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Completion of the Proof of the Contradiction of Set Theory

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Abstract

The article is devoted to completing the proof of the inconsistency of set theory. In this article and in the two preceding ones, all steps of the proof are based on generally accepted informal set-theoretic reasoning, but consider the prohibitions that were included in axiomatic set theories in order to overcome the difficulties encountered by the naive Cantor set theory. Therefore, in fact, the articles are about proving the inconsistency of existing axiomatic set theories, in particular, the ZFC theory.

Keywords

Set Theory, Inconsistency, Tree, Strange Tree, Through Way, Almost through Way, Isomorphism, Almost Isomorphism, Isomorphism Tree, Place Plane, Superposition (Imposition) of Trees on the Place Plane, Disposition of Trees on the Place Plane

1. Introduction

In the twentieth century, there were crises in mathematics, which led first to its complete axiomatization (in particular, axiomatic set theories appeared), and then to the famous theorems of Gödel on the incompleteness and impossibility of proving the consistency of an axiomatic theory by means formalized within the theory itself, and to the outstanding proofs of Gödel and Cohen that the generalized continuum hypothesis does not depend on the axioms of ZFC (the most popular axiomatic set theory), see [1]. That is, it cannot be either proven or refuted in ZFC (if ZFC is consistent).

The general opinion of mathematicians now boils down to the fact that Peano's arithmetic is certainly consistent, and set theory is almost certainly consistent. This point of view actually means a rejection of the principle of scientific knowability, which Hilbert, apparently considered the most important principle in the work of a scientist.

The papers [2] [3] are devoted to proving that existing axiomatic set theories (in particular, ZFC theory) contain a contradiction. The present paper completes these articles, filling the gap in the proof at the end of [3]. Accordingly, the present paper assumes that the reader is familiar with the papers [2] [3]. In particular, all the notations used in the paper are borrowed from [1] [2]. The numbering of lemmas and theorems given in these papers is also preserved. To help the reader, a list of notations (common to all three papers) is given at the end of the paper.

It should be noted that the study of the inconsistency-consistency of existing axiomatic set theories has a meaning that goes far beyond the framework of "mathematical purism". In fact, this question has a deep general mathematical and even physical, meaning. In connection with the latter, see, for example, the work [4], where set theory is used in models describing the phenomena of quantum physics.

2. Completion of the Proof of the Inconsistency of Set Theory

To make the reader's work easier, we will first give a brief summary of the main results obtained in the works [2] [3]. The brief summary will allow the reader to better navigate the material presented in these works.

The concept of a tree and the operation of cutting trees are introduced. A tree contains vertices at each *level* except possibly the *upper* one. If an ordinal m denotes the level with the number m in a tree T, then $cut(T,m)$ means the tree obtained by cutting T at the level m.

The concept of a *strange* tree is introduced as a tree T_w without through paths in which from each vertex w^k at level k there is a path to some vertex w^m of any *level* m less than the upper one, and considerations are given due to which the tree T_w is indeed strange (not consistent with intuitive ideas). It is shown that for $\propto < \omega_1$ strange trees $T^{\propto}$ of height $\propto$ do not exist, but there is a strange homogeneous tree $T_{str}^{\omega_1}$. Theorems 1 and 2 are proved. By theorem 2 (theorem 1 is a weakened version of theorem 2), for each almost homogeneous tree T_a of height ω_1 (in particular, T_a can be a strange tree), there exists a through almost homogeneous tree T_b of height ω_1 that is almost isomorphic to T_a. If $T_a = T_{str}^{\omega_1}$, then T_b is not isomorphic to T_a, but is almost isomorphic to it.

Almost isomorphism means that the trees $cut(T_a,m)$ and $cut(T_b,m)$ (cuts of trees T_a and T_b at level m) are isomorphic for $m < \omega_1$. Theorem 2 underlies the approach to proving the inconsistency of set theory developed in [1] [2]. If it is proved that the isomorphism of $cut(T_a,m)$ and $cut(T_b,m)$ for all $m < \omega_1$ implies the isomorphism of T_a and T_b, then a contradiction in set theory will follow from this.

A tree T_w is called *homogeneous* if for each two of its vertices w_i^k, w_j^k at level k the trees generated by them are isomorphic.

A tree T_w is called *almost homogeneous* if $cut(T_w,m)$ is homogeneous for $m < \propto$, where $\propto$ is the height of the tree.

Next, we introduce a class-set of almost through almost homogeneous trees of height $\propto = \omega_1$ (notations: $T_w = T_w^{\propto}$, $T_t = T_t^{\propto}$, etc.) almost isomorphic to each other, including, together with the trees T_w, T_t, their cuts at any level $m \le \propto$: $T_w^m = cut(T_w, m)$, $T_t^m = cut(T_t, m)$, containing both trees without through paths and with through paths (called the *first class*). By theorem 2, the first class of trees exists. We introduce trees with *double vertices* T_{wt}^m, representing isomorphisms of the trees T_w^m, T_t^m (and, for simplicity, identified with them). A tree T_t is fixed and various overlays (isomorphisms) of T_{wt} (and their cuttings) are considered.

For each two trees T_w^m, T_t^m from the first class of height m, the *tree of overlays* (*isomorphisms*) $\boldsymbol{T}_{wt}^m$ is introduced, whose vertices at level k are all possible isomorphisms T_{wt}^k. Thus, the *second class* of trees is introduced into consideration - the class of trees of isomorphisms of trees of the first class. In this class, there are also trees without through paths, as well as through trees. The former are obtained if we take two non-isomorphic trees of the first class of height $\propto = \omega_1$, and the latter - in all other cases. For $m < \propto$, the tree $\boldsymbol{T}_{wt}^m$ is always through and homogeneous.

As it was said already, the tree T_t is fixed as a tree on which trees T_w of the first class are superimposed (when this can be done). For each level k in T_t, a numbering of automorphisms T_{tt}^k is fixed in a certain way and a notation P_{ti}^k is introduced for numbered automorphisms. We assume that P_{t0}^k is the identity automorphism, and P_{ti}^k for $i < \overline{\beta}_k$ is a non-final automorphism. An operation of multiplication of the isomorphism T_{wt}^k by the automorphism P_{ti}^k is introduced, the result of which is a new isomorphism.

For each $k \le m$, we choose some non-final isomorphism T_{wt}^k as the *main* isomorphism for a given k and give it the number 0. After that, we introduce the numbering of vertex-isomorphisms: $T_{wt,i}^k = T_{wt,0}^k \times P_{ti}^k$. In this case, we will have: $T_{wt,i}^k \times P_{tj}^k = T_{wt,0}^k \times \left(P_{ti}^k \times P_{tj}^k\right)$ and $T_{wt,0}^k = T_{wt,i}^k \times \left(P_{ti}^k\right)^{-1}$.

Each numbering is determined by the choice of $T_{wt,0}^k$. The introduction of a numbering allows us to place the isomorphism trees $\boldsymbol{T}_{wt}^m$ on the plane of places in various ways. We assume that when placed on the plane of places, the isomorphism $T_{wt,i}^k$ is imposed on the place n_i^k. The same applies to P_{ti}^k.

The fundamental property of an isomorphism tree $\boldsymbol{T}_{wt}^m$ is the property reflected in lemma 25.

Lemma 25. The isomorphism tree $\boldsymbol{T}_{wt}^m$ satisfies the *decomposition rule*:

$$cut\left(T_{wt,i}^l \times P_{tj}^l, k\right) = cut\left(T_{wt,i}^l, k\right) \times cut\left(P_{tj}^l, k\right), k \le l \le m.$$

Also

$$cut\left(P_{ti}^l \times P_{tj}^l, k\right) = cut\left(P_{ti}^l, k\right) \times cut\left(P_{tj}^l, k\right).$$

Lemma 25 implies lemma 27. The last lemma plays a significant role throughout the paper.

Lemma 27. Let $I^m = \left(i_k, k \le m\right)$ be a sequence of numbers for which $i_k < \overline{\beta}_k$. For isomorphic T_t^m, T_w^m there exists a through tree $\boldsymbol{T}_{wt}^m$ for which $\left(T_{wt,i_k}^k, k \le m\right)$

is a through path. By specifying a pair $\left(T_{wt}^{m}, I^{m}\right)$, where $T_{wt}^{m} = T_{wt,i_m}^{m}$ is a non-final isomorphism, the tree $\boldsymbol{T}_{wt}^{m}$ is uniquely determined.

Let a sequence of place numbers $I^{m} = \left(n_{i_k}^{k}, k \le m\right)$ be given. If the isomorphism tree $\boldsymbol{T}_{wt}^{m}$ has a through path $W^{m} = \left(T_{wt,i_k}^{k} = cut\left(T_{wt,i_m}^{m}, k\right), k \le m\right)$, then we say that I^{m} *determines* this path. For $k < m$, $i_k < \overline{\beta}_k$, and $i_m < \overline{\beta}_m$ for *non-final* I^{m}, determining non-final paths $\left(T_{wt,i_k}^{k}, k \le m\right)$. The operation of multiplication of places n_i^k by automorphisms P_{tj}^{k} is introduced: $n_0^k \times P_{ti}^{k} = n_i^k$, $n_i^k \times P_{tj}^{k} = n_0^k \times \left(P_{ti}^{k} \times P_{tj}^{k}\right)$, in which the sets $\left(n_i^k, i < \beta_k\right)$, $\left(P_{ti}^{k}, i < \beta_k\right)$ and $\left(T_{wt,i}^{k}, i < \beta_k\right)$ are *trivially isomorphic* with respect to the operation of multiplication of set elements by P_{tj}^{k}.

Let PI^{m} denote the continuing sequence of automorphisms $\left(P_{tj_k}^{k}, k \le m\right)$ imposed on the plane of places. By definition, $I^{m} \times PI^{m} = \left(n_{i_k}^{k} \times P_{tj_k}^{k}, k \le m\right)$, while $W^{m} \times PI^{m} = \left(T_{wt,i_k}^{k} \times P_{tj_k}^{k}, k \le m\right)$. $PI^{l} = \left(P_{tj_k}^{k}, k \le l\right) = cut\left(PI^{m}, l\right)$, $l \le m$. In the continuing sequence of automorphisms $\left(P_{tj_k}^{k}, k \le m\right)$, all $P_{tj_k}^{k}$ for $k < m$ are non-final. If $P_{tj_m}^{m}$ is not final, then PI^{m} will be called non-final. Otherwise, we will talk about final PI^{m}. Also by definition, $PI_a^{m} \times PI_b^{m} = \left(P_{ti_k}^{k} \times P_{tj_k}^{k}, k \le m\right)$ if $PI_a^{m} = \left(P_{ti_k}^{k}, k \le m\right)$, $PI_b^{m} = \left(P_{tj_k}^{k}, k \le m\right)$. The multiplication operation "×" turns the set $\left(PI_i^{m}, i < \beta_m\right)$ into a group, and the set $\left(PI_i^{m}, i < \overline{\beta}_m\right)$ into a subgroup of this group.

The sequence $PI_0^{m} = \left(P_{t0}^{k}, k \le m\right)$ is a continuing sequence of identical automorphisms.

Lemma 29. For any non-final sequence $I^{m} = \left(n_{i_k}^{k}, k \le m\right)$ there exists a through tree $\boldsymbol{T}_{wt}^{m}$ for which I^{m} defines a strongly through path $W^{m} = \left(T_{wt,i_k}^{k}, k \le m\right)$. If I^{m} defines a path W^{m} in the tree $\boldsymbol{T}_{wt}^{m}$, then $I^{m} \times PI^{m}$ defines the path $W^{m} \times PI^{m}$.

Each through tree $\boldsymbol{T}_{wt}^{m}$ is uniquely determined by the set of its through paths $\boldsymbol{W}^{m} = \left(W_i^{m}, i = 0,1,\cdots\right)$, and each path is a continuing sequence of tree vertices. Therefore, it is convenient and clearly to characterize the order relations in a tree using through paths and keeping in mind that each sequence $PI^{m} = \left(P_{ti_k}^{k}, k \le m\right)$ is completely concretized by its upper term $P_{ti_m}^{m}$: $P_{ti_k}^{k} = cut\left(P_{ti_m}^{m}, k\right)$.

The vector notation of the decomposition rule (see above) provides great convenience for a more compact formulation of the results. This notation is reflected in lemma 30.

Lemma 30. Let W^{m} be a path in $\boldsymbol{T}_{wt}^{m}$. Then $W^{m} \times PJ^{m}$ is a path in $\boldsymbol{T}_{wt}^{m}$ if and only if $PJ^{m} = \left(P_{ti_k}^{k}, k \le m\right)$ is a continuing sequence of automorphisms

($P_{ti_k}^k = cut\left(P_{ti_m}^m, k\right)$) for all $k \le m$.

We will call the sequences I^m defining paths W^m in the tree $\boldsymbol{T}_{wt}^m$ the *prototypes* of these paths. Due to the presence of the trivial isomorphism discussed above, this name is justified, and we can treat the prototypes of paths in almost the same way as the paths themselves. In particular, we have: if I^m is the prototype of a path, then $I^m \times PI^m$ is also the prototype of a path. The following two lemmas are true, emphasizing the significance of the introduced concept of "being the prototype of a path".

Lemma 31. Let $I^m = \left(n_{i_k}^k, k \le m\right)$ be the prototype of a non-final path in a through tree $\boldsymbol{T}_{wt}^m$. Then $I^m \times PI_i^m$, when PI_i^m runs over all sequences (non-final sequences) of automorphisms, forms the set of all prototypes of paths (non-final paths) in $\boldsymbol{T}_{wt}^m$. Given $I^m = \left(n_{i_k}^k, i_k < \overline{\beta}_k, k \le m\right)$ and a non-final isomorphism T_{wt}^m, their collection uniquely determines a tree $\boldsymbol{T}_{wt}^m$ for which I^m is the prototype of a through non-final path, and the isomorphism T_{wt}^m is placed on $n_{i_m}^m$.

Lemma 32. Let I_a^m, I_b^m be the prototypes of through non-final paths in a through tree $\boldsymbol{T}_{wt}^m$. The one-to-one correspondence $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$, when PI_i^m runs over all continuing sequences of automorphisms, defines a strong automorphism of the tree $\boldsymbol{T}_{wt}^m$ under which I_a^m goes to I_b^m. Conversely, every strong automorphism of the through tree $\boldsymbol{T}_{wt}^m$ is determined by the pair of prototypers of non-final paths I_a^m, I_b^m according to the formula $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$, when PI_i^m runs over all continuing sequences of automorphisms. Fixing I_a^m and varying I_b^m (or vice versa) yields all existing strong automorphisms of $\boldsymbol{T}_{wt}^m$.

It is easy to see that the situation is similar if in lemma 32 the prototype of the through path I_a^m is replaced by the through path W_a^m itself, and the correspondence $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$ is replaced by the correspondence $W_a^m \times PI_i^m \to I_b^m \times PI_i^m$. Lemmas 31 and 32 characterize the situation with automorphisms of the tree $\boldsymbol{T}_{wt}^m$ and play a fundamental role in the work.

We will call sequences of places I_a^m, I_b^m *related* if $I_b^m = I_a^m \times PI^m$ with some PI^m. We will call them *strongly related* if $I_b^m = I_a^m \times PI^m$ with some non-final PI^m. We will talk about classes of related (strongly related) I^m, meaning non-complementable sets of sequences of places I^m that are related (strongly related) to each other. We will introduce the notations $\boldsymbol{I}^m$ and $\overline{\boldsymbol{I}}^m$ for the classes of related and strongly related sequences of places. The set of non-final sequences of places is one of the classes of strongly related sequences. We will give a special notation to this class - $\boldsymbol{J}^m$. For non-limit m we assume that $\boldsymbol{I}^m = \boldsymbol{J}^m$. $\boldsymbol{I}^m = \left(I_i^m = I_0^m \times PI_i^m, i < \beta_m\right)$. $\boldsymbol{J}^m = \left(I_i^m = I_0^m \times PI_i^m, i < \overline{\beta}_m\right)$. The operation of cutting preserves the relationship to be related and strongly related: if I_a^m, I_b^m are related (strongly related), then $cut\left(I_a^m, l\right), cut\left(I_b^m, l\right)$ are also related (strongly related).

The order relation between places in $\boldsymbol{I}^m$ induces a through *tree of places* T_n^m for which $\boldsymbol{I}^m$ is simultaneously the set of through paths and the set of prototypes of through paths. For $m < \propto$, T_n^m is always isomorphic to $\boldsymbol{T}_{wt}^m$.

Lemma 36. For a given m, any class of related sequences of places is a union of disjoint classes of strongly related sequences: $\boldsymbol{I}^m = \bigcup_i \overline{\boldsymbol{I}}_i^m$, $\overline{\boldsymbol{I}}_i^m \cap \overline{\boldsymbol{I}}_j^m = \varnothing$, если $i \neq j$.

The definition of related (strongly related) sequences can be preserved for the case where $m-0$ it is taken instead of m (m is the limit ordinal). In this case, the analogue of lemma 36 is satisfied.

Lemma 37. For every through $\boldsymbol{T}_{wt}^m$, the set of prototypes of through paths is a class of related sequences of places.

Lemma 38. For any $\boldsymbol{I}^m$ and isomorphic T_w^m, T_t^m, there exists a through tree $\boldsymbol{T}_{wt}^m$ for which $\boldsymbol{I}^m$ is the set of prototipes of through paths.

The concept of a continuing sequence $\left(\boldsymbol{I}^k, k \leq m\right)$ is introduced in a natural way. $\boldsymbol{I}^m$ continues $\boldsymbol{I}^l$ for $l < m$ if $cut\left(\boldsymbol{I}^m, l\right) = \boldsymbol{J}^l$.

In [3] it is proved

Lemma 43. Let $\boldsymbol{T}_{wt}^m$ be a tree of isomorphisms imposed on the plane of places. The tree $\boldsymbol{T}_{wt}^m$ introduces a continuing sequence $\left(\boldsymbol{I}^k, k \leq m\right)$ in which each $\boldsymbol{I}^k$ is the set of prototypes of through paths of the tree $\boldsymbol{T}_{wt}^k = cut\left(\boldsymbol{T}_{wt}^m, k\right)$.

An immediate consequence of lemma 43 is

Theorem 4. Among the continuing sequences $\left(\boldsymbol{I}^m, m < \propto\right)$ there are both sequences with through paths and sequences without through paths.

Therefore, if it is proved that a continuing sequence $\left(\boldsymbol{I}^m = \left(I_i^m, i < \beta_m\right), m < \propto\right)$ without through paths does not exist, then the inconsistency of set theory will follow.

We assume that the mathematical objects under study, the trees of isomorphisms $\boldsymbol{T}_{wt}^m$, are located on a section of the homogeneous plane of places (see the beginning of section 4 in [3]). This placement is determined by specifying for all $k \leq m$ the numbering of isomorphisms $T_{wt,i}^k = T_{wt,0}^k \times P_{ti}^k$, where P_{ti}^k is the *i*-th automorphism of the tree T_t^k, and is therefore entirely determined by the choice of the main isomorphisms. We assume that $T_{wt,i}^k$ and P_{ti}^k are superimposed on n_i^k.

For greater clarity (specifying what was said in [3]), we will assume that for each tree $\boldsymbol{T}_{wt}$ for each k, a *primary numbering* $\overline{T}_{wt,i}^k = \overline{T}_{wt,0}^k \times P_{ti}^k$ is first introduced and fixed (simply as a way of denoting different isomorphism trees), and the *secondary* (*working*) numbering $T_{wt,i}^k = T_{wt,0}^k \times P_{ti}^k$ is determined through the choice of the main isomorphism $T_{wt,0}^k = \overline{T}_{wt,r_k}^k$, $T_{wt,i}^k = T_{wt,0}^k \times P_{ti}^k$. When we speak about a tree $\boldsymbol{T}_{wt}^m$ imposed on the plane of places, we mean that it is imposed in accordance with the introduced working numbering of isomorphism trees. For different working numberings (with different choices of r_k) different superpositions will take place. Of course, the primary numbering can be considered as a special case of the secondary (with $r_k = 0$).

The superposition (imposition) of $\boldsymbol{T}_{wt}^m$ on the plane of places defines some isomorphism of $\boldsymbol{T}_{wt}^m$ on the tree of places T_n^m, where the set of through paths in the tree of places T_n^m is the set of prototypes of through paths in $\boldsymbol{T}_{wt}^m$, and is determined by the working numbering of the vertices in the tree $\boldsymbol{T}_{wt}^m$. By the *disposition* of $\boldsymbol{T}_{wt}^m$ on the plane of places we mean the set of all possible superpositions of $\boldsymbol{T}_{wt}^m$ for the same set $\boldsymbol{I}^m = \left(I_i^m, i < \beta_m\right)$ of prototypes of through paths of the tree $\boldsymbol{T}_{wt}^m$. The disposition is uniquely determined by specifying any one of the impositions, the set of which forms it. The asignment of the sequence of main isomorphisms $\left(T_{wt,0}^k, k \le m\right)$ uniquely determines some superposition and thereby determines some disposition $\boldsymbol{T}_{wt}^m$ on the plane of places. The disposition $\boldsymbol{T}_{wt}^m$ continues the disposition $\boldsymbol{T}_{wt}^l$ if and only if the set $\boldsymbol{I}^m$ defining the disposition $\boldsymbol{T}_{wt}^m$ continees the set $\boldsymbol{I}^l$ defining the disposition $\boldsymbol{T}_{wt}^l$.

Let there be a continuing sequence of sets of related sequences of places $\left(\boldsymbol{I}^m, m < \propto\right)$. The tree $\boldsymbol{T}_{wt}^m$ for $m < \propto$ can always be strongly imposed on the set $\boldsymbol{I}^m = \left(I_i^m = I_0^m \times PI_i^m, i < \beta_m\right)$ in different ways, the totality of which determines the disposition of $\boldsymbol{T}_{wt}^m$ on the plane of places, for which $\boldsymbol{I}^m$ is the set of prototypes of through paths of the tree $\boldsymbol{T}_{wt}^m$ (lemmas 32, 37). In this case, we will speak of an $\boldsymbol{I}^m$-disposition. Each continuing sequence $\left(\boldsymbol{I}^m = \left(I_i^m = I_0^m \times PI_i^m, i < \beta_m\right), m < \propto\right)$ determines a continuing sequence of $\boldsymbol{I}^m$-dispositions of $\boldsymbol{T}_{wt}^m$ $\left(0 \le m < \propto\right)$ on the plane of places (see [3]).

Let us move on to the final part of the article. We want to prove that any continuing sequence of sets of related sequences of places $\left(\boldsymbol{I}^k = \left(I_i^k, i < \beta_k\right), k < \propto\right)$ has a through path. This will imply a contradiction in set theory (theorem 4).

Let T_w be isomorphic to T_t and $\boldsymbol{T}_{wt}$ be a through homogeneous tree. Let $\boldsymbol{W}^m = \left(W_i^m, i < \beta_m\right)$ be the set of through paths in the tree $\boldsymbol{T}_{wt}^m$. $\boldsymbol{W}^m$ completely determines the tree $\boldsymbol{T}_{wt}^m$ (and for simplicity can be identified with it).

Each (isomorphic) mapping of $\boldsymbol{W}^m$ onto $\boldsymbol{I}^m$ uniquely determines a mapping of $\boldsymbol{T}_{wt}^m$ onto $\boldsymbol{I}^m$ and is uniquely determined by (any) leading pair $\left(W_i^m, I_j^m\right)$, where $i, j < \overline{\beta}_m$. We denote this mapping by $\boldsymbol{WI}_{ij}^m$. In this case, we have the following individual path mappings: $\left(W_i^m \times PI_r^m, I_j^m \times PI_r^m\right), r < \beta_m$ (see lemma 32). If we fix $i = 0$ and allow j to take on all admissible values, we obtain the mappings $\boldsymbol{WI}_{0j}^m = \left(\left(W_r^m, I_j^m \times PI_r^m\right), r < \beta_m\right)$, $j < \overline{\beta}_m$, covering all possible $\mathbf{W}\boldsymbol{I}^m$-mappings. Therefore, we obtain a $\mathbf{W}\boldsymbol{I}^m$-disposition of $\boldsymbol{T}_{wt}^m$ on the plane of places. For clarity, we note that we regard trees obtained by mapping trees onto other ones as trees with *double vertices* (see section 3 of [2]). In view of what was said above, we can also speak of trees with *double paths*. A mapping of $\boldsymbol{W}^m$ onto $\boldsymbol{I}^m$ continues a mapping of $\boldsymbol{W}^l$ onto $\boldsymbol{I}^l$ if and only if the double paths of the first mapping continue the double paths of the second one. This means that if in the first mapping the path W_i^m is mapped onto I_j^m, then in the second mapping $cut\left(W_i^m, l\right)$ is mapped onto $cut\left(I_j^m, l\right)$.

Note that in a similar way, each (automorphic) mapping of $\boldsymbol{W}^m$ onto itself uniquely determines a mapping of $\boldsymbol{T}_{wt}^m$ onto itself (an automorphism of $\boldsymbol{T}_{wt}^m$) and is uniquely determined by (any) leading pair $\left(W_i^m, W_j^m\right)$, where $i, j < \overline{\beta}_m$. In this case, the following individual mappings of paths take place: $\left(\left(W_i^m \times PI_r^m, W_j^m \times PI_r^m\right), r < \beta_m\right)$.

Recall that we assume (and this does not reduce the generality of the results) that in the trees T_w^m at all levels with non-limit, m there are no final vertices. So, at these levels $\overline{\beta}_m = \beta_m$. Thus, $\overline{\beta}_m < \beta_m$ it can only take place in the case of limit m.

Let $m < \infty$ and a non-final continuing sequence of place sequences $\left(I^k, k \le m\right)$ be fixed. The sequence $I^{\boldsymbol{m}}$ defines mappings of $\boldsymbol{W}^m$ on the place plane $\left(\left(W_i^m \times PI_r^m, I^m \times PI_r^m\right), r < \beta_m\right)$, where i varies, taking values less than $\overline{\beta}_m$. In the mapping $\left(\left(W_i^m \times PI_r^m, I^m \times PI_r^m\right), r < \beta_m\right)$, the leading pair is the pair $\left(W_i^m, I^m\right)$. The set of these mappings forms a disposition of $\boldsymbol{W}^m$ on the place plane, where $\boldsymbol{I}^m = \left(\left(I^m \times PI_r^m\right), r < \beta_m\right)$. Note that pairs $\left(W_i^m, I^m\right)$ with different W_i^m belong to different impositions. Next, we introduce in an admissible way a sequence of places I^{m+1}, continuing the sequence $I^{\boldsymbol{m}}$ (this means that we add n_p^{m+1} to I^m with $p < \overline{\beta}_{m+1}$). As a result, we obtain the superpositions on the plane of places $\left(\left(W_i^{m+1} \times PI_r^{m+1}, I^{m+1} \times PI_r^{m+1}\right), r < \beta_{m+1}\right)$, $i < \overline{\beta}_{m+1}$, continuing the corresponding superpositions $\left(\left(W_i^m \times PI_r^m, I^m \times PI_r^m\right), r < \beta_m\right)$, $i < \overline{\beta}_m$. Consequently, we obtain the disposition $\boldsymbol{W}^{m+1}$ on the plane of places, continuing the disposition of $\boldsymbol{W}^m$.

At the same time, for any $\boldsymbol{WI}^m$-disposition, taking any non-final sequence $I^m \in \boldsymbol{I}^m$, we arrive at the same $\boldsymbol{WI}^m$-disposition.

It is easy to see that this consideration will cover all dispositions of $\boldsymbol{W}^m$ on the plane of places and all cases where the disposition of $\boldsymbol{W}^{m+1}$ continues the disposition of $\boldsymbol{W}^m$.

The same holds when m is a limit ordinal and all continuing sequences of dispositions of $\boldsymbol{W}^k, k < m$ on the plane of places have already been constructed. An admissible sequence of places I^{m-0} generates a class of related sequences of places $\boldsymbol{I}^{m-0} = \left(I^{m-0} \times cut\left(PI_r^m, m-0\right), r < \beta_m\right) = \left(I_r^{m-0}, r < \beta_m\right)$. Let I^m continue I^{m-0} in an admissible way (this means that we add n_p^m with $p < \overline{\beta}_m$ to I^{m-0}). Then I^m generates a disposition of $\boldsymbol{W}^m$, which continues the disposition $\boldsymbol{W}^{m-0}$ generated by I^{m-0}. For clarity, note that the choice of I^{m-0} determines the choice of a class of non-final strongly related sequences of places

$\overline{\boldsymbol{I}}^{m-0} = \left(I^{m-0} \times cut\left(PI_r^m, m-0\right), r < \overline{\beta}_m\right)$, the continuations of which form a set $\boldsymbol{I}^m$, which has the final part in $\overline{\beta}_m < \beta_m$.

The above considerations make convincing the assertion that every continuing sequence of dispositions $\boldsymbol{T}_{wt}^m$ is generated by some continuing sequence of places $\left(I^k, k < \propto\right)$ and hence has as its limit some disposition of $\boldsymbol{T}_{wt}$ on the plane of places determined by the sequence $I^{\propto-0}$, which is the limit of the sequence $\left(I^k, k < \propto\right)$. Therefore, every time the continuing sequence of sets $\left(\boldsymbol{I}^m, m < \propto\right)$ (determining some continuing sequence of dispositions $\boldsymbol{T}_{wt}^m$) has a through path, and we obtain a contradiction in set theory.

3. Conclusion

The work presented in this article complements the works [2] [3], filling the gap contained at the end of the work [3]. All steps of the proof in these works are based on generally accepted informal set-theoretic reasoning, but take into account the prohibitions that were included in axiomatic set theories in order to overcome the difficulties encountered by the naive Cantor set theory. So, due to these works, existing axiomatic set theories (in particular, ZFC theory) are inconsistent.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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List of Basic Notations

$cut(T,k)$ —tree obtained by cutting tree T at level k (has height $k+1$);

$height(T)$ —height of tree T;

n_i^k —place with number i in row with number k on plane of places;

P_{ti}^m —automorphism of tree T_t^m with number i in primary numbering;

$T, T_w, T_v, T_t, T_w^m, T_v^m, T_t^m, T_{wt}^m, \boldsymbol{T}_{wt}^m$ - notations for trees;

T^m, T^{m-0} —tree of height m and tree obtained from T^m by deleting vertices of the upper level m;

T_w^m —tree of height m with vertices w_i^k;

T_{wt}^m —tree representing isomorphism of trees T_w^m and T_t^m (double-vertex tree);

$\boldsymbol{T}_{wt}^m$ —isomorphism tree for trees T_w^m and T_t^m (at level k, tree vertices are trees T_{wt}^k representing isomorphisms of trees T_w^k and T_t^k);

$\boldsymbol{T}_{tt}^m$ —automorphism tree for tree T_t^m;

$(T^k, k \le m)$ when $T^k = cut(T^m, k)$ —continuing sequence of trees;

$T^k \le T^l$ means that $T^k = cut(T^l, k)$;

T_{str} —a strange tree from section 4 [2];

$(v^k, v^{k+1}, \cdots)$ —continuing sequence of vertices (path) in the tree T: for all k,l with $k<l$ $v^k \le v^l$ holds;

w_i^k —the vertice with number i at level with number k in the tree T_w^m;

$\boldsymbol{W}^m$ —the set of through paths in the tree T^m under consideration (gives a complete picture of the tree T^m and can be identified with it);

W_i^m —the path with number i in the set $\boldsymbol{W}^m$.