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On Fuzzy Regularly Closed Filters in Michálek Fuzzy Topological Space

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Abstract

C.L. Chang's introduction of fuzzy topology in 1981 opened up new avenues for parallel theories in topology. However, Chang's work appears to focus more on the topology of fuzzy sets rather than fuzzy topology itself. In 1975, Michálek presented a functional definition of ordinary topology and later developed fuzzy topology as a distinct extension of this idea, setting it apart from Chang's approach. While there has been significant research on Chang's fuzzy topology, Michálek's version has not received as much attention. This paper introduces the concept of fuzzy regularly closed filters, or FRC_M filters, within Michálek's fuzzy topological space and explores some properties of FRC_M ultrafilters.

Keywords

Topology, Fuzzy Topology, Michálek's Fuzzy Topological Space, Fuzzy Closed, Fuzzy Open, Filters

1. Introduction

A topological space Y that incorporates a dense subspace X is referred to as an **extension of** X. This concept is pivotal in topology, as it allows researchers to explore the properties of X within a larger framework. The study of extensions, particularly through the lens of **compactification** [1], has garnered significant attention among topological researchers.

The construction of extension points often leverages ultrafilters on various lattices associated with X. An ultrafilter can be thought of as a special type of filter that captures the notion of "largeness" in a precise way, facilitating the identification of limit points and compactifications. This area of study gained momentum following the introduction of Stone Ĉech Compactification in the 1930s, a landmark result that set the stage for extensive research into different forms of compactification and extension. Scholars have since built on this foundation, exploring various methods and implications of extending spaces.

In parallel with extensions, the concept of **Absolutes** [2] has emerged as a crucial dual notion in topology. Construction of absolutes of space often involves the use of ultrafilters on lattices connected to X. While extensions and Absolutes may appear to be fundamentally opposite, they share a common toolset. Both are constructed using ultrafilters on lattices, illustrating the rich interplay between these concepts.

In 1968, C. L. Chang [3] introduced the notion of Fuzzy Topology, marking a significant shift in topological theory. Fuzzy topology provides a framework to incorporate the concept of vagueness into topological spaces, thus allowing for a more nuanced understanding of continuity and convergence. This innovation opened avenues for researchers to parallelly develop theories that adapt regular topology principles to accommodate fuzzy sets and spaces [4]-[7].

Subsequently, in 1975, J. Michálek [8] developed and studied a more generalized form of fuzzy topological space that deviates from Chang's original definition. Michálek's approach sought to broaden the conceptual landscape of fuzzy topology, introducing a new paradigm that defines fuzzy topological spaces in terms of functions, which allowed for greater flexibility and application in various contexts.

In 2001, Francisco Galligo Lupranez [9] explored several features of Michálek's fuzzy topological space, contributing to a deeper understanding of its properties and potential applications. Additionally, the concept of fuzzy filters was first proposed by M. A. De Prade [10] in 1981, building on Chang's fuzzy topology. This idea was further examined by Francisco Galligo Lupranez [11], who analysed filters within the framework established by Michálek.

As a prerequisite for constructing the Fuzzy Absolute in the Michálek sense, we introduce FRC_M filters and FRC_M ultrafilters. P. R. Neethu [12] has already introduced FRC_M sets, laying the groundwork for further research. This ongoing exploration continues to enrich the field of topology, blending traditional ideas with innovative approaches to expand our understanding of complex spatial relationships.

2. Preliminaries

This section includes the basic ideas required for the subsequent results in the present article.

Michálek made notable contributions by developing a more generalized form of fuzzy topological space that diverged from Chang's original definition. Michálek's framework was designed as an extension of conventional topological spaces, introducing a novel approach that defines fuzzy topological spaces in terms of functions. This new perspective allowed for greater flexibility and broadened the applications of fuzzy topology.

Definition 2.1 [8]: Consider a nonvoid set X and its power set $P(X)$, the function $u: P(X) \to P(X)$ satisfies the conditions below

1) If a subset A of X is either empty or singleton, then $uA = A$.

2) If A_1, A_2 are two different subsets of X, then $u(A_1 \cup A_2) = uA_1 \cup uA_2$

Then the pair (X, u) is a topological space.

Definition 2.2 [8]: Let F be the collection of all fuzzy sets on X and u, a function from $P(X)$ to F such that

1) $uA(x) = 1$ for every x in A, A is any subset of X.

2) $uA(x) = \chi_A(x)$ where subset A of X contains at most one element.

3) $u(A \cup B)(x) = Max\{uA(x), uB(x)\}$, A, B are any two subsets of X.

Then the pair (X, u) is a fuzzy topological space.

Note 2.3: Unless otherwise stated (X, u) or simply X is a Michálek fuzzy topological space or in short *M*-fuzzy topological space.

Definition 2.4 [8]: Consider *M*-fuzzy topological space and the subset A of X. Then the

1) Fuzzy Interior of A denoted by μ_{A^o} is defined as

$\mu_{A^o}(x) = 1 - (uA^C)(x)$.

2) Fuzzy Boundary of A denoted by, $\mu_{\delta A}$ is defined as

$\mu_{\delta A}(x) = Min\{uA(x), uA^c(x)\}$.

Definition 2.5 [8]: An ordinary subset A of X is said to be Fuzzy closed in X if $\chi_A(x) \geq Min\{uA(x), uA^c(x)\}$ for every x in X and A is Fuzzy open in X if $Min\{\chi_A(x), \mu_{\delta A}(x)\} = 0$ forevery x in X.

Note 2.6 [8]: A subset C of X is fuzzy closed whenever $uC = \chi_C$ and fuzzy open whenever $\mu_{C^o} = \chi_C$.

Definition 2.7 [12]: Consider the *M*-fuzzy topological space (X, u). Then

1) $\mu \in I^X$ is fuzzy closed whenever $\mu = u_A$ for some fuzzy closed subset A of X.

2) $\mu \in I^X$ is fuzzy open whenever $\mu = u_B$ for some fuzzy open subset B of X.

Definition 2.8 [12]:

1) for $\mu \in I^X$, closure of $\mu(\bar{\mu})$ is defined as

$\bar{\mu} = \inf\{\gamma \in I^X : \gamma \geq \mu, \gamma \text{ is fuzzy closed}\}$ which is equivalent to

$\bar{\mu}(x) = Min\{u_A(x) : u_A(x) \geq \mu(x)\}$ to each x in X where A is fuzzy closed in X.

2) for $\mu \in I^X$, interior of $\mu(\mu^o)$ is defined as,

$\mu^o = \sup\{\delta \in I^X : \delta \leq \mu, \delta \text{ is fuzzy open}\}$ which is equivalent to

$\mu^o(x) = Max\{uA(x) : uA(x) \leq \mu(x)\}$ to each x in X where A isfuzzy open in X.

Definition 2.9 [12]: For any fuzzy closed subset A of X

1) *M*-fuzzy closure of A denoted by $\mu_{\bar{A}}$ is defined as

$\mu_{\bar{A}}(x) = Min\{u_A(x) : u_A(x) \geq \chi_A(x)\}$.

2) A is *M*-fuzzy regularly closed if it satisfies the condition $fcl(f\,intA) = \chi_A$.

Note 2.10 [12]**:** Set of all *M*-fuzzy regularly closed set in X is denoted as $FRC_M(X)$.

Definition 2.11 [12]**:** For any fuzzy open subset A of X, A is *M*-fuzzy regularly open whenever $f\,int(f\,clA) = \chi_A$.

Note 2.12 [12]**:** Set of all *M*-fuzzy regularly open sets in X is denoted as $FRO_M(X)$.

Definition 2.13 [12]**:** For any two members A, B of $FRC_M(X)$, $A \subseteq B$ if it satisfies the condition $u_A(x) \leq u_B(x)$ for each x in X. Then "$\subseteq$" gives a partial order on $FRC_M(X)$ and hence $FRC_M(X)$ is a poset.

If C and D are any two *M*-fuzzy regularly closed sets in X then $C \cup D$ is *M*-fuzzy regularly closed but their intersection need not be *M*-fuzzy regularly closed.

Theorem 2.14 [12]**:** $(FRC_M(X), \subseteq)$ is a complemented lattice with

$B^C = fcl(X - B) = X - (f\,int(B))$ for $B \in FRC_M(X)$.

Theorem 2.15 [12]**:** $FRC_M(X)$ is a complete lattice.

Definition 2.16 [11]**:** A nonvoid family F of $P(X)$ is a filter on X whenever F satisfies the conditions below

i) All members of F be nonvoid.

ii) Intersection of any two sets in F is in F.

iii) Supersets of members in F is in F.

3. FRC_M Filters

Definition 3.1: A non-empty subset $\mathfrak{F}$ of $FRC_M(X)$ is said to be an *M*-fuzzy regularly closed filter or in short FRC_M filter if it satisfies the conditions below

i) $A \in \mathfrak{F}$ and $B \in FRC_M(X)$ such that $B \supset A$ imply $B \in \mathfrak{F}$;

ii) $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$;

iii) $0 \notin \mathfrak{F}$.

Definition 3.2: If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are FRC_M filters on X, $\mathfrak{F}_1$ is finer than $\mathfrak{F}_2$ if and only if $\mathfrak{F}_1 \supset \mathfrak{F}_2$. An FRC_M filter $\mathfrak{F}$ on X is an FRC_M-ultrafilter if there is no other FRC_M-filter finer than $\mathfrak{F}$.

Note 3.3: Zorns lemma guaranteed the existence of FRC_M ultra filters.

Definition 3.3: Suppose $\mathfrak{F}$ is an FRC_M-filter on X. Then a crisp regularly closed subset H of X is said to be subsumed in $\mathfrak{F}$ whenever every *M*-fuzzy regularly closed subset of X with support H belongs to $\mathfrak{F}$.

Theorem 3.4: If $\mathfrak{F}$ is an FRC_M-filter on X then the below conditions imply each other.

i) $\mathfrak{F}$ is an FRC_M ultrafilter.

ii) Let $A \in FRC_M(X)$. If $A \notin \mathfrak{F}$ then there is some $B \in \mathfrak{F}$ such that $A \cap B = 0$.

iii) Let U be a regularly closed subset of X. Then either U or U^c is subsumed in $\mathfrak{F}$ where $U^c = cl(X/U)$.

Proof.

i) $\Rightarrow$ ii)

Let $A \notin \mathfrak{F}$. If $A \cap B \neq 0$ for every $B \in \mathfrak{F}$ the collection $V = \{A \cap B : B \in \mathfrak{F}\}$ is a base for an FRC_M-filter which is finer than $\mathfrak{F}$, contradicting i). Therefore there exists at least one $B \in \mathfrak{F}$ such that $A \cap B = 0$.

ii) $\Rightarrow$ iii)

Let U be a regularly closed subset of X. Suppose both U and U^C were not subsumed in $\mathfrak{F}$. Therefore, there exist $A, B \in FRC_M(X)$ with supports U and U^C respectively such that both do not belong to $\mathfrak{F}$. By part ii) if $A \notin \mathfrak{F}$, there exist a $C \in \mathfrak{F}$ such that $A \cap C = 0$. Similarly, there is a $D \in \mathfrak{F}$ such that $B \cap D = 0$. Suppose $A = U$. Therefore $A(x) \neq 0$ for every $X \in U$. So $C(x) = 0$ for every $X \in U$. Suppose $B = U^c$.

Therefore $B(x) \neq 0$ for every $X \in U^c$. So $D(x) = 0$ for every $X \in U^c$. Therefore $(C \cap D)(x) = 0$ for every $x \in X$. That is, $C \cap D = 0$ which is not possible since $C, D \in \mathfrak{F}$. Therefore either U or U^c is subsumed in $\mathfrak{F}$.

iii) $\Rightarrow$ i)

If $\mathfrak{F}$ is not an FRC_M ultrafilter, let $G \supset \mathfrak{F}$.

Let $B \in G$ such that $B \notin F$ and suppose $B = U$.

That is $U \notin \mathfrak{F}$ and hence U is not subsumed in $\mathfrak{F}$. Hence by iii) U^c is subsumed in $\mathfrak{F}$. That is any FRC_M set C with support U^c belongs to $\mathfrak{F}$ and hence in G. Therefore $B \cap C \in G$ which is not possible since $B \cap C = 0$. Therefore $\mathfrak{F}$ is an FRC_M ultrafilter.

That is $U^c(x) \neq 0$ $\forall x$ in X.

4. Application

Some features of topology are consistently maintained through perfect continuous mappings. Therefore, when a space X exhibits a new topological property P, we need to identify a space that is either a perfect continuous image of X or can be mapped onto X by a perfect continuous surjection. If we find such a space, we can investigate its properties and infer characteristics of X from it. The absolute of X is a well-structured space that facilitates this process, as there exists a perfect surjection from the absolute to X. In constructing the absolute, the points correspond to ultrafilters on the lattice of regularly closed subsets of X. A similar approach can be applied in fuzzy topology, as introduced by Michálek, where FRC_M ultrafilters serve as the points of fuzzy absolutes in Michálek's framework, creating a space that can be easily handled effectively.

5. Conclusion

By investigating absolutes and ultrafilters, mathematicians gain valuable insights into the characteristics of X and its properties. The introduction of fuzzy topology adds further complexity and depth, enhancing the dynamic nature of the study of topology. Additionally, filters and ultrafilters play a crucial role not only in theoretical topology but also as practical tools for constructing and analysing diverse topological spaces. Their applications in areas such as convergence, compactness, and quotient topology highlight their versatility and importance in the field. This ongoing exploration not only deepens our mathematical knowledge but also opens up practical avenues in fields such as topological data analysis.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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