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Existence of Positive Solutions to Boundary Value Problems for Fractional Differential Equation with P-Laplacian Operators

Junrui Yue

Computer Information Engineering Institute, Shanxi Technology and Business University, Taiyuan, China
Email: yuejunrui@163.com

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Abstract

For the study of some complex physical, biological and other phenomena, it is difficult for the traditional integer differential equations to describe these processes accurately. In this paper, by means of the Guo-Krasnoselskii fixed point theorem, we study the existence of positive solutions to boundary value problems for a class of fractional differential equations with p-Laplacian operators.

$$\begin{cases} \left(D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)\right) = f\left(t,u(t)\right), \ t\in[0,1], \\ \left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)' = \left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right), \\ u(0)=u(1)=u'(0)=u'(1)=0, \end{cases}$$

where $\beta\in(1,2]$, $\alpha\in(3,4]$ are real numbers, both D_{0+}^{β} and D_{0+}^{α} are in the range of standard Riemann-Liouville derivatives, $\phi_p(s)=|s|^{p-2}s$, $p\in(1,+\infty)$, $\phi_p^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$, and $f\in C\left([0,1]\times[0,+\infty)\to[0,+\infty)\right)$.

Keywords

Fractional Differential Equation, P-Laplacian Operator, Fixed Point Index, Positive Solution

1. Introduction

In the field of modern mathematical analysis, the theory of differential equations likes a shining pearl, continuously exuding its fascinating charm and guiding numerous scientific researchers to explore its depth and breadth. With the continuous development of science and technology, when people study some complex physical, chemical, biological and other phenomena, they find that traditional integer differential equations are difficult to describe these processes accurately [1]-[3]. Boundary value problems is a key of differential equations, the study of boundary value problems of fractional differential equations not only greatly enriches the theoretical system of differential equations but also provides a powerful mathematical tool for solving practical problems. For example, in the heat conduction problem, by setting boundary value conditions such as the boundary temperature or heat flux density and solving the boundary value problem of the fractional heat conduction equation, can accurately master the temperature distribution inside the object, which is of great significance for the analysis of the thermal properties of materials and the optimization of industrial heating and cooling processes [4] [5].

Positive solutions often have clear physical meaning in practical applications, among numerous boundary value conditions, the boundary value conditions with the p-Laplacian operator are unique [6]-[14]. Currently, the research on the existence of positive solutions to the boundary value problems of fractional differential equations with boundary value conditions of the p-Laplacian operator is still relatively scarce. Existing research methods face many challenges in dealing with the coupling problem of such complex boundary value conditions and fractional differential equations, and there is an urgent need to develop new mathematical analysis methods and techniques. Therefore, fractional differential equations and differential equations with p-Laplacian operators have attracted extensive attention from mathematicians, and a large number of special studies have been conducted on various problems of fractional differential equations [15]-[23].

In [9], the existence of positive solutions to a class of fractional order differential equations with p-Laplacian operators is investigated, and using the iterative method of monotone boundary value problems, results are obtained for the existence of positive solutions:

$$\begin{cases} D^{\gamma}\left(\phi_p\left(D^{\alpha}u(t)\right)\right)=f\left(t,u(t)\right),\ t\in(0,1), \\ u(0)=D^{\alpha}u(0)=0, \\ D^{\beta}u(1)=aD^{\beta}u(\xi), \\ D^{\alpha}u(1)=bD^{\alpha}u(\eta), \end{cases}$$

where $\phi_p(s)=|s|^{p-2}s$, $p\in(1,+\infty)$, $1<\alpha\le 2$, $0<\beta\le\alpha-1$, $0<\xi<1$, $0<\eta<1$, $a\ge 0$, $b\ge 0$ and $1-a\xi^{\alpha-\beta-1}>0$, $1-b^{p-1}\eta^{\gamma-1}>0$, f is continuous on $[0,1]\times(0,+\infty)\to(0,+\infty)$, D^{α} is the Riemann-Liouville fractional derivative.

The differential equations discussed in the literature [9] contain fractional order derivative terms, based on which in this paper we consider differential equations with both integer solution derivative terms and fractional derivative terms of the p-Laplacian operator. Higher-order problems make derivatives more complex and impose higher requirements on the regularity of solutions. In this paper, by constructing a new cone, which require the construction of a new Banach space to satisfy the Guo-Krasnoselskii fixed point theorem. While the term p-Laplacian operator fractional order derivative does not appear in the above literature, in this paper we consider the following boundary value problem:

$$\begin{cases} \left(D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)\right) = f\left(t,u(t)\right),\ t\in[0,1], \\ \left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)' = \left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right), \\ u(0)=u(1)=u'(0)=u'(1)=0, \end{cases} \tag{1}$$

where $1<\beta\le 2$, $3<\alpha\le 4$ are real numbers, D_{0+}^{β}, D_{0+}^{α} are standard Riemann-Liouville derivatives, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

In order to obtain our main result, we neeed the following Guo-Krasnoselskii fixed point theorem[1] [2].

Theorem 1 *Let E be a Banach space and let P be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $\Omega_1\subset\Omega_2$, and let $A:P\cap(\bar{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that either*

(1) $\|Ax\|\le\|x\|$ for $x\in P\cap\partial\Omega_1$ and $\|Ax\|\ge\|x\|$ for $x\in P\cap\partial\Omega_2$ or

(2) $\|Ax\|\ge\|x\|$ for $x\in P\cap\partial\Omega_1$ and $\|Ax\|\le\|x\|$ for $x\in P\cap\partial\Omega_2$.

Then A has a fixed point in $P\cap(\bar{\Omega}_2\setminus\Omega_1)$.

2. Preliminaries

For convenience, in this section, we will give some basic theory of fractional order calculus.

Definition 1 [15] *The Riemann-Liouville fractional integral $I_{0+}^{\alpha}u$ of order α is defined by*

$$\left(I_{0+}^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{u(s)\mathrm{d}s}{(t-s)^{1-\alpha}},\ t>0.$$

Definition 2 [16] *The Riemann-Liouville fractional derivative $D_{0+}^{\alpha}u$ of order α is defined by*

$$\left(D_{0+}^{\alpha}u\right)(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t\frac{u(s)\mathrm{d}s}{(t-s)^{\alpha-n+1}},\ n\ge[\alpha]+1,\ t>0.$$

Lemma 1 [3] *Let $\alpha>0$, $u\in C(0,1)\cap u\in L(0,1)$. Then fractional differential equation $D_{0+}^{\alpha}u(t)=0$ has unique solution*

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},c_i\in R,\ i=0,1,\cdots,n,$$

where $n\ge[\alpha]+1$.

Lemma 2 [24] *Let $u\in C(0,1)\cap u\in L(0,1)$ and $D_{0+}^{\alpha}u(t)\in C(0,1)\cap L(0,1)$, $\alpha>0$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},c_i\in R,\ i=0,1,\cdots,n,$$

where $n\ge[\alpha]+1$.

3. Main Results

Let $E = C[0,1]$ be a complete space and define norm $\|u\| = \max_{0 \le t \le 1} |u(t)|$ and $P = \{u \in E \mid u(t) \ge 0\}$ on E. Clearly E is a Banach space and P is a cone of E.

For convenience later on, the operator T is first defined on the cone P as

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau) f(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s.$$

Lemma 3 *Let* y *is a given continuous function on* $[0,1]$, $1 < \beta \le 2$, $3 < \alpha \le 4$, *then the BVP*

$$\begin{cases} \left(D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)\right) = y(t),\ t \in (0,1), \\ \left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)' = \left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right) = 0, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

has a positive solution of

$$u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau) y(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, 0 \le s \le t \le 1. \\ t^{\alpha-1}(1-s)^{\alpha-2}, 0 \le t \le s \le 1. \end{cases} \tag{3}$$

$$H(s,\tau) = \frac{1}{\Gamma(\beta)}\begin{cases} (s-s\tau)^{\beta-1} - (s-\tau)^{\beta-1}, 0 \le \tau \le s \le 1. \\ (s-s\tau)^{\beta-1}, 0 \le s \le \tau \le 1. \end{cases} \tag{4}$$

Proof 1 *The* β *order integrals are obtained for both sides of equation* $D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right) = y(t)$, *we know that*

$$\phi_p\left(D_{0+}^{\alpha}u(t)\right) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau + c_1 t^{\beta-1} + c_2 t^{\beta-2}, c_1, c_2 \in R$$

and

$$\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)' = \frac{1}{\Gamma(\beta-1)}\int_0^t (t-\tau)^{\beta-2} y(\tau)\mathrm{d}\tau + c_1(\beta-1)t^{\beta-2} + c_2(\beta-2)t^{\beta-3}, \tag{5}$$

which together with the boundary condition

$$\left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)' = \left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right) = 0,$$

implies that

$$c_1 = -\frac{1}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau, c_2 = 0.$$

Put c_1, c_2 into (5), we know

$$\begin{aligned}\phi_p\left(D_{0+}^{\alpha}u(t)\right) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau - \frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau \\ &= -\frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau \\ &= -\int_0^t \left(\frac{t^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)} - \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}\right) y(\tau)\mathrm{d}\tau + \int_t^1 \frac{t^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)} y(\tau)\mathrm{d}\tau \\ &= -\int_0^1 H(t,\tau) y(\tau)\mathrm{d}\tau.\end{aligned}$$

Since $\phi_p^{-1} = \phi_q$, let $\phi\left(\int_0^1 H(t,\tau)y(\tau)\mathrm{d}\tau\right) = v(t)$, we have

$$D_{0+}^{\alpha}u(t) = -v(t).$$

The α order integrals for both of equation $D_{0+}^{\alpha}u(t) = -v(t)$, then

$$\begin{aligned}u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)\mathrm{d}s + d_1 t^{\alpha-1} \\ + d_2 t^{\alpha-2} + d_3 t^{\alpha-3} + d_4 t^{\alpha-4},\ d_i \in R(i=1,2,3,4).\end{aligned} \tag{6}$$

Combine the boundary value conditions $u(0) = u(1) = u'(0) = u'(1) = 0$, we have

$$d_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s,$$

$$d_2 = d_3 = d_4 = 0.$$

Therefore, BVP (2) has a unique solution:

$$\begin{aligned}u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2} v(s)\mathrm{d}s \\ &= \int_0^t \left(\frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}\right) v(s)\mathrm{d}s + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} v(s)\mathrm{d}s \\ &= \int_0^1 G(t,s) v(s)\mathrm{d}s.\end{aligned}$$

Lemma 4 *The function* $G(t,s), H(t,s)$ *satisfy following properties:*

(1) $(t,s) \in [0,1]\times[0,1]$,

$$G(t,s) \ge 0, H(t,s) \ge 0; \tag{7}$$

(2) $(t,s) \in [0,1]\times[0,1)$,

$$G(t,s) \le G(1,s), H(t,s) \le H(s,s); \tag{8}$$

(3) exists two positive function $k(s), q(s) \in C(0,1)$, satisfy

$$\min_{\frac{1}{3}\le t\le \frac{2}{3}} G(t,s) \ge q(s)\max_{0\le t\le 1} G(t,s) = q(s)G(1,s), s \in (0,1); \tag{9}$$

$$\min_{\frac{1}{3}\le t\le \frac{2}{3}} H(t,s) \ge k(s)\max_{0\le t\le 1} H(t,s) = k(s)H(s,s), s \in (0,1). \tag{10}$$

Proof 2 (1) *According to the expression for the Green's function, property* (1) *holds and we only need to verify the other two properties.*

(2) First, under the known conditions, if $0 \le s \le t \le 1$, then

$$\frac{\partial G(t,s)}{\partial t}=\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}$$

$$=\frac{\left[t(1-s)\right]^{\alpha-2}-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}$$

$$=\frac{(t-ts)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha-1)}\geq 0,$$

and if $0\leq t\leq s\leq 1$, then

$$\frac{\partial G(t,s)}{\partial t}=\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)}\geq 0.$$

So, $G(t,s)$ is increasing on t. Lemma 4 (1) and the monotony of $G(t,s)$, we have

$$G(t,s)\leq \max_{0\leq t\leq 1} G(t,s)=G(1,s).$$

Next, we study the property of $H(t,s)$, if $0\leq s\leq t\leq 1$, then

$$\frac{\partial H(t,s)}{\partial t}=\frac{(\beta-1)(t-ts)^{\beta-2}(1-s)-(\beta-1)(t-s)^{\beta-2}}{\Gamma(\beta)}$$

$$=\frac{(t-ts)^{\beta-2}(1-s)-(t-s)^{\beta-2}}{\Gamma(\beta-1)}$$

$$=\frac{t^{\beta-2}(1-s)^{\beta-1}-(t-s)^{\beta-2}}{\Gamma(\beta-1)}$$

$$\leq\frac{t^{\beta-2}(1-s)^{\beta-2}-(t-s)^{\beta-2}}{\Gamma(\beta-1)}$$

$$=\frac{(t-ts)^{\beta-2}-(t-s)^{\beta-2}}{\Gamma(\beta-1)}\leq 0.$$

So, $H(t,s)$ is decreasing on t, therefore $H(t,s)\leq H(s,s)$.

If $0\leq t\leq s\leq 1$, then

$$\frac{\partial H(t,s)}{\partial t}=\frac{(\beta-1)t^{\beta-2}(1-s)^{\beta-2}}{\Gamma(\beta)}\geq 0.$$

So, $H(t,s)$ is monotonically increasing with respect to t, therefore $H(t,s)\leq H(s,s)$, that is (8) stands.

(3) Utilizing the monotonic property possessed by the Green's function $G(t,s)$, let

$$g_1(t,s)=\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)},$$

$$g_2(t,s)=\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)},$$

then

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} G(t,s)=\begin{cases} g_1\left(\frac{1}{3},s\right), & s\in\left(0,\frac{1}{3}\right],\\ g_2\left(\frac{1}{3},s\right), & s\in\left[\frac{1}{3},1\right),\end{cases} \tag{11}$$

where

$$g_1\left(\frac{1}{3},s\right)=\frac{\left(\frac{1}{3}\right)^{\alpha-1}(1-s)^{\alpha-2}-\left(\frac{1}{3}-s\right)^{\alpha-1}}{\Gamma(\alpha)},$$

$$g_2\left(\frac{1}{3},s\right)=\frac{\left(\frac{1}{3}\right)^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)},$$

Utilizing the monotonic property possessed by the Green's function $G(t,s)$, we have

$$\max_{0\le t\le 1} G(t,s)=G(1,s)=\frac{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}{\Gamma(\alpha)},$$

let

$$q(s)=\begin{cases} \dfrac{\left(\frac{1}{3}\right)^{\alpha-1}(1-s)^{\alpha-2}-\left(\frac{1}{3}-s\right)^{\alpha-1}}{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}, & s\in\left(0,\frac{1}{3}\right],\\ \dfrac{\left(\frac{1}{3}\right)^{\alpha-1}(1-s)^{\alpha-2}}{(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}, & s\in\left[\frac{1}{3},1\right),\end{cases} \tag{12}$$

then, (9) stands.

Utilizing the monotonic property possessed by the $H(t,s)$, let

$$h_1(t,s)=\frac{(t-ts)^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)},$$

$$h_2(t,s)=\frac{(t-ts)^{\beta-1}}{\Gamma(\beta)},$$

then

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} H(t,s)=\begin{cases} h_1\left(\frac{2}{3},s\right), & s\in\left(0,\frac{1}{3}\right],\\ \min\left\{h_1\left(\frac{2}{3},s\right),h_2\left(\frac{1}{3},s\right)\right\}, & s\in\left[\frac{1}{3},\frac{2}{3}\right),\\ g_2\left(\frac{1}{3},s\right), & s\in\left[\frac{1}{3},1\right),\end{cases} \tag{13}$$

$$=\begin{cases} h_1\left(\frac{2}{3},s\right), & s\in(0,r],\\ g_2\left(\frac{1}{3},s\right), & s\in[r,1),\frac{1}{4}<r<\frac{3}{4},\end{cases} \tag{14}$$

where

$$h_1\left(\frac{2}{3},s\right)=\frac{\left[\frac{2}{3}(1-s)\right]^{\beta-1}-\left(\frac{2}{3}-s\right)^{\beta-1}}{\Gamma(\beta)},s\in[0,r],$$

$$h_2\left(\frac{1}{3},s\right)=\frac{\left[\frac{1}{3}(1-s)\right]^{\beta-1}}{\Gamma(\beta)},s\in[r,1].$$

Utilizing the monotonic property possessed by the $H(t,s)$, we have

$$\max_{0\le t\le 1}H(t,s)=H(s,s)=\frac{s(1-s)^{\beta-1}}{\Gamma(\beta)},$$

let

$$k(s)=\begin{cases}\dfrac{\left[\frac{2}{3}(1-s)\right]^{\beta-1}-\left(\frac{2}{3}-s\right)^{\beta-1}}{\left[s(1-s)\right]^{\beta-1}}, & s\in(0,r],\\ \left(\frac{1}{3}\right)^{\beta-1}\dfrac{1}{s^{\beta-1}}, & s\in[r,1),\end{cases}$$

then, (10) stands.

Theorem 2 *$T:P\to P$ is completely continuous.*

Proof 3 *First, for any For any u in P, according to the continuity of G,H,f, it can be concluded that T is a continuous function. Takes a bounded subset Ω of P, it is always possible to find a constant B that is nonnegative and satisfies $\|u\|\le B$, $u\in\Omega$, let $L=\max_{0\le t\le 1,0\le u\le B}|f(t,u(t))|+1$, we have*

$$\begin{aligned}(Tu)(t)&=\left|\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{ds}\right|\\&\le L^{q-1}\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)\mathrm{d}\tau\right)\mathrm{ds}\\&\le L^{q-1}\int_0^1 G(1,s)\phi_q\left(\int_0^1 H(\tau,\tau)\mathrm{d}\tau\right)\mathrm{ds}+\infty,\end{aligned}$$

which indicates that $T(\Omega)$ is uniformly bounded.

Next, for any $(s,t)\in[0,1]\times[0,1]$, since $G(t,s)$ is consistent continuity, so fixing s on [0, 1], then for any $\epsilon>0$, there exists $\delta>0$ such that when $t_1,t_2\in[0,1]$, $t_1<t_2$, $|t_2-t_1|<\delta$, we have

$$|G(t_2,s)-G(t_1,s)|=\frac{\epsilon}{L^{q-1}\phi_q\left(\int_0^1 H(\tau,\tau)\mathrm{d}\tau\right)},$$

that is

$$\begin{aligned}&|(Tu)(t_2)-(Tu)(t_1)|\\&\le\int_0^1\left|[G(t_2,s)-G(t_1,s)]\phi_q\cdot\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\right|\mathrm{ds}\\&\le\int_0^1|G(t_2,s)-G(t_1,s)|\phi_q\cdot\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{ds}\\&\le L^{q-1}\int_0^1|G(t_2,s)-G(t_1,s)|\phi_q\cdot\left(\int_0^1 H(\tau,\tau)\mathrm{d}\tau\right)\mathrm{ds}\\&<\epsilon,\end{aligned}$$

which indicates that $T(\Omega)$ is equicontinuous. By Aezela-Ascoli theorem, we know that $T: P \to P$ is completely continuous.

Easy to read, we remember

$$M = \left(\int_0^1 G(1,s)\,\mathrm{d}s\,\phi_q\left(\int_0^1 H(\tau,\tau)\,\mathrm{d}\tau\right)\right)^{-1},$$

$$N = \left(\int_{\frac{1}{3}}^{\frac{2}{3}} q(s)G(1,s)\,\mathrm{d}s \cdot \phi_q\left(\int_{\frac{1}{3}}^{\frac{2}{3}} k(\tau)H(\tau,\tau)\,\mathrm{d}\tau\right)\right)^{-1}.$$

Theorem 3 *Now let* $f \in C([0,1]\times[0,+\infty)\to[0,+\infty))$. *At the same time two different positive numbers* r_1, r_2 *and* $r_2 > r_1 > 0$ *satisfy the following two conditions,*

(H_1) $f(t,u(t)) \ge (Nr_1)^{p-1}, (t,u(t)) \in \left[\frac{1}{3},\frac{2}{3}\right]\times[0,r_1]$;

(H_2) $f(t,u(t)) \ge (Mr_2)^{p-1}, (t,u(t)) \in [0,1]\times[0,r_2]$.

The BVP (1) has only one positive solution u, and $r_1 \le \|u\| \le r_2$.

Proof 4 *Let*

$$\Omega_1 = \{u \in P \mid \|u\| < r_1\},$$

$$\Omega_2 = \{u \in P \mid \|u\| < r_2\}.$$

Then for any $u \in \partial\Omega_1$, we get $0 \le u(t) \le r_1, t \in [0,1]$, which together with ($H_1$) and Lemma 3 implies that

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau) f(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\ge \min_{\frac{1}{3}\le t\le\frac{2}{3}} Nr_1 \int_{\frac{1}{3}}^{\frac{2}{3}} q(s)G(1,s)\,\mathrm{d}s\,\phi_q \int_{\frac{1}{3}}^{\frac{2}{3}} k(\tau)H(\tau,\tau)\,\mathrm{d}\tau \\ &= r_1 = \|u\|.\end{aligned}$$

This implies that for any $u \in \partial\Omega_1$, there is $\|Tu\| \le \|u\|$.

For any $u \in \partial\Omega_2$, we get $0 \le u(t) \le r_2, t \in [0,1]$ which together with (H_2) and Lemma 3 implies that

$$\begin{aligned}\|(Tu)(t)\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau) f(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\ &\le Mr_2 \int_0^1 G(1,s)\,\mathrm{d}s\,\phi_q\left(\int_0^1 H(\tau,\tau)\,\mathrm{d}\tau\right) \\ &= r_2 = \|u\|.\end{aligned}$$

This indicates that $\|Tu\| \ge \|u\|$, for $u \in \partial\Omega_1$.

Therefore, it follows from Theorem 1 that the operator T has a fixed point u, which is a desired positive solution of BVP (1).

4. Conclusions

Based on the theory of fractional differential equations, which is widely used in physics, engineering and other fields for describing complex systems memory and non-local characteristics. In this paper, we conduct an in-depth discussion on the

Riemann-Liouville boundary value problem of fractional differential equations with a multi-term p-Laplacian operator fractional differential equation. Compared with the literature [9], this paper considers both integer and fractional orders for the derivative term.

Firstly, we derive the corresponding Greens function through strict deduction and analyze its non-negativity, monotonicity and boundedness, which lay the foundation for subsequent positive solution research.

Secondly, we transform the original boundary value problem into an equivalent integral equation, converting the differential equation solution problem into an integral operator fixed point problem.

Finally, combined with the fixed point theorem on cones, the existence of a positive solution for the boundary value problem (1) is obtained. The results of this paper not only provide a new theoretical perspective for the study of boundary value problems of fractional differential equations with p-Laplacian operators, but also enrich the work of existing literature, which is great theoretical significance for the qualitative analysis of boundary value problems for multiple fractional differential equations. We expect that future research can further explore on this basis.

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Declaration of Competing Interest

The author declares that we have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors. Informed consent was obtained from all individual participants included in the study.

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