

# Forbidden Subgraphs for the Existence of an Even Factor with Exactly Two Components in 2-Edge-Connected Graphs

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## Abstract

In this paper, we mainly consider characterize all the pairs  $\{R, S\}$  of 2-edge-connected graphs  $G$  such that every  $\{R, S\}$ -free graph  $G$  has an even factor with exactly two components if and only if  $\delta(G) \geq 2$  and every odd branch-bond of  $G$  has an edge branch.

## Keywords

Even Factor, Odd Branch-Bond, Forbidden Pair, Edge-Branch

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## 1. Introduction

Throughout this paper, we consider finite undirected simple graphs and follow the notation and terminology of [1].

Let  $G = (V, E)$  be a graph and we use  $\delta(G)$ ,  $\Delta(G)$  and  $\omega(G)$  to denote the minimum degree, the maximum degree and the number of components of the graph  $G$ , respectively. For a vertex  $v \in V(G)$ , we use  $N_G(v)$  to denote the neighborhood of  $v$  in  $G$ . Let  $S$  be a subset of  $V(G)$ . We use  $G[S]$  to denote the subgraph of  $G$  induced on  $S$ , and also use  $P_n$ ,  $K_n$  and  $C_n$  to denote the path, the complete graph and the cycle of order  $n$ , respectively, where  $K_3$  is called *triangle*. Let  $K_{m,n}$  is the complete bipartite graph with partition sets of size  $m$  and  $n$ . We call  $K_{1,3}$  and  $K_{1,k}$  a *claw* and a *star*, respectively.

Let  $H$  be a given graph. A graph  $G$  is said to be  $H$ -free if  $G$  does not contain  $H$  as an induced subgraph. For a set of connected graphs  $\mathcal{H}$ . A graph  $G$  is said to be  $\mathcal{H}$ -free if  $G$  does not contain  $H$  as an induced subgraph for all  $H \in \mathcal{H}$ . The set  $\mathcal{H}$  was defined as the set of *forbidden subgraphs* of  $G$ . We call  $\mathcal{H}$  a forbidden pair if  $|\mathcal{H}| = 2$ . A subgraph containing all vertices of  $G$  is

called a *spanning subgraph* of  $G$ . If  $G$  contains a spanning cycle, then  $G$  is *Hamiltonian*. A subgraph of  $G$  is called *even* if its degrees are all positive even. A subgraph of  $G$  is called an *even-factor* if it is an even spanning subgraphs. A subgraph of  $G$  is called *2-factor* if it is an even factor its degrees are all two. A graph is called *supereulerian* if it contains an even factor with exactly one component. For a subset  $X \subseteq E(G)$ , the *contraction*  $G/X$  is the graph obtained  $G$  by identifying the ends of each edge in  $X$  and then deleting the resulting loops. Note that the edges in  $E(G/X)$  can be regarded as edges in  $E(G)$ . If  $H$  is a subgraph, then we use  $G/H$  for  $G/E(H)$ . A graph  $H$  is a *minor* if it is isomorphic to the contraction image of a subgraph of  $G$ . A graph  $H$  is called an *induced minor* if it is isomorphic to a contraction image of an induced subgraph of  $G$ .

In 1972, Richard M. Karp [2] proved that the Hamiltonian problem is NP-*hard*. After that, Pulleyblank pointed out in [3] that determining whether a graph is supereulerian is NP-*hard*, even when the problem is restricted to planar graphs. In most cases, when studying the Hamiltonian problem or the Supereulerian problem, we tend to start with the conditions of forbidden subgraphs.

In 1991, Bedrossian first gave all connected forbidden pairs for hamiltonicity of 2-connected graphs in [4]. In 1997, Faudree and Gould further proved this result of Bedrossian in [5] by completely characterizing all forbidden pairs for hamiltonicity 2-connected graphs with sufficiently large order.

**Theorem 1.** (Faudree and Gould [5]) Let  $R, S$  be connected graphs of order at least three other than  $P_3$ . Then every 2-connected  $\{R, S\}$ -free graph with order at least  $n_0$  is hamiltonian if and only if  $R = K_{1,3}$  and  $S$  is an induced subgraph of  $P_6, B_{1,2}, N_{1,1,1}, Z_3$ .

Meanwhile, they also characterized the case of a forbidden subgraph for hamiltonicity of 2-connected graphs.

**Theorem 2.** (Faudree and Gould [5]) Let  $H$  be a connected graph of order at least 3 and  $G$  be a 2-connected graph. Then  $G$  is  $H$ -free implies  $G$  is hamiltonian if and only if  $H = P_3$ .

A wheel  $W_n$  is the graph obtained from the cycle  $C_n = u_1u_2 \cdots u_n$ , and an additional vertex  $u$  by joining  $u$  with  $u_i$  for  $u \in \{1, 2, \dots, n\}$ . Define  $W'_n$  to be the graph obtained from  $W_n$  by subdividing each edge of  $E(C_n)$  once. Let  $W'_3$  be the graph from  $C_7$  and  $K_1$  by joining  $K_1$  to three non-adjacent vertices of  $C_7$ . Let  $F_2$  be the graph from  $C_4$  and  $P_3$  by identifying a vertex of  $C_4$  with an endvertex of  $P_3$ . Let  $Z_i$  be the graph by identifying end vertices of one path of length  $i$  to the one vertex of triangle. Let  $B_{i,j}$  be the graph by identifying end vertices of two disjoint paths of lengths  $i, j$  to the two vertices of triangle. Let  $N_{i,j,k}$  be the graph by identifying end vertices of three disjoint paths of lengths  $i, j, k$  to the three vertices of the triangle. Here  $Z_i, B_{i,j}, N_{i,j,k}$  with  $\{i, j, k\} \geq 1$ , see **Figure 1**.

In 1995, Lai [6] considered the same problem and obtained the following result.

**Theorem 3.** (Lai [6]) Let  $G$  be a 2-edge-connected graph. Then every 2-edge-

connected graph induced subgraph of  $G$  is supereulerian if and only if  $G$  has no induced minor isomorphic to a member in  $\mathcal{W}$ .

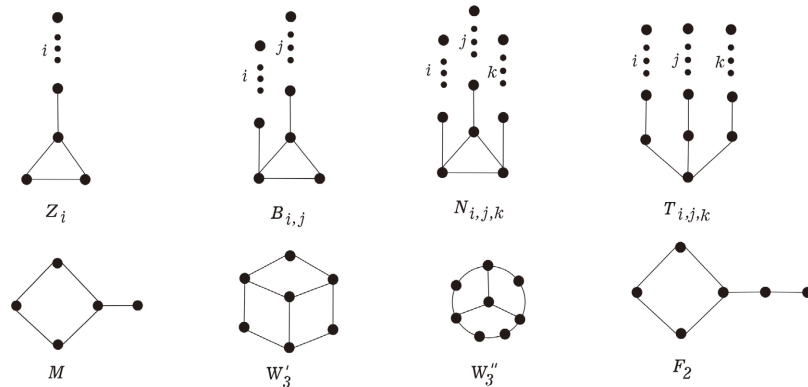


Figure 1. Some common induced subgraphs and  $W'_3$ ,  $W''_3$ .

Over time, many researchers have shifted their focus from the study of Hamiltonian problems to the exploration of graph factorization problems. In [7], Tutte proved that a graph  $G$  has a 1-factor if and only if the deletion of an arbitrary vertex set  $S$  makes  $G - S$  have at most  $|S|$  odd branches. Later, Gallai in [8] began to focus on  $k$ -factors. In 2007, Plummer published a survey article on graph factorization in [9], which is widely regarded as the most authoritative literature on graph factorization problems.

In 2008, Xiong [10] gave a necessary and sufficient condition for a claw-free graph  $G$  to have an even factor.

**Theorem 4.** (Xiong [10]) Let  $G$  be a claw-free graph. Then  $G$  has an even factor if and only if  $\delta(G) \geq 2$  and every odd branch-bond of  $G$  has an edge branch.

Furthermore, in [11], he extended the above result and characterized all forbidden subgraphs  $H$  such that every  $H$ -free graph  $G$  has an even factor if and only if  $\delta(G) \geq 2$  and every odd branch-bond of  $G$  has an edge branch.

**Theorem 5.** (Xiong [11]) Let  $H$  be a connected graph of order at least three and  $G$  be a graph other than  $W'_3$  such that  $\delta(G) \geq 2$  and every odd branch-bond of  $G$  has an edge branch. Then every  $H$ -free graph  $G$  has an even factor if and only if  $H$  is an induced subgraph of  $P_5$  or  $T_{2,1,1}$ .

It is easy to see that if there exists an even factor in a graph  $G$ , then  $G$  satisfies the following conditions:

- 1)  $\delta(G) \geq 2$ ;
- 2) every odd branch-bond of  $G$  has an edge branch.

In 2017, Lv and Xiong characterized all forbidden pairs of connected subgraphs for supereulerian of 2-connected graphs with sufficiently large order in [12].

**Theorem 6.** (Lv and Xiong [12]) Let  $R, S$  be a pair of graphs and  $G$  be a 2-connected graph with  $n \geq 10$ . Then every  $\{R, S\}$ -free graph is supereulerian, if and only if (up to symmetry):

- 1)  $R = K_{1,4}$  and  $S$  is an induced subgraph of  $P_5$ ;

- 2)  $R = K_{1,3}$  and  $S$  is an induced subgraph of  $P_7, Z_4, N_{1,1,3}$ ;
- 3)  $R = C_4$  and  $S$  is an induced subgraph of  $P_5$ .

It is obviously that a supereulerian graph has an even factor and a Hamiltonian graph must be supereulerian. In other words, a necessary condition for a graph to be supereulerian is that it has an even factor and a necessary condition for a graph to be Hamiltonian is that it is supereulerian.

In order to illustrate some results better, we still need some definitions. A non-trivial path is called a *branch* if it has only internal vertices of degree two and end vertices of degree not two. The *length* of a branch is the number of its edges. Note that a branch of length one has no internal vertex, it is called an *edge branch*. We denote by  $\mathcal{B}(G)$  the set of branches of  $G$ . For any subset  $S$  of  $\mathcal{B}(G)$ , we denote by  $G-S$  the subgraph obtained from  $G$  by deleting all edges and internal vertices of branches of  $S$ . A subset  $S$  of  $\mathcal{B}(G)$  is called a *branch cut* if  $G-S$  has more components than  $G$ . A minimal branch cut is called a *branch-bond*. A branch-bond is called odd if it has an odd numbers of branches.

In 2021, Yang, Du and Xiong [13] further considered the forbidden subgraphs for supereulerian of 2-edge-connected graphs satisfying that every odd branch bond of  $G$  has an edge branch, and characterized all forbidden subgraphs for supereulerian.

**Theorem 7.** (Yang, Du and Xiong [13]) Let  $H$  be a connected graph of order at least three. Then every 2-edge-connected  $H$ -free graph  $G$  satisfying that every odd branch bond of  $G$  has an edge branch is supereulerian if and only if  $H$  is an induced subgraph of  $P_4$ .

**Theorem 8.** (Yang, Du and Xiong [13]) Let  $R, S$  be a pair of graphs and  $G$  be a 2-edge-connected graph satisfying that every odd branch bond of  $G$  has an edge branch. Then every  $\{R, S\}$ -free graph  $G$  of order at least 11 is supereulerian if and only if  $\{R, S\} \preceq \{T_{1,1,2}, H_1\}, \{T_{1,1,2}, Z_6\}, \{T_{1,1,2}, B_{1,5}\}, \{T_{1,1,2}, B_{3,3}\}, \{T_{1,1,2}, N_{1,2,3}\}, \{T_{1,1,3}, Z_2\}$  or  $\{P_5, W\}$ .

The number of components of an even factor of a graph is a key parameter characterizing the sparsity of the even factor: when the number of components is 1, the even factor degenerates into an even cycle; when the number of components is at least 3, the structure tends to be complex. An even factor with exactly two components serves as an important intermediate case connecting single even cycles and even factors with multiple components, and studying its existence conditions can fill the gap in the classification research on the number of components of even factors. Meanwhile, it can serve as an intermediate target for algorithm optimization, and can also be used to design more efficient algorithms for determining the existence of an arbitrary even factor, thereby having direct application value for network reliability.

Motivated by their results, we characterize all forbidden subgraphs for the existence of an even factor with exactly two components in a 2-edge-connected graph. The results as follows:

**Theorem 9.** Let  $H$  be a connected graph of order at least three, and  $G$  be a

2-edge-connected graph such that  $\delta(G) \geq 2$  satisfying that every odd branch-bond of  $G$  has an edge branch. Then every  $H$ -free graph  $G$  has an even factor with exactly two components if and only if  $H$  is an induced subgraph of  $P_5$ .

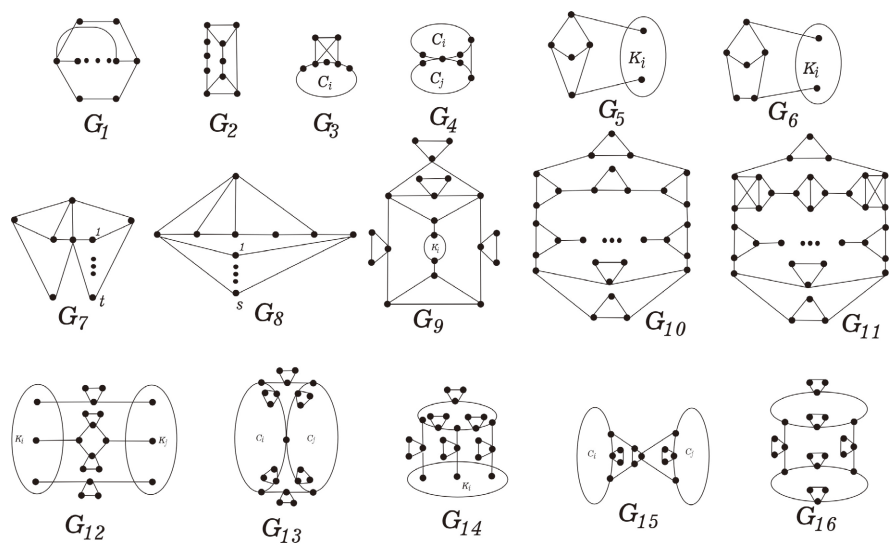
**Theorem 10.** Let  $R, S$  be connected graphs such that  $\{R, S\} \not\cong \{P_5\}$ . Let  $G$  be a graph of order at least 10 such that  $\delta(G) \geq 2$  and every odd branch-bond of  $G$  has an edge branch. Then every  $\{R, S\}$ -free graph  $G$  has an even factor with exactly two components if and only if (up to symmetry):

- 1)  $R = F_2$  and  $S$  is an induced subgraph of  $P_8$ ;
- 2)  $R = T_{2,1,1}$  and  $S$  is an induced subgraph of  $P_8, Z_5, B_{2,3}$  or  $N_{1,1,3}$ .

In what follows, we will prove the necessity and sufficiency parts of Theorems 9 and 10 in sections 2 and 3, respectively.

## 2. The Proof of Necessity Part of Theorems 9 and 10

In this section, we construct sixteen classes of connected graphs which satisfy that  $\delta(G) \geq 2$  and every odd branch bond has an edge branch, but they contain no even factor with exactly two components, see **Figure 2**.



**Figure 2.** Some classes of graphs without an even factor with exactly two components.

**Proof of the necessity of Theorem 9.** It is obviously that each of  $G_1$  to  $G_{16}$  in **Figure 2** contains  $H$  as an induced subgraph and satisfies  $\delta(G) \geq 2$  and every odd branch bond has an edge branch but it has no an even factor with exactly two components. Since the graphs  $G_1, G_2, \dots, G_{16}$  have no common induced cycle, without loss of generality, we assume that  $H$  is an induced subgraph of  $G_7$ , then  $H$  is  $K_{1,t} (t \geq 3)$  or  $P_i (i \geq 3)$ . Note that  $G_2$  is  $K_{1,3}$ -free,  $H$  is  $P_i (i \geq 3)$ . Since the largest common induced path of  $G_1$  to  $G_{16}$  is  $P_5$ ,  $H$  is an induced subgraph of  $P_5$ . This completes proof of the necessity of Theorem 9.

**Proof of the necessity of Theorem 10.** Let  $R, S$  be a pair of connected graphs of order at least three other than  $P_5$ . It is obviously that each of  $G_1$  to

$G_{16}$  in **Figure 2** satisfies  $\delta(G) \geq 2$  and every odd branch bond has an edge branch but that it has no an even factor with exactly two components. Each of them contains at least one of  $R$  and  $S$  as an induced subgraph, without loss of generality, we assume that  $G_7$  contains  $R$  as an induced subgraph. Then  $R$  is a tree or contains  $C_3$  or  $C_4$ . Now we distinguish the following two cases.

**Case 1:**  $R$  is a tree.

First suppose that  $R$  is a path. We note that the largest common induced path of  $G_5, G_7$  is  $P_5$ , then  $R$  is an induced subgraph of  $P_5$  if  $R$  is a path, which contradicts our assumption.

Next we suppose that  $R$  is  $K_{1,i}$  ( $i \geq 3$ ). If  $R$  is  $K_{1,4}$  ( $i \geq 4$ ), then we note that the graphs  $G_2, G_3, G_5, G_6, G_{10}, G_{11}, G_{12}, G_{14}, G_{15}, G_{16}$  are 2-edge-connected and  $K_{1,4}$ -free, they contain  $S$  as an induced subgraph. Since the graphs  $G_2, G_3, G_5, G_6, G_{10}, G_{11}, G_{12}, G_{14}, G_{15}, G_{16}$  have no common cycle and  $G_2$  is  $K_{1,3}$ -free,  $S$  is a path. Note that the largest induced path of  $G_5$  is  $P_5$ , then  $S$  is an induced subgraph of  $P_5$ , contradiction. Thus,  $R = K_{1,3}$ . Since the graph  $G_2$  is 2-edge-connected and  $K_{1,3}$ -free, it contains  $S$  as an induced subgraph, thus  $S$  is a path or contains a cycle. Since the largest induced path of  $G_2$  is  $P_8$ , then  $S$  is an induced subgraph of  $P_8$ . Next we consider  $S$  contains a cycle. Note that the maximal induced cycle of  $G_2$  is triangle,  $S$  contains a triangle. Furthermore,  $S$  contains at most one triangle. Note that the maximal induced subgraph containing triangle in  $G_2$  is  $Z_5, B_{3,2}$  or  $N_{1,1,3}$ ,  $S$  is an induced subgraph of  $Z_5, B_{3,2}$  or  $N_{1,1,3}$ .

Finally suppose that  $R$  is a tree. Since the maximal induced subgraph of  $G_7$  is  $T_{3,1,1}$ , we have  $R = T_{3,1,1}$ . Note that the graphs  $G_2, G_5, G_6$  are 2-edge-connected and  $T_{3,1,1}$ -free, they must contain  $S$  as a common induced subgraph. Since the graphs  $G_2, G_5, G_6$  have no common induced cycle and  $G_2$  is  $K_{1,3}$ -free,  $S$  must be a path. Since the maximal induced path of  $G_5$  is  $P_5$ , then the maximal common induced subgraph of  $G_2, G_5, G_6$  is  $P_5$ ,  $S$  is an induced subgraph of  $P_5$ , contradiction. Then  $R = T_{2,1,1}$ . Since the graph  $G_2$  is 2-edge-connected and  $T_{2,1,1}$ -free, it contains  $S$  as an induced subgraph, thus  $S$  is a path or contains a cycle. Since the largest induced path of  $G_2$  is  $P_8$ , then  $S$  is an induced subgraph of  $P_8$ . Next we consider  $S$  contains a cycle. Note that the maximal induced cycle of  $G_2$  is triangle,  $S$  contains a triangle. Furthermore,  $S$  contains at most one triangle. Note that the maximal induced subgraph containing triangle in  $G_2$  is  $Z_5, B_{3,2}$  or  $N_{1,1,3}$ ,  $S$  is an induced subgraph of  $Z_5, B_{3,2}$  or  $N_{1,1,3}$ . In fact,  $K_{1,3}$  is an induced subgraph of  $T_{2,1,1}$ ,  $K_{1,3}$ -free implies  $T_{2,1,1}$ -free. Thus,  $\{R, S\} \preceq \{T_{2,1,1}, P_8\}, \{T_{2,1,1}, Z_5\}, \{T_{2,1,1}, B_{3,2}\}, \{T_{2,1,1}, N_{1,1,3}\}$ .

**Case 2:**  $R$  contains an induced  $C_4$  or  $C_3$ .

First assume that  $R$  contains an induced  $C_4$ . Note that the graphs  $G_1, G_2, G_4, G_{10}, G_{13}, G_{14}$  are 2-edge-connected and  $C_4$ -free, they must contain  $S$  as a common induced subgraph. Since the graphs  $G_1, G_2, G_4, G_{10}, G_{13}, G_{14}$  have no common induced cycle and  $G_2$  is  $K_{1,3}$ -free,  $S$  must be a path. Note that the maximal common induced path of  $G_1, G_2, G_4, G_{10}, G_{13}, G_{14}$  is  $P_8$ ,

$S$  is an induced subgraph of  $P_8$ . On the other hand, assume  $S = P_8$ . Since the graphs  $G_5, G_6, G_7, G_8, G_{12}$  are  $P_8$ -free and their maximal common induced subgraph containing  $C_4$  is  $F_2$ ,  $R$  is an induced subgraph of  $F_2$ . Thus,  $\{R, S\} \preceq \{F_2, P_8\}$ .

Finally suppose that  $R$  contains an induced  $C_3$ . Note that the graphs  $G_1, G_4, G_5, G_6$  are 2-edge-connected and  $C_3$ -free, they must contain  $S$  as a common induced subgraph. Since the graphs  $G_1, G_4, G_5, G_6$  have no common induced cycle,  $S$  is a tree. Since the maximal induced path of  $G_5$  is  $P_5$ , if  $S$  is a path,  $S$  is an induced subgraph of  $P_5$ , contradiction. Then  $S$  is a tree. We note that the largest common induced subgraph the graphs  $G_1, G_4, G_5, G_6$  are  $K_{1,3}$  or  $T_{1,1,2}$ . We assume that  $S$  is  $T_{1,1,2}$ . Note that the graph  $G_2$  is  $T_{1,1,2}$ -free,  $R$  is an induced subgraph  $G_2$ . We note that the maximal induced subgraph containing a triangle in  $G_2$  is  $Z_5, B_{3,2}$  or  $N_{1,1,3}$ . Thus,  $\{R, S\} \preceq \{T_{2,1,1}, P_8\}, \{T_{2,1,1}, Z_5\}, \{T_{2,1,1}, B_{3,2}\}, \{T_{2,1,1}, N_{3,1,1}\}$ . This completes proof of the necessity of Theorem 10.

### 3. The Proof of Sufficiency Part of Theorems 9 and 10

Before the start of this section, we still need some notations. Let  $G_1$  and  $G_2$  be two spanning subgraphs of  $G$ , and we can form the spanning subgraph  $G_1 \Delta G_2$  of  $G$ , whose vertex set is  $V(G_1) \cup V(G_2)$  and edge set is  $E(G_1) \Delta E(G_2)$ . This graph is called the *symmetric difference* of  $G_1$  and  $G_2$ . Note that the symmetric difference of two even graphs remains an even graph. We denote the edge set of  $\{ux, uy, vx, vy\}$  by  $E(\{u, v\}, \{x, y\})$ .

Let  $G$  be a connected graph admitting an even factor  $F$  with components  $Q_1, Q_2, \dots, Q_t$  ( $t \geq 2$ ). Let  $\widehat{\Theta}_F = \{e \in E(G) \setminus E(F)\}$ , where  $e$  is an edge with two end vertices in different components of  $F$ . Let  $C$  be an induced cycle of  $G$  with at least one edge of  $\widehat{\Theta}_F$ . For any edge  $e = ab \in E(C)$  contained in a triangle, denoted by  $T(ab)$  and exactly one vertex in  $T(ab)$  that is not in  $C$ . Let

$$\Theta_F^\Delta = \{e \in E(C) : e \text{ is contained in a triangle } T(e)\}.$$

We also define an operation on the triangle  $T(ab)$ :

$$F_C(T(ab)) = \begin{cases} F \Delta E(C) & \text{if either } ab \notin E(F) \text{ or } E(T(ab)) \subseteq E(F), \\ F \Delta ((E(C) \cup E(T(ab))) \setminus \{ab\}) & \text{if } ab \in E(F) \text{ and } |E(T(ab)) \cap E(F)| \leq 2. \end{cases}$$

Then  $F_C(T(ab))$  is an even factor of  $G$  such that  $a, b$  are in the same components of  $F_C(T(ab))$ .

An edge of  $G$  is called a *singular* edge if it is not contained in any triangle. Otherwise, it is called *nonsingular*.

**Lemma 11.** Let  $G$  be a connected graph admitting an even factor. Let  $F$  be an even factor of  $G$  with components  $Q_1, Q_2, \dots, Q_t$  ( $t \geq 2$ ) such that it contains as few components as possible. Then for any edge  $uv \in \widehat{\Theta}_F$ ,  $u \in Q_i, v \in Q_j$ ,

it holds that:

- 1)  $uv$  is singular;
- 2)  $E(N(u) \cap V(Q_i), N(v) \cap V(Q_j)) = \emptyset$ .

**Proof of Lemma 11. 1)** We choose an even factor  $F$  of  $G$  with components  $Q_1, Q_2, \dots, Q_t$  ( $t \geq 2$ ) such that it contains components as few as possible. Then  $u \in Q_i, v \in Q_j$ . Assume that  $uv$  is a nonsingular edge,  $uv$  is contained in a triangle  $uvw$ . Then  $F\Delta E(uvw)$  is an even factor with fewer components than  $F$ , contradiction.

2) Conversely, we assume there exist two distinct vertices  $u_1 \in N(u) \cap V(Q_i)$  and  $v_1 \in N(v) \cap V(Q_j)$  such that  $u_1v_1 \in E(G)$ . Then  $F\Delta E(uv_1u_1)$  is an even factor with fewer components than  $F$ , contradiction.  $\square$

**Proof of the sufficiency of Theorem 9.** Let  $G$  be a  $\{P_5\}$ -free graph. We choose an even factor  $F$  of  $G$  with components  $Q_1, Q_2, \dots, Q_t$  ( $t \geq 2$ ) such that it contains components as few as possible. Suppose that  $G$  has no even factor with exactly two components. Then the number of components  $t$  of any even factor satisfies  $t = 1$  or  $t \geq 3$ . If  $t = 1$ , then  $G = F$  is an even factor with exactly one component, so  $E \setminus F = \emptyset$ . Then we may choose an induced cycle  $C = xx_1\omega_1y_1yx$  of  $G$ . Since  $G$  is an even subgraph, the degree of  $x$  is even. Thus, there exist  $x_2 \in N(x) \setminus \{x_1, y\}$  and  $y_2 \in N(y) \setminus \{y_1, x\}$ . However,  $G[\{x_2, x, x_1, \omega_1, y_1\}] \cong P_5$ , a contradiction. Thus,  $t \neq 1$ , which implies  $t \geq 2$ .

To prove that  $t = 2$ , we assume that  $t \geq 3$ . Then there is an edge  $xy \in \widehat{\Theta}_F$  such that  $x \in Q_i, y \in Q_j$  and  $\{i, j\} \in \{1, 2, \dots, t\}$ . Since  $G$  is 2-edge-connected, such an induced cycle  $C$  always exists. Then we choose an induced cycle  $C = xzz_1\omega_1 \dots \omega_l y_1 yx$  of  $G$  such that  $xy \in \widehat{\Theta}_F \cap E(C)$  and  $xz \in \widehat{\Theta}_F \cap E(C)$ . Let  $\{x_1\} \subseteq N(x) \cap V(Q_i), \{y_1\} \subseteq N(y) \cap V(Q_j)$  and  $\{z_1\} \subseteq N(z) \cap V(Q_k)$ . By Lemma 11,  $|V(C)| \geq 5$ . Since  $G$  is  $P_5$ -free,  $|V(C)| \leq 5$ . Then  $|V(C)| = 5$ . If  $\omega_1 \notin V(Q_i) \cup V(Q_j) \cup V(Q_k)$ , then  $F = F\Delta E(C)$  is an even factor with fewer components than  $F$ , such that  $Q_i, Q_j, Q_k$  contains  $\omega_1$  in the same component, a contradiction. Then  $\omega_1 \in V(Q_i) \cup V(Q_j) \cup V(Q_k)$ . Without loss of generality, assume that  $\omega_1 \in V(Q_k)$ . By lemma 11,  $xy$  and  $xz$  are singular, we can obtain

$E(\{x_1, x_2\}, \{y_1, y, y_2\}) = \emptyset, E(\{x\}, \{y_1, y_2\}) = \emptyset,$   
 $E(\{x_1, x_2\}, \{z_1, z, z_2\}) = \emptyset$  and  $E(\{x\}, \{z_1, z_2\}) = \emptyset$ , where  
 $\{y_2\} \subseteq N(y) \cap V(Q_j) \setminus \{y_1\}, \{x_2\} \subseteq N(x) \cap V(Q_i) \setminus \{x_1\}$  and  
 $\{z_2\} \subseteq N(z) \cap V(Q_k) \setminus \{z_1\}$ . By the choice of  $C, \{x\omega_1, z\omega_1\} \cap E(G) = \emptyset$ . We claim that  $z_1z_2 \notin E(G)$ . Otherwise, let  $T(z\omega_1) = zz_1\omega_1z$ . If  $z\omega_1 \notin E(F)$ , then  $F_C(T(z\omega_1)) = F\Delta E(xzz_1\omega_1yx)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $z\omega_1 \in E(F)$  and  $|\{zz_1, z\omega_1, z_1\omega_1\} \cap E(G)| \leq 2$ , then  $F_C(T(z\omega_1)) = F\Delta E(xzz_1\omega_1yx)\Delta E(zz_1\omega_1z) \setminus \{z\omega_1\}$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(z\omega_1))$  has fewer components than  $F$ , a contradiction. However,

$G[\{y_1, y, \omega_1, z_1, z\}] \cong P_5$ , a contradiction. Then, without loss of generality, we assume that  $\omega_1 \in V(Q_j)$ . We claim that  $z_1z_2 \notin E(G)$ . Otherwise, let  $T(y\omega_1) = yy_1\omega_1y$ . If  $y\omega_1 \notin E(F)$ , then  $F_C(T(y\omega_1)) = F\Delta E(xzz_1\omega_1yx)$  is an

even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $y\omega_1 \in E(F)$  and  $|\{yy_1, y\omega_1, y_1\omega_1\} \cap E(G)| \leq 2$ , then  $F_C(T(y\omega_1)) = F\Delta E(xzz_1\omega_1yx)\Delta E(yy_1\omega_1y) \setminus \{y\omega_1\}$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(y\omega_1))$  has fewer components than  $F$ , a contradiction. However,  $G[\{y_1, y, \omega_1, z_1, z\}] \cong P_5$ , a contradiction. This completes proof of the sufficiency of Theorem 9.  $\square$

**Proof of the sufficiency of Theorem 10.** Let  $G$  be a 2-edge-connected graph. We may choose an even factor  $F$  of  $G$  with components  $Q_1, Q_2, \dots, Q_t$  ( $t \geq 2$ ) such that it contains components as few as possible. Suppose that  $G$  has no even factor with exactly two components. Then the number of components  $t$  of any even factor satisfies  $t=1$  or  $t \geq 3$ . Before present the proofs 1), 2), we note that a cycle always existis, since  $G$  is 2-edge-connected and  $C$  is an induced cycle. Then we can choose an induced cycle  $C = xzz_1\omega_1 \dots \omega_l y_1 yx$ , ( $l \geq 4$ ) of  $G$  such that  $xy \in \widehat{\Theta}_F \cap E(C)$ ,  $xz \in \widehat{\Theta}_F \cap E(C)$  and  $z_1 \in N_C(z) \cap V(Q_k)$  and  $y_1 \in N_C(y) \cap V(Q_j)$ . Next, we will prove parts 1) to 2) of Theorem 11, respectively.

**1):** Let  $G$  be a 2-edge-connected and  $\{F_2, P_8\}$ -free graph. Since  $G$  is  $P_8$ -free,  $|V(C)| \leq 8$ . By lemma 11,  $|V(C)| \geq 5$ . If  $t=1$ , then  $G=F$  is an even factor with exactly one component, so  $E \setminus F = \emptyset$ . Then we may choose an induced cycle  $C = xx_1\omega_1 y_1 yx$  of  $G$ . Since  $G$  is an even subgraph, the degree of  $x$  is even. Thus, there exist  $x_2 \in N(x) \setminus \{x_1, y\}$  and  $y_2 \in N(y) \setminus \{y_1, x\}$ . Since  $|V(G)| \geq 10$ , there exist at least three additional distinct vertices such that  $y_3 \in N(y_1) \setminus \{\omega_1, y\}$ ,  $y_4 \in N(y_3) \setminus \{y_1\}$ ,  $y_5 \in N(y_4) \setminus \{y_3\}$ . Since  $G[\{x_2, x, x_1, \omega_1, y_1, y_3, y_4, y_5\}] \not\cong P_8$ ,  $\{\omega_1 x_2, y_2 y_1\} \subseteq E(G)$ . However,  $G[\{x_2, \omega_1, x_1, x, y, y_2\}] \cong F_2$ , a contradiction. Thus,  $t \neq 1$ , which implies  $t \geq 2$ . Now we distinguish the following cases:

To prove that  $t=2$ , we assume that  $t \geq 3$ .

**Case (1):**  $|V(C)| = 5$ .

If  $|V(C)| = 5$ , then there are two edges  $xy \in \widehat{\Theta}_F$  and  $xz \in \widehat{\Theta}_F$  such that  $x \in Q_i, y \in Q_j, z \in Q_k$  and  $\{i, j, k\} \in \{1, 2, \dots, t\}$ . Let  $\{x_1, x_2\} \subseteq N(x) \cap V(Q_i)$ ,  $\{y_1, y_2\} \subseteq N(y) \cap V(Q_j)$  and  $\{z_1, z_2\} \subseteq N(z) \cap V(Q_k)$ . By applying Lemma 11 to  $xy$  and  $xz$ , we can obtain  $E(\{x_1, x_2\}, \{y_1, y, y_2\}) = \emptyset$ ,  $E(\{x\}, \{y_1, y_2\}) = \emptyset$ ,  $E(\{x_1, x_2\}, \{z_1, z, z_2\}) = \emptyset$  and  $E(\{x\}, \{z_1, z_2\}) = \emptyset$ . Then we can choose an induced cycle  $C = xzz_1\omega_1 yx$ . Since  $Q_i, Q_j$  and  $Q_k$  are different components in  $F$ , then at least one of  $\{z_1\omega_1, \omega_1 y\}$  is in  $\widehat{\Theta}_F$ . If  $\omega_1 \notin V(Q_i) \cup V(Q_j) \cup V(Q_k)$ , then  $F\Delta E(C)$  is an even factor with fewer components than  $F$  such that  $Q_i, Q_j, Q_k$  which contains  $\omega_1$  are all in the same component of  $F\Delta E(C)$ , a contradiction. Then  $\omega_1 \in V(Q_i) \cup V(Q_j) \cup V(Q_k)$ , without loss of generality, we assume that  $\omega_1 \in V(Q_k)$ .

**Claim 1:**  $\omega_1$  is not adjacent to any vertex in  $(N(z) \cap V(Q_k)) \setminus \{z_1\}$ .

**Proof:** By contradiction, we first assume that there exists a vertex  $z_0 \in (N(z) \cap V(Q_k)) \setminus \{z_1\}$  such that  $z_0\omega_1 \in E(G)$ . Applying lemma 11(1) to  $xz$ , we have  $E(\{z_0\}, \{x_1, x_2, x\}) = \emptyset$ . Since  $C$  is an induced cycle,

$\{x\omega_1\} \cap E(G) = \emptyset$ . Since  $G[\{\omega_1, z_1, z, z_0, x, y\}] \cong F_2$ ,  $z_0z_1 \in E(G)$ . Let  $T(z z_1) = z z_1 z_0 z$ . If  $z z_1 \notin E(F)$ , then  $F_C(T(z z_1)) = F\Delta E(xy\omega_1 z_1 z x)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $z z_1 \in E(F)$  and  $|\{z z_1, z z_0, z_1 z_0\} \cap E(G)| \leq 2$ , then  $F_C(T(z z_1)) = F\Delta E(xy\omega_1 z_1 z x) \cup E(z z_1 z_0 z) \setminus \{z z_1\}$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(z z_1))$  has fewer components than  $F$ , which is a contradiction. This proves Claim 1.  $\square$

Since  $|V(G)| \geq 10$ , there exist additional distinct vertices such that  $z_3 \in N(z_2) \setminus \{z\}$ ,  $z_4 \in N(z_3) \setminus \{z_2\}$ ,  $z_5 \in N(z_4) \setminus \{z_3\}$  such that  $E(\{z_3\}, \{z, z_1, \omega_1\}) = \emptyset$ . First suppose that  $z_3 z_1 \in E(G)$ , then  $G[\{z_1, z, z_2, z_3, z_4, z_5\}] \cong F_2$ , a contradiction. Next suppose that  $z_3 z_2 \in E(G)$ . Let  $T(z z_3) = z z_2 z_3 z$ . If  $z z_3 \notin E(F)$ , then  $F_C(T(z z_3)) = F\Delta E(xy\omega_1 z_1 z x)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $z z_3 \in E(F)$  and  $|\{z z_3, z z_2, z_2 z_3\} \cap E(G)| \leq 2$ , then  $F_C(T(z z_3)) = F\Delta E(xy\omega_1 z_1 z x) \Delta E(z z_2 z_3 z)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(z z_3))$  has fewer components than  $F$ , a contradiction. By claim 1, we have  $\{z_3\omega_1, z_4\omega_1, z_5\omega_1\} \subsetneq E(G)$ . Since  $G[\{z_4, z_3, z_2, z, z_1, \omega_1\}] \cong F_2$ ,  $z z_4 \notin E(G)$ . Since  $G[\{y_1, y, x, z, z_2, z_3, z_4, z_5\}] \cong P_8$ ,  $z z_5 \in E(G)$ . Since  $G[\{y_1, y, x, z, z_2, z_3, z_4, z_1\}] \cong P_8$ ,  $z_1 z_4 \notin E(G)$ . We suppose that there exists a vertex  $\{y_0\} \subseteq N(y_1) \cap V(Q_j) \setminus \{y_2\}$ . We claim that  $y_0 y \notin E(G)$ . Otherwise, let  $T(y y_0) = y y_0 y_1 y$ . If  $y y_0 \notin E(F)$ , then  $F_C(T(y y_0)) = F\Delta E(xy\omega_1 z_1 z x)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $y y_0 \in E(F)$  and  $|\{y y_1, y y_0, y_1 y_0\} \cap E(G)| \leq 2$ , then  $F_C(T(y y_0)) = F\Delta E(xy\omega_1 z_1 z x) \Delta E(y y_0 y_1 y)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(y y_0))$  has fewer components than  $F$ , a contradiction. Since  $G[\{y_0, y_1, y, x, z, z_2, z_3, z_4\}] \cong P_8$ ,  $z_2\omega_1 \in E(G)$ . However,  $G[\{z_2, z, z_1, \omega_1, y, y_1\}] \cong F_2$ , a contradiction.

Then, without loss of generality, we assume that  $\omega_1 \in V(Q_j)$ . By the choice of  $C$ , we have  $z\omega_1 \notin E(G)$ . Since  $G[\{y_1, y, \omega_1, z_1, z, z_2, z_3, z_4\}] \cong P_8$ ,  $\{y_1\omega_1, z_3 z_1\} \subseteq E(G)$ . However,  $G[\{z_3, z_2, z, z_1, \omega_1, y\}] \cong F_2$ , a contradiction.

**Case (2):**  $|V(C)| = 6$ .

If  $|V(C)| = 6$ , then we can choose an induced cycle  $C = x z z_1 \omega_1 y_1 y x$ . Since  $Q_i, Q_j$  and  $Q_k$  are different components in  $F$ , then at least one of  $\{\omega_1 z_1, \omega_1 y_1\}$  is in  $\widehat{\Theta}_F$ . First suppose that  $\omega_1 y_1 \in \widehat{\Theta}_F$ . By applying Lemma 11 to  $\omega_1 y_1$ , we have  $\{z_1 y_1, z_1 y_0, \omega_1 y_0, \omega_1 y\} \subsetneq E(G)$ . Since  $G[\{y_0, y_1, \omega_1, z_1, z, z_2, z_3, z_4\}] \cong P_8$ ,  $z_2\omega_1 \in E(G)$ . However,  $G[\{z_2, z, z_1, \omega_1, y_1, y_0\}] \cong F_2$ , a contradiction. Then  $\omega_1 z_1 \in \widehat{\Theta}_F$ . By applying Lemma 11 to  $\omega_1 z_1$ , we have  $\{z y_1, z\omega_1\} \subsetneq E(G)$ . Since  $G[\{z_3, z_2, z, z_1, \omega_1, y_1, y, y_2\}] \cong P_8$ ,  $z_3 z_1 \in E(G)$ . However,  $G[\{z_3, z_2, z, z_1, \omega_1, y_1\}] \cong F_2$ , a contradiction.

**Case (3):**  $|V(C)| = 7$ .

If  $|V(C)| = 7$ , then we can choose an induced cycle  $C = x z z_1 \omega_1 \omega_2 y_1 y x$ . By the choice of  $C$ , without loss of generality, we assume that at least one of  $\{\omega_1, \omega_2\}$  is in  $V(Q_j) \cup V(Q_k)$ . If  $\{\omega_1, \omega_2\} \notin E(G)$ , then  $F\Delta E(C)$  is an even factor with

fewer components than  $F$ , a contradiction. Thus, both  $\omega_1$  and  $\omega_2$  are in  $V(Q_j) \cup V(Q_k)$ . We assume that  $y_2y_1 \notin E(G)$ . Conversely, let  $T(yy_1) = yy_1y_2y$ . If  $yy_1 \notin E(F)$ , then  $F_C(T(yy_1)) = F\Delta E(xzz_1\omega_1\omega_2y_1yx)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $yy_1 \in E(F)$  and  $|\{yy_1, yy_2, y_1y_2\} \cap E(G)| \leq 2$ , then  $F_C(T(yy_1)) = F\Delta E(xzz_1\omega_1\omega_2y_1yx)\Delta E(yy_1y_2y) \setminus \{yy_1\}$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(yy_1))$  has fewer components than  $F$ , a contradiction. Next, suppose that  $\omega_1 \in V(Q_k)$  and  $\omega_2 \notin V(Q_i) \cup V(Q_j) \cup V(Q_k)$ . Since  $|V(G)| \geq 10$ , there exist additional distinct vertices such that  $y_3 \in N(y_2) \setminus \{y\}$ . Since  $G[\{y_3, y_2, y, y_1, x, z\}] \not\cong F_2$ ,  $y_3y_1 \notin E(G)$ . Since  $y_3y \notin E(G)$ . Otherwise, let  $T(yy_3) = yy_2y_3y$ . If  $yy_3 \notin E(F)$ , then  $F_C(T(yy_3)) = F\Delta E(xzz_1\omega_1\omega_2y_1yx)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $yy_3 \in E(F)$  and  $|\{yy_3, yy_2, y_2y_3\} \cap E(G)| \leq 2$ , then  $F_C(T(yy_3)) = F\Delta E(xzz_1\omega_1\omega_2y_1yx)\Delta E(yy_2y_3y)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(yy_3))$  has fewer components than  $F$ , a contradiction. Since  $G[\{y_3, y_2, y, y_1, \omega_2, \omega_1, z_1, z\}] \not\cong P_8$ ,  $y_3y_1 \in E(G)$ . However,  $G[\{y_3, y_2, y, y_1, \omega_1, \omega_2\}] \cong F_2$ , a contradiction. Then  $\{\omega_1, \omega_2\} \subseteq E(G)$ .

First suppose that  $\omega_1, \omega_2$  are in the same component such that  $\{\omega_1, \omega_2\} \subseteq E(G)$ . Since  $G[\{y_2, y, y_1, \omega_2, \omega_1, z_1, z, z_2\}] \not\cong P_8$ ,  $z\omega_2 \in E(G)$ . By applying lemma 12 to  $y_1\omega_2, \{\omega_1y_1, \omega_1y\} \subsetneq E(G)$ . However,  $G[\{y, y_1, z, z_1, \omega_1, \omega_2\}] \cong F_2$ , a contradiction. Next suppose that  $\omega_1, \omega_2$  are in the different component such that  $\omega_1 \in V(Q_k)$  and  $\omega_2 \in V(Q_j)$ . Since  $G[\{y_2, y, y_1, \omega_2, \omega_1, z_1, z, z_2\}] \not\cong P_8$ ,  $y\omega_2 \in E(G)$ . However,  $F_C(T(y\omega_2))$  has fewer components than  $F$ , a contradiction.

**Case (4):**  $|V(C)| = 8$ .

If  $|V(C)| = 8$ , then at least one of  $\{z_1\omega_1, \omega_1\omega_2, \omega_2\omega_3, \omega_3y_1\}$  is in  $\widehat{\Theta}_F$ . By symmetry, if  $\omega_2\omega_3 \in \widehat{\Theta}_F$ . By applying Lemma 12 to  $\omega_2\omega_3$ , we have  $\{\omega_1y_1, \omega_1\omega_3\} \subsetneq E(G)$ . Since  $G[\{y_2, y, y_1, \omega_3, \omega_2, \omega_1, z_1, z\}] \not\cong P_8$ ,  $\{\omega_2z, y\omega_3\} \subseteq E(G)$ . However,  $G[\{y_1, \omega_3, \omega_2, \omega_1, z_1, z\}] \cong F_2$ , a contradiction. Then, up to symmetry, that  $z_1\omega_1 \in \widehat{\Theta}_F$ . By applying Lemma 11 to  $z_1\omega_1$ , we have  $\{z\omega_1, z\omega_2\} \subsetneq E(G)$ . Since  $G[\{z_2, z, z_1, \omega_1, \omega_2, \omega_3, y_1, y\}] \not\cong P_8$ ,  $z_2z_1 \in E(G)$ . However, let  $T(zz_1) = zz_1z_2z$ . If  $zz_1 \notin E(F)$ , then

$F_C(T(zz_1)) = F\Delta E(xzz_1\omega_1\omega_2\omega_3y_1yx)$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. If  $zz_1 \in E(F)$  and  $|\{zz_1, zz_2, z_1z_2\} \cap E(G)| \leq 2$ , then  $F_C(T(zz_1)) = F\Delta E(xzz_1\omega_1\omega_2\omega_3y_1yx)\Delta E(zz_1z_2z) \setminus \{zz_1\}$  is an even factor of  $G$  such that  $Q_i, Q_j, Q_k$  are in the same component. In this case,  $F_C(T(zz_1))$  has fewer components than  $F$ , a contradiction.

**2):** Let  $G$  be a 2-edge-connected and  $\{T_{2,1,1}, S\}$ -free graph, where  $S$  is  $P_8, Z_5, B_{2,3}$  or  $N_{1,1,3}$ . Since  $G$  is  $P_8$ -free,  $|V(C)| \leq 8$ . By lemma 11,  $|V(C)| \geq 5$ . If  $t=1$ , then  $G=F$  is an even factor with exactly one component, so  $E \setminus F = \emptyset$ . Then we may choose an induced cycle  $C = xx_1\omega_1\omega_2\omega_3\omega_4y_1yx$  of  $G$ .

Since  $G$  is an even subgraph, the degree of  $x$  is even. Thus, there exist  $x_2 \in N(x) \setminus \{x_1, y\}$  and  $y_2 \in N(y) \setminus \{y_1, x\}$ . Since  $|V(G)| \geq 10$ , there exist additional distinct vertices such that  $y_3 \in N(y_1) \setminus \{\omega_1, y\}$  and  $y_4 \in N(y_3) \setminus \{y_1\}$ . Since  $G[\{x, x_1, \omega_1, x_2, y\}] \not\cong T_{2,1,1}$ ,  $x_2x_1 \in E(G)$ . Since  $G[\{y, y_1, y_3, y_2, x\}] \not\cong T_{2,1,1}$ ,  $y_2y_1 \in E(G)$ . Since  $G[\{y_1, \omega_1, \omega_3, y_3, y\}] \not\cong T_{2,1,1}$ ,  $yy_3 \in E(G)$ . However,  $G[\{x_2, x_1, \omega_1, \omega_2, \omega_3, \omega_4, y_1, y_3\}] \cong P_8$ ;  
 $G[\{y, y_1, y_2, \omega_4, \omega_3, \omega_2, \omega_1, x_1\}] \cong Z_5$ ;  $G[\{x, x_1, x_2, \omega_1, \omega_2, \omega_3, y, y_3\}] \cong B_{3,2}$ ;  
 $G[\{y_1, y, y_3, \omega_4, \omega_3, \omega_2, x, y_4\}] \cong N_{1,1,3}$ , a contradiction. Thus,  $t \neq 1$ , which implies  $t \geq 2$ .

We prove that  $t = 2$ , by contradiction, assume that  $t \geq 3$ . Then we choose an induced cycle  $C = xz z_1 \omega_1 \cdots \omega_l y_1 y x$  of  $G$  such that  $xy \in \widehat{\Theta}_F \cap E(C)$  and  $xz \in \widehat{\Theta}_F \cap E(C)$ . By the choice of  $C$ , without loss of generality, we assume that  $z_1 \in V(Q_k) \cap N_C(z)$ . Suppose that  $y_1 \in V(F) \setminus (V(Q_i) \cup V(Q_j) \cup V(Q_k))$ . Let  $x_1 \subseteq N(x) \cap V(Q_i)$ ,  $y_2 \subseteq N(y) \cap V(Q_j)$  and  $z_2 \subseteq N(z) \cap V(Q_k) \setminus \{z_1\}$ . By lemma 11 to  $xy$  and  $xz$ , we have that  $E(\{x_1\}, \{y, y_2\}) = \emptyset$ ,  $E(\{x_1\}, \{z_1, z, z_2\}) = \emptyset$  and  $E(\{x\}, \{z_1, z_2, y_2\}) = \emptyset$ . Since  $|V(C)|$  is minimized and  $|\{u \in V(C), d(u) = 2\}|$  is also minimized. Then  $\{zy_1, z_1y_1\} \subsetneq E(G)$ . However,  $G[\{y, y_2, y_1, x, z\}] \cong T_{2,1,1}$ , a contradiction. Now suppose that  $y_1 \in V(Q_k) \cap N_C(z)$ . Since  $G[\{z, x, z_2, z_1, y_1\}] \not\cong T_{2,1,1}$ ,  $z_1z_2 \in E(G)$ . Then  $F_C(T(z z_1))$  has fewer components than  $F$ , a contradiction. Thus,  $y_1 \in V(Q_j) \cap N_C(y)$  and  $|V(C)| = 5$ . Now we distinguish the following claim.

**Claim 2:**  $G[N(x) \setminus \{y\}]$  is a clique, for any singular edge  $xy \in E(G)$  with  $d_G(x) \geq 3$ .

**Proof:** By contradiction, we assume that there exist two distinct vertices  $x'$  and  $x''$  such that  $\{x', x''\} \subseteq N(x) \setminus \{y\}$ . Since  $G[\{x, x', x'', y, y'\}] \not\cong T_{2,1,1}$ ,  $x'y' \in E(G)$  or  $x''y' \in E(G)$ . First assume that  $x'y' \in E(G)$ . By the choice of  $C$ , we have  $y'' \in N(y') \setminus \{y\}$  and  $y''' \in N(y'') \setminus \{y'\}$ . However,  $G[\{y', x', y, y'', y'''\}] \cong T_{2,1,1}$ , a contradiction. Next assume that  $x''y' \in E(G)$ . However,  $G[\{y', x'', y, y'', y'''\}] \cong T_{2,1,1}$ , a contradiction. Thus,  $G[N(x) \setminus \{y\}]$  is a clique, which implies  $x'x'' \in E(G)$ .  $\square$

By lemma 11,  $|V(C)| \geq 5$ . By claim 2,  $x_1x_2, y_1y_2, z_1z_2 \in E(G)$ . If  $d_G(\omega_s) \geq 3$  for any  $\omega_s \in V(C)$ , where  $s \in \{1, 2, \dots, l\}$ . By the choice of  $C$ , it is known that there are at most  $l-1$  vertices with degree two in  $\{\omega_1, \dots, \omega_l\}$ , which implies  $0 \leq |V_2(G) \cap V(C)| \leq l-1$ . If  $|V_2(G) \cap V(C)| = l-1$ . Then  $d(\omega_1) = \dots = d(\omega_{l-1}) = 2$  or  $d(\omega_2) = \dots = d(\omega_l) = 2$ . Thus,  $\{\omega_1, \dots, \omega_l\} \subseteq V(Q_j)$  or  $\{\omega_1, \dots, \omega_l\} \subseteq V(Q_k)$  and  $\omega_l y_1 \in \widehat{\Theta}_F$ . By claim 2, we have a vertex  $\omega_u \in (N(\omega_l) \cap V(Q_k) \setminus \{\omega_{l-1}\})$  and  $\omega_u \omega_{l-1} \in E(G)$ , which implies  $d_G(\omega_{l-1}) \geq 3$ . This contradicts that  $d_G(\omega_{l-1}) = 2$ . Then  $0 \leq |V_2(G) \cap V(C)| \leq l-2$ . Now we distinguish the following cases.

**Case (1):**  $|V(C)| = 6$ .

If  $|V(C)| = 6$ , since  $Q_i, Q_j$  and  $Q_k$  are different components in  $F$ , then at least one of  $\{z_1\omega_1, \omega_1y_1\}$  is in  $\widehat{\Theta}_F$ . First assume that  $z_1\omega_1 \in \widehat{\Theta}_F$ . By claim 2,  $z_1z_2$  is in a triangle  $T(z_1z_2)$ . Then  $F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2))$

is an even factor with fewer components than  $F$ , a contradiction.

**Case (2):**  $|V(C)| = 7$ .

If  $|V(C)| = 7$ , since  $Q_i, Q_j$  and  $Q_k$  are different components in  $F$ , then at least one of  $\{z_1\omega_1, \omega_1\omega_2, \omega_2y_1\}$  is in  $\widehat{\Theta}_F$ . Since  $0 \leq |V_2(G) \cap V(C)| \leq l - 2 = 0$ , which implies  $d_G(\omega_s) \geq 3$  for each  $s \in \{1, 2\}$ . First assume that  $z_1\omega_1 \in \widehat{\Theta}_F$ . By claim 2,  $\omega_1\omega_2$  is in a triangle  $T(\omega_1\omega_2)$ . Then

$F_C(T(\omega_1\omega_2) \cup F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  is an even factor with fewer components than  $F$ , a contradiction. Next assume that  $\omega_1\omega_2 \in \widehat{\Theta}_F$ . Furthermore,  $\omega_1 \in V(Q_k)$  and  $\omega_2 \in V(Q_j)$ . Then

$F_C(T(z_1z_2) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  is an even factor with fewer components than  $F$ , a contradiction. Then  $\omega_2y_1 \in \widehat{\Theta}_F$ . Thus,

$F_C(T(\omega_1\omega_2) \cup F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  is an even factor with fewer components than  $F$ , a contradiction.

**Case (3):**  $|V(C)| = 8$ .

Since  $0 \leq |V_2(G) \cap V(C)| \leq l - 2 = 1$ . We distinguish the following two cases.

**Subcase (1):**  $|V_2(G) \cap V(C)| = 0$ .

Since  $Q_i, Q_j$  and  $Q_k$  are different components in  $F$ , by the choice of  $C$ , then at least one of  $\{z_1\omega_1, \omega_1\omega_2, \omega_2\omega_3, \omega_3y_1\}$  is in  $\widehat{\Theta}_F$ . First suppose that  $z_1\omega_1 \in \widehat{\Theta}_F$ . By claim 2,  $\omega_1\omega_2$  is in a triangle  $T(\omega_1\omega_2)$ . Then

$F_C(T(\omega_1\omega_2) \cup F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  is an even factor with fewer components than  $F$ , a contradiction. Furthermore, at least one of  $\{\omega_2\omega_3, \omega_3y_1\}$  is nonsingular, this implies that there exist triangles  $T(\omega_2\omega_3)$  and  $T(\omega_3y_1)$ . Then

$F_C(T(\omega_1\omega_2) \cup F_C(T(\omega_2\omega_3) \cup F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  or  $F_C(T(\omega_1\omega_2) \cup F_C(T(\omega_3y_1) \cup F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  is an

even factor with fewer components than  $F$ , a contradiction. Next suppose that  $\omega_1\omega_2 \in \widehat{\Theta}_F$ . By claim 2,  $\omega_2\omega_3$  is in a triangle  $T(\omega_2\omega_3)$ . Then

$F_C(T(\omega_2\omega_3) \cup F_C(T(z_1z_2)) \cup F_C(T(y_1y_2)) \cup F_C(T(x_1x_2)))$  is an even factor with fewer components than  $F$ , a contradiction.

**Subcase (1):**  $|V_2(G) \cap V(C)| = 1$ .

It suffices to consider when  $|V_2(G) \cap V(C)| = 1 = \{\omega_1\}$  or  $\{\omega_2\}$ . First suppose that  $|V_2(G) \cap V(C)| = \{\omega_2\}$ , then at least one of  $\{z_1\omega_1, \omega_3y_1\}$  is in  $\widehat{\Theta}_F$  and  $\{\omega_1, \omega_2, \omega_3\}$  are in the same component  $Q_j$  or  $Q_k$ . We assume that  $z_1\omega_1 \in \widehat{\Theta}_F$ . Then there exists a vertex  $\omega'_1 \in (N(\omega_1) \cap V(Q_j)) \setminus \{\omega_2\}$ . By claim 2, we have  $\omega'_1\omega_2 \in E(G)$ . This contradiction implies that  $d_G(\omega_2) \neq 2$ . Then  $\omega_3y_1 \in \widehat{\Theta}_F$  and there exists a vertex  $\omega'_3 \in (N(\omega_3) \cap V(Q_k)) \setminus \{\omega_2\}$ . By claim 2, we have  $\omega'_3\omega_2 \in E(G)$ , contradicting  $d_G(\omega_2) = 2$ .

Now suppose that  $|V_2(G) \cap V(C)| = \{\omega_1\}$ . Then at least one of  $\{\omega_2\omega_3, \omega_3y_1\}$  is in  $\widehat{\Theta}_F$ . If  $\omega_2\omega_3 \in \widehat{\Theta}_F$ , then there exists a vertex  $\omega'_2 \in (N(\omega_2) \cap V(Q_k)) \setminus \{\omega_1\}$ . By claim 2, we have  $\omega'_2\omega_1 \in E(G)$ . This contradiction implies that  $d_G(\omega_1) \neq 2$ . Thus,  $\omega_2\omega_3 \notin \widehat{\Theta}_F$ . Then  $\omega_3y_1 \in \widehat{\Theta}_F$  and  $\{\omega_1, \omega_2, \omega_3\} \subseteq V(Q_k)$ . By claim 2, we have  $\{x_1x_2, y_1y_2, z_1z_2\} \subseteq E(G)$ . Take a vertex  $y'_2 \in (N(y_2) \cap V(Q_j)) \setminus \{y\}$  and  $\omega_3$  has no neighbors other than  $\omega_2$  and  $y_1$ . Otherwise, we assume that there

exist a vertex  $\omega'$  such that  $\omega' \in N(\omega_3) \setminus \{\omega_2\}$ . Since  $G[\{\omega_3, \omega', \omega_2, y_1, y'\}] \not\cong T_{2,1,1}$ ,  $\omega' \omega_2 \in E(G)$ , contradicting  $d_G(\omega_2) = 2$ . However,  $G[\{x_1, x, z, z_1, \omega_1, \omega_2, \omega_3, y_1\}] \cong P_8$ ;  $G[\{z_2, z, z_1, \omega_1, \omega_2, \omega_3, y_1, y_2\}] \cong Z_5$ ;  $G[\{z_2, z, z_1, \omega_1, \omega_2, \omega_3, x, y\}] \cong B_{3,2}$ ;  $G[\{y_1, y, y_2, y'_2, \omega_3, x, z, z_1\}] \cong N_{1,1,3}$ , a contradiction. Thus,  $t \neq 3$ , which implies  $t = 2$ . This completes proof of the sufficiency of Theorem 10.  $\square$

#### 4. Concluding Remarks

Determining whether a graph is supereulerian is NP-hard, so we often study this problem through even factors. It is obviously that graphs  $G$  containing an even factor satisfy  $\delta(G) \geq 2$  and every odd branch-bond of  $G$  has an edge branch. What conditions must a graph satisfy for the converse to hold? In this paper, we study this converse by forbidden subgraphs, and characterize all forbidden subgraphs for the existence of an even factor with exactly two components in a 2-edge-connected graph.

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#### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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