Applied Mathematics, 2025, 16(12), 867-876
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

The Independent Cascade Graph Burning for Operation Graphs

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How to cite this paper: Zhu, Q.Y., Wu, J.Q. and Li, Y.K. (2025) The Independent Cascade Graph Burning for Operation Graphs. *Applied Mathematics*, **16**, 867-876.
https://doi.org/10.4236/am.2025.1612045

Received: November 21, 2025
Accepted: December 19, 2025
Published: December 22, 2025

Abstract

Graph burning is a model to describe the spread of social influence. In 2023, Song *et al.* proposed the Independent Cascade Graph Burning model, where a vertex v can be burned by its burning neighbors u and the influence that u gives to v is larger than a given threshold β. The minimum number of time steps that can be chosen as rounds to burn the whole graph G with the Independent Cascade Graph Burning model called the IC burning number $b_\beta(G)$. In this paper, we determined the IC burning number for some graphs and operation graphs.

Keywords

IC Burning Number, Binary Tree, Spider, Sunflower Graph

1. Introduction

Graph burning is a discrete-time process on graphs [1], Bonato *et al.* introduced the concept of the burning number to measures the speed of contagion spread on a graph and denoted the burning number of graph G by $b(G)$ in [2], some special classes of graphs has been studied, such as spider graphs [3], path forest [4], generalized Petersen graphs [5], theta graphs [6], caterpillars [7] and fence graphs [8]. For a survey on graph burning see [9]. Later, Li *et al.* [10] generalized the burning number and introduced the generalized burning number $b_r(G)$ of G for $r \geq 1$. Follow these, Song *et al.* [11] propose the Independent Cascade graph burning model of G, where a burned vertex v can burn its neighbor w only if the influence that v exerts on w is larger than a given threshold β. Note that when $\beta = 0$, it is a traditional burning problem. The task is still to find the minimum sequence of vertices that can burn the whole graph. The minimum number of time steps is IC burning number $b_\beta(G)$ of graph G. In the burn

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process of IC model of graph G, we call a vertex is fire source, it is selected to burn. For a given threshold β, the x_i is the i-th fire source in the β-burning process of G. If G are burned after k time steps, we call the fire source sequence $(x_1, x_2, \cdots, x_k)$ a β-burning sequence of G. Clearly, the IC burning number $b_\beta(G)$ is the minimum length among all β-burning sequences of graph G.

In reality, the influence u receives from its neighbor w is an arbitrary value in $[0,1]$, we easily know whether a vertex can be burned by its neighbor depends heavily on its degree. For $v, u \in V(G)$, the distance between them is denoted by $d(u,v)$. The open neighborhood $N(v)$ is the set of vertices at distance one from a vertex v. Clearly, the closed neighborhood $N[v] = N(v) \cup \{v\}$. Given a positive integer k and fraction $\beta \in [0,1]$, the k-th closed β-neighborhood of u is a set $\{v \in V(G) : d(u,v) \le k, f(u) \ge \beta\}$ and is denoted by $N_\beta^k[v]$, where $f(u) = \frac{1}{d(u)}$.

For $h \ge 1$, $k \ge 2$, a perfect k-ary tree with height h, denoted T_k^h, is a tree with k^h leaves and a root vertex with degree k whose distance to all the leaves is h and all other internal vertices have degree $k+1$. The height of a vertex is the number of edges present in the path connecting that vertex to a leaf vertex. We call the internal vertices that are the parents of leaves as parent-leaves.

A spider is a tree contain one vertex called the spider head with degree at least 3. In a spider graph, every leaf is connected to the head by a path which called an arm. we denote such a spider graph by $SP(s,r)$ if all the arms of the spider graph with maximum degree s are of the same length r.

A $(n,1)$-lollipop graph $L_{n,1}$ is a graph with $V(L_{n,1}) = \{v, u_1, u_2, \cdots, u_n\}$ and $E(L_{n,1}) = \{u_i u_j \mid i, j = 1, 2, \cdots, n, i \ne j\} \cup \{v u_1\}$, see **Figure 1(a)**. A corona graph of K_n and K_1, denoted by $K_n \circ K_1$, is graph with
$V(K_n \circ K_1) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and
$E(K_n \circ K_1) = \{u_i u_j \mid i, j = 1, 2, \cdots, n, i \ne j\} \cup \{v_i u_i \mid i = 1, 2, \cdots, n\}$, see **Figure 1(b)**.

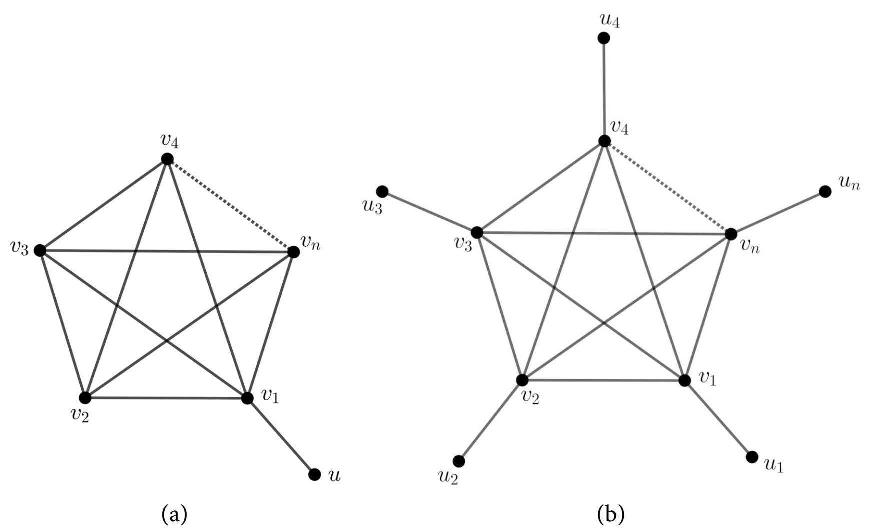

Figure 1. $(n,1)$-lollipop graph $L_{n,1}$ and corona graph $K_n \circ K_1$.

All graphs considered in this paper are finite and simple. We use book [12] for notation and terminology not defined here. In this paper, we study the β-burning problem on several graph including complete k-ary tree, spider graphs, $(n,1)$-lollipop graph, corona graph of K_n and K_1 and other graphs such as sunflower graph, friendship graph and Dutch windmill graph.

2. Primarilies

Proposition 2.1. [11] *G is a connected graph with n vertices and $f(v_i)<\beta$ for $1\le i\le k$, where $0<\beta\le 1$. Then, $k\le b_\beta(G)\le n$.*

Proposition 2.2. [11] *G is a graph with n vertices and $\Delta(G)$ is the maximum degree. Then, (x_1,x_2) is an optimum β-burning sequence for G if and only if one of the following conditions is met:*

1) $\Delta(G)=d(x_1)=n-1$, and $f(w)\ge\beta$ for all $\forall w\in N(x_1)\setminus\{x_2\}$.
2) $\Delta(G)\ge d(x_1)=n-2$, and $f(w)\ge\beta$ for all $\forall w\in N(x_1)$.

In [13], Bonato *et al.* provide a number of properties of the burning number.

Proposition 2.3. [13] *If T is a tree and H is a subtree of T, then we have that $b(H)\le b(T)$.*

A subgraph H of graph G is called an isometric subgraph if we have $d_H(u,v)=d_G(u,v)$.

Proposition 2.4. [13] *Let H be an isometric subgraph of a graph G and for any node $x\in V(G)\setminus V(H)$, and any positive integer k, there exist a node $f_k(x)\in V(H)$ satisfies $N_k[x]\cap V(H)\subseteq N_k^H[f_k(x)]$. Then we have that $b(H)\le b(G)$.*

Proposition 2.5. [13] *For any graph G with radius r and diameter d, we have that*

$$\left\lceil\sqrt{d+1}\right\rceil\le b(G)\le r+1.$$

Proposition 2.6. [13] *For a graph G, we have that*

$$b(G)=\min\{b(T):T \text{ is a spanning subtree of } G\}.$$

3. Main Results

In this section, we determined the IC burning number of some operation graphs. First, we consider the perfect binary tree, spider graphs, $(n,1)$-lollipop graph, corona graph $K_n\circ K_1$.

Theorem 3.1. *Let T_2^h be a perfect binary tree of height h, where $h\ge 2$. Then*

$$b_\beta(T_2^h)=\begin{cases} h+1, & \text{If } 0\le\beta\le\frac{1}{3};\\ \begin{cases}3, & \text{If } h=2;\\ 2^h-2, & \text{If } h\ge 3.\end{cases} & \text{If } \frac{1}{3}<\beta\le\frac{1}{2};\\ 2^h-1, & \text{If } \frac{1}{2}<\beta\le 1.\end{cases}$$

Proof. Let s be the root of T_2^h and the leaves of T_2^h be u_i for $1\le i\le 2^h$.

DOI: 10.4236/am.2025.1612045

Suppose the parent-leaves of T_2^h be v_i for $1 \le i \le 2^{h-1}$.

Case 1: $0 \le \beta \le \frac{1}{3}$

The T_2^h has total $h+1$ levels, since T_2^h has height h. Suppose we let $x_1 = s$. At the t-th step $(t = h+1)$, the fire completes the burning of the t-th level of T_2^h. Therefore, $b_\beta(T_2^h) \le h+1$. Now, suppose $S = (x_1, x_2, \cdots, x_k)$ is a β-burning sequence of T_2^h, where $k < h+1$. The fire source x_i can spread to the vertices of $N_{k-i}[x_i]$. For $1 \le i \le k$, $N_{k-i}[x_i]$ can contain at most 2^{k-i} leaf vertices of T_2^h. Therefore, we burned at most $\sum_{i=1}^{k} 2^{k-i} = 2^k - 1$ leaf vertices. Since the total number of leaves of T_2^h is 2^h. Now, $2^h \ge 2^k > 2^k - 1$, a contradiction, we have $b_\beta(T_2^h) \ge h+1$. Thus $b_\beta(T_2^h) = h+1$.

Case 2: $\frac{1}{3} < \beta \le \frac{1}{2}$

Let two children of s be s_1 and s_2. Suppose $S = \{s, u_1, u_2, \cdots, u_{2^h}\}$, $Y = \{s_1, s_2, v_1, v_2, \cdots, v_{2^{h-1}}\}$ and $X = S \cup Y$. The case for $h = 2$, by simple checking, we know that $b_\beta(T_2^h) = 3$. Next, the case $h \ge 3$. Since $\frac{1}{3} < \beta \le \frac{1}{2}$, thus $T_2^h \setminus S$ must be fire sources. Let $x_1 = s_1$, $x_2 = s_2$ and $x_i = v_{i-2}$ for $3 \le i \le 2^{h-1} + 2$. Further, we choose $T_2^h \setminus X$ as remain fire sources. Obviously, $(x_1, x_2, \cdots, x_{2^h - 2})$ is a β-burning sequence of T_2^h and thus $b_\beta(T_2^h) \le 2^h - 2$. Now, we show $b_\beta(T_2^h) \ge 2^h - 2$. As $f(V(T_2^h) \setminus S) < \beta$ for $1 \le i \le 2^h - 2$. By Proposition 2.1, we get $2^h - 2 \le b_\beta(T_2^h)$. Then, $b_\beta(T_2^h) = 2^h - 2$.

Case 3: $\frac{1}{2} < \beta \le 1$

Let $U = \{u_i \mid 1 \le i \le 2^h\}$, $V = \{v_i \mid 1 \le i \le 2^{h-1}\}$ and $H = U \cup V$. We choose $x_i = v_i$ for $3 \le i \le 2^{h-1}$ and $V(T_2^h) \setminus H$ as remain fire sources. Thus, $b_\beta(T_2^h) \le 2^h - 1$. On the other hand, $f(V(T_2^h) \setminus U) < \beta$ for $1 \le i \le 2^h - 1$, combine Proposition 2.1, we have $2^h - 1 \le b_\beta(T_2^h)$. Therefore, $b_\beta(T_2^h) = 2^h - 1$. ☐

Corollary 3.2. *Let T_k^h be a perfect k-ary tree of height h and order n, where $h \ge 3$ and $k \ge 3$. Then*

$$b_\beta(T_k^h) = \begin{cases} h+1, & \text{If } 0 \le \beta \le \frac{1}{k+1}; \\ n - k^h - 1, & \text{If } \frac{1}{k+1} < \beta \le \frac{1}{k}; \\ n - k^h, & \text{If } \frac{1}{k} < \beta \le 1. \end{cases}$$

Proof. Let s be the root of T_k^h and the leaves of T_k^h be u_i for $1 \le i \le k^h$. Suppose the parent-leaves of T_k^h be v_i for $1 \le i \le k^{h-1}$.

For the case $0 \le \beta \le \frac{1}{k+1}$, consider T_2^h is a subtree of T_k^h. Combine Proposition 2.3 and Theorem 3.1, we directly get $b_\beta(T_k^h) \ge b_\beta(T_2^h) = h+1$. On the other hand, the fact that $b_\beta(T_k^h) \le h+1$ can be determined by let $x_1 = s$. Then,

$b_\beta\left(T_k^h\right)=h+1$.

The case for $\frac{1}{k+1}<\beta\le\frac{1}{k}$, let the children of s be s_i for $1\le i\le k$. Suppose $S=\left\{u_i \mid 1\le i\le k^h\right\}\cup\{s\}$, $Y=\left\{s_i \mid 1\le i\le k\right\}\cup\left\{v_j \mid 1\le j\le k^{h-1}\right\}$ and $X=S\cup Y$. Similarly with Theorem 3.1, we have that $b_\beta\left(T_k^h\right)\le n-k^h-1$. Further, $f\left(V\left(T_k^h\right)\setminus S\right)<\beta$, by Proposition 2.1, we directly get $n-k^h-1\le b_\beta\left(T_k^h\right)$. Then, $b_\beta\left(T_k^h\right)=n-k^h-1$.

Consider that $\frac{1}{k}<\beta\le 1$, let $U=\left\{u_i \mid 1\le i\le k^h\right\}$, $V=\left\{v_i \mid 1\le i\le k^{h-1}\right\}$ and $H=U\cup V$. Similarly, we choose $V\left(T_k^h\right)\setminus U$ be fire sources. Clearly, $\left(x_1,x_2,\cdots,x_{n-k^h}\right)$ is a β-burning sequence of T_k^h. So, $b_\beta\left(T_k^h\right)\le n-k^h$. On the other hand, since $f\left(V\left(T_k^h\right)\setminus U\right)<\beta$, combine Proposition 2.1, we have that $n-k^h\le b_\beta\left(T_k^h\right)$. Therefore, $b_\beta\left(T_k^h\right)=n-k^h$. □

Theorem 3.3. *Let* $SP(s,r)$ *be a spider graph of order* n *for* $s\ge r$. *Then*

$$b_\beta\left(SP(s,r)\right)=\begin{cases} r+1, & \text{If } 0\le\beta\le\frac{1}{2};\\ n-s, & \text{If } \frac{1}{2}<\beta\le 1.\end{cases}$$

Proof. Let leaf vertices of $SP(s,r)$ be $Y=\left\{v_i \mid 1\le i\le s\right\}$. Suppose the neighbour vertices of leaf vertices are $U=\left\{u_i \mid 1\le i\le s\right\}$ and $X=Y\cup U$.

The case for $0\le\beta\le\frac{1}{2}$, we first discuss $SP(r,r)$. The radius of $SP(r,r)$ is r. So by Proposition 2.5, we have that $b_\beta\left(SP(r,r)\right)\le r+1$. Now we show that $b_\beta\left(SP(r,r)\right)\ge r+1$. Assume $b_\beta\left(SP(r,r)\right)\le r$ and $S=\left(x_1,x_2,\cdots,x_t\right)$ is a β-burning sequence of $SP(r,r)$, where $t\le r$. No fire source in S shall be able to burn more than 1 leaf vertex in $SP(r,r)$, since $d\left(v_i,v_j\right)=2r$ for $i,j=1,2,\cdots,s$ and $i\ne j$. Thus S will be able to burn at most t leaf vertices. This implies that at least 1 leaf vertex will be left unburned, a contradiction, thus we have that $b_\beta\left(SP(r,r)\right)\ge r+1$. Therefore, $b_\beta\left(SP(r,r)\right)=r+1$. As $SP(r,r)$ is a subtree of $SP(s,r)$, by Proposition 2.4 and Proposition 2.5, we have $b_\beta\left(SP(s,r)\right)\le r+1$.

For the case $\frac{1}{2}<\beta\le 1$. Consider that $f\left(V\left(SP(s,r)\right)\setminus Y\right)<\beta$, by Proposition 2.1, we have that $n-s\le b_\beta\left(SP(s,r)\right)$. Now, we show that $b_\beta\left(SP(s,r)\right)\le n-s$. Let $x_i=u_i$ for $1\le i\le s$ and $V\left(SP(s,r)\right)\setminus X$ be remain fire sources. Obviously, $\left(x_1,x_2,\cdots,x_{n-s}\right)$ is a β-burning sequence of $SP(s,r)$ and then $b_\beta\left(SP(s,r)\right)\le n-s$. Further, we have that $b_\beta\left(SP(s,r)\right)=n-s$ for $\frac{1}{2}<\beta\le 1$.

□

Theorem 3.4. *Let* $K_n\circ K_1$ *be a corona graph of* K_n *and* K_1 *with order* $2n(n\ge 2)$. *Then*

$$b_\beta\left(K_n\circ K_1\right)=\begin{cases} 3, & \text{If } 0\le\beta\le\frac{1}{n};\\ n, & \text{If } \frac{1}{n}<\beta\le 1.\end{cases}$$

Proof. Let $V(K_n \circ K_1) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and $E(K_n \circ K_1) = \{u_i u_j \mid i, j = 1, 2, \cdots, n, i \neq j\} \cup \{v_i u_i \mid i = 1, 2, \cdots, n\}$. The case for $0 \leq \beta \leq \frac{1}{n}$, combine Proposition 2.2, we directly get $b_\beta(K_n \circ K_1) \geq 3$. Now, we show $b_\beta(K_n \circ K_1) \leq 3$. Let $x_1 = u_1$, $x_2 = v_1$ and $x_3 = v_n$. Clearly, (x_1, x_2, x_3) is a β-burning sequence of $K_n \circ K_1$ and thus $b_\beta(K_n \circ K_1) \leq 3$. Therefore, $b_\beta(K_n \circ K_1) = 3$. Consider the case $\frac{1}{n} < \beta \leq 1$, let $x_i = u_i$ for $1 \leq i \leq n$. Clearly, $(x_1, x_2, \cdots, x_n)$ is a β-burning sequence of $K_n \circ K_1$. Thus $b_\beta(K_n \circ K_1) \leq n$. Now, we show $b_\beta(K_n \circ K_1) \geq n$. Obviously, $f(u_i) < \beta$ for $1 \leq i \leq n$. By Proposition 2.1, we get $n \leq b_\beta(K_n \circ K_1)$. Then, $b_\beta(K_n \circ K_1) = n$. □

Corollary 3.5. *Let* $L_{n,1}$ *be a* $(n,1)$*-lollipop of order* $n+1$*. Then*

$$b_\beta(L_{n,1}) = \begin{cases} 2, & \text{If } 0 \leq \beta \leq \frac{1}{n-1}; \\ n, & \text{If } \frac{1}{n-1} < \beta \leq 1. \end{cases}$$

Proof. Let $V(L_{n,1}) = \{u, v_1, v_2, \cdots, v_n\}$ and $E(L_{n,1}) = \{u_i u_j \mid i, j = 1, 2, \cdots, n, i \neq j\} \cup \{v u_1\}$. The case for $0 \leq \beta \leq \frac{1}{n-1}$, combining Proposition 2.2, we get $b_\beta(L_{n,1}) = 2$. For the case $\frac{1}{n-1} < \beta \leq 1$, similar as Theorem 3.4, the details omitted here, we have $b_\beta(L_{n,1}) = n$. □

Next, we determined the IC burning number of the sunflower graph, friendship graph and Dutch windmill graph.

A sunflower Sf_n is graph with $V(Sf_n) = \{v, u_1, u_2, \cdots, u_n, w_1, w_2, \cdots, w_n\}$ and $E(Sf_n) = \{vu_i \mid i = 1, 2, \cdots, n\} \cup \{u_i w_i \mid i = 1, 2, \cdots, n\} \cup \{w_i u_{i+1} \mid i = 1, 2, \cdots, n\} \cup \{u_i u_{i+1} \mid i = 1, 2, \cdots, n\}$, where edge $u_n u_{n+1}$ is $u_n u_1$ and $w_n u_{n+1}$ is $w_n u_1$, see Figure 2(a).

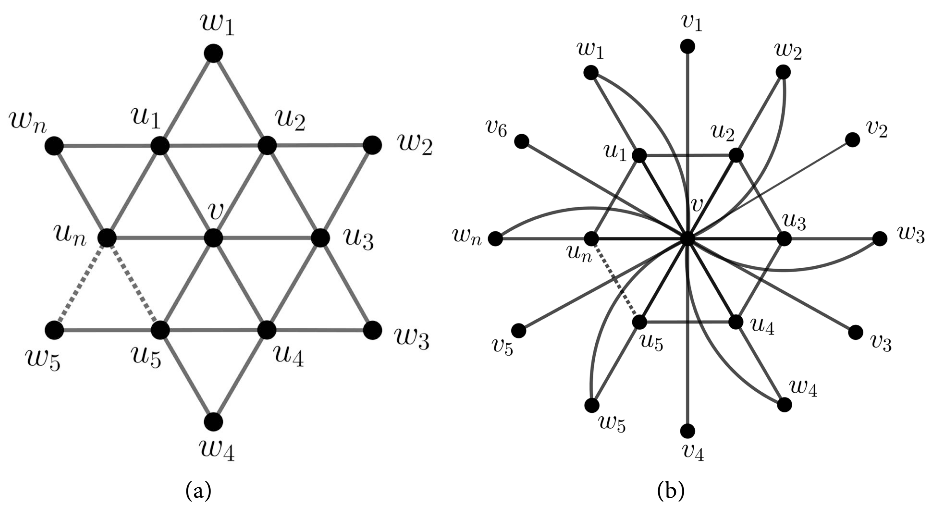

Figure 2. Sf_n with order $2n+1$ and Sf_n^I with order $3n+1$.

An I type sunflower Sf_n^I is a graph with $V\left(Sf_n^I\right)=\{v,u_1,\cdots,u_n,w_1,\cdots,w_n\}$ and $$E\left(Sf_n^I\right)=\{vu_i \mid i=1,2,\cdots,n\}\cup\{vw_i \mid i=1,2,\cdots,n\}\cup\{vv_i \mid i=1,2,\cdots,n\}\cup\{u_iw_i \mid i=1,2,\cdots,n\}\cup\{u_iu_{i+1} \mid i=1,2,\cdots,n\},$$ where edge u_nu_{n+1} is u_nu_1, see **Figure 2(b)**.

A friendship graph F_n obtained $V\left(F_n\right)=\{v,v_1,v_2,\cdots,v_n,u_1,u_2,\cdots,u_n\}$ and $E\left(F_n\right)=\{vv_i \mid i=1,2,\cdots,n\}\cup\{vu_i \mid i=1,2,\cdots,n\}\cup\{u_iv_i \mid i=1,2,\cdots,n\}$, see **Figure 3(a)**.

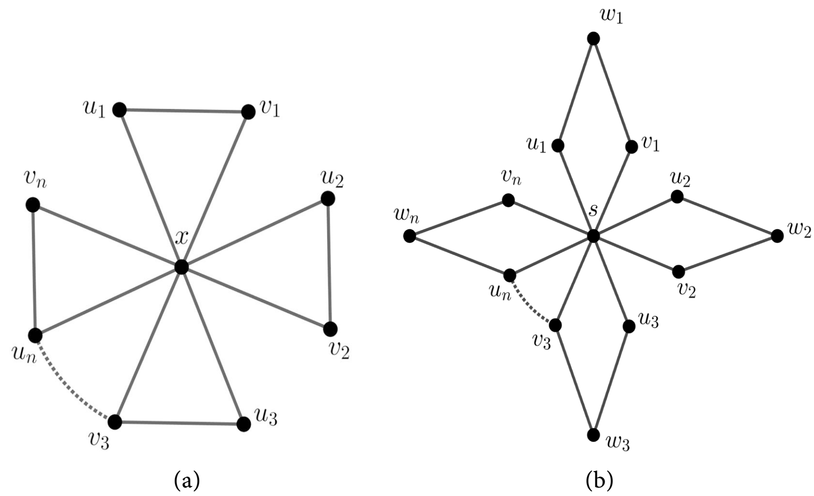

Figure 3. F_n with order $2n+1$ and D_4^n with order $3n+1$.

A Dutch windmill graph D_4^n satisfied $V\left(D_4^n\right)=\{s\}\cup\{v_i,u_i,w_i \mid 1\le i\le n\}$ and $$E\left(D_4^n\right)=\{su_i \mid i=1,2,\cdots,n\}\cup\{sv_i \mid i=1,2,\cdots,n\}\cup\{w_iu_i \mid i=1,2,\cdots,n\}\cup\{w_iv_i \mid i=1,2,\cdots,n\},$$ see **Figure 3(b)**.

Lemma 3.6. *For a connected graph G of order n and $\delta(G)$ is the minimum degree of G. Then, $b_\beta(G)=n$ if and only if $\beta>\frac{1}{\delta(G)}$.*

Proof. If $\beta>\frac{1}{\delta(G)}$, then $f\left(V(G)\right)<\beta$. By Proposition 2.1, we have that $n\le b_\beta(G)\le n$. Therefore, $b_\beta(G)=n$.

If $b_\beta(G)=n$, suppose $\beta\le\frac{1}{\delta(G)}$. Consider the minmum degree of G is $\delta(G)$ and $\beta\le\frac{1}{\delta(G)}$, then any vertex u of G receives influence from a neighbour is $f(u)=\frac{1}{d(u)}\ge\beta$, thus we can selecte $n-1$ fire source to burn the whole graph G, a contradiction, thus $\beta>\frac{1}{\delta(G)}$. □

Theorem 3.7. *Let Sf_n be a sunflower graph of order $2n+1$, where $n\ge5$. Then*

$$b_\beta(Sf_n)=\begin{cases}3, & \text{If } 0\le\beta\le\frac{1}{5};\\ n+1, & \text{If } \frac{1}{5}<\beta\le\frac{1}{2};\\ 2n+1, & \text{If } \frac{1}{2}<\beta\le 1.\end{cases}$$

Proof. Let $V(Sf_n)=\{v,u_1,u_2,\cdots,u_n,w_1,w_2,\cdots,w_n\}$ (see **Figure 2(a)**). The case for $0\le\beta\le\frac{1}{5}$, by Proposition 2.2, we directly get $b_\beta(Sf_n)\ge 3$. On the other hand, let $x_1=v$, $x_2=u_1$ and $x_3=w_1$. Clearly, (x_1,x_2,x_3) is a β-burning sequence of Sf_n and thus $b_\beta(Sf_n)\le 3$. Then, $b_\beta(Sf_n)=3$. Consider the case $\frac{1}{5}<\beta\le\frac{1}{2}$, let $X=\{u_i\,|\,1\le i\le n\}\cup\{v\}$. Consider $f(X)<\beta$, by Proposition 2.1, we have that $b_\beta(Sf_n)\ge n+1$. Now we show that $b_\beta(Sf_n)\le n+1$, let $x_{n+1}=v$ and $x_i=u_i$ for $1\le i\le n$. Clearly, $(x_1,x_2,\cdots,x_{n+1})$ is a β-burning sequence of Sf_n and thus $b_\beta(Sf_n)\le n+1$. Thus, $b_\beta(Sf_n)=n+1$. For the case $\frac{1}{2}<\beta\le 1$, by Lemma 3.6, $\delta(G)=2$ and $\beta>\frac{1}{\delta(G)}$, thus we directly get $b_\beta(Sf_n)=2n+1$. □

Theorem 3.8. *Let* Sf_n^I *be an* I *type sunflower graph of order* $3n+1$. *Then*

$$b_\beta(Sf_n)=\begin{cases}2, & \text{If } 0\le\beta\le\frac{1}{4};\\ n+1, & \text{If } \frac{1}{4}<\beta\le\frac{1}{2};\\ 3n+1, & \text{If } \frac{1}{2}<\beta\le 1.\end{cases}$$

Proof. Let $V(Sf_n^I)=\{v,u_1,u_2,\cdots,u_n,w_1,w_2,\cdots,w_n\}$. The case for $0\le\beta\le\frac{1}{4}$, consider $d(v)=3n$, by Proposition 2.2, we directly get $b_\beta(Sf_n^I)=2$. The case for $\frac{1}{4}<\beta\le\frac{1}{2}$, let $X=\{u_i\,|\,1\le i\le n\}\cup\{v\}$. Consider $f(X)<\beta$, by Proposition 2.1, we have that $b_\beta(Sf_n^I)\ge n+1$. Now we show that $b_\beta(Sf_n)\le n+1$. let $x_{n+1}=v$ and $x_i=u_i$ for $1\le i\le n$. Clearly, $(x_1,x_2,\cdots,x_{n+1})$ is a β-burning sequence of Sf_n^I and thus $b_\beta(Sf_n)\le n+1$. Thus, $b_\beta(Sf_n^I)=n+1$. For the case $\frac{1}{2}<\beta\le 1$, similar as Theorem 3.7. We have $b_\beta(Sf_n^I)=n+1$. □

Theorem 3.9. *Let* F_n *be a friendship graph of order* $2n+1$. *Then*

$$b_\beta(F_n)=\begin{cases}2, & \text{If } 0\le\beta\le\frac{1}{2};\\ 2n+1, & \text{If } \frac{1}{2}<\beta\le 1.\end{cases}$$

Proof. Firstly, this case $0\le\beta\le\frac{1}{2}$. Consider $d(v)=2n$ and all $f(V(F_n))\ge\beta$, by Proposition 2.2, we have that $b_\beta(F_n)=2$. Now, consider $\frac{1}{2}<\beta\le 1$. Clearly, if $\frac{1}{2}<\beta\le 1$, we know that $f(V(F_n))<\beta$. Combine with Lemma 3.6, we directly get $b_\beta(F_n)=2n+1$. □

Theorem 3.10. *Let* D_4^n *be a Dutch windmill graph of order* $3n+1$*. Then*

$$b_\beta(D_4^n)=\begin{cases}\begin{cases}2, & \text{If } n=1;\\ 3, & \text{If } n\ge 2.\end{cases} & \text{If } 0<\beta\le\frac{1}{2};\\ 3n+1, & \text{If } \frac{1}{2}<\beta\le 1.\end{cases}$$

Proof. For the case $0\le\beta\le\frac{1}{2}$. If $n=1$, by Theorem 3.8, we have that $b_\beta(D_4^n)=2$. If $n\ge 2$, let $x_1=v$, $x_2=u_1$ and $x_3=w_1$. Clearly, (x_1,x_2,x_3) is a burning sequence of D_4^n and thus $b_\beta(D_4^n)\le 3$. Consider $b_\beta(D_4^n)\ge 3$. When $0<\beta\le\frac{1}{2}$, all $f(V(F_n))\ge\beta$. By Proposition 2.2, we gain $b_\beta(D_4^n)\ge 3$ and thus $b_\beta(D_4^n)=3$. Now, consider the case $\frac{1}{2}<\beta\le 1$. Clearly, if $\frac{1}{2}<\beta\le 1$, we know that $f(V(D_4^n))<\beta$. by Lemma 3.6, we have $b_\beta(D_4^n)=3n+1$. □

Theorem 3.11. *For a graph* G*, let* H *is a spanning subgraph of* G *and* $\forall v\in H$, $d_H(v)=d_G(v)$. *Then* $b_\beta(G)\le b_\beta(H)$.

Proof. If $\beta=0$, then it turns a traditional burning problem, combining Proposition 2.6, we have $b(G)\le b(H)$. Now we consider $0<\beta\le 1$, assume $b_\beta(H)=k$, $(x_1,x_2,\cdots,x_k)$ is a β-burning sequence of H. As $f(v)=\frac{1}{d(v)}$ and $d_H(v)=d_G(v)$, therefore, for any vertex v, the influence it receives from a neighbour is $f(v)=\frac{1}{d_H(v)}=\frac{1}{d_G(v)}$, clearly, $(x_1,x_2,\cdots,x_k)$ also can be β-burning sequence of G. We get $b_\beta(G)\le k$, thus $b_\beta(G)\le b_\beta(H)$. □

Acknowledgements

This research was supported by NSFC No.12371352.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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