

Fractional Integral Operators on Generalized Campanato Spaces with Variable Exponents

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How to cite this paper: Niu, S.J. and Tao, S.P. (2025) Fractional Integral Operators on Generalized Campanato Spaces with Variable Exponents. *Applied Mathematics*, 16, 851-865.

<https://doi.org/10.4236/am.2025.1611044>

Received: October 25, 2025

Accepted: November 21, 2025

Published: November 24, 2025

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Abstract

This paper offers a novel portrayal of the generalized Campanato space with variable exponents. By introducing the position-dependent weight function $\omega(x, r)$, the generalized variable exponent Campanato space significantly increases its flexibility and power to characterize complex phenomena. The boundedness of the fractional integrals and their commutators is obtained on the generalized Campanato space with variable exponents.

Keywords

Generalized, Campanato Space, Fractional Integral, Commutator

1. Introduction

In the last several decades, research on Morrey-Campanato-type spaces and their applications has seen a remarkable increase, with each advancing the other. Campanato spaces, as a generalization of Morrey spaces that are occasionally referred to as Morrey-Campanato spaces, were introduced by S. Campanato in 1963 [1]. These spaces also showed up in the paper by G. Stampacchia [2]. They generalize the *BMO* spaces that were introduced by F. John and L. Nirenberg [3] and defined as follows.

$$\|f\|_{BMO} := \sup_{B(x,r)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x) - f_B| dx, \quad (1.1)$$

where $f_B = \frac{1}{|B|} \int_B f(y) dy$, and what follows $|B|$ is the Lebesgue measure of measurable set B in \mathbb{R}^n .

The interpolation features of Morrey-Campanato-type spaces were given in [2], [4] and [5]. In truth, they derived the result for a more comprehensive space that is now termed the Campanato spaces.

Let Ω be an open set in \mathbb{R}^n , we denote $B'(x, r) = B(x, r) \cap \Omega$, $1 \leq q < \infty$ and $\lambda \geq 0$, $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. The Morrey space $\mathcal{M}^{q, \lambda}(\Omega)$ is defined as

$$\mathcal{M}^{q, \lambda}(\Omega) = \left\{ f \in L_{\text{loc}}^q(\Omega) : \sup_{\substack{x \in \Omega \\ r > 0}} \frac{1}{r^\lambda} \int_{B'(x, r)} |f(x)|^q dx < \infty \right\}. \quad (1.2)$$

This is a Banach space with respect to the norm

$$\|f\|_{\mathcal{M}^{q, \lambda}(\Omega)} := \sup_{\substack{x \in \Omega \\ r > 0}} \left(\frac{1}{r^\lambda} \int_{B'(x, r)} |f(x)|^q dx \right)^{\frac{1}{q}}. \quad (1.3)$$

Campanato spaces are useful tools in the regularity theory of PDEs as a result of their better structures, which allow us to give an integral characterization of the spaces of Hölder continuous functions. The Campanato space $\mathcal{C}^{q, \lambda}(\Omega)$ is defined as

$$\mathcal{C}^{q, \lambda}(\Omega) = \left\{ f \in L_{\text{loc}}^q(\Omega) : \|f\|_{\mathcal{C}^{q, \lambda}(\Omega)} < \infty \right\}. \quad (1.4)$$

The Campanato semi norm is given as

$$\|f\|_{\mathcal{C}^{q, \lambda}(\Omega)} := \sup_{\substack{x \in \Omega \\ r > 0}} \left(\frac{1}{r^\lambda} \int_{B'(x, r)} |f(x) - f_{B'(x, r)}|^q dx \right)^{\frac{1}{q}}. \quad (1.5)$$

The importance of Campanato spaces stems from the fact that, as $n < \lambda < n + q$, they coincide with the spaces of Hölder continuous functions, providing an integral characterization of such functions. While in the case λ less than n they coincide with Morrey spaces. For some recent results for research on Morrey-Campanato-type space and their applications, see [6]-[12].

The main purpose of this paper is to discuss the boundedness of the fractional integrals and their commutators on the generalized Campanato space with variable exponents given in Section 3.

The paper is organized as follows. Some main definitions and some auxiliary results are provided in Section 2. In Section 3, we prove the main boundedness results and provide some corollaries.

2. Some Preliminaries

2.1. Fractional Integral Operator

The commutator generated by a linear operator T and a locally integrable function b is defined by

$$T_b(f) := bT(f) - T(bf). \quad (2.1)$$

The investigation into the operator T_b commenced with the groundbreaking studied by Coifman-Rochberg-Weiss in [13]. Two primary motivations underlie the consideration of commutators in this context. First, the boundedness of commutators provides valuable insights into the characterization of function spaces [14]-[20]. Second, commutator theory is of great importance in analyzing the reg-

ularity of solutions to second-order elliptic and parabolic partial differential equations [21]-[23]. The well-posedness of solutions to many PDEs depends on the boundedness of commutators for integral operators [24]-[26]. Fractional integrals play an important role in harmonic analysis and PDEs.

Let $0 < \alpha < n$. Then the fractional integral is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-\alpha}} dy \quad (2.2)$$

Let b be a locally integrable function. $I_{\alpha,b}^m$ is the m -th commutator generated by I_α and b , which is defined by

$$I_{\alpha,b}^m(f) = \int_{\mathbb{R}^n} (b(x) - b(y))^m \frac{f(y)}{|x-y|^{n-\alpha}} dy \quad (2.3)$$

In [27], Chanillo studied the characterization of BMO space via boundedness of $I_{\alpha,b}$. Paluszynski derived the characterization of the $Lip_\beta(\mathbb{R}^n)$ space, which involves the boundedness of $I_{\alpha,b}$ [19]. In contrast to the extensive and profound results regarding commutators with symbol functions in BMO spaces and Lipschitz spaces, research on Morrey spaces has been relatively scarce. However, recent developments have emerged, providing innovative characterizations of Morrey spaces through the analysis of the boundedness of commutators associated with Calderón-Zygmund singular integral operators. These developments utilize novel methodologies, moving away from the traditional dependence on the sharp maximal function theorem [28]. This previous breakthrough marks the initial exploration of commutator studies where the symbol function belongs to Morrey spaces.

Let f be a locally integrable function on \mathbb{R}^n . Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy. \quad (2.4)$$

given α , $0 < \alpha < n$, the fractional maximal operator M_α is defined by

$$M_\alpha f(x) := \sup_{x \in B} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| dy. \quad (2.5)$$

where the supremum is again taken over all balls B which contain x . In the limiting case $\alpha = 0$, the fractional maximal operator reduces to the Hardy-Littlewood maximal operator.

2.2. Variable Exponent Lebesgue Spaces

Let $p(\cdot)$ be a measurable function on \mathbb{R}^n taking values in $[1, \infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable on } \mathbb{R}^n : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}. \quad (2.6)$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \tag{2.7}$$

The space $L^{p(\cdot)}_{loc}(\mathbb{R}^n)$ is defined by $L^{p(\cdot)}_{loc}(\mathbb{R}^n) := \{f : f \chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n\}$, where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$, we denote $p^- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$, $p^+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 1$ and $p^+ < \infty$; $\mathcal{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 0$ and $p^+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as above for $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Let $p'(\cdot)$ be the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Basic facts concerning Lebesgue spaces with variable exponent can be found in [29] [30]

Lemma 2.1 see [31] Let f belong to the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and g belong to $L^{p'(\cdot)}(\mathbb{R}^n)$, where $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, the product function fg is integrable over \mathbb{R}^n , and the following inequality holds.

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}, \tag{2.8}$$

where r_p is defined as $r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

Theorem 2.2 see [32] Let Ω be an open set in \mathbb{R}^n , and α , $0 < \alpha < n$, Let $p(\cdot) : \Omega \rightarrow [1, \infty)$, be such that $1 < p_- \leq p_+ < n/\alpha$. Suppose further that $p(\cdot)$ satisfies

(i) For any $x, y \in \Omega$, $|x - y| < 1/2$, if

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|}, \tag{2.9}$$

(ii) For any $x, y \in \Omega$, $|y| \geq |x|$, if

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}, \tag{2.10}$$

Define $q(\cdot) : \Omega \rightarrow [1, \infty)$, $x \in \Omega$, by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n} \tag{2.11}$$

Then the fractional maximal operator M_α is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$.

Theorem 2.3 see [32] Let $\Omega, \alpha, p(\cdot)$ and $q(\cdot)$ are as in Theorem 2.2, then the fractional integral operator I_α is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$.

3. Boundedness of Fractional Integral Operators

3.1. Variable Exponent Generalized Morrey Spaces

The Morrey spaces $\mathcal{M}^{p(\cdot), \lambda(\cdot)}(\Omega)$ with variable exponents $p(\cdot)$ and $\lambda(\cdot)$

over an open set $\Omega \subset \mathbb{R}^n$ have recently been introduced almost simultaneously by different authors. Relevant works can be found in [33]-[37].

Let $\lambda(\cdot)$ be a measurable function on Ω with the values in $[0, n]$. The variable exponents Morrey space $\mathcal{M}^{p(\cdot), \lambda(\cdot)}(\Omega)$ is defined [33] as the set of measurable functions f on Ω with the finite norm

$$\|f\|_{\mathcal{M}^{p(\cdot), \lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, r > 0} r^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{B(x, r)}\|_{L^{p(\cdot)}(\Omega)}. \quad (3.1)$$

The Campanato space $\mathcal{C}^{p(\cdot), \lambda(\cdot)}(\Omega)$ of variable order, in the Euclidean case, are defined [38] via the norm

$$\|f\|_{\mathcal{C}^{p(\cdot), \lambda(\cdot)}(\Omega)} := \|f\|_{L^1(\Omega)} + \sup_{x \in \Omega, r > 0} r^{-\frac{\lambda(x)}{p(x)}} \|f - f_{B(x, r)}\|_{L^{p(\cdot)}(\Omega)}. \quad (3.2)$$

Let $p(\cdot): \mathbb{R}^n \rightarrow [1, \infty) < \infty$ and $\omega(x, \cdot): (0, \infty) \rightarrow (0, \infty)$. $\omega(x, r)$ satisfies the doubling condition

$$\omega(x, 2r) \leq C\omega(x, r), \text{ for any } x \in \Omega, r > 0,$$

where $C > 0$ is a constant independent of x and r .

Let B be the ball of centre at x and radius r . The generalized variable Morrey space $\mathcal{M}^{p(\cdot), \omega}$ was introduced in [39] as the space of functions equipped with the norm

$$\|f\|_{\mathcal{M}^{p(\cdot), \omega}} = \sup_B \frac{1}{\omega(x, r)} r^{-\frac{n}{p(x)}} \|f \chi_{B(x, r)}\|_{L^{p(\cdot)}}. \quad (3.3)$$

The generalized variable Campanato space $\mathcal{C}^{p(\cdot), \omega}$ is defined by the norm

$$\|f\|_{\mathcal{C}^{p(\cdot), \omega}} = \sup_B \frac{1}{\omega(x, r)} r^{-\frac{n}{p(x)}} \|(f - f_B) \chi_{B(x, r)}\|_{L^{p(\cdot)}}. \quad (3.4)$$

The generalized Campanato spaces with variable exponents are a key link in the “space-operator-equation” chain in functional analysis. They not only generalize the theory of classical spaces but also provide a powerful functional analysis framework for dealing with problems of non-uniform regularity, such as analysis under variable coefficients and inhomogeneous media.

In [40], a characterization of Campanato spaces was provided via the boundedness of $I_{\alpha, b}$ under the assumption that b satisfies a mean value inequality. In [41], authors respectively characterized the strong and weak type boundedness of $I_{\alpha, b}$ on the generalized Morrey spaces. Inspired by these results, in this paper, we obtain a characterization of $\mathcal{C}^{p(\cdot), \omega}$ via the boundedness of $I_{\alpha, b}$ under a suitable assumption that b satisfies the mean value inequality. A function f is said to satisfy the well-known mean value inequality if there exists a constant $C > 0$ such that for any ball B in \mathbb{R}^n ,

$$\sup_B |f(x) - f_B| \leq \frac{C}{|B|} \int_B |f(x) - f_B| dx. \quad (3.5)$$

(3.5) is also called the reverse Hölder class which contains many kinds of func-

tions. For example, if $P(x)$ is a polynomial and $\xi > 0$, then $f(x) = |P(x)|^\xi$ satisfies (3.5) [42]. Besides polynomial functions, the mean value equalities also characterize harmonic functions [43]. For more theories about the reverse Hölder classes see [44] and [45] for example.

Our main results can be stated as follows.

Theorem 3.1 Let $0 < \alpha < n$, $p(\cdot) : \Omega \rightarrow [1, \infty) < \infty$, $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$, ω is

a non-negative measurable function on $\Omega \times (0, \ell)$, $\ell = \text{diam} \Omega$, and ω satisfies the doubling condition. Then the fractional integral I_α is a bounded operator from $\mathcal{M}^{p(\cdot), \omega}$ to $\mathcal{M}^{q(\cdot), \omega}$.

Theorem 3.2 Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty) < \infty$. $\omega(x, r)$ satisfies the doubling condition, $\omega = \omega_1 \omega_2$, $0 < \alpha < n$, b satisfy (3.5). Then the following statements are equivalent.

- (a) $b \in \mathcal{C}^{p_1(\cdot), \omega_1}$;
- (b) $I_{\alpha, b}$ is a bounded operator from $\mathcal{M}^{q(\cdot), \omega_2}$ to $\mathcal{C}^{p(\cdot), \omega}$;
- (c) For any ball $B \subset \mathbb{R}^n$ and $m \in \mathbb{N}^+$, there exists a constant $C > 0$ such that

$$\|I_{\alpha, b}^m f\|_{\mathcal{C}^{p(\cdot), \omega}} \leq C \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}}$$

As mentioned in a prior study, solutions to a substantial category of elliptic second-order PDEs fulfill the mean value inequality (3.5). In this manner, our primary findings can provide characterizations for the spaces of solutions to some second-order elliptic PDEs. Let b be a solution to the following Laplace equation

$$\Delta u = 0 \tag{3.6}$$

where Δ is the Laplace operator and u is a function defined on a bounded domain $\Omega \subset \mathbb{R}^n$. As shown in a previous study, b satisfies (3.5) and is bounded [38]. Thus, the corresponding commutator is bounded from $\mathcal{M}^{q(\cdot), \omega_2}$ to $\mathcal{C}^{p(\cdot), \omega}$. Then the space of solutions to (3.8) is $\mathcal{C}^{p(\cdot), \omega}$. The regularity of solutions to some elliptic PDEs with smooth boundaries can be credited to the boundedness of corresponding commutators with smooth kernels. The question of what occurs when the boundary conditions are relaxed can be addressed by studying the boundedness of commutators with rough kernels. For this purpose, the results are extended to the case of rough kernels.

This can be justified by simply stating that the proof follows that of Theorem 3.2 with minor adjustments to the estimates to account for the kernel $\Omega(x - y)$. we can claim that Theorem 3.2 also holds true for fractional integrals with homogeneous kernels, which were defined [40] by

$$T_{\Omega, \alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|y - x|^{n - \alpha}} dy, 0 < \alpha < n. \tag{3.7}$$

where $\Omega(x) \in C^\infty(S^{n-1})$ is homogeneous of degree 0 and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{3.8}$$

Here $d\sigma$ is the normalized Lebesgue measure and $x' = x/|x|$. When $\Omega = 1$, $T_{\Omega, \alpha}$ is the same as the fractional integral I_α . If $\alpha = 0$, then $T_{\Omega, \alpha}$ becomes a Calderón-Zygmund singular integral operator in the sense of the principal value Cauchy integral.

Corollary 3.3 Let $p(\cdot), \omega, \omega_1, \omega_2, \alpha$ and b be as Theorem 3.2. Then the following conditions are equivalent.

- (1) $b \in \mathcal{C}^{p_1(\cdot), \omega_1}$;
- (2) $T_{\Omega, \alpha, b}$ is a bounded operator from $\mathcal{M}^{q(\cdot), \omega_2}$ to $\mathcal{C}^{p(\cdot), \omega}$;
- (3) For any ball $B \subset \mathbb{R}^n$ and $m \in \mathbb{N}^+$, there exists a constant $C > 0$ such that

$$\|T_{\Omega, \alpha, b}^m f\|_{\mathcal{C}^{p(\cdot), \omega}} \leq C \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}}$$

3.2. Proof of Our Main Results

This section begins with some lemmas which will be used in the proof of our results.

Lemma 3.4 Let $p(\cdot), \omega, \omega_1, \omega_2, \alpha$ and b be as Theorem 3.2, and

$$\frac{1}{s(x)} = \frac{1}{p_1(x)} + \frac{1}{q(x)}. \text{ Then}$$

$$\|(b - b_B) f \chi_B\|_{L^s(\cdot)} \leq C \omega(x, r) r^{\frac{n}{p(x)}} \|b\|_{\mathcal{C}^{p_1(\cdot), \omega_1}} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}}. \quad (3.9)$$

Proof. By applying the Hölder inequality we get

$$\begin{aligned} \|(b - b_B) f \chi_B\|_{L^s(\cdot)} &\leq C \|(b - b_B) \chi_B\|_{L^{p_1(\cdot)}} \|f \chi_B\|_{L^{q(\cdot)}} \\ &\leq C \omega_1(x, r) r^{\frac{n}{p_1(x)}} \|b\|_{\mathcal{C}^{p_1(\cdot), \omega_1}} \omega_2(x, r) r^{\frac{n}{q(x)}} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \\ &\leq C \omega(x, r) r^{\frac{n}{p(x)}} \|b\|_{\mathcal{C}^{p_1(\cdot), \omega_1}} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \end{aligned}$$

where $\frac{1}{q(x)} = \frac{1}{p_2(x)} + \frac{\alpha}{n}$.

Lemma 3.5 Let $B_* \subset B$, $b \in \mathcal{C}^{p_1(\cdot), \omega_1}$. Then the following inequality holds.

$$|b_{B_*} - b_B| \leq C \omega_1(x, r) \|b\|_{\mathcal{C}^{p_1(\cdot), \omega_1}}. \quad (3.10)$$

Proof. We suppose two cases.

Case 1. When $B_* \subset B \subseteq 2B_*$. By using Hölder inequality

$$\begin{aligned} |b_{B_*} - b_B| &\leq \frac{1}{|B_*|} \int_{B_*} |b(x) - b_B| dx + \frac{1}{|B|} \int_B |b(x) - b_B| dx \\ &\leq \frac{C}{|B|} \int_B |b(x) - b_B| dx \\ &\leq C |B|^{-\frac{1}{p_1(x)}} \|b - b_B\|_{L^{p_1(x)}} \\ &\leq C \omega_1(x, r) \|b\|_{\mathcal{C}^{p_1(\cdot), \omega_1}}. \end{aligned}$$

Case 2. When $2B_* \subset B$. Choose a sequence of nested cubes

$$B_* := B_1 \subset B_2 \subset \dots \subset B =: B_{m+1}$$

Similarly to case 1, we have

$$\begin{aligned} |b_{B_*} - b_B| &= |b_{B_1} - b_{B_2} + b_{B_2} - b_{B_3} + \dots + b_{B_m} - b_{B_{m+1}}| \\ &\leq \sum_{i=1}^{\infty} |b_{B_i} - b_{B_{i+1}}|, \\ &\leq C\omega_1(x, r) \|b\|_{C^{p_1(\cdot), 2q_1}}. \end{aligned}$$

Proof of Theorem 3.1. Let f be a function in $\mathcal{M}^{p(\cdot), \omega}(\Omega)$. For any fixed $r > 0$, denote $B(0, r)$ by B . $B' = B \cap \Omega$. For any ball B , let $2B = B(0, 2r)$. We write f as $f = f_1 + f_2$, where $f_1 = f \chi_{2B}$ and $f_2 = f - f_1$. By the Minkowski inequality, we have

$$\begin{aligned} \|I_\alpha f\|_{\mathcal{M}^{q(\cdot), \omega}} &\leq \|I_\alpha f_1\|_{\mathcal{M}^{q(\cdot), \omega}} + \|I_\alpha f_2\|_{\mathcal{M}^{q(\cdot), \omega}} : \\ &= I_1 + I_2 \end{aligned}$$

We first estimate I_1 .

Note that $f_1 \in L^{p(\cdot)}$ and I_α is bounded from $L^{p(\cdot)}$ to $L^{q(\cdot)}$, we get

$$\begin{aligned} \|I_\alpha f_1 \chi_{B'(x,r)}\|_{L^{q(\cdot)}} &\leq \|I_\alpha f_1\|_{L^{q(\cdot)}} \lesssim \|f_1\|_{L^{p(\cdot)}} \\ &= \|f \chi_{2B}\|_{L^{p(\cdot)}}. \end{aligned}$$

Thus,

$$\begin{aligned} \|I_\alpha f_1\|_{\mathcal{M}^{q(\cdot), \omega}} &= \sup_B \frac{1}{\omega(x, r)} r^{-\frac{n}{q(x)}} \|I_\alpha f_1 \chi_{B'(x,r)}\|_{L^{q(\cdot)}} \\ &\lesssim \frac{1}{\omega(x, r)} r^{-\frac{n}{q(x)}} \|f \chi_{2B}\|_{L^{p(\cdot)}} \\ &\lesssim \frac{1}{\omega(x, r)} r^{-\frac{n}{q(x)}} \omega(x, 2r) (2r)^{\frac{n}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot), \omega}} \\ &\lesssim r^\alpha \|f\|_{\mathcal{M}^{p(\cdot), \omega}}. \end{aligned}$$

Now we turn to prove I_2 . For $x \in B$ and $y \in 2^{k+1}B \setminus 2^k B$, we have $|x - y| \approx |2^k B|^{1/n}$, together with the Hölder inequality, we obtain

$$\begin{aligned} |I_\alpha f_2(x)| &= \left| I_\alpha \left(f \chi_{(2B)^c} \right)(x) \right| \\ &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f(y)|}{|y-x|^{n-\alpha}} dy \\ &\lesssim \sum_{k=1}^{\infty} |2^k B|^{-1+\frac{\alpha}{n}} \int_{2^{k+1}B} |f(y)| dy \\ &\lesssim \sum_{k=1}^{\infty} |2^k B|^{-1+\frac{\alpha}{n}} \|\chi_{2^{k+1}B}\|_{L^{p(\cdot)}} \|f \chi_{2^{k+1}B}\|_{L^{p(\cdot)}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\|I_\alpha f_2\|_{\mathcal{M}^{q(\cdot),\omega}} &= \sup_B \frac{1}{\omega(x,r)} r^{-\frac{n}{q(x)}} \left\| I_\alpha f_2 \chi_{B'(x,r)} \right\|_{L^{q(\cdot)}} \\
&\lesssim \frac{1}{\omega(x,r)} r^{-\frac{n}{q(x)}} \sum_{k=1}^{\infty} |2^k B|^{-1+\frac{\alpha}{n}} \left\| \chi_{2^{k+1}B} \right\|_{L^{p(\cdot)}} \left\| f \chi_{2^{k+1}B} \right\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{q(\cdot)}} \\
&\lesssim \frac{1}{\omega(x,r)} r^{-\frac{n}{q(x)}} \sum_{k=1}^{\infty} |2^k B|^{-1+\frac{\alpha}{n}} \left\| \chi_{2^{k+1}B} \right\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{q(\cdot)}} \omega(x, 2^{k+1}r) \\
&\quad \times (2^{k+1}r)^{\frac{n}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot),\omega}} \\
&\lesssim \frac{1}{\omega(x,r)} r^{-\frac{n}{q(x)}} \sum_{k=1}^{\infty} |2^k B|^{-1+\frac{\alpha}{n}} |2^{k+1}B|^{1-\frac{1}{p_-}} |B|^{\frac{1}{q_-}} \omega(x, 2^{k+1}r) \\
&\quad \times (2^{k+1}r)^{\frac{n}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot),\omega}} \\
&\lesssim \|f\|_{\mathcal{M}^{p(\cdot),\omega}}.
\end{aligned}$$

This completes the proof.

Proof of Theorem 3.2. Let f be a function in $\mathcal{M}^{q(\cdot),\omega_2}$. For any fixed $r > 0$, denote $B(0, r)$ by B . It suffices to show that the fact

$$\left\| (I_{\alpha,b} f - (I_{\alpha,b} f)_B) \chi_B \right\|_{L^{p(\cdot)}} \leq C \omega(x, r) r^{\frac{n}{p(x)}} \|f\|_{\mathcal{M}^{q(\cdot),\omega_2}}$$

For any ball B , let $2B = B(0, 2r)$. We write f as $f = f_1 + f_2$, where $f_1 = f \chi_{2B}$ and $f_2 = f - f_1$. By using the facts $I_{\alpha,b} f = I_{\alpha,(b-b_B)} f$, and

$$\left| I_{\alpha,(b-b_B)} f(y) - (I_{\alpha,(b-b_B)} f)_B \right| \leq 2 \left| I_{\alpha,(b-b_B)} f(y) - (I_\alpha(b-b_B) f_2)(x_B) \right| \quad ([19], \text{ p. 5},$$

[40], p. 6) imply the following.

$$\begin{aligned}
&\left\| (I_{\alpha,b} f - (I_{\alpha,b} f)_B) \chi_B \right\|_{L^{p(\cdot)}} \\
&= \left\| (I_{\alpha,(b-b_B)} f - (I_{\alpha,(b-b_B)} f)_B) \chi_B \right\|_{L^{p(\cdot)}} \\
&\leq 2 \left\| (I_{\alpha,(b-b_B)} f - (I_\alpha(b-b_B) f_2)(x_B)) \chi_B \right\|_{L^{p(\cdot)}} \\
&\leq 2 \left\| ((b-b_B) I_\alpha f) \chi_B \right\|_{L^{p(\cdot)}} + 2 \left\| (I_\alpha(b-b_B) f_1) \chi_B \right\|_{L^{p(\cdot)}} \\
&\quad + 2 \left\| I_\alpha(b-b_B) f_2 - (I_\alpha((b-b_B) f_2)(x_B)) \chi_B \right\|_{L^{p(\cdot)}} \\
&=: E_1 + E_2 + E_3
\end{aligned}$$

Now we first estimate E_1 ,

$$\begin{aligned}
\left\| ((b-b_B) I_\alpha f) \chi_B \right\|_{L^{p(\cdot)}} &\lesssim \left\| (b-b_B) \chi_B \right\|_{L^{p(\cdot)}} \left\| (I_\alpha f) \chi_B \right\|_{L^{p_2(\cdot)}} \\
&\lesssim \omega_1(x, r) r^{\frac{n}{p_1(x)}} \|b\|_{C^{p_1(\cdot),\omega_1}} \|f \chi_B\|_{L^{q(\cdot)}} \\
&\lesssim \omega_1(x, r) r^{\frac{n}{p_1(x)}} \|b\|_{C^{p_1(\cdot),\omega_1}} \omega_2(x, r) r^{\frac{n}{q(x)}} \|f\|_{\mathcal{M}^{q(\cdot),\omega_2}} \\
&\lesssim \omega(x, r) r^{\frac{n}{p(x)}} \|b\|_{C^{p_1(\cdot),\omega_1}} \|f\|_{\mathcal{M}^{q(\cdot),\omega_2}}.
\end{aligned}$$

where $\frac{1}{q(x)} = \frac{1}{p_2(x)} + \frac{\alpha}{n}$, $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$.

Next we will find the estimate for E_2

$$\begin{aligned} (I_\alpha(b-b_B)f_1)\chi_{B_{L^{p(\cdot)}}} &\lesssim \|((b-b_B)f_1)\chi_B\|_{L^{q(\cdot)}} \\ &\lesssim \|(b-b_B)\chi_B\|_{L^{p_1(\cdot)}} \|f\chi_{2B}\|_{L^{q(\cdot)}} \\ &\lesssim \omega_1(x,r)r^{\frac{n}{p_1(x)}} \|b\|_{C^{p_1(\cdot),\omega_1}} \omega_2(x,2r)(2r)^{\frac{n}{q(x)}} \|f\|_{\mathcal{M}^{q(\cdot),\omega_2}} \\ &\lesssim \omega(x,r)r^{\frac{n}{p(x)}} \|b\|_{C^{p_1(\cdot),\omega_1}} \|f\|_{\mathcal{M}^{q(\cdot),\omega_2}}. \end{aligned}$$

where $\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{\alpha}{n}$, $\frac{1}{s(x)} = \frac{1}{p_1(x)} + \frac{1}{q(x)}$.

Now we will find the estimate for the term E_3 ,

$$\begin{aligned} &|I_\alpha(b-b_B)f_2(y) - (I_\alpha((b-b_B)f_2))(x_B)| \\ &\leq \int_{(2B)^c} \frac{|y-x_B|}{|x_B-z|^{n+1-\alpha}} |b(z)-b_B| |f(z)| dz \\ &\leq \sum_{k=2}^\infty \int_{2^k B \setminus 2^{k-1} B} 2^{-k} |2^k B|^{-1+\frac{\alpha}{n}} |b(z)-b_B| |f(z)| dz \\ &\lesssim \sum_{k=2}^\infty \frac{1}{2^k |2^k B|^{1-\frac{\alpha}{n}}} \int_{2^k B} |b(z)-b_{2^k B}| |f(z)| dz \\ &\quad + \sum_{k=2}^\infty \frac{1}{2^k |2^k B|^{1-\frac{\alpha}{n}}} \int_{2^k B} |b_B - b_{2^k B}| |f(z)| dz \\ E_3 &\lesssim \left\| \sum_{k=2}^\infty \frac{1}{2^k |2^k B|^{1-\frac{\alpha}{n}}} \int_{2^k B} |b(z)-b_{2^k B}| |f(z)| dz \right\|_{L^{p(\cdot)}} \\ &\quad + \left\| \sum_{k=2}^\infty \frac{1}{2^k |2^k B|^{1-\frac{\alpha}{n}}} \int_{2^k B} |b_B - b_{2^k B}| |f(z)| dz \right\|_{L^{p(\cdot)}} \\ &=: I_1 + I_2 \\ I_1 &\lesssim \left\| \sum_{k=2}^\infty \frac{1}{2^k} \int_{2^k B} \frac{1}{|2^k B|^{1-\frac{\alpha}{n}}} |b(z)-b_{2^k B}| |f(z)| dz \right\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{k=2}^\infty \frac{1}{2^k} \|M_\alpha |b-b_{2^k B}| |f|\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{k=2}^\infty \frac{1}{2^k} \| |b-b_{2^k B}| |f|\|_{L^{s(\cdot)}} \\ &\lesssim \omega(x,r)r^{\frac{n}{p(x)}} \|b\|_{C^{p_1(\cdot),\omega_1}} \|f\|_{\mathcal{M}^{q(\cdot),\omega_2}}. \end{aligned}$$

Now we find the estimate for I_1 , by using the boundedness of M_α (Theorem

2.2) and Lemma 3.4, we have.

Now we find the estimate for I_2 . By using the boundedness of M_α (Theorem 2.2) and Lemma 3.5, we have.

$$\begin{aligned} I_2 &\lesssim \left\| \sum_{k=2}^{\infty} \frac{1}{2^k} \int_{2^k B} \frac{1}{|2^k B|^{1-\frac{\alpha}{n}}} |b_B - b_{2^k B}| |f(z)| dz \right\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{k=2}^{\infty} \frac{1}{2^k} |b_B - b_{2^k B}| \|M_\alpha f \chi_{2^k B}\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{k=2}^{\infty} \frac{1}{2^k} \omega_1(x, r) \|b\|_{C^{p_1(\cdot), \omega_1}} \|f \chi_{2^k B}\|_{L^{p(\cdot)}} \\ &\lesssim \sum_{k=2}^{\infty} \frac{1}{2^k} \omega_1(x, r) \|b\|_{C^{p_1(\cdot), \omega_1}} \|\chi_{2^k B}\|_{L^{p_1(\cdot)}} \|f \chi_{2^k B}\|_{L^{q(\cdot)}} \\ &\lesssim \omega(x, r) r^{\frac{n}{p(x)}} \|b\|_{C^{p_1(\cdot), \omega_1}} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \end{aligned}$$

which completes our desired results.

Now we will prove that (b) \Rightarrow (c). In this case, the induction can be performed on m . $m = 1$ is trivial, so it can be assumed that for any $b \in C^{p_1(\cdot), \omega_1}$,

$$\|I_{\alpha, b}^{m-1} f\|_{C^{p(\cdot), \omega}} \leq C \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}}.$$

Next, the case m can be shown. The following can be confirmed easily.

$$\begin{aligned} |I_{\alpha, b}^m f(x) \chi_B| &= \left| \int_B (b(x) - b_y)^{m-1} \frac{|f(y)|}{|y-x|^{n-\alpha}} (b(x) - b_y) dy \right| \\ &\leq \left| \int_B (b(x) - b_y)^{m-1} \frac{|f(y)|}{|y-x|^{n-\alpha}} (b(x) - b_B) dy \right| \\ &\quad + \left| \int_B (b(x) - b_y)^{m-1} \frac{|f(y)|}{|y-x|^{n-\alpha}} (b(y) - b_B) dy \right| \\ &\leq (|b - b_B| |I_{\alpha, b}^{m-1} f|)(x) + |I_{\alpha, b}^{m-1} ((b - b_B) f)(x)| \\ &=: J_1 + J_2 \end{aligned}$$

The condition (3.5) and Hölder inequality allow the user to estimate J_1 as follows.

$$\begin{aligned} \|J_1\|_{C^{p(\cdot), \omega}} &\leq \| |b - b_B| |I_{\alpha, b}^{m-1} f| \|_{C^{p(\cdot), \omega}} \\ &\lesssim \| |b - b_B| \|_{L^\infty} \| I_{\alpha, b}^{m-1} f \|_{C^{p(\cdot), \omega}} \\ &\lesssim \frac{C}{|B|} \int_B |b(x) - b_B| dx \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \\ &\lesssim \|b\|_{C^{p_1(\cdot), \omega_1}} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \end{aligned}$$

$$\begin{aligned} \|J_2\|_{C^{p(\cdot), \omega}} &= \| I_{\alpha, b}^{m-1} (b - b_B) f \|_{C^{p(\cdot), \omega}} \\ &\lesssim \| (b - b_B) f \|_{\mathcal{M}^{q(\cdot), \omega_2}} \\ &\lesssim \| |b - b_B| \|_{L^\infty} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \cdot \\ &\lesssim \|b\|_{C^{p_1(\cdot), \omega_1}} \|f\|_{\mathcal{M}^{q(\cdot), \omega_2}} \end{aligned}$$

Similarly, the following can be shown.

This produces the following result.

$$\|I_{\alpha,b}^m f\|_{C^{p(\cdot),\omega}} \leq C \|f\|_{M^{q(\cdot),\omega_2}}$$

(c) \Rightarrow (a), this proof consists of the construction of proper commutator [28]. This depends on the smoothness of the kernel function $K(x) = 1/|x|^{n-\alpha}$. Choosing $z_0 \neq 0$ and $\delta > 0$ such that $|z|^{n-\alpha}$ can be expressed in the neighborhood $|z - z_0| < \sqrt{n}\delta$ as an absolute convergent Fourier series

$$|z|^{n-\alpha} = \sum a_n e^{iv_n \cdot z},$$

where the exact form of the vectors v_n is irrelevant. Set $z_1 = \delta^{-1}z_0$. If $|z - z_1| < \sqrt{n}$,

$$|z|^{n-\alpha} = \delta^{\alpha-n} |z_1|^{n-\alpha} = \delta^{\alpha-n} \sum a_n e^{iv_n \cdot z_1}.$$

Choose now any ball $B = B(x_0, r)$. Set $y_0 = x_0 - rz_1$ and $B' = B(y_0, r)$. Thus, if $x \in B$ and $y \in B'$, then

$$\left| \frac{x-y}{r} - z_1 \right| \leq \left| \frac{x-x_0}{r} - \frac{y-y_0}{r} \right| \leq \sqrt{n}.$$

Denote $s(x) = \text{sgn}(b(x) - b_{B'})^{p_1(x)}$, and using Hölder inequality, we have.

$$\begin{aligned} \|b - b_{B'}\|_{L^{p(\cdot)}(B)} &\leq \left\| \frac{1}{|B'|} \int_{B'} |b(x) - b(y)| dy \cdot s \right\|_{L^{p(\cdot)}} \\ &\lesssim \sum_m a_m \frac{1}{|B|} \left\| \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{g_n(y)}{|x-y|^{n-\alpha}} dy h_n \right\|_{L^{p(\cdot)}} \\ &\lesssim \|I_{\alpha,b} g_n\|_{L^{p(\cdot)}} \\ &\lesssim \omega_1(x, r) r^{\frac{n}{p_1(x)}} \|I_{\alpha,b} g_n\|_{C^{p(\cdot),\omega_1}} \\ &\lesssim \omega_1(x, r) r^{\frac{n}{p_1(x)}} \|g_n\|_{M^{q(\cdot),\omega_2}} \end{aligned}$$

where $\frac{1}{q(x)} = \frac{1}{p_1(x)} + \frac{\alpha}{n}$.

Hence $b \in C^{p_1(\cdot),\omega_1}$ yields the desired results.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant No. 12361018).

Conflicts of Interest

The authors declare that they have no competing interests.

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