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Existence of a Nontrivial Solution for a Class of Superquadratic Elliptic Problems*

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ABSTRACT

We consider the existence of a nontrivial solution for the Dirichlet boundary value problem

$$\begin{cases} -\Delta u + a(x)u = g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We prove an abstract result on the existence of a critical point for the functional f on a Hilbert space via the local linking theorem. Different from the works in the literature, the new theorem is constructed under the $(C)^*$ condition instead of $(PS)^*$ condition.

Keywords: Elliptic Problems; Local Linking Theorem; $(C)^*$ Condition; Superquadratic

1. Introduction and Main Results

Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u + a(x)u = g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $a(x) \in L^p(\Omega), p > N/2, g \in C(\bar{\Omega} \times R, R)$ and $\Omega \subset R^N (N \ge 3)$ is a bounded domain whose boundary is a smooth manifold.

We assume that G, where $G(x,u) = \int_0^u g(x,s)\mathrm{d}s$. In [1], Li and Willem established the existence of a nontrivial solution for problem (1) under the following well-known Ambrosetti-Rabinowitz superlinearity condition: there exists $\mu > 2$ and $L > 0$ such that

$$0 < \mu G(x,u) < ug(x,u) \tag{AR}$$

for all $|u| \ge L$ and $x \in \Omega$, which has been used extensively in the literature; see [1-4] and the references therein. It is easy to see that condition (*AR*) does not include some superquadratic nonlinearity like

$$G(x,u) = |u|^2 \left(\ln\left(\frac{1}{3}|u|^4 - |u|^2 + 1 \right) \right)^3 \tag{G0}$$

In [5], Qin Jiang and Chunlei Tang completed the Theorem 4 in [1], and obtained the existence of a nontrivial solution for problem (1) under a new superquadratic condition which covered the case of (G0). The conditions are as follows:

(G1) $G(x,u)/|u|^2 \to +\infty$, *as* $|u| \to +\infty$ *uniformly on* Ω,

(G2) $G(x,u)/|u|^2 \to 0$, *as* $|u| \to 0$ *uniformly on* Ω,

(G3) *There are constants* $1 < \lambda < \frac{N+2}{N-2}$ *and* $a_1 > 0$ *such that*

$$|g(x,u)| \le a_1 \left(|u|^\lambda + 1 \right)$$

for all $(x,u) \in \Omega \times R$,

(G4) *There are constants* $\beta > \frac{2N}{N+2}\lambda$, $a_2 > 0$ *and* $L > 0$ *such that*

$$ug(x,u) - 2G(x,u) \ge a_2 |u|^\beta$$

for all $|u| \ge L$ *and* $x \in \Omega$,

If 0 is an eigenvalue of $-\Delta + a$ (with Dirichlet boundary condition) assume also the condition that:

(G5) *There exists* $\delta > 0$ *such that*:

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1) $G(x,u) \ge 0$, *for all* $|u| \le \delta$, $x \in \Omega$; *or*
2) $G(x,u) \le 0$, *for all* $|u| \le \delta$, $x \in \Omega$.

Note that (G4) is also (AR) type condition.

The aim of this paper is to consider the nontrivial solution of problem (1) in a more general sense. Without the Ambrosetti-Rabinowitz superlinearity condition (*AR*) or (*G*4), the superlinear problems become more complicated. We do not know in our situations whether the (*PS*) or $(PS)^*$ sequence are bounded. However, we can check that any Cerami (or $(C)^*$) sequence is bounded. The definition of $(PS)^*$ (or $(C)^*$) sequence can be found in [6].

We will obtain the same conclusion under the $(C)^*$ condition instead of $(PS)^*$ condition. So we only need the following conditions instead of (*G*3) (*G*4):

(*G*3') Let $\tilde{G} = \frac{1}{2} g(x,u)u - G(x,u)$ satisfying

1) $\tilde{G} \ge a_3 |u|^{\beta}$ if $|u| \ge R$,

2) $|g(x,u)|^{\sigma} / |u|^{\sigma} \le a_4 \tilde{G}(x,u)$ if $|u| \ge R$, where

$a_3, a_4 > 0$, $\sigma > \frac{N}{2} + 1, q = \frac{\sigma+1}{\sigma-1}, \beta > q+1$.

It is easy to see that function

$$G(x,u) = \frac{1}{2}|u|^{\frac{8}{3}} + |u|^2 \ln\left(1+|u|^2\right)$$

satisfies conditions of (*G*1) (*G*2) (*G*5) and (*G*3').

Our main result is the following theorem:

Theorem 1.1. *Suppose that* G *satisfies* (*G*1) (*G*2) (*G*5) *and* (*G*3'). *If* 0 *is an eigenvalue of* $-\Delta + a$ (*with Dirichlet boundary condition*). *Then problem* (1) *has at least one nontrivial solution.*

Remark 1. There are many functions which are superlinear but it is not necessary to satisfy Ambrosetti-Rabinowitz condition. For example,

$$f(x,u) = \theta |u|^{\theta-2} u + (\theta-1)|u|^{\theta-3} u \sin^2 u + |u|^{\theta-1} \sin 2u,$$
$$u \in R \setminus \{0\}$$

where $\mu > 2$. Then it is easy to check that (*AR*) does not hold. On the other hand, in order to verify (*AR*), it usually is an annoying task to compute the primitive function of f and sometimes it is almost impossible. For example,

$$f(x,u) = |u| u \left(1 + e^{(1+|\sin u|)^{\alpha}} + |\cos u|^{\alpha}\right), \quad u \in R$$

where $\alpha > 0$.

Remark 2. Our condition is much weaker than (*AR*) type condition (cf. [6]).

2. Proof of Theorem

Define a functional f in the space $E = H_0^1(\Omega)$ by

$$f(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\Omega} G(x,u)\mathrm{d}x,$$

where $u^- \in E^-, u^+ \in E^+$, $E^+(E^-)$ is the space spanned by the eigenvectors corresponding to negative (positive) eigenvalue of $-\Delta + a$.

In this paper, we shall use the following local linking theorem (Lemma 2.1) to prove our Theorem . Let X be a real Banach space with $X = X^1 \oplus X^2$ and $X_0^j \subset X_1^j \subset \cdots \subset X^j$ such that $X^j = \bigcup_{n \in N} X_n^j$, $j = 1,2$. For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, let $X_\alpha = X_{\alpha_1}^1 \oplus X_{\alpha_2}^2$. We know that $\alpha \le \beta \Leftrightarrow \alpha_1 \le \beta_1$, $\alpha_2 \le \beta_2$. A sequence $(\alpha_n) \subset N^2$ is admissible if for every $\alpha \in N^2$ there is $m \in N$ such that $n \ge m \Rightarrow \alpha_n \ge \alpha$. We say $f \in C^1(X,R)$ satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and satisfies

$$u_{\alpha_n} \in X_{\alpha_n}, supf\left(u_{\alpha_n}\right) < \infty, \left(1+\|u_{\alpha_n}\|\right) f'\left(u_{\alpha_n}\right) \to 0$$

contains a subsequence which converges to a critical point of f.

Lemma 2.1. ([6]) *Suppose that* $f \in C^1(X,R)$ *satisfies the following assumptions*:

(f_1) f has a local linking at 0,

(f_2) f satisfies $(C)^*$ condition,

(f_3) f maps bounded sets into bounded sets,

(f_4) For every $m \in N$, $f(u) \to -\infty$ as $\|u\| \to \infty$, on $u \in X_m^1 \oplus X^2$.

Then f has at least two critical points.

Proof of Theorem 1. We shall apply Lemma 2.1 to the functional f associated with (1), we consider the case where 0 is an eigenvalue of $-\Delta + a$ and

$$G(x,u) \le 0 \text{ for } |u| \le \delta. \tag{2}$$

The other case are similar.

1) $f \in C^1(X,R)$ and f maps bounded sets into bounded sets.

Let $X^2 = X^-$, $X^1 = X^+ \oplus X^0$. Choose Hilbertian basis $(e_n)_{n\ge1}$ for X^1 and $(e_n)_{n\le-1}$ for X^2, define

$$X_n^1 := span\{e_1, \cdots, e_{n+1}\}, n \in N;$$

$$X_n^2 := span\{e_{-1}, \ldots, e_{-n-1}\}, n \in N.$$

Assumption (*G*3') implies there are constants $C_1, C_2 > 0$ such that

$$|g(x,u)| \le C_1 + C_2 |u|^q, \tag{3}$$

so

$$\begin{aligned}|G(x,u)| &= \left|\int_0^1 g(x,su)u\mathrm{d}s\right| \\ &\le \int_0^1 \left(C_1|u| + C_2|s|^q|u|^{q+1}\right)\mathrm{d}s \\ &\le C_3 + C_4|u|^{q+1},\end{aligned}$$

where $q=\dfrac{\sigma+1}{\sigma-1}>1$.

Hence $f\in C^1(X,R)$ and maps bounded sets into bounded sets.

In fact,

$$\begin{aligned}|f(u)|&=\left|\frac{1}{2}\|u^+\|^2-\frac{1}{2}\|u^-\|^2-\int_\Omega G(x,u)\,\mathrm{d}x\right|\\&\le\frac{1}{2}\|u\|^2+\int_\Omega\left(C_3+C_4|u|^{q+1}\right)\\&\le\frac{1}{2}\|u\|^2+|\Omega|C_3+C_5\|u\|,\end{aligned}$$

so (f_3) holds.

2) f has a local linking at 0.

It follows from ($G2$) and ($G3$) that, for any $\varepsilon>0$, there exists $C_\varepsilon>0$, such that

$$|G(x,u)|\le\varepsilon|u|^2+C_\varepsilon|u|^{q+1},\tag{4}$$

we obtain, on X^2, for some $c>0$,

$$\begin{aligned}f(u)&\le-\frac{1}{2}\|u\|^2+\varepsilon\int_\Omega|u|^2+C_\varepsilon\int_\Omega|u|^{q+1}\\&\le-\frac{1}{2}\|u\|^2+\varepsilon C_6\|u\|^2+C_7\|u\|^{q+1}\\&=\left(-\frac{1}{2}+\varepsilon C_6+C_7\|u\|^{q-1}\right)\|u\|^2,\end{aligned}$$

choosing $r_1=\dfrac{1}{\sqrt[q-1]{4C_7}}$, then $f(u)\le 0,\ u\in X^2,\ \|u\|\le r_1$.

Decompose X^1 into $V+W$ when $V=ker(-\Delta+a)$, $W=\left(X^2+V\right)^\perp$. Also set $u=v+w,u\in X^1,v\in V,w\in W$. Since V is a finite-dimensional space, there exists $C>0$, such that

$$\|v\|_\infty\le C\|v\|,\forall v\in V.\tag{5}$$

First we set $\|u\|\le\delta/2C$ and

$$\Omega_1=\left\{x\in\Omega\,\middle|\,|w(x)|\le\delta/2\right\},\Omega_2=\Omega\setminus\Omega_1.$$

On Ω_1, we have, by (5)

$$|u(x)|\le|v(x)|+|w(x)|\le\|v\|_\infty+\delta/2\le\delta,$$

hence, by (2)

$$\int_{\Omega_1}G(x,u)\,\mathrm{d}x\le 0.$$

On Ω_2, we have also by (5)

$$|u(x)|\le|v(x)|+|w(x)|\le 2|w(x)|,$$

hence, by (4)

$$\begin{aligned}G(x,u)&\le\varepsilon u^2+C_\varepsilon|u|^{q+1}\\&\le 4\varepsilon w^2+2^{q+1}C_\varepsilon|w|^{q+1},\end{aligned}$$

and for some $c>0$

$$\int_{\Omega_2}G(x,u)\,\mathrm{d}x\le\int_{\Omega_2}\left(4\varepsilon w^2+c|w|^{q+1}\right).$$

Therefore we deduce that

$$\begin{aligned}f(u)&\ge\frac{1}{2}\|w\|^2-\int_{\Omega_2}\left(4\varepsilon w^2-c|w|^{q+1}\right)-\int_{\Omega_1}G(x,u)\\&\ge\frac{1}{2}\|w\|^2-\varepsilon C_8\|w\|^2-C_9\|w\|^{q+1}\\&=\left(\frac{1}{2}-\varepsilon C_8-C_9\|w\|^{q-1}\right)\|w\|^2,\end{aligned}$$

choosing $r_2=\dfrac{1}{\sqrt[q-1]{4C_9}}$, then $f(u)\ge 0,\ u\in X^1,\ \|u\|\le r_2$,

Let $r=\min\left\{\dfrac{\delta}{2C},r_1,r_2\right\}$, then ($f_1$) holds.

3) f satisfies $(C)^*$ condition.

Consider a sequence (u_{α_n}) such that (α_n) is admissible and

$$u_{\alpha_n}\in X_{\alpha_n},c:=\sup_n f(u_{\alpha_n})<\infty,$$

$$\left(1+\|u_{\alpha_n}\|\right)f'(u_{\alpha_n})\to 0.$$

I) (u_{α_n}) is bounded.

For n large, from assumption ($G3'$), with $u_n=u_{\alpha_n}$, for some $c>0,m>0$,

$$\begin{aligned}c&\ge f(u_n)-\frac{1}{2}f'(u_n)u_n=\int_\Omega\tilde{G}(x,u_n)\\&\ge\int_{\Omega_R}a_3u_n^\beta+\int_{\Omega_R^c}\tilde{G}(x,u_n)\\&\ge\int_{\Omega_R}a_3u_n^2+\int_{\Omega_R^c}\tilde{G}(x,u_n)\ge\int_{\Omega_R}a_3u_n^2+m,\end{aligned}$$

where $\Omega_R=\left\{x\in\Omega\,\middle|\,|u(x)|>R\right\},\Omega_R^c=\Omega\setminus\Omega_R$.

So

$$\int_\Omega u_n^2=\int_{\Omega_R}u_n^2+\int_{\Omega_R^c}u_n^2\le c.\tag{6}$$

Arguing indirectly, assume $\|u_n\|\to\infty$. Set $v_n=u_n/\|u_n\|$, Then $\|v_n\|=1$ and $|v_n|_s\le C_s$ for all $s\in[1,\infty)$. In addition, using (6)

$$\int_\Omega v_n^2=\frac{1}{\|u_n\|^2}\int_\Omega u_n^2\le\frac{c}{\|u_n\|^2}\to 0,$$

hence by Interpolation inequality for L^p norms, for $s\in[1,\infty)$

$$\int_\Omega|v_n|^s\le\left(\int_\Omega|v_n|^t\right)^{(1-\theta)s}\left(\int_\Omega|v_n|^2\right)^{s\theta}\to 0,\tag{7}$$

where $\dfrac{1}{s}=\dfrac{\theta}{2}+\dfrac{1-\theta}{t},1\le t\le s\le 2$ or $2\le s\le t\le\infty$.

Since $dimker(-\Delta+a)<\infty$,

$$
\begin{aligned}
& f'(u_n)(u_n^+ - u_n^-) \\
&= \|u_n^+ + u_n^-\|^2 - \int_\Omega g(x,u_n)(u_n^+ - u_n^-)\mathrm{d}x \\
&= \|u_n\|^2 - \|u_n^0\|^2 - \int_\Omega g(x,u_n)(u_n^+ - u_n^-)\mathrm{d}x \\
&= \|u_n\|^2 \left(1 - \int_\Omega \frac{g(x,u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\mathrm{d}x\right) - \|u_n^0\|^2 \\
&\ge \|u_n\|^2 \left(1 - \int_\Omega \frac{g(x,u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\mathrm{d}x\right) - c|u_n^0|_2^2,
\end{aligned}
$$

so

$$1 - \int_\Omega \frac{g(x,u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\mathrm{d}x = o(1).$$

From (7) for some $c > 0$

$$
\begin{aligned}
&\left|\int_\Omega \frac{g(x,u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\right| \le 2\int_\Omega \frac{g(x,u_n)}{|u_n|}|v_n|^2 \\
&\le c\left(\int_\Omega \left(\frac{g(x,u_n)}{|u_n|}\right)^\sigma\right)^{\frac{1}{\sigma}} \left(\int_\Omega |v_n|^{2\sigma'}\right)^{\frac{1}{\sigma'}} \\
&\le c\left(\int_\Omega |v_n|^{2\sigma'}\right)^{\frac{1}{\sigma'}} \to 0,
\end{aligned}
$$

as $n \to \infty$, therefore, $1 = o(1)$, a contradiction. Hence $(C)^*$ sequence is bounded.

II) From (I) we see that (u_n) is bounded in X, going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in X. Since $dimker(-\Delta + a) < \infty$, $u_n^0 \to u^0$ in X.

$$
\begin{aligned}
\|u_n^+ - u^+\|^2 &= (f'(u_n) - f'(u))(u_n^+ - u^+) \\
&\quad + \int_\Omega (g(x,u_n) - g(x,u))(u_n^+ - u^+)
\end{aligned}
$$

which implies that $u_n^+ \to u^+$ in X. Similarly, $u_n^- \to u^-$ in X. It follows then that $u_n \to u$ in X and $f'(u) = 0$.

4) Finally, we claim that, for every $m \in N$,

$$f(u) \to \infty, \|u\| \to \infty, u \in X_m^1 \oplus X^2.$$

Indeed, by (G1) we have, there exists $M > 0, R > 0$ such that

$$G(x,u) \ge M|u|^2, |u| \ge R,$$

so on $X_m^1 \oplus X^2$,

$$
\begin{aligned}
f(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_\Omega G(x,u) \\
&\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - M|u^+|_2^2 - M|u^0|_2^2 \\
&\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - MC\|u^+\|^2 - MC\|u^0\|^2 \to -\infty, \\
&\|u\| \to \infty
\end{aligned}
$$

where $C > 0, MC > 0$. □

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