

A Conserved Phase-Field Model with Dirichlet Boundary Condition and Regular Potentials

Narcisse Batangouna

Institut Supérieur d'Architecture, Urbanisme, Bâtiment et Travaux Publics, Université DENIS SASSOU-N'GUESSO, Kintélé,
République du Congo
Email: banarcissess@gmail.com

How to cite this paper: Batangouna, N. (2026) A Conserved Phase-Field Model with Dirichlet Boundary Condition and Regular Potentials. *Applied Mathematics*, 17, 153-163.

<https://doi.org/10.4236/am.2026.173009>

Received: October 10, 2025

Accepted: February 28, 2026

Published: March 3, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Phase field models are a fundamental tool in describing phase transition phenomena. In this context, our study focuses on a conserved phase field model, subject to Dirichlet boundary conditions and incorporating regular potentials. The main objective of this article is to rigorously establish the existence and uniqueness of solutions for the model under consideration. The Dirichlet boundary conditions impose constraints on the behavior of the phase field at the boundary of the domain.

Keywords

A Conserved Phase-Field Model Phase-Field, Regular Potential, Dirichlet Boundary Conditions

1. Introduction

The Caginalp proposed in [1] [2] two phase-field systems, namely,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \quad (1)$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \quad (2)$$

called non conserved system, and

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \theta \quad (3)$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \quad (4)$$

called conserved system (in the sense that, when endowed with Dirichlet Boundary conditions, the spatial average of u is conserved).

In this paper, we consider the following model:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta(\varphi - \Delta\varphi), \quad (5)$$

$$\frac{\partial \varphi}{\partial t} - \Delta \frac{\partial \varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t}, \quad (6)$$

derived from the theory of heat conduction involving two temperatures (see, for example, [3]-[7] and references therein), in $\theta = \varphi - \Delta\varphi$, this highlights the complexity of thermal phenomena in non ideal systems. In these materials, thermal dissipation can be influenced by various mechanics, such as internal structure, the presence of impurities, or interactions between particles. These factors can lead to thermal behaviors that cannot be described by classical models, where a single temperature is generally sufficient [8]-[13].

Indeed the distinction between conductive temperature and thermodynamic temperature allows for a better understanding of heat dynamics in complex materials [14]-[20].

For example in systems where relaxation processes or non-linear effects are present, the two temperatures can diverge, resulting in unexpected effects in heat transfer [21]-[23].

This approach also paves the way for new research on heat management in advanced materials, particularly for applications in engineering and technology where precise control of thermal conductivity is crucial [24]-[26].

If you have any questions or would like to discuss a specific aspect of this topic further, feel free to let me know by considering the linear approximation theory of heat conduction and the total free energy

$$\psi(u, \varphi) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + F(u) + u\Delta\varphi - u\varphi - \frac{1}{2} \varphi^2 \right) dx,$$

where Ω is the domain occupied by the material and F is an anti-derivative of f . Finally, the governing equation for a variational derivative [27]-[32]. The governing equation for φ is obtained by linearization of the energy equation in the isotropic case, which yields (see [33]-[40])

$$\frac{\partial \varphi}{\partial t} - \Delta \frac{\partial \varphi}{\partial t} - \Delta\varphi = r.$$

r being the heat supplied from the external world, see [41] [42], and then, taking $r = -\frac{\partial u}{\partial t}$, we obtain equation (5) and (6). We endow this model with Dirichlet Boundary Conditions and initial conditions. Then, we are led to the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta(\varphi - \Delta\varphi), \quad \text{in } \Omega, \quad (7)$$

$$\frac{\partial \varphi}{\partial t} - \Delta \frac{\partial \varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t}, \quad \text{in } \Omega, \quad (8)$$

$$u|_{\Gamma} = \Delta u|_{\Gamma} = \varphi|_{\Gamma} = 0, \quad \text{on } \partial\Omega, \quad (9)$$

$$u|_{t=0} = u_0, \varphi|_{t=0} = \varphi_0, \tag{10}$$

As far as the nonlinear term f is concerned, we assume that

$$f \in C^1(\mathbb{R}), f(0) = 0 \tag{11}$$

$$f' \geq -c_0, c_0 \geq 0, \tag{12}$$

$$f(s)s \geq c_1 F(s) - c_2 \geq -c_3, \quad c_1 > 0, \quad c_2, c_3 \geq 0, s \in \mathbb{R}, \tag{13}$$

where $F(s) = \int_0^s f(\xi) d\xi$. In particular, the usual cubic non linear term $f(s) = s^3 - s$ satisfies these assumptions. We further assume that

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$$

Remark 1.1. We take here, for simplicity, Dirichlet Boundary Conditions. However, we can obtain the same results for Neumann Boundary Conditions, namely,

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \frac{\partial \varphi}{\partial \nu} \quad \text{on } \Gamma \tag{14}$$

where ν denotes the unit outer normal to Γ . To do so, we rewrite, owing to (7) and (8), the equations in the form

$$\frac{\partial \bar{u}}{\partial t} + \Delta^2 \bar{u} - \Delta(f(u) - \langle f(u) \rangle) = -\Delta(\bar{\varphi} - \Delta \bar{\varphi})$$

$$\frac{\partial \bar{\varphi}}{\partial t} - \Delta \frac{\partial \bar{\varphi}}{\partial t} - \Delta \bar{\varphi} = -\frac{\partial \bar{u}}{\partial t},$$

where $\bar{\nu} = \nu - \langle \nu \rangle$, $|\langle \nu_0 \rangle| \leq M_1$, $|\langle \nu_0 \rangle| \leq M_2$, for fixed positive constants M_1 and M_2 . Then, we note that

$$\nu \rightarrow \left(\left\| (-\Delta)^{-\frac{1}{2}} \nu \right\|^2 + \langle \nu \rangle^2 \right)^{\frac{1}{2}}$$

here, $-\Delta$ denotes the minus Laplace operator with Neumann boundary conditions and acting on functions with null average and where it is understood that (see [43]-[52])

$$\langle \cdot \rangle = \frac{1}{\text{vol}(\Omega)} \langle \cdot, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)}$$

Furthermore

$$\nu \mapsto \left(\|\bar{\nu}\|^2 + \langle \nu \rangle^2 \right)^{\frac{1}{2}},$$

$$\nu \mapsto \left(\|\nabla \nu\|^2 + \langle \nu \rangle^2 \right)^{\frac{1}{2}},$$

$$\nu \mapsto \left(\|\Delta \nu\|^2 + \langle \nu \rangle^2 \right)^{\frac{1}{2}},$$

$$\nu \mapsto \left(\|\nabla \Delta \nu\|^2 + \langle \nu \rangle^2 \right)^{\frac{1}{2}},$$

are norms in $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$ and $H^3(\Omega)$, respectively, which are equivalent to the usual ones.

We further assume that

$$|f(s)| \leq \varepsilon F(s) + c_\varepsilon, \quad \forall \varepsilon > 0, \quad s \in \mathbb{R}, \tag{15}$$

which allows to deal with term $\langle f(u) \rangle$.

2. Notations

We denote by $\|\cdot\|$ the usual L^2 -norm (with associated product scalar (\cdot, \cdot)) and set $\|\cdot\|_{-1} = \left\| (-\Delta)^{-\frac{1}{2}} \cdot \right\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet Boundary Conditions. More generally, $\|\cdot\|_X$ denote the norm of Banach space X .

Throughout this paper, the same letters c_1, c_2 and c_3 denote (generally positive) constants which may change from line to line, or even a same line.

3. The Priori Estimates

The estimates derived in this section are formal, but they can easily be justified within a Galerkin scheme.

We rewrite (7) in the equivalent form

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u) = \varphi - \Delta \varphi. \tag{16}$$

We multiply (16) by $\frac{\partial u}{\partial t}$ and have, integrating over Ω and by parts;

$$\frac{d}{dt} \left(\|\nabla u\|^2 + 2 \int_{\Omega} F(u) dx \right) + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 = 2 \left(\varphi - \Delta \varphi, \frac{\partial u}{\partial t} \right) \tag{17}$$

We then multiply (8) by $\varphi - \Delta \varphi$ and obtain

$$\frac{d}{dt} \left(\|\varphi\|^2 + 2 \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 \right) + 2 \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 = -2 \left(\varphi - \Delta \varphi, \frac{\partial u}{\partial t} \right). \tag{18}$$

Summing (17) and (18), we find the differential equality

$$\frac{d}{dt} E_1 + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2 \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 = 0 \tag{19}$$

where

$$E_1 = \|\nabla u\|^2 + 2 \int_{\Omega} F(u) dx + \|\varphi\|^2 + 2 \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 \tag{20}$$

satisfies

$$E_1 \geq c \left(\|u\|_{H^1}^2 + \|\varphi\|_{H^2}^2 + \int_{\Omega} F(u) dx \right) \tag{21}$$

Next, we multiply (7) by u and have, integrating over Ω

$$\frac{1}{2} \frac{d}{dt} \|u\|_{-1}^2 + \|\nabla u\|^2 + \int_{\Omega} F(u) dx = (\varphi - \Delta \varphi, u) \tag{22}$$

and by Poincare inequalities $\|u\|_{-1} \leq c \|\nabla u\|$

$$\frac{d}{dt} \|u\|_{-1}^2 + c \left(\|\nabla u\|^2 + \int_{\Omega} F(u) dx \right) \leq c' \|\nabla \varphi\|^2 + c'' \tag{23}$$

Multiplying also (8) by $\frac{\partial \varphi}{\partial t}$, we have

$$\frac{d}{dt} \|\nabla \varphi\|^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 \leq \left\| \frac{\partial u}{\partial t} \right\|^2 \tag{24}$$

Summing (19), γ_1 (23) and (24), where $\gamma_1 > 0$, we find a differential inequality of the form

$$\frac{d}{dt} E_2 + c \left(E_2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 \right) \leq c', \quad c > 0. \tag{25}$$

where

$$E_2 = E_1 + \|\nabla \varphi\|^2 + \gamma_1 \|\nabla u\|^2 \tag{26}$$

satisfies

$$E_2 \geq c \left(\|u\|_{H^1}^2 + \|\varphi\|_{H^2}^2 + \int_{\Omega} F(u) dx \right) \tag{27}$$

We multiply (7) by $-\Delta u$ and have, owing to (12), where $\|\nabla u\|_{-1} \leq c \|\nabla u\|$

$$\frac{d}{dt} \|\nabla u\|_{-1}^2 + \|\Delta u\|^2 \leq c_0 \|\nabla u\|^2 + \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 \tag{28}$$

Summing (25) and γ_2 times (28), where $\gamma_2 > 0$ is small enough, we obtain a differential inequality of the form

$$\frac{d}{dt} E_3 + c \left(E_3 + \|\Delta u\|^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 \right) \leq c'' \tag{29}$$

where

$$E_3 = E_2 + \gamma_2 \|\nabla u\|_{-1}^2$$

satisfies

$$E_3 \geq c \left(\|u\|_{H^1}^2 + \|\varphi\|_{H^2}^2 + \int_{\Omega} F(u) dx \right)$$

Multiply (16) by $-\Delta \frac{\partial u}{\partial t}$ and integrate over. We have

$$\frac{d}{dt} \|\Delta u\|^2 + 2 \left\| \frac{\partial u}{\partial t} \right\|^2 = -2 \int_{\Omega} |f'(u)| |\nabla u| \left| \nabla \frac{\partial u}{\partial t} \right| dx + 2 \left(\nabla \varphi, \nabla \frac{\partial u}{\partial t} \right) + 2 \left(\Delta \varphi, \Delta \frac{\partial u}{\partial t} \right). \tag{30}$$

We deduce the following inequality

$$\frac{d}{dt} \|\Delta u\|^2 + 2 \left\| \frac{\partial u}{\partial t} \right\|^2 \leq 2 \int_{\Omega} |f'(u)| |\nabla u| \left| \nabla \frac{\partial u}{\partial t} \right| dx + 2 \left(\nabla \varphi, \nabla \frac{\partial u}{\partial t} \right) + 2 \left(\Delta \varphi, \Delta \frac{\partial u}{\partial t} \right). \tag{31}$$

Thanks to use $f'(s)$, we find the estimate

$$\begin{aligned} 2 \int_{\Omega} |f'(u)| |\nabla u| \left| \nabla \frac{\partial u}{\partial t} \right| dx &\leq \int_{\Omega} |3u^2 - 1| |\nabla u| \left| \nabla \frac{\partial u}{\partial t} \right| dx \\ &\leq \int_{\Omega} |3u^2 + 1| |\nabla u| \left| \nabla \frac{\partial u}{\partial t} \right| dx \\ &\leq c \left(\|u\|_{L^6}^4 + 1 \right) \|\nabla u\|_{L^6}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 \end{aligned}$$

Since $u \in L^\infty(0, T, H_0^1(\Omega))$, then the estimate (31) implies

$$\frac{d}{dt} \|\Delta u\|^2 + 2 \left\| \frac{\partial u}{\partial t} \right\|^2 \leq c \|\Delta u\|^2 + \left\| \frac{\nabla \partial u}{\partial t} \right\|^2 + 2 \left(\nabla \varphi, \nabla \frac{\partial u}{\partial t} \right) + 2 \left(\Delta \varphi, \Delta \frac{\partial u}{\partial t} \right) \quad (32)$$

Multiplying (8) by $\Delta^2 \varphi$ and integrating over Ω , we get

$$\frac{d}{dt} \left(\|\Delta \varphi\|^2 + \|\nabla \Delta \varphi\|^2 \right) + 2 \|\nabla \Delta \varphi\|^2 = -2 \left(\Delta \varphi, \Delta \frac{\partial u}{\partial t} \right) \quad (33)$$

Summing (32) and (33), we obtain

$$\frac{d}{dt} E_4 + 2 \frac{\partial u}{\partial t} + 2 \|\nabla \Delta \varphi\|^2 \leq c \|\Delta u\|^2 + 2 \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \|\nabla \varphi\|^2 \quad (34)$$

where

$$E_4 = \|\Delta u\|^2 + \|\Delta \varphi\|^2 + \|\nabla \Delta \varphi\|^2$$

Summing (29) and γ_3 (34)

$$\frac{d}{dt} E_5 + c \left(E_5 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial \varphi}{\partial t} \right\| + \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 \right) \leq c \|\nabla \varphi\|^2 + c'' \quad (35)$$

where

$$E_5 = E_3 + \gamma_3 E_4$$

satisfies

$$E_5 = c \left(\|u\|_{H^2}^2 + \|\varphi\|_{H^3}^2 + \int_{\Omega} F(u) \, dx \right)$$

We multiply (8) by $-\Delta \frac{\partial \varphi}{\partial t}$ and have, integrating over Ω and by parts,

$$\frac{d}{dt} \|\Delta \varphi\|^2 + \left\| \Delta \frac{\partial \varphi}{\partial t} \right\|^2 + 2 \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 \leq \left\| \frac{\partial u}{\partial t} \right\|^2$$

hence, noting that $\left\| \frac{\partial u}{\partial t} \right\|^2 \leq c \left\| \frac{\partial u}{\partial t} \right\|_{-1} \left\| \nabla \frac{\partial u}{\partial t} \right\|$,

$$\frac{d}{dt} \|\Delta \varphi\|^2 + \left\| \Delta \frac{\partial \varphi}{\partial t} \right\|^2 + 2 \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 \leq c \left\| \frac{\partial u}{\partial t} \right\|_{-1} + \frac{1}{2} \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 \quad (36)$$

Summing (35) and γ_4 (37), we find a differential inequality of the form

$$\frac{d}{dt} E_6 + c \left(E_6 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial \varphi}{\partial t} \right\| + \left\| \nabla \frac{\partial \varphi}{\partial t} \right\|^2 + \left\| \Delta \frac{\partial \varphi}{\partial t} \right\|^2 \right) \leq c \|\nabla \varphi\|^2 + c'' \quad (37)$$

where

$$E_6 = E_5 + \gamma_4 \|\Delta \varphi\|^2$$

satisfies

$$E_6 \geq c \left(\|u\|_{H^2}^2 + \|\varphi\|_{H^3}^2 + \int_{\Omega} dx \right)$$

4. Existence and Uniqueness of Solutions

Theorem 4.1. (Existence) We assume $(u_0, \varphi_0) \in (H_0^1 \cap H^2) \times (H_0^1 \cap H^3)$, the

problem (7)-(10) possesses at least one solution (u, φ) such that

$$u \in L^\infty(R_+; H_0^1 \cap H^2) \cap L^2(0, T; H_0^1 \cap H^2); \varphi \in (R_+; H_0^1 \cap H^3)$$

$$\frac{\partial u}{\partial t} \in L^2(0, T; H_0^1), \frac{\partial \varphi}{\partial t} \in L^2(0, T; H^2 \cap H_0^1), \text{ for } T > 0.$$

Theorem 4.2. The system (7)-(10) possesses a unique solution with the above regularity.

Proof. There only remains to prove the uniqueness.

Let $(u^{(1)}, \varphi^{(1)})$ and $(u^{(2)}, \varphi^{(2)})$ be two solutions (7)-(10) with initial data $(u_0^{(1)}, \varphi_0^1)$ and $(u_0^{(2)}, \varphi_0^2)$, respectively. We set

$$(u, \varphi) = (u^{(1)}, \varphi^{(1)}) - (u^{(2)}, \varphi^{(2)})$$

and

$$(u_0, \varphi_0) = (u_0^{(1)}, \varphi_0^1) - (u_0^{(2)}, \varphi_0^2)$$

Then, (u, φ) satisfies

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u^{(1)}) - f(u^{(2)})) = -\Delta(\varphi - \Delta\varphi), \text{ in } \Omega, \tag{38}$$

$$\frac{\partial \varphi}{\partial t} - \Delta \frac{\partial \varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t}, \text{ in } \Omega, \tag{39}$$

$$u|_\Gamma = \Delta u|_\Gamma = \varphi|_\Gamma = 0, \text{ on } \partial\Omega, \tag{40}$$

$$u|_{t=0} = u_0, \varphi|_{t=0} = \varphi_0, \tag{41}$$

Multiplying (38) by $(-\Delta)^{-1} \frac{\partial u}{\partial t}$, we have

$$\frac{d}{dt} \|u\|^2 + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2 \left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) = 2 \left(\frac{\partial u}{\partial t}, \varphi - \Delta\varphi \right) \tag{42}$$

Multiplying (39) by $\varphi - \Delta\varphi$, we obtain

$$\frac{d}{dt} (\|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2) + 2\|\nabla\varphi\|^2 + 2\|\Delta\varphi\|^2 = -2 \left(\frac{\partial u}{\partial t}, \varphi - \Delta\varphi \right) \tag{43}$$

Summing (42) and (43), we find

$$\begin{aligned} & \frac{d}{dt} (\|u\|^2 + \|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2) + 2 \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + 2\|\nabla\varphi\|^2 + 2\|\Delta\varphi\|^2 \\ & = -2 \left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \end{aligned}$$

we have

$$\begin{aligned} 2 \left| \left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right| & \leq 2 \left\| (-\Delta)^{\frac{1}{2}} (f(u^{(1)}) - f(u^{(2)})) \right\| \left\| (-\Delta)^{-\frac{1}{2}} \frac{\partial u}{\partial t} \right\| \\ & \leq \left\| (-\Delta)^{\frac{1}{2}} (f(u^{(1)}) - f(u^{(2)})) \right\|^2 + \left\| (-\Delta)^{-\frac{1}{2}} \frac{\partial u}{\partial t} \right\|^2 \\ & \leq Q \|u\|^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \end{aligned}$$

where, here and below

$$Q = Q\left(\|u_0^{(1)}\|_{H^0}, \|\varphi_0^{(1)}\|_{H^2}, \|u_0^{(2)}\|_{H^0}, \|\varphi_0^{(2)}\|_{H^2}\right)$$

There

$$\frac{d}{dt}\left(\|u\|^2 + \|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2\right) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + 2\|\nabla\varphi\|^2 + 2\|\Delta\varphi\|^2 \leq Q\|u\|^2 \quad (44)$$

In particular

$$\frac{d}{dt}E_6 + c\left(E_6 + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right) \leq Q\|u\|^2 \quad (45)$$

It thus follows from (45) and Gronwall's lemma that

$$\|u\|_{H^0}^2 + \|\varphi\|_{H^2}^2 \leq ce^{Qt}\left(\|u_0\|_{H^0}^2 + \|\varphi_0\|_{H^2}^2\right), \quad t > 0$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data. \square

5. Conclusions

Our study focuses on the mathematical analysis of the phase field model proposed by Caginalp. We specifically concentrate on the questions of existence and uniqueness of solutions, considering a polynomial potential under Dirichlet boundary conditions.

In this context, it is pertinent to explore a numerical approach to complement our theoretical analysis.

Implementing numerical methods, such as finite element methods or finite difference methods, could allow us to solve the partial differential equations associated with the model. Furthermore, numerical simulation serves as a valuable tool for visualizing and studying the dynamic behaviors of the system, particularly phase transitions and equilibrium configurations.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Altundas, Y.B. and Caginalp, G. (2005) Velocity Selection in 3D Dendrites: Phase Field Computations and Microgravity Experiments. *Nonlinear Analysis: Theory, Methods & Applications*, **62**, 467-481. <https://doi.org/10.1016/j.na.2005.02.122>
- [2] Babin, A.V. and Vishik, M.I. (1992) Attractors of Evolution Equations, Vol. 25 of Studies in Mathematics and Its Applications. North-Holland Publishing Co.
- [3] Aristotelous, A.C., Karakashian, O.A. and Wise, S.M. (2015) Adaptive, Second-Order in Time, Primitive-Variable Discontinuous Galerkin Schemes for a Cahn-Hilliard Equation with a Mass Source. *IMA Journal of Numerical Analysis*, **35**, 1167-1198. <https://doi.org/10.1093/imanum/dru035>
- [4] Batangouna, N. and Pierre, M. (2018) Convergence of Exponential Attractors for a

- Time Splitting Approximation of the Caginalp Phase-Field System. *Communications on Pure and Applied Analysis*, **17**, 1-19. <https://doi.org/10.3934/cpaa.2018001>
- [5] Bai, F., Elliott, C.M., Gardiner, A., Spence, A. and Stuart, A.M. (1995) The Viscous Cahn-Hilliard Equation. I. Computations. *Nonlinearity*, **8**, 131-160. <https://doi.org/10.1088/0951-7715/8/2/002>
- [6] Bates, P.W. and Zheng, S.M. (1992) Inertial Manifolds and Inertial Sets for the Phase-Field Equations. *Journal of Dynamics and Differential Equations*, **4**, 375-398. <https://doi.org/10.1007/BF01049391>
- [7] Brezis, H. (1973) Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Publishing Co.
- [8] Brochet, D., Hilhorst, D. and Chen, X. (1993) Finite-Dimensional Exponential Attractor for the Phase Field Model. *Applicable Analysis*, **49**, 197-212. <https://doi.org/10.1080/00036819108840173>
- [9] Brokate, M. and Sprekels, J. (1996) Hysteresis and Phase Transitions, Vol. 121 of Applied Mathematical Sciences. Springer. <https://doi.org/10.1007/978-1-4612-4048-8>
- [10] Landau, L.D. and Lifshitz, E.M. (1980) Statistical Physics I. 3rd Edition, Butterworth-Heinemann.
- [11] Caginalp, G. and Socolovsky, E.A. (1989) Efficient Computation of a Sharp Interface by Spreading via Phase Field Methods. *Applied Mathematics Letters*, **2**, 117-120. [https://doi.org/10.1016/0893-9659\(89\)90002-5](https://doi.org/10.1016/0893-9659(89)90002-5)
- [12] Caginalp, G. (1986) An Analysis of a Phase Field Model of a Free Boundary. *Archive for Rational Mechanics and Analysis*, **92**, 205-245. <https://doi.org/10.1007/BF00254827>
- [13] Caginalp, G. and Chen, X. (1998) Convergence of the Phase Field Model to Its Sharp Interface Limits. *European Journal of Applied Mathematics*, **9**, 417-445.
- [14] Cavaterra, C., Rocca, E. and Wu, H. (2017) Optimal Boundary Control of a Simplified Ericksen-Leslie System for Nematic Liquid Crystal Flows in 2D. *Archive for Rational Mechanics and Analysis*, **224**, 1037-1086. <https://doi.org/10.1007/s00205-017-1095-2>
- [15] Chen, P.J., Gurtin, M.E. and Williams, W.O. (1968) A Note on Non-Simple Heat Conduction. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, **19**, 969-970. <https://doi.org/10.1007/BF01602278>
- [16] Chepyzhov, V.V. and Vishik, M.I. (2002) Attractors for Equations of Mathematical Physics, Vol. 49 of American Mathematical Society Colloquium Publications. American Mathematical Society.
- [17] Cherfils, L. and Miranville, A. (2007) Some Results on the Asymptotic Behavior of the Caginalp System with Singular Potentials. *Advances in Mathematical Sciences and Applications*, **17**, 107-129.
- [18] Cherfils, L. and Miranville, A. (2009) On the Caginalp System with Dynamic Boundary Conditions and Singular Potentials. *Applications of Mathematics*, **54**, 89-115. <https://doi.org/10.1007/s10492-009-0008-6>
- [19] Chill, R., Fařangová, E. and Prüss, J. (2006) Convergence to Steady State of Solutions of the Cahn-Hilliard and Caginalp Equations with Dynamic Boundary Conditions. *Mathematische Nachrichten*, **279**, 1448-1462. <https://doi.org/10.1002/mana.200410431>
- [20] Chorin, A.J. (1968) Numerical Solution of the Navier-Stokes Equations. *Mathematics of Computation*, **22**, 745-762. <https://doi.org/10.1090/S0025-5718-1968-0242392-2>
- [21] Dupaix, C., Hilhorst, D. and Kostin, I.N. (1999) The Viscous Cahn-Hilliard Equation as a Limit of the Phase Field Model: Lower Semicontinuity of the Attractor. *Journal*

- of Dynamics and Differential Equations*, **11**, 333-353.
- [22] Doumbe, B. (2013) Etude de modeles de champ de phase de type Caginalp. PhD Thesis, Université de Poitiers.
- [23] Eden, A., Foias, C., Nicolaenko, B. and Temam, R. (1994) Exponential Attractors for Dissipative Evolution Equations, Vol. 37 of RAM: Research in Applied Mathematics. Wiley.
- [24] Efendiev, M. and Miranville, A. (2003) The Dimension of the Global Attractor for Dissipative Reaction-Diffusion Systems. *Applied Mathematics Letters*, **16**, 351-355. [https://doi.org/10.1016/S0893-9659\(03\)80056-3](https://doi.org/10.1016/S0893-9659(03)80056-3)
- [25] Efendiev, M., Miranville, A. and Zelik, S. (2000) Exponential Attractors for a Nonlinear Reaction-Diffusion System in R^3 . *Comptes Rendus de l'Académie des Sciences Series I. Mathematics*, **330**, 713-718.
- [26] Efendiev, M., Miranville, A. and Zelik, S. (2004) Exponential Attractors for a Singularly Perturbed Cahn-Hilliard System. *Mathematische Nachrichten*, **272**, 11-31. <https://doi.org/10.1002/mana.200310186>
- [27] Elliott, C.M. and Stuart, A.M. (1993) The Global Dynamics of Discrete Semilinear Parabolic Equations. *SIAM Journal on Numerical Analysis*, **30**, 1622-1663. <https://doi.org/10.1137/0730084>
- [28] Eyre, D. (1998) An Unconditionally Stable One-Step Scheme for Gradient Systems. *IEEE Transactions on Image Processing*, Article 117273508.
- [29] Fabrie, P., Galusinski, C. and Miranville, A. (2000) Uniform Inertial Sets for Damped Wave Equations. *Discrete and Continuous Dynamical Systems*, **6**, 393-418. <https://doi.org/10.3934/dcds.2000.6.393>
- [30] Galusinski, C. (1996) Perturbations singulières de problèmes dissipatifs: Etude dynamique via l'existence et la continuité d'attracteurs exponentiels. Ph.D. Thesis, Université de Bordeaux.
- [31] Grasselli, M., Miranville, A. and Schimperna, G. (2010) The Caginalp Phase-Field System with Coupled Dynamic Boundary Conditions and Singular Potentials. *Discrete and Continuous Dynamical Systems*, **28**, 67-98. <https://doi.org/10.3934/dcds.2010.28.67>
- [32] Grasselli, M., Petzeltová, H. and Schimperna, G. (2006) Long Time Behavior of Solutions to the Caginalp System with Singular Potential. *Zeitschrift für Analysis und ihre Anwendungen*, **25**, 51-72.
- [33] Miranville, A. and Zelik, S. (2008) Attractors for Dissipative Partial Differential Equations in Bounded and Unbounded Domains. In: *Handbook of Differential Equations: Evolutionary Equations*, Elsevier, 103-200.
- [34] Miranville, A. and Quintanilla, R. (2009) Some Generalizations of the Caginalp Phase-Field System. *Applicable Analysis*, **88**, 877-894. <https://doi.org/10.1080/00036810903042182>
- [35] Miranville, A. (2014) Some Mathematical Models in Phase Transition. *Discrete and Continuous Dynamical Systems*, **7**, 271-306. <https://doi.org/10.3934/dcdss.2014.7.271>
- [36] Miranville, A. and Quintanilla, R. (2016) A Caginalp Phase-Field System Based on Type III Heat Conduction with Two Temperatures, *Quart. Applied Mathematics*, **74**, 375-398.
- [37] Penrose, O. and Fife, P.C. (1990) Thermodynamically Consistent Models of Phase-Field Type for the Kinetics of Phase Transitions. *Physics D*, **43**, 44-62. [https://doi.org/10.1016/0167-2789\(90\)90015-H](https://doi.org/10.1016/0167-2789(90)90015-H)

- [38] Quintanilla, R. (2009) A Well-Posed Problem for the Three-Dual-Phase-Lag Heat Conduction. *Journal of Thermal Stresses*, **32**, 1270-1278. <https://doi.org/10.1080/01495730903310599>
- [39] Raugel, G. (2002) Global Attractors in Partial Differential Equations. In: *Handbook of Dynamical Systems, Vol. 2*, North-Holland, 885-982. [https://doi.org/10.1016/S1874-575X\(02\)80038-8](https://doi.org/10.1016/S1874-575X(02)80038-8)
- [40] Stuart, A.M. and Humphries, A.R. (1996) *Dynamical Systems and Numerical Analysis*, Vol. 2 of Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press.
- [41] Temam, R. (1969) Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires. II. *Archive for Rational Mechanics and Analysis*, **33**, 377-385. <https://doi.org/10.1007/BF00247696>
- [42] Temam, R. (1997) *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. 2nd Edition, Springer. <https://doi.org/10.1007/978-1-4612-0645-3>
- [43] Mavoungou, U.C. (2016) Existence and Uniqueness of Solution for Caginalp Hyperbolic Phase-Field System with a Singular Potential. *Nonlinear Analysis and Differential Equations*, **4**, 151-160.
- [44] Fakh, H. (2015) A Cahn-Hilliard Equation with a Proliferation Term for Biological and Chemical Applications. *Asymptotic Analysis*, **94**, 71-104. <https://doi.org/10.3233/ASY-151306>
- [45] Xie, W. (1998) The Classical Stefan Model as the Singular Limit of Phase Field Equations. *Discrete and Continuous Dynamical Systems*, **2**, 288-302.
- [46] Yan, Y. (1992) Dimensions of Attractors for Discretizations for Navier-Stokes Equations. *Journal of Dynamics and Differential Equations*, **4**, 275-340. <https://doi.org/10.1007/BF01049389>
- [47] Ntsokongo, A.J. and Batangouna, N. (2016) Existence and Uniqueness of Solutions for a Conserved Phase-Field Type Model. *AIMS Mathematics*, **1**, 144-155. <https://doi.org/10.3934/Math.2016.2.144>
- [48] Zhang, Z. (2005) Asymptotic Behavior of Solutions to the Phase-Field Equations with Neumann Boundary Conditions. *Communications on Pure and Applied Analysis*, **4**, 683-693. <https://doi.org/10.3934/cpaa.2005.4.683>
- [49] Zhu, C. (2015) Attractor of a Semi-Discrete Benjamin-Bona-Mahony Equation on \mathbb{R}^1 . *Annales Polonici Mathematici*, **115**, 219-234. <https://doi.org/10.4064/ap115-3-2>
- [50] Thiele, U. and Knobloch, E. (2004) Thin Liquid Films on a Slightly Inclined Heated Plate. *Physics D*, **190**, 213-248. <https://doi.org/10.1016/j.physd.2003.09.048>
- [51] Tremaine, S. (2003) On the Origin of Irregular Structure in Saturns Rings. *The Astronomical Journal*, **125**, 894-901. <https://doi.org/10.1086/345963>
- [52] Villain-Guillot, S. (2010) Phases Modules et dynamique de Cahn-Hilliard. Habilitation Thesis, Université Bordeaux.