal{R}_{i_1+\cdots+i_k-k}(+\infty)\text{ of any order }n\right\}$ *with its i-th derivative in the class* $\left\{\mathcal{R}_{i_1+\cdots+i_k-k-1}(+\infty)\text{ of any order }n\right\}$. *Hence*:

$$\begin{cases}\phi_4^{(k)}/\phi_4\in\left\{\mathcal{R}_0(+\infty)\text{ of any order }n\right\},\\ \left(\phi_4^{(k)}/\phi_4\right)'\in\left\{\mathcal{R}_{-1}(+\infty)\text{ of any order }n\right\},\ i\ge 1,\end{cases} \quad (4.35)$$

and in this case all the indexes of the entries in the Wronskian in (4.2) *coincide so preventing the direct application of Lemma* 3.1.

Some devices are now appropriate. For each k the quantity $\phi_4^{(k)}(x)/\phi_4(x)$ has an expression of type

$$\begin{aligned}\phi_4^{(k)}(x)/\phi_4(x)&=c^k\left(\log x\right)^{k\beta}+\sum_{i=1}^{m}c_{0,i}\left(\log x\right)^{a_i}+x^{-1}\sum_{i=1}^{m_1}c_{1,i}\left(\log x\right)^{a_{1,i}}+\cdots\\&\quad+x^{1-k}\sum_{i=1}^{m_{k-1}}c_{k-1,i}\left(\log x\right)^{a_{k-1,i}},\ \left(k\beta>a_1>a_2>\cdots>a_m\right),\end{aligned} \quad (4.36)$$

whence

$$\begin{aligned}\left(\phi_4^{(k)}(x)/\phi_4(x)\right)'&=c^kk\beta x^{-1}\left(\log x\right)^{k\beta-1}+x^{-1}\sum_{i=1}^{m}d_i\left(\log x\right)^{a_i-1}\\&\quad+x^{-2}\sum_i d_{1,i}\left(\log x\right)^{b_{1,i}}+\cdots+x^{-k}\sum_i d_{k,i}\left(\log x\right)^{b_{k,i}}\\&\equiv\beta x^{-1}\left(\log x\right)^{-1}f_k(x),\end{aligned} \quad (4.37)$$

where

$$\begin{aligned}f_k(x)&:=kc^k\left(\log x\right)^{k\beta}+\sum_{i=1}^{m}\tilde d_i\left(\log x\right)^{a_i}+x^{-1}\sum_i\tilde d_{1,i}\left(\log x\right)^{\tilde b_{1,i}}\\&\quad+\cdots+x^{-k+1}\sum_i\tilde d_{k,i}\left(\log x\right)^{\tilde b_{k,i}},\ \left(k\beta>a_1>a_2>\cdots>a_m\right),\end{aligned} \quad (4.38)$$

with certain coefficients $c_{j,i},a_i,a_{j,i},d_i,d_{j,i},b_{j,i},\tilde d_i,\tilde d_{j,i},\tilde b_{j,i}$ depending on k. Using (1.38) we get from (4.2):

$$\begin{aligned}H_n\left[\exp\left(cx\left(\log x\right)^{\beta}\right)\right]&=\beta^{n-1}\cdot\exp\left(cnx\left(\log x\right)^{\beta}\right)\cdot\left(x\log x\right)^{1-n}\\&\quad\times W\left(f_1(x),\cdots,f_{n-1}(x)\right).\end{aligned} \quad (4.39)$$

In the last Wronskian we make the change of variable $y = \log x$, a device highlighted in ([2], §4), so getting, by (1.39):

$$\begin{cases} u_k(y) := kc^k y^{k\beta} + \sum_{i=1}^{m} \tilde{d}_i y^{a_i} + \mathrm{e}^{-y} \sum_i \tilde{d}_{1,i} y^{\tilde{b}_{1,i}} + \cdots + \mathrm{e}^{-(k-1)y} \sum_i \tilde{d}_{k,i} y^{\tilde{b}_{k,i}} \\ \qquad \sim kc^k y^{k\beta},\ y \to +\infty, (k\beta > a_1 > a_2 > \cdots > a_m); \\ f_k(x) := u_k(\log x); \\ W(f_1(x), \cdots, f_{n-1}(x)) \equiv W(u_1(\log x), \cdots, u_{n-1}(\log x)) \\ \qquad = x^{-(n-1)(n-2)/2} \cdot \left[W(u_1(y), \cdots, u_{n-1}(y))\right]_{y=\log x}. \end{cases} \tag{4.40}$$

Now Proposition 2.8-(III) implies that "$u_k \in \{\mathcal{SR}_{k\beta}(+\infty)$ of any order $n\}$" for each $k \in \mathbb{N}$, hence Lemma 3.1 applies and

$$\begin{cases} W(u_1(y), \cdots, u_{n-1}(y)) \\ \sim V(\beta, 2\beta, \cdots, (n-1)\beta) \cdot (n-1)!\, c^{n(n-1)/2} \times y^{\beta + 2\beta + \cdots + (n-1)\beta - [(n-1)(n-2)]/2} \\ = \beta^{(n-1)(n-2)/2} \cdot \left(\prod_{i=0}^{n-1} i! \right) \cdot c^{n(n-1)/2} \cdot y^{(n-1)[(\beta-1)n+2]/2},\ y \to +\infty. \end{cases} \tag{4.41}$$

Coming back to (4.40) and (4.39), we get the final relation:

$$\begin{cases} H_n\left[\exp\left(cx(\log x)^\beta\right)\right] \\ \sim c^{n(n-1)/2} \cdot \beta^{n(n-1)/2} \cdot \left(\prod_{i=0}^{n-1} i! \right) \times x^{-n(n-1)/2} \cdot (\log x)^{n(n-1)(\beta-1)/2} \cdot \exp\left(cnx(\log x)^\beta\right), \\ x \to +\infty, (c, \beta \neq 0; n \geq 2). \end{cases} \tag{4.42}$$

Concluding comments on the examples. In each of the preceding examples the text in italics contains a direct elementary argument leading to the existence and evaluation of the index of variation for the linear combination generated by the sum in (4.5a) and this is not, in general, an easy task. In examples 4.2, 4.3 we had resort to a special elementary result without being involved in direct calculations. Example 4.4 required the additional step of changing the variable. The whole matter reported in §2 provides the appropriate means for obtaining general results avoiding, in addition, unwelcome restrictions on exceptional values of some parameters.

5. Procedure and Results for the Exponential of a Regularly-Varying Function

5.1. Sketch of the Procedure

We are going to investigate the asymptotic behavior of the Hankelians of a function of type in (4.3) with R regularly varying assuming, more precisely, that:

$$\phi(x) := \exp(R(x)) \text{ where } \left|R^{(k)}\right| \in \mathcal{R}_{\gamma-k}(+\infty), (\gamma > 0; 0 \leq k \leq 2n-3); \tag{5.1}$$

which means that R is regularly varying of order $2n-2$ with well-specified indexes of variation for the involved derivatives. The positivity of the index γ grants the hypothesis "$|R(+\infty)| = +\infty$", essential in the subsequent calculations.

We need to know the indexes of the entries in the Wronskian in (4.2) guessing that they may be regularly varying. Direct calculations for the first few derivatives give:

$$\phi'/\phi = R' \in \mathcal{R}_{\gamma-1}(+\infty) \text{ and } (\phi'/\phi)' = R'' \in \mathcal{R}_{\gamma-2}(+\infty) \text{ by hypothesis;} \tag{5.2}$$

$$\begin{cases} \phi''/\phi = (R')^2 + R'' \in \mathcal{R}_{2(\gamma-1)}(+\infty), \\ (\phi''/\phi)' = 2R'R'' + R''' \in \mathcal{R}_{2\gamma-3}(+\infty), \end{cases} \tag{5.3}$$

because each of the terms in the expressions has a different index of variation, ([3], Prop. 2.1, formula (2.27));

$$\phi'''/\phi = (R')^3 + 3R'R'' + R''' \in \mathcal{R}_{3\gamma-3}(+\infty), \text{ for the same reason.} \tag{5.4}$$

But no immediate conclusion can be drawn for the expression

$$(\phi'''/\phi)' = 3(R')^2 R'' + 3(R'')^2 + 3R'R''' + R^{(4)} \in ??$$

because the second and third terms have the same index of variation, $2\gamma - 4$, and not necessarily the same signs so that it is not legitimate to automatically infer that their sum has an index of variation at all. Similar situations occur for the other derivatives which are polynomials in $R', R'', \cdots, R^{(i)}$. Proposition 2.6 provides a key to get a general result. *For the sake of immediacy in reading the relations in* (5.2)-(5.4), *we omitted the absolute values as required by* (1.19) *and wrote down the essential information*: *the first relation in* (5.2) *replaces the correct relation* $|R'| \in \mathcal{R}_{\gamma-1}(+\infty)$ *and, in the present context, even the more complete information* $R' \in \{\mathcal{R}_{\gamma-1}(+\infty) \text{ of order } 2n-3\}$. *Similar shortened notations will be used in this section.*

A sketch of the adopted procedure for the proofs in this section.

We use the shortened notation in (4.2), $\psi_k(x) := \phi^{(k)}(x)/\phi(x)$.

First step. Direct estimates of each quantity in (4.5a), $R^{(i)}(x)$, show that $\psi_k(x) \sim (R'(x))^k$; and an application of Proposition 2.6-(II) implies that $\psi_k \in \mathcal{SR}_{k(\gamma-1)}(+\infty)$ of a suitable order.

Second (immediate) step. The further relation $\psi_k' \in \mathcal{SR}_{k(\gamma-1)-1}(+\infty)$ follows at once from the second inference in (1.26) with no additional restrictions on the values of γ: no direct calculations are needed.

Third step. The principal parts of ψ_k' and their indexes of variation allow application of Theorem 3.2-(I) so getting the final asymptotic relation.

5.2. Two Main Results

The two results in this subsection involve the function ϕ in (5.1) with $\gamma > 0$, and also with $\gamma = 0$ under a special restriction. The more complicated case $\gamma = 1$ will be the last to be studied in this section.

Theorem 5.1. (The case $\gamma > 0$). *For the function* ϕ *in* (5.1) *with the restrictions* "$\gamma > 0$, $\gamma \neq 1$" *the following relation holds true*:

$$\begin{cases} H_n\left[\exp\left(R(x)\right)\right] \sim V\left(\gamma-2, 2\gamma-3, \cdots, (n-1)\gamma-n\right)\cdot(n-1)!\gamma^{n(n-1)/2} \\ \times(\gamma-1)^{n-1}\cdot x^{-n(n-1)}\cdot\left(R(x)\right)^{n(n-1)/2}\cdot\exp\left(nR(x)\right),\ x\to+\infty, (n\ge 2). \end{cases} \tag{5.5}$$

Proof. As in Example 4.1 we first show that the index of smooth variation of $\phi^{(k)}/\phi$ is that of the first term in (4.5a). From relation (1.23) with f replaced by R' we get:

$$R^{(i)}(x) = O\left(x^{-i+1}R'(x)\right),\ x\to+\infty,\ 2\le i\le 2n-2; \tag{5.6}$$

with no further restriction on γ. These relations imply that for each term into the summation symbol in (4.5a) we have the estimate:

$$\begin{cases} \left(R'(x)\right)^{i_1}\cdot\left(R''(x)\right)^{i_2}\cdots\left(R^{(k)}(x)\right)^{i_k} \\ = O\left(x^{-i_2-2i_3-\cdots-(k-1)i_k}\left(R'(x)\right)^{i_1+\cdots+i_k}\right) \\ = O\left(x^{(i_2+i_3+\cdots+i_k)-(2i_2+3i_3+\cdots+ki_k)}\left(R'(x)\right)^{i_1+\cdots+i_k}\right) \\ = O\left(x^{i_1+i_2+\cdots+i_k-k}\left(R'(x)\right)^{i_1+\cdots+i_k}\right). \end{cases} \tag{5.7}$$

Here the situation is similar to that of estimating the first quantity in (4.24), Example 4.3, for arbitrary values of the parameter $\beta\ne 0$ in that example. In the present case one may write:

$$\begin{cases} O\left(x^{i_1+i_2+\cdots+i_k-k}\left(R'(x)\right)^{i_1+\cdots+i_k}\right) \equiv O\left(x^{-k}\left(xR'(x)\right)^{i_1+\ldots+i_k}\right) = \\ \cdots \text{by the hypotheses "}xR'(x)\sim\gamma R(x)=\pm\infty(1)\text{" and "}i_1+\cdots+i_k<k\text{"} \cdots \\ = o\left(x^{-k}\left(xR'(x)\right)^k\right) = o\left(\left(R'(x)\right)^k\right); \end{cases} \tag{5.8}$$

so inferring that

$$\begin{cases} \psi_k(x) := \phi^{(k)}(x)/\phi(x) \sim \left(R'(x)\right)^k \\ \cdots \text{by the assumption } |R|\in\mathcal{R}_\gamma(+\infty)\cdots \\ \sim \gamma^k x^{-k}\left(R(x)\right)^k,\ x\to+\infty, (1\le k\le n-1). \end{cases} \tag{5.9}$$

(We point out that the last asymptotic relation in (5.9) may be inferred from the relations in (1.31) and a general result in ([4], Prop. 7.6-(I), p. 827) which last implies that ϕ is rapidly varying of order $2n-2$. The above direct estimates may be appreciated by the reader, as they give a more convincing proof of the not-so-elementary results we are exhibiting. Moreover, we shall refer to the estimates in (5.8) in the sequel.)

Now, thanks to Proposition 2.6-(II) applied to representation (4.5a) and with no further restriction on γ, we infer that the function ψ_k belongs to the same class of smooth variation of $\left(R'(x)\right)^k$, namely

$$\psi_k \in \left\{\mathcal{SR}_{k(\gamma-1)}(+\infty)\ \textit{of order}\ 2n-3\right\}, (1\le k\le n-1), \tag{5.10}$$

and, by the second inference in (1.26):

$$\psi_k' \in \mathcal{R}_{k(\gamma-1)-1}(+\infty), \tag{5.11}$$

which is the needed piece of information. Moreover:

$$\begin{cases} \psi_k'(x) \overset{\text{by }(5.10)}{\sim} k(\gamma-1)x^{-1}\psi_k(x) \\ \qquad\quad \overset{\text{by }(5.9)}{\sim} k\gamma^k(\gamma-1)x^{-k-1}(R(x))^k, \; x\to+\infty, (1\le k\le n-1). \end{cases} \tag{5.12}$$

The situation is similar to that in Example 4.1 with α replaced by γ, hence the arguments of the Wronskian in (4.2) have indexes of smooth variation $[k(\gamma-1)-1]$, $1\le k\le n-1$, and

$$\prod_{k=1}^{n-1}\psi_k'(x) \sim (n-1)!\gamma^{n(n-1)/2}(\gamma-1)^{n-1}\cdot x^{-(n+2)(n-1)/2}\cdot(R(x))^{n(n-1)/2}. \tag{5.13}$$

Relation (3.4) applied to (4.2) yields the final relation in (5.5) noticing that the number of entries in our determinat is $n-1$ and that the Vandermondian is the same as in (4.9). □

Before studying the case $\gamma=0$ let us see what happens for $n=2$. The explicit expression of the Hankelian in (1.9), "$H_2[\exp(R(x))] = R''(x)\cdot\exp(2R(x))$", and the regular variations of $|R|\in\mathcal{R}_\gamma(+\infty)$ and $|R'|\in\mathcal{R}_{\gamma-1}(+\infty)$ yield as $x\to+\infty$:

$$H_2[\exp(R(x))] \sim \begin{cases} \gamma(\gamma-1)x^{-2}R(x)\cdot\exp(2R(x)), & \text{if } \gamma>0, \gamma\ne1; \\ (\gamma-1)x^{-1}R'(x)\cdot\exp(2R(x)), & \text{if } \gamma\ge0, \gamma\ne1; \end{cases} \tag{5.14}$$

and these make evident a difference between the two cases. For $\gamma=0$ the principal part must be expressed in terms of R' because relation "$R''(x)\sim -x^{-1}R'(x) = o(x^{-2}R(x))$" would only give a *o*-relation, whereas for $\gamma>0$ the principal part may be expressed in terms of either R or R'. Relation (5.5) may be expressed in terms of R' as well using relation "$R(x)\sim(1/\gamma)xR'(x)$".

The excluded case $\gamma=0$ in Theorem 5.1 can be treated exploiting the preceding proof and using properties of regular variation of R' plus an explicit condition on the growth of $xR'(x)$.

Theorem 5.2. (The case $\gamma=0$). *Under the following conditions*:

$$\left|R^{(k)}\right|\in\mathcal{R}_{-k}(+\infty), (1\le k\le 2n-3); \; xR'(x) = \pm\infty(1), \; x\to+\infty; \tag{5.15}$$

we have the relation:

$$H_n[\exp(R(x))] \sim (-1)^{n(n-1)/2}\left(\prod_{i=0}^{n-1} i!\right)\cdot x^{-n(n-1)/2} \times (R'(x))^{n(n-1)/2}\cdot\exp(nR(x)), \; x\to+\infty, (n\ge2). \tag{5.16}$$

(See Remark 2 after the proof for the missing assumption on R.)

Proof. The previous calculations remain essentially valid as the function R is in fact absent in our calculations. In (5.8) we needed the indirectly-inferred condition "$xR'(x)\gg1$" which is now explicitly assumed. Using also (1.20) and (1.24) we get:

$$\begin{cases} \psi_k \in \{\mathcal{SR}_{-k}(+\infty) \text{ of order } 2n-2\}, \\ \psi_k' \in \{\mathcal{SR}_{-k-1}(+\infty) \text{ of order } 2n-3\}, \ (1 \le k \le n-1); \\ \psi_k'(x) \sim -kx^{-1}(R'(x))^k, \ x \to +\infty, \end{cases} \tag{5.17}$$

$$\prod_{k=1}^{n-1} \psi_k'(x) \sim (-1)^{n-1}(n-1)!x^{1-n} \cdot (R'(x))^{n(n-1)/2}; \tag{5.18}$$

$$\begin{cases} V(-2,-3,\cdots,-n) = (-1)^{(n-1)(n-2)/2} V(0,1,\cdots,n-2) \\ \qquad = (-1)^{(n-1)(n-2)/2} \prod_{i=0}^{n-2} i!, \end{cases} \tag{5.19}$$

by the formulas in ([1], Lem. 4 , p. 9). And (5.16) follows by applying (3.4) to (4.2). □

Remarks. Some explanations may be useful to properly understand assumptions and results in the above two theorems.

1) Example 4.1 is included in Theorem 5.1 as is the more general function in (1.35) with $0 < \alpha \ne 1$; and Example 4.2 is included in Theorem 5.2 as is the more general function:

$$\exp\left[\sum_{i=1}^{p} c_i (\log x)^{\beta_i}\right], \ c_i \ne 0, \ \beta_1 > 1, \ \beta_i \in \mathbb{R} \ (2 \le i \le p). \tag{5.20}$$

2) Condition "$xR'(x) \gg 1$, $x \to +\infty$," has been used in the calculations in §4, including the examples, and in the proofs of Theorems 5.1-5.2. Concerning Theorem 5.2, this condition obviously implies "$\int^{+\infty} |R'| = +\infty$", and by L'Hospital's rule:

$$\lim_{x\to+\infty} \frac{xR'(x)}{\int_T^x R'} \overset{\text{H}}{=} \lim_{x\to+\infty} \left[1 + \frac{xR''(x)}{R'(x)}\right] = 0 \ \text{ as } R''(x) \sim -x^{-1}R'(x);$$

which means that the function "$\left|\int_T^x R'\right| \in \mathcal{R}_0(+\infty)$" for an arbitrarily-fixed T large enough; hence $|R| \in \mathcal{R}_0(+\infty)$ as well, ([3], Prop. 2.4-(I)).

On the contrary, a condition such as, e.g., "$xR'(x) = c + o(1)$, with $c \in \mathbb{R}$" would imply "$\exp(R(x)) \in \mathcal{R}_c(+\infty)$", the case treated in §3.

3) Regarding the first condition in (5.15) notice that condition "$R' \in \mathcal{R}_{-1}(+\infty)$" is the only possible in the present context in the sense that condition "$R' \in \mathcal{R}_{\gamma_1}(+\infty)$ with $\gamma_1 > -1$" is inconsistent with condition "$R \in \mathcal{R}_0(+\infty)$" by ([3], Prop. 2.6-(I)), whereas condition "$R' \in \mathcal{R}_{\gamma_1}(+\infty)$ with $\gamma_1 < -1$" is inconsistent with $xR'(x) \gg 1$ by ([3], property in (2.19)).

5.3 A Related Result

The following result is the simplest case of an "asymptotic factorization" involving Hankelians. It has a theoretical interest in the present context and will be included in a larger class of results in Part II.

Proposition 5.3. *For the function*

$$\psi(x) := P(x)\exp(R(x)) \text{ where } \begin{cases} \left|P^{(k)}\right| \in \mathcal{R}_{\alpha-k}(+\infty),\ 0 \le k \le 2n-3; \\ \left|R^{(k)}\right| \in \mathcal{R}_{\gamma-k}(+\infty),\ 0 \le k \le 2n-3; \\ R \text{ as in } (5.1);\ \gamma > 0,\ \gamma \ne 1; \\ \text{no restriction on } \alpha, \end{cases} \quad (5.21)$$

the following relation holds true:

$$H_n\left[P(x)\exp(R(x))\right] \sim (P(x))^n \cdot H_n\left[\exp(R(x))\right],\ x \to +\infty,\ (n \ge 2). \quad (5.22)$$

This same relation holds true if in (5.21) *we have* $\gamma = 0$ *under the additional restriction* "$xR'(x) \gg 1$" *as in* (5.15).

(Asymptotic relations of the kind in (5.22), with a factorized principal part, are those that will be interpreted and studied in Part II of the present work, §9.4.)

First (*direct*) *proof for any* α. We put $\psi(x) := P(x)\phi(x)$ with ϕ as in Theorem 5.1; for the ratio $P^{(i)}/P$ we have the estimates in (1.23)

$$P^{(i)}(x)/P(x) = O\left(x^{-i}\right),\ x \to +\infty; \quad (5.23)$$

and for ϕ we have all the estimates in the proof of Theorem 5.1 taking account of the essential condition $\left|xR'(x)\right| = +\infty(1)$. Now:

$$\frac{\psi'(x)}{\psi(x)} = \frac{\phi'(x)}{\phi(x)} + \frac{P'(x)}{P(x)} \sim \frac{\phi'(x)}{\phi(x)} \equiv R'(x),\ \text{because} \begin{cases} P'(x)/P(x) = O\left(x^{-1}\right), \\ R'(x) \sim \gamma x^{-1} R(x) \gg x^{-1}; \end{cases} \quad (5.24)$$

$$\begin{aligned} \frac{\psi^{(k)}(x)}{\psi(x)} &= \frac{\phi^{(k)}(x)}{\phi(x)} + \sum_{i=1}^{k}\binom{k}{i}\frac{P^{(i)}(x)}{P(x)}\cdot\frac{\phi^{(k-i)}(x)}{\phi(x)} \\ &\cdots \text{by } (5.9) \text{ and } (5.23) \cdots \\ &= \frac{\phi^{(k)}(x)}{\phi(x)} + \sum_{i=1}^{k} O\left(x^{-i}\left(R'(x)\right)^{k-i}\right) \\ &= \frac{\phi^{(k)}(x)}{\phi(x)} + \left(R'(x)\right)^k \sum_{i=1}^{k} O\left(\left(xR'(x)\right)^{-i}\right) \\ &= \frac{\phi^{(k)}(x)}{\phi(x)} + \left(R'(x)\right)^k \cdot o(1) \sim \frac{\phi^{(k)}(x)}{\phi(x)} \sim \left(R'(x)\right)^k,\ (k \ge 2); \end{aligned} \quad (5.25)$$

and in the same way as for (5.10) we conclude that:

$$\psi^{(k)}/\psi \in \left\{\mathcal{SR}_{k(\gamma-1)}(+\infty) \text{ of order } 2n-3\right\},\ (1 \le k \le n-1). \quad (5.26)$$

Hence the indexes of regular variation coincide with those of $\left(\phi^{(k)}/\phi\right)'$ and we have the asymptotic equivalences in (5.24)-(5.25), so that relation (3.4) implies:

$$\begin{aligned} &W\left(\left(\frac{\psi'(x)}{\psi(x)}\right)', \left(\frac{\psi''(x)}{\psi(x)}\right)', \ldots, \left(\frac{\psi^{(n-1)}(x)}{\psi(x)}\right)'\right) \\ &\sim W\left(\left(\frac{\phi'(x)}{\phi(x)}\right)', \left(\frac{\phi''(x)}{\phi(x)}\right)', \ldots, \left(\frac{\phi^{(n-1)}(x)}{\phi(x)}\right)'\right),\ x \to +\infty; \end{aligned} \quad (5.27)$$

and according to (4.2) we have only to add the factor $\left(P(x)\right)^n$ in the right-hand side of (5.5) so getting (5.22). For $\gamma=0$ the restriction $R'(x)\gg x^{-1}$ is needed in (5.24) and the remaining calculations are still valid.

Remark: A possible second (indirect) proof. Some reader might observe: Why not write $\psi(x)=\exp\left(R(x)+\log P(x)\right)$ and work on this expression? Cannot we easily infer from known results that

$$\left|\left(R+\log P\right)^{(k)}\right| \in \mathcal{R}_{\gamma-k}(+\infty) \text{ for all the admissible values of } k, \tag{5.28a}$$

whereas relation

$$\log P(x)=o\left(R(x)\right),\ x\to+\infty, \tag{5.28b}$$

trivially follows from "$\log P\in\mathcal{R}_0(+\infty)$" and "$R\in\mathcal{R}_\gamma(+\infty)$ with $\gamma>0$"? The situation is as follows.

First: the sign of $P(x)$ is no problem because it is strict for x large enough and we may assume $P(x)>0$, while writing $P(x)=-\left|P(x)\right|$ and using (1.38) in the other case; hence $\log P(x)$ becomes meaningful.

Second: $\alpha>0$ implies $P(+\infty)=+\infty$ and we may study the composition $\log P$ having recourse to ([4], Prop. 7.5-(III)) and noticing that, in this case, the quantity "$\alpha_1\cdot\beta+\beta_1$" appearing in ([4], formula (7.41)) equals "$(-1)\alpha+\alpha-1=-1$" so that the cited proposition yields $\left|\left(\log P\right)^{(k)}\right|\in\mathcal{R}_{-k}(+\infty)$. Together with the assumption on R, namely $\left|R^{(k)}\right|\in\mathcal{R}_{\gamma-k}(+\infty)$, $\gamma>0$, we get relations in (5.28a) by the elementary case $n=1$ of Lemma 2.5. Relations in (5.28a), in turn, allow applying Theorem 5.1 to the function $\exp\left(R+\log P\right)$ and (5.5) yields (5.22) using the relation in (5.28b).

For the remaining cases $\alpha\le 0$ relations in (5.28a) require direct proofs not reported in previous papers and, for this reason, we opted for the direct proof valid for all cases.

Example 5.1. From (4.11) with α replaced by γ, and from (5.5) we get:

$$\begin{cases} H_n\left[x^\alpha \exp\left(cx^\gamma\right)\right] \sim V_\gamma\cdot(n-1)!(\gamma-1)^{n-1}\cdot(c\gamma)^{n(n-1)/2} \\ \qquad\qquad \times x^{n\alpha+(\gamma-2)n(n-1)/2}\cdot\exp\left(ncx^\gamma\right),\ x\to+\infty, \\ \left(n\ge 2;\ c\ne 0;\ 0<\gamma\ne 1;\ \alpha\in\mathbb{R}\right), \end{cases} \tag{5.29a}$$

where

$$V_\gamma := V\left(\gamma-2, 2\gamma-3,\cdots,(n-1)\gamma-n\right). \tag{5.29b}$$

For $\gamma=1$ there is a closed formula to be reported in Part II of the present work.

Example 5.2. Analogously, from (4.29) with α replaced by γ, and from (5.5) we get:

$$\begin{cases} H_n\left[x^\alpha \exp\left(cx^\gamma (\log x)^\beta\right)\right] \sim V_\gamma \cdot (n-1)!(\gamma-1)^{n-1}(c\gamma)^{n(n-1)/2} \\ \times x^{n(n-1)(\gamma-2)/2} \cdot (\log x)^{\beta n(n-1)/2} \cdot \exp\left(ncx^\gamma (\log x)^\beta\right),\ x \to +\infty, \\ \left(n \ge 2;\ c \ne 0;\ \beta \ne 0;\ 0 < \gamma \ne 1;\ \alpha \in \mathbb{R}\right). \end{cases} \tag{5.30}$$

5.4. The Exceptional Case $\gamma = 1$: Preliminaries

For the case $\gamma = 1$ in Theorem 5.1 and under the assumptions on R, R' in (5.1), no a-priori asymptotic relation may be anticipated as it depends on the type of asymptotic variation of R'' usually unrelated to that of R: think of $R(x) := x + x^{-\alpha}$, $\alpha > 0$,, or $R(x) := x + e^{-x}$. Consider, e.g., $H_n\left[e^{cx}\right] \equiv 0$ and the Example 4.4 whose complete treatment required some devices. Some preliminary calculations, collected in the following lemma, shall show that a satisfying general result is not to be expected.

Lemma 5.4. *For the functions* ψ_k *in* (4.2) *associated to the function*

$$\phi(x) := \exp\left(x\mathcal{L}(x)\right) \text{ where } \begin{cases} \left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty),\ 0 \le k \le 2n-3, \\ \mathcal{L}(+\infty) = \pm + \infty, \end{cases} \tag{5.31}$$

we have the following types of aymptotic variation:

$$\begin{cases} \psi_k \in \left\{\mathcal{SR}_0(+\infty) \text{ of order } 2n-3\right\}, (1 \le k \le n-1), \\ \psi'_k \in \left\{\mathcal{SR}_{-1}(+\infty) \text{ of order } 2n-4\right\}, (1 \le k \le n-1). \end{cases} \tag{5.32}$$

Proof. In this case:

$$\begin{cases} R(x) := x\mathcal{L}(x);\ R'(x) = \mathcal{L}(x) + x\mathcal{L}'(x) \sim \mathcal{L}(x); \\ \left|R^{(i)}(x)\right| \in \mathcal{R}_{1-i}(+\infty),\ 0 \le i \le 2n-3; \\ R^{(i)}(x) = O\left(x^{1-i}R'(x)\right) = O\left(x^{1-i}\mathcal{L}(x)\right),\ 2 \le i \le 2n-2, \text{ by } (1.20); \end{cases} \tag{5.33}$$

and (5.7) now reads:

$$\begin{aligned} &\left(R'(x)\right)^{i_1} \cdot \left(R''(x)\right)^{i_2} \cdots \left(R^{(k)}(x)\right)^{i_k} = O\left(x^{i_1+i_2+\cdots+i_k-k}\left(\mathcal{L}(x)\right)^{i_1+\cdots+i_k}\right) \\ &= o\left(\left(x\mathcal{L}(x)\right)^{i_1+\cdots+i_k-k}\left(\mathcal{L}(x)\right)^k\right) \overset{\text{by (4.5b)}}{=} o\left(\left(\mathcal{L}(x)\right)^k\right). \end{aligned} \tag{5.34}$$

Hence:

$$\psi_k(x) := \phi^{(k)}(x)/\phi(x) \sim \left(\mathcal{L}(x)\right)^k,\ x \to +\infty,\ 1 \le k \le n-1, \tag{5.35}$$

and, as in the proof of Theorem 5.1, Proposition 2.6-(II) would lead to the first relation in (5.32) preventing any conclusion about the index of ψ'. But we may resort to a result concerning product of higher-order regularly-varying functions. In fact, applying ([4], Prop. 7.3-(I), p. 821), we may infer, as a *first step*, that

$$\left|\left(R^{(i)}\right)^{i_j}\right| \in \left\{\mathcal{R}_{(1-i)i_j}(+\infty) \text{ of the same order as } R^{(i)}\right\} \text{ because } (1-i)i_j \notin \mathbb{N}; \tag{5.36}$$

(which is the special case of an iterated product: f^k with $k \in \mathbb{N}$); and, as a *second step*, that both properties

$$\begin{cases} \left| \left(R^{(i)}\right)^{i_j} \right| \in \mathcal{R}_{(1-i)i_j}(+\infty) \text{ and} \\ \sum_{j=1}^{k}(1-j)i_j \overset{\text{as in (5.7)}}{=} i_1+\cdots+i_k-k \le -1, \left(\text{hence} \notin \mathbb{N}\right), \end{cases} \tag{5.37}$$

imply that each single term into the summation symbol in (4.5a) belongs to the class $\mathcal{R}_{i_1+\cdots+i_k-k}(+\infty)$ with its derivative belonging to the class $\mathcal{R}_{i_1+\cdots+i_k-k-1}(+\infty)$ where $i_1+\cdots+i_k-k-1 \le -2$. As the derivative of the first isolated term in (4.5a) belongs to the class $\mathcal{R}_{-1}(+\infty)$ we infer that $\psi_k' \in \mathcal{R}_{-1}(+\infty)$. Together with the first relation in (5.32), this implies the second relation in (5.32).

The next step in Example 4.4 was examining the algebraic structures of ψ_k, ψ_k'; and in the situation of the preceding lemma we have that, for $R(x) := x\mathcal{L}(x)$,

$$R^{(i)}(x) = i\mathcal{L}^{(i-1)}(x) + x\mathcal{L}^{(i)}(x),\ i \ge 1, \tag{5.38}$$

and from (4.5a):

$$\begin{cases} \psi_1(x) = \mathcal{L}(x) + x\mathcal{L}'(x); \\ \psi_k(x) = \left[\mathcal{L}(x) + x\mathcal{L}'(x)\right]^k + \sum_{i_1,\cdots,i_k}^{\cdots} a_{i_1,\cdots,i_k} P_{i_1,\cdots,i_k}(x),\ k \ge 2; \\ P_{i_1,\cdots,i_k}(x) := \left[\mathcal{L}(x) + x\mathcal{L}'(x)\right]^{i_1}\left[2\mathcal{L}'(x) + x\mathcal{L}''(x)\right]^{i_2} \\ \qquad \times\cdots\times\left[k\mathcal{L}^{(k-1)}(x) + x\mathcal{L}^{(k)}(x)\right]^{i_k},\ 1 \le i_1+\cdots+i_k \le k-1. \end{cases} \tag{5.39}$$

Unlike the situation in Example 4.4 one realizes that it is impossible to pick out a change of variable working well for a generic slowly-varying $\mathcal{L}$; so we limit ourselves to working out a special case.

Proposition 5.5. (Slight generalization of Example 4.4). *For the function*

$$\phi(x) := \exp\left(x\tilde{R}(\log x)\right) \text{ with } \left|\tilde{R}^{(i)}\right| \in \mathcal{R}_{\alpha-i}(+\infty),\ 0 \le i \le 2n-3,\ (\alpha \ne 0), \tag{5.40}$$

the following relation holds true:

$$\begin{aligned} &H_n\left[\exp\left(x\tilde{R}(\log x)\right)\right] \sim \alpha^{(n-1)(n-2)/2} \cdot \left(\prod_{i=0}^{n-1} i!\right) \cdot x^{-n(n-1)/2} \cdot (\log x)^{-(n-1)(n-2)/2} \\ &\times\left(\tilde{R}(\log x)\right)^{(n-1)(n-2)/2} \cdot \left(\tilde{R}'(\log x)\right)^{n-1} \cdot \exp\left(nx\tilde{R}(\log x)\right),\ x \to +\infty,\ (n \ge 2). \end{aligned} \tag{5.41}$$

5.5. The Exceptional Case $\gamma = 1$: Proof

Proof of Proposition 5.5. Notice that, according to the definitions in §1, the assumptions imply that

$$\tilde{R} \in \left\{\mathcal{SR}_\alpha(+\infty) \text{ of order } 2n-2\right\} \tag{5.42}$$

because also the pertinent relation for $\tilde{R}^{(2n-2)}$ is satisfied. In this case, for the function R in (5.1) we have:

$$\begin{cases} R(x) := x\tilde{R}(\log x);\ \tilde{R}'(y) \sim \alpha y^{-1}\tilde{R}(y),\ y \to +\infty; \\ R'(x) = \tilde{R}(\log x) + \tilde{R}'(\log x) \sim \tilde{R}(\log x),\ x \to +\infty; \end{cases} \tag{5.43}$$

and for the higher derivatives:

$$\begin{cases} R''(x) = x^{-1}\left[\tilde{R}'(\log x) + \tilde{R}''(\log x)\right]; \\ R^{(i)}(x) = x^{i-1}\sum_{j=1}^{i} c_{i,j}\tilde{R}^{(j)}(\log x),\ i \geq 1, \left(\text{with suitable } c_{i,j}\right). \end{cases} \tag{5.44}$$

For $n = 2$ the simple expression in (5.41) follows at once from (1.9) and "$R''(x) \sim x^{-1}\tilde{R}'(\log x)$, $x \to +\infty$." Assuming $n \geq 3$ we have:

$$\psi_k(x) = \left[\tilde{R}(\log x) + \tilde{R}'(\log x)\right]^k + \sum_{\cdots} a_{i_1,\cdots,i_k} P_{i_1,\cdots,i_k}(x),\ 1 \leq k \leq n-1, \tag{5.45}$$

where

$$\begin{aligned} P_{i_1,\cdots,i_k}(x) &:= \prod_{j=1}^{k}\left(R^{(j)}(x)\right)^{i_j} = x^{i_1+\cdots+i_k-k}\prod_{j=1}^{k}\left(\sum_{p=1}^{j} c_{j,p}\tilde{R}^{(p)}(\log x)\right)^{i_j} \\ &\equiv x^{i_1+\cdots+i_k-k}\tilde{P}_{i_1,\cdots,i_k}(\log x), \end{aligned} \tag{5.46}$$

and we rewrite (5.45) as:

$$\psi_k(x) = \left[\tilde{R}(\log x) + \tilde{R}'(\log x)\right]^k + \sum_{\cdots} a_{i_1,\cdots,i_k} x^{i_1+\cdots+i_k-k}\tilde{P}_{i_1,\cdots,i_k}(\log x). \tag{5.47}$$

As for ψ_k' we have:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left[\tilde{R}(\log x) + \tilde{R}'(\log x)\right]^k \\ &= kx^{-1}\left[\tilde{R}'(\log x) + \tilde{R}''(\log x)\right]\left[\tilde{R}(\log x) + \tilde{R}'(\log x)\right]^{k-1} \\ &= kx^{-1}\left[\tilde{R}'(\log x) + \tilde{R}''(\log x)\right] \\ &\quad\times\left[\left(\tilde{R}(\log x)\right)^{k-1} + \sum_{i=1}^{k-1}\binom{k-1}{i}\left(\tilde{R}(\log x)\right)^{k-1-i}\left(\tilde{R}'(\log x)\right)^{i}\right] \\ &\equiv x^{-1}Q_k(\log x); \end{aligned} \tag{5.48}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\sum_{\cdots} a_{i_1,\cdots,i_k} x^{i_1+\cdots+i_k-k}\tilde{P}_{i_1,\cdots,i_k}(\log x) \\ &= \sum_{\cdots} a_{i_1,\cdots,i_k} x^{i_1+\cdots+i_k-k-1}\left[(i_1+\cdots+i_k-k)\tilde{P}_{i_1,\cdots,i_k}(\log x) + \tilde{P}'_{i_1,\cdots,i_k}(\log x)\right] \\ &\equiv x^{-1}T_k(\log x). \end{aligned} \tag{5.49}$$

Hence:

$$\begin{cases} \psi_k'(x) = x^{-1}\left[Q_k(\log x) + T_k(\log x)\right], \\ W\left(\psi_1'(x),\cdots,\psi_{n-1}'(x)\right) = x^{1-n}\cdot W\left(\left\{Q_k(\log x) + T_k(\log x)\right\}_{k=1,\cdots,n-1}\right), \end{cases} \tag{5.50}$$

with Q_k, T_k implicitly defined in (5.48), (5.49). At this point we make the change of variable $y = \log x$ so getting, by (1.38):

$$\begin{cases} u_k(y) := Q_k(y) + T_k(y); \\ W\left(\left\{Q_k(\log x) + T_k(\log x)\right\}_{k=1,\ldots,n-1}\right) \\ = x^{-(n-1)(n-2)/2}\cdot\left[W\left(u_1(y),\cdots,u_{n-1}(y)\right)\right]_{y=\log x}. \end{cases} \tag{5.51}$$

With a view to applying Lemma 3.1 we shall show that

$$u_k \in \left\{\mathcal{SR}_{k\alpha-1}(+\infty) \text{ of order } n-2\right\},\ 1 \leq k \leq n-1. \tag{5.52}$$

First:

$$Q_k(y) := k\left[\tilde{R}'(y)+\tilde{R}''(y)\right]\cdot\left[\left(\tilde{R}(y)\right)^{k-1}+\sum_{i=1}^{k-1}\binom{k-1}{i}\left(\tilde{R}(y)\right)^{k-1-i}\left(\tilde{R}'(y)\right)^{i}\right], \quad (5.53)$$

where, by Proposition 2.6-(II):

$$\begin{cases}\tilde{R}'+\tilde{R}''\in\left\{\mathcal{SR}_{\alpha-1}(+\infty)\text{ of order }2n-4\right\};\\ \tilde{R}^{k-1}+\sum_{i=1}^{k-1}\binom{k-1}{i}\tilde{R}^{k-1-i}\left(\tilde{R}'\right)^{i}\in\left\{\mathcal{SR}_{\alpha(k-1)}(+\infty)\text{ of order }2n-3\right\};\end{cases} \quad (5.54)$$

and Proposition 2.4-(I) implies

$$Q_k\in\left\{\mathcal{SR}_{k\alpha-1}(+\infty)\text{ of order }2n-4\right\},\ 1\le k\le n-1. \quad (5.55)$$

Second:

$$\begin{aligned}T_k(y) := \sum_{\cdots} a_{i_1,\cdots,i_k}&\exp\left((i_1+\cdots+i_k-k)y\right)\\ &\times\left[(i_1+\cdots+i_k-k)\tilde{P}_{i_1,\cdots,i_k}(y)+\tilde{P}'_{i_1,\cdots,i_k}(y)\right],\end{aligned} \quad (5.56)$$

where for $\tilde{P}_{i_1,\cdots,i_k}(y)$, implicitly defined in (5.46), we have:

$$\begin{cases}\tilde{R}^{(p)}\in\left\{\mathcal{SR}_{\alpha-p}(+\infty)\text{ of order }2n-2-p\right\}\text{ by assumption;}\\ \sum_{p=1}^{j}c_{j,p}\tilde{R}^{(p)}(y)\begin{cases}\in\left\{\mathcal{SR}_{\alpha}(+\infty)\text{ of order }2n-2-j\right\}\\ \text{by Proposition 2.6-(II);}\end{cases}\\ \left(\sum_{p=1}^{j}c_{j,p}\tilde{R}^{(p)}(y)\right)^{i_j}\begin{cases}\in\left\{\mathcal{SR}_{\alpha i_j}(+\infty)\text{ of order }2n-2-j\right\}\\ \quad\text{by ([4], Prop. 7.3-(I)) if }i_j\ne0;\\ \equiv1\text{ if }i_j=0,\text{ hence having no import;}\end{cases}\\ \prod_{j=1}^{k}\left(\sum_{p=1}^{j}c_{j,p}\tilde{R}^{(p)}(y)\right)^{i_j}\in\left\{\mathcal{SR}_{\alpha(i_1+\cdots+i_k)}(+\infty)\text{ of order }2n-2-k\right\}\\ \text{by the same reason because }\alpha(i_1+\cdots+i_k)\ne0;\end{cases} \quad (5.57)$$

whence:

$$\begin{cases}\tilde{P}_{i_1,\cdots,i_k}\in\left\{\mathcal{SR}_{\alpha(i_1+\cdots+i_k)}(+\infty)\text{ of order }2n-2-k\right\},\ 1\le k\le n-1,\\ \tilde{P}'_{i_1,\cdots,i_k}\in\left\{\mathcal{SR}_{\alpha(i_1+\cdots+i_k)-1}(+\infty)\text{ of order }2n-3-k\right\},\ 1\le k\le n-1.\end{cases} \quad (5.58)$$

Moreover:

(i) the exponential in (5.56) obviously belongs to $\left\{\mathcal{R}_{-\infty}(+\infty)\text{ of any order }n\in\mathbb{N}\right\}$ because "$i_1+\cdots+i_k-k\le-1$", hence each term into this sum belongs to the class $\left\{\mathcal{R}_{-\infty}(+\infty)\text{ of order }2n-3-k\right\}$ by Proposition 2.4-(III);

(ii) all the terms in (5.56) can be arranged as to form an asymptotic scale so that the whole sum belongs to the class $\left\{\mathcal{SR}_{-\infty}(+\infty)\text{ of order }2n-3-k\right\}$ by Prop 2.8-(I).

Hence Proposition 2.8-(III) implies that

$$T_k\in\left\{\mathcal{SR}_{k\alpha-1}(+\infty)\text{ of order }2n-3-k\right\},\ 1\le k\le n-1. \quad (5.59)$$

It follows from (5.55) and (5.59) that each u_k belongs to the class in (5.59) and the greatest common order for all the u_k's is "$2n-3-(n-1)=n-2$": and (5.52) is proved.

We also have the following asymptotic relations. Using "$\tilde{R}'(y)=o(\tilde{R}(y))$", inferred from (5.43), into (5.53) we get $Q_k(y)\sim k\tilde{R}'(y)(\tilde{R}(y))^{k-1}$ with both $\tilde{R}(y),\tilde{R}'(y)$ asymptotically bounded from below by some power of y, ([3], Prop. 2.1-(ii)); obviously $T_k(y)$ tends exponentially to zero and we infer:

$$u_k(y)\sim k(\tilde{R}(y))^{k-1}\tilde{R}'(y),\ y\to+\infty,(1\le k\le n-1). \tag{5.60}$$

We may now apply Lemma 3.1. The Vandermondian is

$$V(\alpha-1,2\alpha-1,\cdots,(n-1)\alpha-1)=V(\alpha,2\alpha,\cdots,(n-1)\alpha)=\alpha^{(n-1)(n-2)/2}\cdot\left(\prod_{i=0}^{n-2}i!\right),$$

and using (5.60) we get relation

$$\begin{aligned}W(u_1(y),\cdots,u_{n-1}(y))&\sim\alpha^{(n-1)(n-2)/2}\cdot\left(\prod_{i=0}^{n-1}i!\right)\cdot y^{-(n-1)(n-2)/2}\\&\times(\tilde{R}(y))^{(n-1)(n-2)/2}(\tilde{R}'(y))^{n-1},\ y\to+\infty,\end{aligned} \tag{5.61}$$

whence (5.41) follows from (4.2), (5.50), (5.51). □

Remark. If $\tilde{R}(\log x)$ is replaced by $\tilde{R}(\ell_p x)$, $p\ge2$, the right substitution is $y=\ell_p(x)$ leading to more complicated formulas, see ([2], §4).

Example 5.3. With $\tilde{R}(y):=y^\alpha(\log y)^\beta$, $\alpha\ne0$, in (5.40) we get the relation:

$$\begin{cases}H_n\left[\exp\left(x(\log x)^\alpha(\ell_2(x))^\beta\right)\right]\sim\alpha^{n(n-1)/2}\cdot\left(\prod_{i=0}^{n-1}i!\right)\cdot x^{-n(n-1)/2}\\ \times(\log x)^{n(n-1)(\alpha-1)/2}\cdot(\ell_2(x))^{\beta n(n-1)/2}\cdot\exp\left(nx(\log x)^\alpha(\ell_2(x))^\beta\right),\\ x\to+\infty,(n\ge2).\end{cases} \tag{5.62}$$

5.6. An Application to Asymptotic Expansions

Theorem 5.6. *Let* ϕ *be the function in* (5.1) *and let* L_n *be the unique linear ordinary differential operator of type* (3.21) *such that:* $\ker L_n=\operatorname{span}(\phi,\phi',\cdots,\phi^{(n-1)})$, $n\ge2$. *Moreover, the relation* $R'\in\mathcal{R}_{\gamma-1}(+\infty)$ *implies that:*

$$R'(x)=\begin{cases}o(1) & \text{if } 0<\gamma<1,\\ +\infty(1) & \text{if } \gamma>1.\end{cases} \tag{5.63}$$

(I) *If* "$0<\gamma<1$" *relations in* (5.9) *imply the asymptotic scale* (1.2), *and a function* $f\in AC^{n-1}[T,+\infty[$ *admits of an asymptotic expansion of type* (3.22) *formally differentiable* $n-1$ *times in the sense of relations* (3.24) *provided that:*

$$\int^{+\infty}t^{2(n-1)}\cdot|R(t)|^{1-n}\cdot\exp(-R(t))\cdot|L[f(t)]|\mathrm{d}t<+\infty. \tag{5.64}$$

(II) *If* "$\gamma>1$" *relations in* (5.9) *imply the asymptotic scale* (1.3), *and a function* $f\in AC^{n-1}[T,+\infty[$ *admits of an asymptotic expansion of type*

$$f(x) = a_0\phi^{(n-1)}(x) + a_1\phi^{(n-2)}(x) + \cdots + a_{n-2}\phi'(x) + a_{n-1}\phi(x) + o(\phi(x)),\ x \to +\infty, \tag{5.65}$$

formally differentiable $n-1$ *times in the sense of* ([1], §6), *provided that*:

$$\int^{+\infty} t^{2(n-1)} \cdot |R(t)|^{1-n} \cdot |R'(t)|^{n-1} \cdot \exp(-R(t)) \cdot |L[f(t)]| \mathrm{d}t < +\infty. \tag{5.66}$$

Formal differentiability in the present context refers to the validity of the following n expansions as $x \to +\infty$:

$$\begin{cases} f = a_0\phi^{(n-1)} + a_1\phi^{(n-2)} + \cdots + a_{n-2}\phi' + \phi\cdot[a_{n-1} + o(1)]; \\ \left(f/\phi^{(n-1)}\right)' = a_1\left(\phi^{(n-2)}/\phi^{(n-1)}\right)' + \cdots + a_{n-2}\left(\phi'/\phi^{(n-1)}\right)' \\ \qquad + \left(\phi/\phi^{(n-1)}\right)' \cdot [a_{n-1} + o(1)]; \\ \left(\left(f/\phi^{(n-1)}\right)' \Big/ \left(\phi^{(n-2)}/\phi^{(n-1)}\right)'\right)' = a_2\left(\left(\phi^{(n-3)}/\phi^{(n-1)}\right)' \Big/ \left(\phi^{(n-2)}/\phi^{(n-1)}\right)'\right)' \\ + \cdots + \left(\left(\phi/\phi^{(n-1)}\right)' \Big/ \left(\phi^{(n-2)}/\phi^{(n-1)}\right)'\right)' \cdot [a_{n-1} + o(1)]; \end{cases} \tag{5.67}$$

and so on, dividing both sides of each expansion by the first term (*constant apart*) *in the right-hand side and then differentiating both sides to obtain the next expansion until differentiating* $n-1$ *times*.

Proof. For Part (I) the ratio of Wronskians, in ([1], formula (198), p. 26), in the present context is $H_{n-1}[\phi(x)]/H_n[\phi(x)]$ as in the proof of Theorem 3.4 and relations (5.5) yield condition (5.64). For Part (II) one must take into account the correct ordering of the functions forming the scale and the ratio of Wronskians is now:

$$W\left(\phi^{(n-1)}, \phi^{(n-2)}, \cdots, \phi'\right) \Big/ W\left(\phi^{(n-1)}, \cdots, \phi', \phi\right) = (-1)^{n-1} H_{n-1}[\phi'(x)]/H_n[\phi(x)]. \tag{5.68}$$

For $H_n[\phi(x)]$ we have relation (5.5), whereas

$$H_{n-1}[\phi'(x)] \equiv H_{n-1}[R'(x)\exp(R(x))] \overset{(5.22)}{\sim} (R'(x))^{n-1} \cdot H_{n-1}[\phi(x)],\ x \to +\infty, \tag{5.69}$$

and (5.66) follows, noticing that the validity of (5.69) requires no additional regularity condition on *R*.

6. Conclusions for Part I

The investigation of the asymptotic behaviors of Wronskians carried out in previous papers highlighted the essential role of the theory of "higher-order types of asymptotic variation" with its equipment of a large number of meticulous (as well as tedious) results on operations with such functions. The treatment in the present paper once again shows the need of the whole apparatus of the cited theory. It is a fact that a first draft of the present paper revealed the lack of many needed preliminary lemmas about products and linear combinations of functions with various types of asymptotic variation, and this urged the author to systematize results of this kind in a separate previously-published paper, [6], results summarized in §2. Hence the study of Hankelians has contributed to completing the general theory of "higher-order types of asymptotic variation" whose applications to differential equations require further attention.

- (*A correction to Proposition* 2.4-(II) in ([6], p. 697-698). In the pertinent statement, in the first line after formula (2.29) it is missing the additional essential assumption that f, g satisfy anyone of the two conditions in (2.28). Anyway, in our Proposition 2.3-(II) in this paper we stated the claim in a clearer way pointing out its direct inference from two previous results which in the original version are ([6], Prop. 2.2 and Prop. 2.4-(I)).
- (*Some typos in* [6]). In the second line after formula (1.24), p. 692, the last words "of exact order $n+1$ " must be read "of exact order $\alpha+1$ ".
- Inequalities " $p+1 \le i \le q$ " must be replaced by " $p+2 \le i \le q$ " in three places:
- 5th line in (3.17); 4th and 5th lines in (3.19).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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