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Optimal Strategies for COVID-19 Control in a Stochastic Process

Bongor Danhree[1*], Koina Rodoumta[2]

[1]Department of Fundamental Sciences, Superior National Public Work School, Djamena, Chad
[2]Superior National Public Work School, Djamena, Chad
Email: *bongordanhrees@gmail.com, *sbongordanhree@yahoo.com

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Abstract

First, the necessary mathematical tools are recalled regarding the concepts of stochastic control processes, including stochastic optimal control and one of its fundamental principles, the minimization principle. Then, a new controlled stochastic model of COVID-19 dynamics is formulated and represented by a stochastic differential equation with a 6-dimensional random vector of state variables (susceptible, exposed, mildly symptomatic, severely symptomatic, recovered for humans and surface concentration of SARS-CoV-2 coronavirus) and a vector of external control functions. The objective is to control the evolution and diffusion of SARS-COV-2 in a stochastic process in order to determine the optimal strategies to combat the spread of COVID-19. The stochastic analysis of the model focuses on the positivity and boundedness of the solutions, as well as the global behavior of the intermediate system, under the sole condition of stability of the disease-free equilibria and the endemic equilibria of the stochastic model. The results of the analysis reveal, provided that the disease-free equilibria of the stochastic model are exponentially p-stable and globally asymptotically stable; finally, that the endemic equilibria are locally asymptotically stable. An optimal control problem is formulated with the aim of eradicating the evolution of COVID-19; using its fundamental principle, the Pontryagin minimum principle, this problem is solved numerically with the optimal strategies to be adopted among the scenarios designed to control COVID-19 in a stochastic process.

Keywords

Stochastic Process, Optimal Control, COVID-19 Model, Stochastic Differential Equation

1. Introduction

A generic analysis suggests that SARS-CoV-2 originated from a bat coronavirus that became infectious to humans after acquiring genes specific to pangolin coronaviruses. Regarding the etiology of the COVID-19 pandemic, this disease is a respiratory infection caused by the SARS-CoV-2 coronavirus, likely originating in China and transmitted by infected bats. COVID-19, whose symptoms resemble those of seasonal flu, is more severe in the elderly and those weakened by certain chronic diseases or lack of treatment. On average, it is asymptomatic in approximately 40% of infected adults. This percentage is an average; it is higher in children and lower in younger adults. The incubation period is the time elapsed between infection and the onset of symptoms (when they appear). The incubation period for COVID-19 is estimated to be 5 days on average. An infected person becomes contagious on the third day of the incubation period. It is estimated that it remains contagious until the seventh day after symptoms disappear. COVID-19 has the particularity of not always causing symptoms. In this case, infected people, unaware of their condition, can secrete the virus and infect other people for a few days. This fact explains the great difficulty in controlling the spread of the COVID-19 pandemic. However, the literature offers mathematical perspectives for the control of COVID-19. Mathematical modeling has proven to be very useful in better understanding the spread and proposing optimal strategies for controlling infectious diseases [1]. In this literature, a deterministic model of COVID-19 was proposed and studied, implementing different controls. It emerged that the best strategy to control the spread of COVID-19 is social distancing and wearing masks and disinfecting the environment.

In stable (human and viral) environments where uncertainties and variability in the dynamic process of human contamination by the virus are absent, deterministic modeling produces acceptable results. However, as soon as uncertainty and variability are present in the dynamic system, deterministic modelling can lead to erroneous results in the long term, as it cannot capture the uncertainties and variability inherent in stochastic dynamics. It is then crucial to be able to quantify, manage, and control contamination risks linked to the unpredictability and complexity of the process or phenomenon. Although deterministic models can be useful in some cases, stochastic models are often preferable for monitoring and controlling complex and uncertain systems, such as managing the risk associated with viral contamination.

In this article, we explore a new stochastic model and we apply the theory of the stochastic optimal control in a model of Coronavirus spread in a human population constituted of the susceptible individuals, the exposed, mildly and severely symptomatic persons and recovered persons. This viral contamination dynamics takes in account, in addition of deterministic evolution, the uncertain diffusion due to noises white demonstrated under form of the movement standard Brownian on a probability filtered space. The optimal strategies of the stochastic process of COVID-19 control are determined by minimizing subject to the stochastic model, the cost functional deriving from the limited development of human infection force λ_1 plus the viral spread force λ_2 because these forces depend on the state variables of exposed persons, mildly and severely symptomatic infected on the one hand and the surface viral concentration on the other hand. The existence of control functions and the optimal characterizations are established with the help of one basic principles of the stochastic control: Pontryagin's Minimum Principle (PMP) that we recall below in the mathematical tool section. This principle serve to convert the minimization problem subject to one stochastic differential equation into one simple minimization problem to be solved from its optimality conditions. Our survey is an extension into the stochastic modelling of a deterministic case in [1]. The important results of this extension are in Sections 3, 4 and 5. The continuation of the paper is structured as follows: Mathematical tools for stochastic process in Section 2.

Presentation in Section 3 of a stochastic model of COVID-19.

Analysis of the stochastic model in Section 4.

Stochastic optimal control of COVID-19 spread in Section 5.

Numerical implementation of data and stochastic model 6.

On finishes by a conclusion in final Section.

2. Mathematical Tools for Stochastic Process

In this Section, before expressing the basic principles of the controlled stochastic process, the mathematical tool recall is necessary to illuminate better and to understand the notion landed in this article.

Thus, we first define, the process and function of control, then the controlled stochastic process and the optimal control problem.

2.1. Definition of Process of Control and Function of Control

- **Process of control:** is a process that influences the behaviour of a stochastic process while modifying at each instant $t \in \mathbb{T} = \left[t_0, t_f\right] \subseteq \mathbb{R}_+$ her dynamics (her evolution and her diffusion) by the action of a function of control denoted $u = u(t) \in \mathcal{U} \subseteq \mathbb{R}^r$.
- **Function of control:** is a progressively measurable function defined on $\mathbb{T} = \left[t_0, t_f\right] \subseteq \mathbf{R}_+$ and taking values in $\mathcal{U} \subseteq \mathbb{R}^r$ designating the set of functions of control.

2.2. Definition of Controlled Stochastic Process

- **Stochastic process:** is a random variable family indexed by the set of time $\mathbb{T}$. It is denoted by $X = \left(X_t\right)_{t\in\mathbb{T}} \in \mathbb{R}^n$. It dynamic can be described by a Stochastic Differential Equation (**SDE**) of Itô given by:

$$\begin{cases} \mathrm{d}X = f(t,X)\,\mathrm{d}t + G(t,X)\,\mathrm{d}W(t), \ \forall X = (X_t)_{t\in\mathbb{T}} \in \mathbb{R}^n, \\ X_{t_0} = X_0 \in \mathbb{R}^n. \end{cases}$$

where, $f(t,X) \in \mathbb{R}^n$ is a vector function designating evolution function; $G(t,X) \in \mathbb{R}^{n\times m}$ is a matrix function designating matrix of diffusion; and $W = \left(W(t)\right)_{t\in\mathbb{T}} \in \mathbb{R}^m$ a m-dimensional Brownian motion defined on a space of probability filtered $(\Omega, \mathcal{F}, \mathcal{F}_{t\in\mathbb{T}}, P)$ and taking values in $\mathbb{R}^m$.

• **Controlled stochastic process:** Let given for all stochastic process $X = (X_t)_{t\in\mathbb{T}} \in \mathbb{R}^n$ and for all process of control $u = u(t) \in \mathcal{U} \subseteq \mathbb{R}^n$, the controlled stochastic process is a random process denoted again by $X = (X_t^u)_{t\in\mathbb{T}} = (X(t), u(t)) \in \mathbb{R}^n \times \mathcal{U}$ where it probabilistic behavior can be influenced by an external function of control $u = u(t) \in \mathcal{U} \subseteq \mathbb{R}^r$ input.

The dynamic (**SDE**) of the stochastic process $X = (X_t)_{t\in\mathbb{T}} \in \mathbb{R}^n$ influenced by the action of external function of control $u = u(t) \in \mathcal{U}$ input. This influenced dynamic can be described generally by a following Controlled Stochastic Differential Equation (**CSDE**):

$$\begin{cases} \mathrm{d}X = f(t, X, u(t))\mathrm{d}t + G(t, X_t, u)\mathrm{d}W, \ \forall X = (X_t)_{t\in\mathbb{T}} \in \mathbb{R}^n, \\ X_{t_0} = X_0 \in \mathbb{R}^n. \end{cases}$$

Remark. In particular, when the matrix of diffusion, multiplicative factor of the random noise is not concerned by actions of control $u = (u_i)_{i=1}^r$, then we have:

$$\begin{cases} \mathrm{d}X = f(t, X, u)\mathrm{d}t + G(t, X)\mathrm{d}W, \ \forall X = (X_t)_{t\in\mathbb{T}} \in \mathbb{R}^n, \\ X_{t_0} = X_0 \in \mathbb{R}^n. \end{cases}$$

So, if the dynamic of the stochastic process influenced by the action of external function of control doesn't take in account the random noise, *i.e.* that the matrix of diffusion is hopeless ($G(t, X_t, u(t)) = 0$), then this evolutionary process is deterministic and it can be described by a following deterministic Controlled Ordinary differential Equation (**CODE**):

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = f(t, X, u), \ \forall X = (X_t)_{t\in\mathbb{T}} \in \mathbb{R}^n, \\ X_{t_0} = X_0 \in \mathbb{R}^n. \end{cases}$$

When the function of control is defined on $[t_0; t_f] \times \mathbb{R}^n$ and takes values in $\mathcal{U} \subseteq \mathbb{R}^r$ by $u = u(t, X(t))$, then it is about a retroactive control or control feedback.

2.3. Basic Principle of the Controlled Stochastic Process

In the deterministic process case [2] as in the stochastic process case [3]-[5], it exists two basic principles of the optimal control: Dynamic Programming Principle (DPP) and Pontryagin's Minimum Principle (PMP). While considering the characteristic of the stochastic model that we are going to control, only the PMP is a priority in this article.

The **PMP:** is a powerful tool in optimal control strategy for a dynamic system (per e.g. CDSE or CODE) that minimizes (or maximizes) a given cost functional. It provides necessary conditions for optimality, transforming the problem into CDSE or CODE) and an optimization problem. The PMP is using here to solve the stochastic optimal control problem formulated with constraint on the state variables as in following Subsection 2.4.

2.4. General Concept of Stochastic Optimal Control Problem

The optimal control problem subject to a controlled EDS is constructed as a minimization problem of the following definite objective function $\mathcal{J}(u)$ according to the mathematical expectation *i.e.* researches a control optimal $u^* \in \mathcal{U}_{ad}$, if it exists such that:

Stochastic optimal control problem:

$$\begin{cases} \mathcal{J}(u^*) = \min_{u \in \mathcal{U}_{ad}} \mathcal{J}(u), \\ \text{Subject to}(\text{CSDE}): \mathrm{d}X(s) = f(s, X(s), u(s))\mathrm{d}s + G(s, X(s), u(s))\mathrm{d}W_s, \\ \qquad\qquad X_s = x \in \mathbb{R}^n. \end{cases}$$

where $\mathcal{J}$ is the functional cost, and for all $\forall u \in \mathcal{U}_{ad}$, we have

$$\mathcal{J}(u) = \mathbb{E}\left[\int_s^{t_f} \varphi(t, X(t), u(t))\mathrm{d}t + \psi(X(t_f)) / X_s = x\right],$$

Hypotheses: We suppose that:

1) $(\mathcal{U}, d)$ is a full metric space separable,

2) f, G, φ and ψ are the measurable functions, and if there exists a real constant $\alpha > 0$ such that for f, G, φ, ψ, $\forall t \in [0, t_f]$, $u, v \in [0;1] \subset \mathcal{U}_{ad}$ and $X, Y \in \mathbb{R}^n$ we have the following conditions to guarantee the existence of an unique solution of CSDE.

$$\|f(t, X, u) - f(Y, u)\| + \|\|G(t, X, u) - G(t, Y, u)\|\| \le \alpha \|X - Y\|$$

$$\|f(t, X, u)\|^2 + \|\|G(t, X, u)\|\|^2 \le \alpha^2 \left(1 + \|X\|^2\right),$$

$$\|\varphi(t, X, u) - \varphi(t, Y, u)\| + \|\psi(X(t_f)) - \psi(Y(t_f))\| \le \alpha \|X - Y\|$$

$$\|\varphi(t, X, u)\|^2 + \|\psi(X(t_f))\|^2 \le \alpha^2 \left(1 + \|X\|^2\right).$$

Theorem 1. Let $(\Omega, \mathcal{F}, P)$ be probability space equipped with a filtration $\{\mathcal{F}_t\}$ and a m-dimensional Brownian motion W. Let $\mathcal{U}_{ad}$ the adapted set of control u. Let $X(0) = X_0$ the random initial condition tacking values in $\mathbb{R}^n$ wich satisfies $\mathbb{E}\left[\|X_0\|^p\right] < \infty$ for some $p \ge 1$. With the guarantee of existence conditions then, there exists an unique solution X of CSDE. Furthermore, for any $t_f > 0$ there exists a constant $C_{t_f}^p > 0$ such that

$$\mathbb{E}\left[\sup_{t \in [0; t_f]} \|X(t)\|^P\right] \le C_{t_f}^p \left(1 + \mathbb{E}\left[\|X_0\|^p\right]\right)$$

and for all $s, t \in [0; t_f]$,

$$\mathbb{E}\left[\|X(t) - X(s)\|^P\right] \le C_{t_f}^p \left(1 + \mathbb{E}\left[\|X_0\|^p\right]\right) |t - s|^{\frac{p}{2}}$$

Proof. [6]

Corollary 1. Let X be a solution of CSDE and $V:[s,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^{1;2}$ such that for $G(X(s),u(s)) = G = (G_{ij})_{i,j=1}^{i=n,j=m}$.

$$\mathbb{E}\int_s^t \left|\frac{\partial V}{\partial x_i}(X(s))G_{ij}(X(s),u(s))\right|^2 < \infty.$$

Then for all $t \in [s,+\infty[$, almost surely, we have

$$\mathbb{E}\left[V(X(t)) - V(X(s))\right] = \mathbb{E}\left[\int_s^t \left(\frac{\partial V}{\partial s}(X(s)) + \mathcal{L}^u V(X(s))\right)\right]$$

where $\mathcal{L}^u(\cdot)$ is an operator defined by

$$\mathcal{L}^u(\cdot) = f(t,x,u)\nabla_x(\cdot) + \frac{1}{2}tr\left(GG^{\mathrm{T}}(t,x,u)H_{ess}(\cdot)\right).$$

Theorem 2. The cost function for using control process u starting from random initial state $X(t) = x$ at time $t \geq 0$ is

$$\mathcal{J}(u) = \mathbb{E}\left[\int_t^{t_f} \varphi(X(t),u)\mathrm{d}t + \psi(X(t_f)) / X(t) = x\right]$$

where φ and ψ the function assumed to be continuous with at most polynomial growth Then the value function of Bellman (in DPP) $V:[0,t_f]\times\mathbb{R}^n \mapsto \mathbb{R}$ is given by

$$V(t,x) = \mathcal{J}(u^*) = \min_{u\in[0;1]\subset\mathcal{U}} \mathcal{J}(u).$$

Using the heuristic arguments in theory of Dynamic Programming Principle (DPP) in [6]-[9], we arrive at the following HJB equation associated with the above optimal control problem:

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \min\limits_{u\in\mathcal{U}}\left[\mathcal{L}^u V(t,x) + \varphi(t,x,u)\right] = 0, \ t\in[0;t_f], x\in\mathbb{R}^n \\ V(t_f,x) = \psi(x). \end{cases}$$

Therefore, the optimal control u^* is expressed by

$$u^* = \arg\min\left[\frac{\partial V}{\partial t}(t,X(t)) + \mathcal{L}^u V(t,X(t)) + \varphi(t,X(t),u(t))\right]$$

Proof. Let's begin the minimal cost by intuition on $[t;t+h]$ then on $[t+h,tf]$ because the minimal cost on $[t;t_f]$ is achieved when running optimally in $[t;t+h]$ and the continue optimally in $[t+h;t_f]$ with $X(t+h)$ as initial random variable. Using this heuristic arguments in order to give a formal derivation of the HJB-equation.

We now start deriving Bellman's dynamic programming principle (or DPP) which read as follows:

$$V(t,x) = \inf_{u\in[0;1]\subset\mathcal{U}} \mathbb{E}\left[\int_t^{t+h}\varphi(X(s),u)\mathrm{d}s + V(s+h,X(s+h)) / X(t) = x\right],$$

then, for all $s\in[t;t+h]$, we have

$$V(t,x) \leq \mathbb{E}\left[\int_t^{t+h}\varphi(X(s),u)\mathrm{d}s + V(s+h,X(s+h)) / X(t) = x\right].$$

Subtracting $V(t,x)$ from both sides and diving by h gives

$$0 \le \frac{1}{h}\mathbb{E}\left[\int_t^{t+h}\varphi(X(s),u)\mathrm{d}s + V(t+h, X(t+h)) - V(t,x)/X(t)=x\right].$$

Let V_1^h and V_2^h are defined as follows

$$\begin{aligned} V_1^h &= \frac{1}{h}\mathbb{E}\left[\int_t^{t+h}\varphi(X(s),u)\mathrm{d}s / X(t)=x\right] \\ &= \frac{1}{h}\int_t^{t+h}\mathbb{E}\left[\varphi(X(s),u)/X(t)=x\right]\mathrm{d}s \end{aligned}$$

and

$$V_2^h = \frac{1}{h}\mathbb{E}\left[V(th, X(t+h)) - V(t,x)/X(t)=x\right].$$

Using equation of Corollary 1 for V_2^h yields

$$\begin{aligned} V_2^h &= \frac{1}{h}\mathbb{E}\left[\int_t^{t+h}\left(\frac{\partial V}{\partial s}(s,X(s)) + \mathcal{L}^u V(s,X(s))\right)\mathrm{d}s / X(s)=x\right] \\ &= \frac{1}{h}\left[\int_t^{t+h}\mathbb{E}\left(\frac{\partial V}{\partial s}(s,X(s)) + \mathcal{L}^u V(s,X(s))\right)/X(s)=x\right]\mathrm{d}s \end{aligned}$$

then $0 \le V_1^h + V_2^h$ implies that

$$0 \le \lim_{h\to 0}\left(V_1^h + V_2^h\right) = \varphi(t, X(t), u(t)) + \frac{\partial V}{\partial t}(t, X(t)) + \mathcal{L}^u V(t, X(t)),$$

thus

$$0 \le \frac{\partial V}{\partial t}(t,X(t)) + \mathcal{L}^u V(t,X(t)) + \varphi(t,X(t),u(t)) \tag{*}$$

If $u^* = u^*(s, X^*(s))$, then $V(t,x) = \mathcal{J}(u^*)$.

Marking similar calculation as those above, one shows that

$$\frac{\partial V}{\partial t}(t,X(t)) + \mathcal{L}^{u^*} V(t,X(t)) + \varphi(t,X(t),u^*(t)) \tag{**}$$

Combining relations ($*$) and ($**$), we prove that relation of HJB equation is satisfied and the searched for optimal control u^* is expressed therefore by:

$$u^* = \arg\min\left[\frac{\partial V}{\partial t}(t,X(t)) + \mathcal{L}^u V(t,X(t)) + \varphi(t,X(t),u(t))\right]$$

It mark the end of this proof □

The **PMP** uses the following Hamiltonian associated to CSDE defined by:

$$\mathcal{H} = \mathcal{H}(t,x,u,p,q) = f(t,x,u)\cdot p + Tr\left[G^{\mathrm{T}}(t,x,u)\cdot q\right] - \varphi(t,x,u),$$

and the ad-joint equations (of first order for control of deterministic term or second order for control of deterministic and stochastic term). They are given respectively by:

$$\begin{cases} \mathrm{d}p = -\left[\nabla_x \mathcal{H}(t,x,u,p,q)\right]\mathrm{d}t + q\mathrm{d}W_t, \\ p(t_f) = -\nabla_x \psi(X(t_f)). \end{cases}$$

$$\begin{cases} \mathrm{d}P = -\left[\nabla_x f^{\mathrm{T}} \cdot P + P \cdot \nabla_x f + \nabla_x G^{\mathrm{T}} P \cdot \nabla_x G + \nabla_x G^{\mathrm{T}} \cdot Q\right]\mathrm{d}t \\ \qquad -\left[Q \cdot \nabla_X G + \mathcal{D}_{XX}^2 \mathcal{H}(t,x,u,P,Q)\right]\mathrm{d}t + Q\mathrm{d}W_t, \\ P(t_f) = -\mathcal{D}_x^2 \psi\left(X(t_f)\right). \end{cases}$$

In the deterministic control case, Pontryagin's Minimum Principle (PMP) applied to problem of optimal control in CODE which consists to find the function of the Feedback control $u = u^*(t,x)$ that minimizes the cost as follows:

$$\begin{cases} \mathcal{J}(u^*) = \min_{u \in \mathcal{U}} \mathcal{J}(u), \\ \text{Subject to}: \dfrac{\mathrm{d}X(s)}{\mathrm{d}s} = f\left(s, X, u(s)\right), \\ \qquad X_s = x, \end{cases}$$

where

$$\mathcal{J}(u) = \int_{t_0}^{t_f} \varphi(t, X, u)\mathrm{d}t + \psi\left(X(t_f)\right)$$

We obtain

$$u^* = \arg\min_{u \in \mathcal{U}} \mathcal{H}\left(t, X, u, \frac{\partial V}{\partial x}\right)$$

such that $\mathcal{H}$ and V verify the following optimality and transversality conditions:

$$\begin{cases} \dfrac{\mathrm{d}X(t)}{\mathrm{d}t} = \dfrac{\partial \mathcal{H}}{\partial p} = f\left(t, X, u^*(t,x,p)\right), \ X(t_0) = x, \\ \dfrac{\mathrm{d}p(t)}{\mathrm{d}t} = -\dfrac{\partial \mathcal{H}}{\partial x}\left(t, X, p, u^*(t,x,p)\right), \ p(t_f) = \dfrac{\partial \psi}{\partial x}(t_f), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \varphi\left(t, x, u^*(t,x,p)\right), \ V(t_f) = \psi\left(X(t_f)\right). \end{cases}$$

This boundary value problem may diverge depending on the initial guess. It is the reason for which PDE of order 1 is not sufficient to solve the problem of optimal control in stochastic case. It is necessary to pass to PDE of order 2 gotten by the application of DPP for CSDE therefore.

The field of stochastic control focuses on finding the best control strategy to achieve the desired objective while dealing with the inherent randomness of the system as propagation of COVID-19 in the human population [1]. This involves using mathematical techniques to model the system, analyze the uncertainty, and determine the optimal control actions.

3. Presentation of a Stochastic Model of COVID-19

In this section, we present a stochastic model of COVID-19 formulated from a deterministic model given in [1] while taking in account the random noise of the propagation of COVID-19.

Stochastic Model of COVID-19 Control

A stochastic differential equation describes the random spread of coronavirus

(SARS-CoV-2 virus precisely) in the human population subdivided into susceptible individuals S, exposed individuals E, infected individuals with middle symptoms I_1, and severe symptoms I_2, and the recovered persons R. The concentration of coronavirus on surfaces is noted by V. The interaction between humans and COVID-19 spread is characterized by parameters of model represented in Figure 1 and described in Table 1.

Table 1. Description of model variable and model parameters.

Random variables	Description
$S = S(t)$	Random variable giving Susceptible individuals number
$E = E(t)$	Random variable giving Exposed individuals number
$I_1 = I_1(t)$	Random variable giving Mildly clinically number
$I_2 = I_2(t)$	Random variable giving Severely symptomatic number
$R = R(t)$	Random variable giving Recovered individuals number
$V = V(t)$	Random variable giving Concentration of virus on surfaces
Para.	**Description [1]**
Λ	Recruitment rate
β	Probability of infection per contact
c	Average contacts of infectious individual per time
θ	Rate of loss of immunity after recovery
ρ	Rate of recovered individuals from COVID-19
δ_1	Rates of progression from E to I_1 and I_2 stage
δ_2	Rates of progression from I_1 to I_2 stage
μ	Natural death rate in humans
μ_d	COVID-19-infected death rate in humans
μ_v	Death rate coronavirus on surfaces
f_1	Proportions of exposed persons
f_2	Proportions of mildly symptomatic immigrants
η_1	Coefficients of infectivity of exposed individuals
η_2	Coefficients of infectivity of severely symptomatic individuals
η_3	Coefficients of viral shedding of severely symptomatic persons
q	Efficacy of quarantine to prevent transmission
β_v	Surface to human transmission probability
ν	Rate of disinfection of the environment
ξ	Viral shedding rate of infected individuals
K	Coronavirus concentration on surfaces

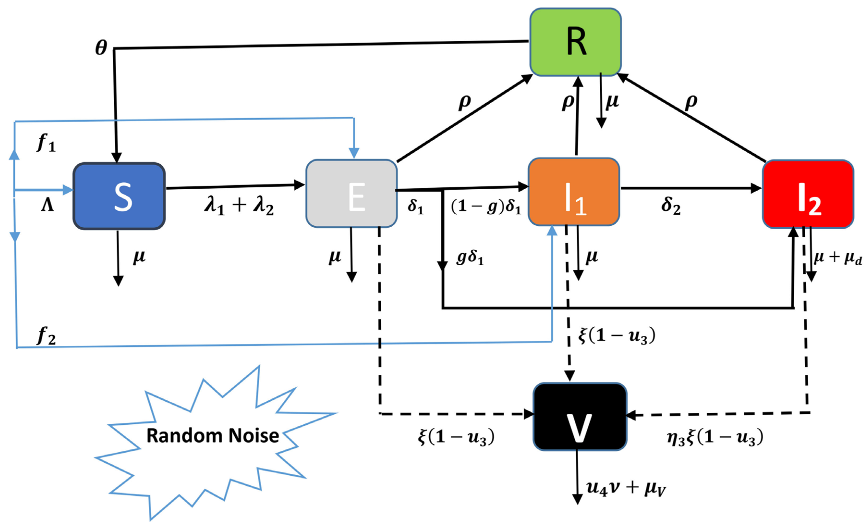


Figure 1. Diagram of stochastic model of COVID-19.

Considering that $\Delta X = (\Delta S \quad \Delta E \quad \Delta I_1 \quad \Delta I_2 \quad \Delta R \quad \Delta V)$ is a random variables vector during a time variation Δt, we have from the deterministic model, the distribution of probabilities denoted P_j of state changes ΔX^j for $j = 1, \cdots, 20$ summarized in Table 2. then from the mean and the variance of ΔX^j, we can formulate the stochastic model while using Poison's law and same formulation technique in literature [10] [11].

Table 2. Distribution of probability state change.

State change (ΔX_j)	Probability P_j	State change (ΔX_j)	Probability P_j
$(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$	$P_1 = \Lambda \Delta t$	$(0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)$	$P_{10} = g\delta_1 \Delta t$
$(-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$	$P_2 = \mu S \Delta t$	$(0 \quad 0 \quad -1 \quad 1 \quad 0 \quad 0)$	$P_{11} = \delta_2 I_1 \Delta t$
$(-1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	$P_3 = \Lambda f_1 \Delta t$	$(0 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0)$	$P_{12} = \mu I_1 \Delta t$
$(-1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	$P_4 = \lambda S \Delta t$	$(0 \quad 0 \quad 0 \quad -1 \quad 1 \quad 0)$	$P_{13} = \rho I_2 \Delta t$
$(-1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$	$P_5 = \Lambda f_2 \Delta t$	$(0 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0)$	$P_{14} = (\mu + \mu_d) I_2 \Delta t$
$(0 \quad -1 \quad 1-g \quad g \quad 0 \quad 0)$	$P_6 = \delta_1 E \Delta t$	$(0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0)$	$P_{15} = \mu R \Delta t$
$(0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0)$	$P_7 = \mu E \Delta t$	$(1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0)$	$P_{16} = \theta R \Delta t$
$(0 \quad -1 \quad 0 \quad 0 \quad 1 \quad 0)$	$P_8 = \rho E \Delta t$	$(0 \quad -1 \quad -1 \quad -1 \quad 0 \quad 3)$	$P_{17} = \xi(1-u_3) I_{EI} \Delta t$
$(0 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0)$	$P_9 = \rho I_1 \Delta t$	$(0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1)$	$P_{18} = (\nu u_4 + \mu_v) V \Delta t$
$(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$	$P_{19} = 1 - \sum_{j=1}^{18}$	Otherwise	$P_j = 0$

wherein $I_{EI} = E + I_1 + \eta_3 I_2$.

The stochastic model of CoVID-19 is represented by a following diagram (Figure 1) and by a stochastic differential Equation (1) written under the compact form.

$$\mathrm{d}X(t) = f(t, X(t), u(t))\mathrm{d}t + G(t, X(t), u(t))\mathrm{d}W(t), \tag{1}$$

where $X = (S, E, I_1, I_2, R, V)^{\mathrm{T}} \in \mathbb{R}_+^{n=6}$ is a 6-dimensional random vector of state variables, $\mathrm{d}X = (\mathrm{d}S\ \mathrm{d}E\ \mathrm{d}I_1\ \mathrm{d}I_2\ \mathrm{d}R\ \mathrm{d}V)^{\mathrm{T}}$, it differential expression, $W = (W^j)^{\mathrm{T}}$, for $j = 1,\cdots,18 = m$ is a 18-dimensional Brownian motion process and is defined on a space $(\mathbb{R}^6, \mathcal{F}, \mathcal{F}_t, P)$, $u = (u_1, u_2, u_3, u_4)$ is a vector of controls, $f(t, X, u)$ and $G(t, X, u)$ are the deterministic function and the random matrix respectively defined by:

$$f(t,X,u) = (f_i)_{i=1}^{6} = \begin{pmatrix} (1-f_1-f_2)\Lambda - (\lambda_1+\lambda_2+\mu)S + \theta R, \\ f_1\Lambda + (\lambda_1+\lambda_2)S - (\rho+\delta_1+\mu)E, \\ f_2\Lambda + \delta_1(1-g)E - (\rho+_2+\mu)I_1, \\ \delta_2 I_1 + g\delta_1 E - (\rho+\mu+\mu_d)I_2, \\ \rho(E+I_1+I_2) - (\theta+\mu)R, \\ \xi(1-u_3)(E+I_1+\eta_3 I_2) - (u_4\nu+\mu_v)V \end{pmatrix},$$

$$G(t,X,u) = \begin{pmatrix} G_{3\times 8} & Q_{3\times 10} \\ O_{3\times 8} & G_{3\times 10} \end{pmatrix} = (G_{ij})_{i,j=1}^{i=6,j=18}, \text{ wherein}$$

$$G_{3\times 8} = \begin{pmatrix} g_1 & -g_2 & -g_3 & -g_4 & -g_5 & 0 & 0 & 0 \\ 0 & 0 & g_3 & g_4 & 0 & -g_6 & -g_7 & -g_8 \\ 0 & 0 & 0 & 0 & g_5 & g_6 & 0 & 0 \end{pmatrix}$$

$$Q_{3\times 10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -g_9 & -g_{10} & -g_{11} & -g_{12} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_{3\times 8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_{3\times 10} = \begin{pmatrix} 0 & g_{10} & g_{11} & 0 & -g_{13} & -g_{14} & 0 & 0 & 0 & 0 \\ g_9 & 0 & 0 & 0 & g_{13} & 0 & -g_{15} & -g_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{17} & -g_{18} \end{pmatrix}$$

with $g_1 = \sqrt{\Lambda}$, $g_2 = \sqrt{\mu S}$, $g_3 = \sqrt{f_1\Lambda}$, $g_4 = \sqrt{(\lambda_1+\lambda_2)S}$, $g_5 = \sqrt{f_2\Lambda}$, $g_6 = \sqrt{\delta_1 E}$, $g_7 = \sqrt{\mu E}$, $g_8 = \sqrt{\rho E}$, $g_9 = \sqrt{\rho I_1}$, $g_{10} = \sqrt{g\delta_1 E}$, $g_{11} = \sqrt{\delta_2 I_1}$, $g_{12} = \sqrt{\mu I_1}$, $g_{13} = \sqrt{\rho I_2}$, $g_{14} = \sqrt{(\mu+\mu_d)I_2}$, $g_{15} = \sqrt{\mu R}$, $g_{16} = \sqrt{\theta R}$, $g_{17} = \sqrt{\xi(1-u_3)(E+I_1+\eta_3 I_2)}$, $g_{18} = \sqrt{(u_4\nu+\mu_v)V}$.

Human infection force λ_1 and viral spread force λ_2 expressed by:

$$\lambda_1 = \frac{\beta c(1-u_1)(\eta_1 E + I_1 + \eta_2 I_2)}{S + (1-q)(E+I_1+I_2) + R + V} \text{ and } \lambda_2 = \frac{\beta_v(1-u_2)V}{K+V}$$

where $\eta_1 > 1$ $\eta_2 \ll 1$, and are modification factors accounting for reduced infection of exposed persons and those with severe symptoms; K is the half-saturation constant of the corona-virus from corona-virus-infected surfaces. Parameter $q \ll 1$ is efficacy of quarantine to prevent transmission.

The parameters β and c are, respectively, the transmission probability and the average number of contacts of the infected per day. The parameter $u_1 \in [0;1] = \mathcal{U}_{ad}$ is taken to be physical distancing control, such that $u_1 = 1$ and $u_1 = 1$ are respectively the perfect observation of preventive protocols and the no-compliance cols. Finally, the parameter $u_2 \in [0;1] = \mathcal{U}_{ad}$ is a control measure accounting for avoidance of touching of infected surfaces and / or washing of hands.

The random variables and the parameters of model are further summarized in Table 1.

Remark. When the random noise is not taken in account in the propagation of COVID-19, *i.e.* that $G(t, X, u) = O_{6\times 18}$ with $g_i = 0, \forall i = 1, \cdots, 18$, then stochastic model (1) formulated amounts to the deterministic model constructs in [1]. This deterministic model is presented briefly by Equation (2) in the following subsection.

4. Analysis of the Stochastic Model of COVID-19

Herein, we present firstly some important and exploratory results about deterministic ordinary differential model (2) in [1].

4.1. Analysis of Deterministic Model

4.1.1. Positivity and Boundedness of Deterministic Model

This deterministic model of CoVID-19 is represented by a mathematical model under the compact form:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = f(t, X, u). \tag{2}$$

where $f(t, X, u)$ is a deterministic function defined previously in (1) with $X = (S, E, I_1, I_2, R, V)^{\mathrm{T}} \in \mathbf{R}_+^6$.

As given in [1], a population of the time-dependant size of $N(t)$ that is subdided into susceptibles, $S(t)$, asymptomatically infected, $E(t)$, clinically infected (those with mild symptoms $I_1(t)$ and those with severe symptoms $I_2(t)$), and recovered, $R(t)$, so that $N(t) = S(t) + E(t) + I_1(t) + I_2(t) + R(t)$, and the concentration of coronavirus on surfaces noted by $V(t)$, a deterministic model to describe the spread of coronavirus in such a population is constructed and analysed mathematically; also the main results has been gotten. The population is increased due to immigrants at rate Λ with proportions f_1, f_2 being asymptomatic and infected with mild symptoms, respectively, and the remainder being susceptible. Susceptible individuals get infected due to effective contact with infectious E, I_1 and I_2 at rate λ_1. The infected persons may deposit the virus on surfaces which can stay for up to 72 hours [12] and may be picked up by susceptibles at rate λ_2.

Lemma 3. Given the set

$$\Omega = \left\{ X = (S, E, I_1, I_2, R, V)^{\mathrm{T}} \in \mathbf{R}_+^6 \,/\, N \le \frac{\Lambda}{\mu} \text{ and } V \le \frac{\xi(1-u_3)\Lambda}{\mu(u_4\nu + \mu_V)} \right\},$$

all solutions of (2) starting in Ω remain in Ω for all $t \ge 0$. Also, the region is a positively invariant set for the deterministic model (2).

Proof. (See [1])

The author shows of it clearly that for

$$t_1 = \{t > 0 / S \ge 0, E \ge 0, I_1 \ge 0, I_2 \ge 0, R \ge 0, V \ge 0\}$$

$S(t_1) \ge 0$, $E(t_1) \ge 0$, $I_1(t_1) \ge 0$, $I_2(t_1) \ge 0$, $R(t_1) \ge 0$ and $V(t_1) \ge 0$, and thus, all solutions with nonnegative initial conditions are nonnegative.

Further, adding the first five subequations of (2)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \Lambda - \mu N - \mu_d I_2 \le \Lambda - \mu N.$$

Thus, $N(t) \le N(0)\mathrm{e}^{\mu t} + \frac{\Lambda}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)$.

Therefore, if $0 \le N(0) \le \frac{\Lambda}{\mu}$, then $0 \le \limsup_{t \to +\infty} N(t) \le \frac{\Lambda}{\mu}$. The last Equation of (2) implies that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \xi(1-u_3)(E + I_1 + \eta_3 I_2) - (\nu u_4 + \mu_V)V \le \frac{\xi(1-u_4)\Lambda}{\mu} - (\nu u_4 + \mu_V)V.$$

So, if $0 \le V(0) \le \frac{\xi(1-u_3)\Lambda}{\mu}$, then $0 \le V(t) \le \frac{\xi(1-u_3)\Lambda}{(\nu u_4 + \mu_V)\mu}$.

Thus, all solutions starting within Ω remain inside Ω. This marks the end of the proof of the Lemma. ☐

4.1.2. Equilibra and Basic Reproduction Number

Lemma 4. (See in [1])

In the absence of immigration of infective individuals, the deterministic COVID-19 model does not have a disease free equilibrium. However, where there are no infective immigrants, the disease free equilibrium is given by $\varepsilon_0 = \left\{\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0\right\}$. Using the Next Generation Method [13], in the absence of infective immigrants, the Basic Reproduction Number is defined by:

$$\mathcal{R}_0 = \frac{\beta c(1-u_1)}{k_1 k_2 k_3} r_h + \frac{\beta_v (1-u_2)(1-u_3)\xi\Lambda}{\mu K k_1 k_2 k_3 k_5} r_v \tag{3}$$

where

$$\begin{aligned}
ck_1 &= \rho + \delta_1 + \mu, \\
k_2 &= \rho + \delta_2 + \mu, \\
k_3 &= \rho + \mu + \mu_d, \\
k_4 &= \theta + \mu, \\
k_5 &= \nu u_4 + \mu_v, \\
k_6 &= g(\rho + \mu) + \delta_2, \\
r_h &= \eta_1 k_2 k_3 + \eta_2 \delta_1 k_6 + (1-g)\delta_1 k_3, \\
r_v &= \delta_1 k_6 (1 + \eta_3) + k_2 k_3.
\end{aligned} \tag{4}$$

Theorem 5. ([1] [13])

In the absence of immigration of infective individuals, the deterministic COVID-19 model has a disease free equilibrium ε_0 which is locally asymptotically stable whenever $\mathcal{R}_0 < 1$ and unstable whenever $\mathcal{R}_0 > 1$.

In the presence of infective individuals, the deterministic model (2) possesses an endemic equilibrium $\varepsilon^* = \left(S^*, E^*, I_1^*, I_2^*, R^*, V^*\right)$, such that:

$$\begin{cases} S^* = \dfrac{\Lambda}{\mu}\left(1 - f_1 - f_2\right) + \dfrac{\theta\left[\pi_0 + \pi_1 + \left(\pi_2 + \pi_3 - k_4 k_3 k_2 k_1\right)\left(k_1 E^* - \Lambda f_1\right)\right]}{\mu k_4 k_3 k_2 k_1}, \\ E^* = \dfrac{1}{k_1}\left(\lambda^* S^* + \Lambda f_1\right), \\ I_1^* = \dfrac{1}{k_2 k_1}\left[\pi_4 + \pi_5\left(k1E^* - f_1\Lambda\right)\right], \\ I_2^* = \dfrac{1}{k_3 k_2 k_1}\left[\pi_0 + \delta_2 \pi_4 + \pi_6\left(k1E^* - f_1\Lambda\right)\right], \\ R^* = \dfrac{\rho}{k_4 k_3 k_2 k_1}\left[\pi_1 + \pi_7 + \left(\pi_2 + \pi_3\right)\left(k1E^* - f_1\Lambda\right)\right], \\ V^* = \dfrac{\xi\left(1 - u_3\right)}{k_5 k_3 k_2 k_1}\left[\pi_1 + \eta_3 \pi_7 + \left(\pi_2 + \eta_3 \pi_3\right)\left(k1E^* - f_1\Lambda\right)\right], \end{cases} \tag{5}$$

where $\pi_i, i = 0, \cdots, 7$ are the auxiliary parameters defined by:

$$\pi_0 = \Lambda f_1\left(1 - g\right)\delta_1 k_2,\ \pi_1 = \Lambda k_3\left(f_1 k_2 + f_2 k_1\right),$$

$$\pi_2 = k_3\left(k_2 + \left(1 - g\right)\delta_1\right),\ \pi_3 = \delta_2 k_1 + g\delta_1 k_2,$$

$$\pi_4 = \Lambda\left(f_1\left(1 - g\right)\delta_1 + f_2 k_1\right),\ \pi_5 = \left(1 - g\right)\delta_1,$$

$$\pi_6 = \pi_5 \delta_2 + g_1 \delta_1 k_2,\ \pi_7 = \pi_0 + \Lambda f_1 \delta_2 k_2;$$

and

$$\lambda^* = \lambda_1^* + \lambda_2^* = \frac{\beta c\left(1 - u_1\right)\left(\eta_1 E^* + I_1^* + \eta_2 I_2^*\right)}{S^* + \left(1 - q\right)\left(E^* + I_1^* + I_2^*\right) + R^* + V^*} + \frac{\beta_v\left(1 - u_2\right)V^*}{K + V^*}$$

is the human infection force plus Viral spread force in equilibria, characteristic parameter of the endemic equilibrium satisfying a polynomial equation of third degree in [1].

4.1.3. Sensitivity Analysis of Deterministic Model

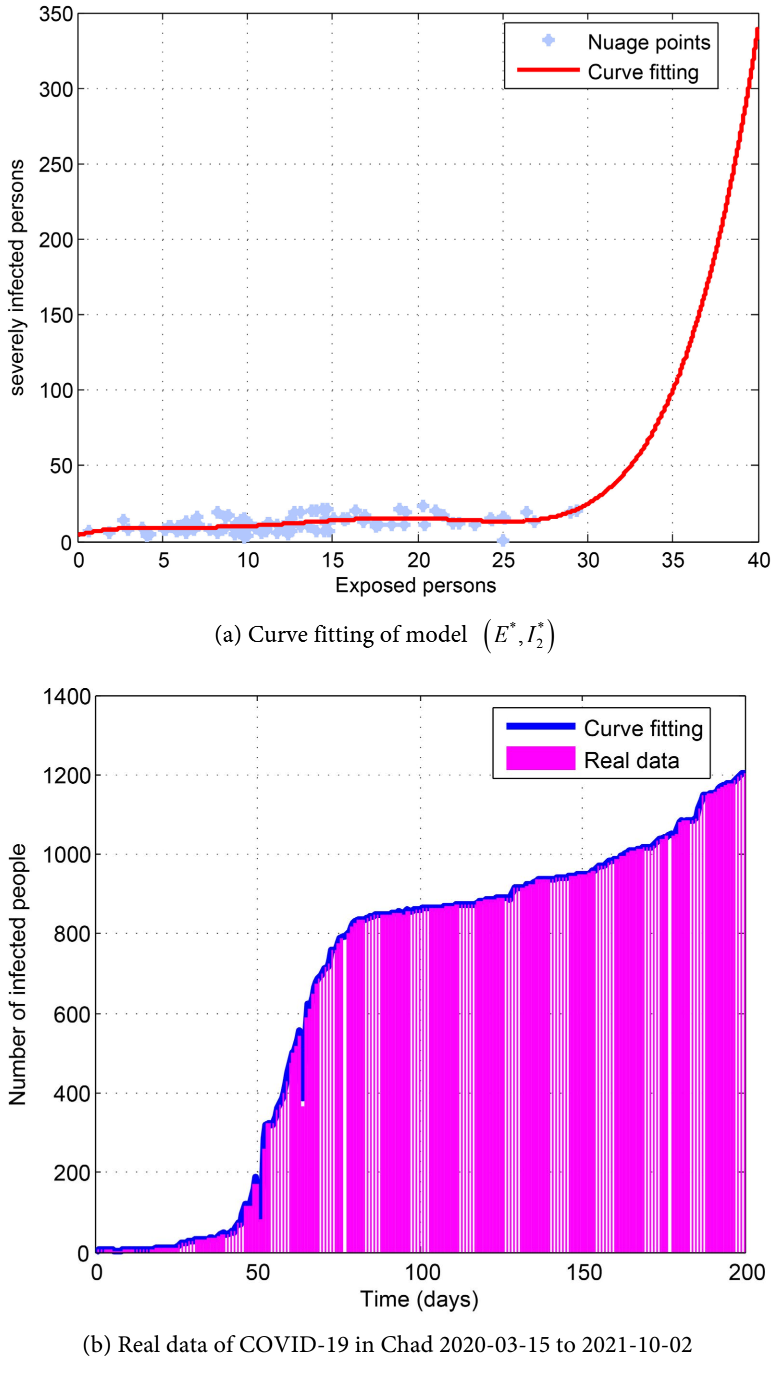
To estimate parameters of Incidence Force $\lambda^* = \lambda_1^* + \lambda_2^*$, Basic Reproduction Number $\mathcal{R}_0$ and endemic equilibrium ε^* as example, the survey of sensitivity of these parameters is very important to determine the conditions necessary of parameter evaluation. The evaluation separated of parameters is possible when the sensitivity of λ^*, $\mathcal{R}_0$ and ε^* don't have a strong interrelationship with respect to each their parameters. e.g. for $u_1 = u_2 = 0$, we have

$$\lambda^* = \lambda_0^* = \frac{\beta c\left(\eta_1 E^* + I_1^* + \eta_2 I_2^*\right)}{S^* + \left(1 - q\right)\left(E^* + I_1^* + I_2^*\right) + R^* + V^*} + \frac{\beta_v V^*}{K + V^*}$$

then, the report of sensitivities of λ_0^* reduced to parameters β, c, η_1, η_2, β_v and K gives

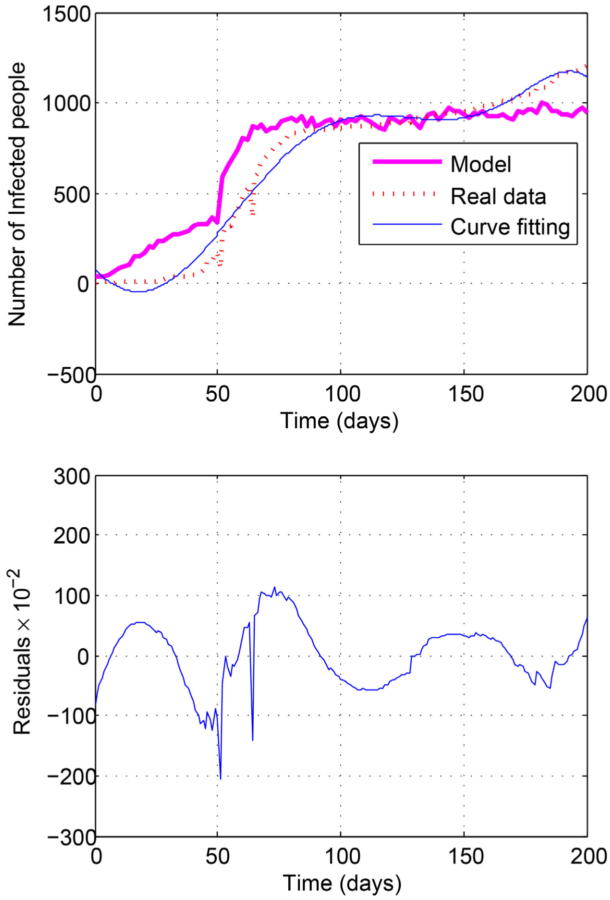
$$\frac{\dfrac{\partial \lambda_0^*}{\partial \beta}}{\dfrac{\partial \lambda_0^*}{\partial c}} = \frac{c}{\beta} = constante \qquad \frac{\dfrac{\partial \lambda_0^*}{\partial \eta_1}}{\dfrac{\partial \lambda_0^*}{\partial \eta_2}} = \frac{E^*}{I_2^*} \qquad \frac{\dfrac{\partial \lambda_0^*}{\partial \beta_v}}{\dfrac{\partial \lambda_0^*}{\partial K}} = -\frac{K + V^*}{K}.$$

In the first case where the report of two parameter sensitivities (β and c) is constant the evaluation of these two parameters is impossible, their values are chosen in [14] and [15]; but in the other cases the evaluation is possible. $\left(\frac{E^*}{I_2^*}\right)$ is simulated by curve fitting under MATLAB Software is represented by **Figure 2** knowing some initial values. Furthermore to the estimations values of parameters η_1 and η_2 are given in **Table 3**. For to estimate parameters of $\mathcal{R}_0$, we use the derivative of the following formula with respect to parameters vector p_i



(a) Curve fitting of model $\left(E^*, I_2^*\right)$

(b) Real data of COVID-19 in Chad 2020-03-15 to 2021-10-02



(c) Model and residuals

Figure 2. Curve fitting $\left(E^{*}, I_2^{*}\right)$ of model; Extract from real data of Chad [19] and Model simulation.

Table 3. Description of model variable and model parameters.

Para.	**values**	References	**Para.**	**values**	References
Λ	150	Assumed	g	0.5	Assumed
β	8.073×10^{-3}	[14]	μ_v	3.33×10^{-1}	[1]
ρ	1/10	[14]	β_v	0.001	Estimated
μ_d	0.015	[14]	f_1	0.10	[1]
c	0.5297	[15]	f_2	0.01	[1]
δ_1	1/5.2	[16]	η_1	1.5	Estimated
δ_2	1/5.8	[16]	η_2	0.10	Estimated
μ	0.0186	[1]	η_3	0.001	[1]
θ	0.0357	[1]	q	0.5	[1]
β_v	0.001	[1]	ν	0.5	Estimated
ξ	100 cells/day	[1]	K	10^3 cells/m^2	[1]

$$\frac{\partial \mathcal{R}_0}{\partial p_i}\left(u_1,u_2,u_3,\beta,c,\theta,\rho,\mu,\mu_d,\mu_v,\delta_1,\delta_2,g,\eta_1,\eta_2,\nu,K\right)$$

wherein $p=(p_i)=(\beta,c,\theta,\rho,\mu,\mu_d,\mu_v,\delta_1,\delta_2,g,\eta_1,\eta_2,\nu,K)$ is the vector parameters of model (2) at which the sensitivity are evaluated. The same technique used high here then sensitivities the evaluation of parameters of $\mathcal{R}_0$ and ε^* can be achieved.

Then there were some important results about the sensitivity of Basic Reproduction Number $\mathcal{R}_0$ and endemic equilibrium ε^* for the deterministic model parameters is discussed and given in [1]. It is important to remember here that, the study of sensitivity of the parameters of a deterministic model is necessary. It consists of analyzing how the variations of the parameters influence $\mathcal{R}_0$ and consequently ε^*. Indeed, this sensitivity analysis made it possible to identify the parameters such that Λ, β, β_v, c, ξ, η_1, η_2, and η_3 whose value variations in growth have the greatest impact on the increase of $\mathcal{R}_0$ (consequently increases E^*, I_1^*, and I_2^*); and the parameters such that ρ, δ_1, μ, μ_d, μ_v and K whose value variations in decrease have the greatest impact on the decrease of $\mathcal{R}_0$ (consequently decreases E^*, I_1^*, and I_2^*). The acceptable values of a model parameters were identified and estimated in **Table 3** wherein values are chosen in [1] [14]-[16], given $\mathcal{R}_0=0.8442$.

4.2. Analysis of Stochastic Model

4.2.1. Positivity and Boundedness of Stochastic Model

Let we denote the following set:

$$\Omega=\left\{X_t=(S,E,I_1,I_2,R,V)\in\mathbb{R}_+^6 \,/\, N_h\le\frac{\Lambda}{\mu};V\le\frac{\xi(1-u_3)}{u_4\nu+\mu_v}\frac{\Lambda}{\mu}\right\},$$

where $N_h=N_h(S(t)+E(t)+I_1(t)+I_2(t)+R(t)$.

Lemma 6. Let $(\Omega,\mathcal{F},\mathcal{F}_{t\ge0},\mathbb{P})$ a filtered probability space. For any positive initial condition $X_0=(S(0),E(0),I_1(0),I_2(0),R(0),V(0))\in\Omega$, the stochastic model (1) admits an unique solution $X_t=(S(t),E(t),I_1(t),I_2(t),R(t),V(t))\in\Omega$ dependent of X_0. Moreover, for all $t\ge0$; this solution $X_t=X(t)\in\Omega$ remains in Ω with probability 1, *i.e.* $\mathbb{P}(X_t\in\Omega)=1$.

Proof. For any positive initial condition X_0, For any positive initial condition X_0, since f and G are locally continuous Lipschitz, then there exists an unique local solution $X_t=(S(t),E(t),I_1(t),I_2(t),R(t),V(t))\in\Omega$, for all $t\in[0;t_e]$, where t_e is the explosion time. This local solution is non negative by Itô's Formula.

Let $N=N(t)=N_h(t)+V(t)$, the random variable giving the total number of humans population and corona-virus on surface to date $t\ge0$. Then we have:

$$\mathrm{d}N=\left(\Lambda-\mu N-\mu_d I_2-\xi(1-u_3)(E+I_1+\eta_3 I_2)-u_4\nu V\right)\mathrm{d}t-g_{17}\mathrm{d}W_t^{17}, \tag{6}$$

what implies that if $X_t\in\mathbb{R}_+^6$, for all $0\le t\le t_e$ almost surely (a.s.), then

$$\frac{\mathrm{d}N}{\mathrm{d}t}(t)\le\Lambda-\mu N(t)-\mu_d I_2-\xi(1-u_3)\left(E(t)+I_1(t)+\eta_3 I_2(t)\right)u_4\nu V(t),$$

So, if $0 \le V(0) \le \xi(1-u_3)\frac{\Lambda}{\mu}$, then $0 \le V(t) \le \frac{\xi(1-u_3)}{u_4\nu+\mu_v}\frac{\Lambda}{\mu}$ for $0 \le t \le t_e$ a.s.

From where

$$\frac{\mathrm{d}N}{\mathrm{d}t}(t) \le \Lambda - \mu N(t) \quad \text{a.s.}$$

While using Gronwall's lemma, we obtain

$$N(t) \le \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right)\mathrm{e}^{-\mu t} \quad \text{a.s.}$$

As initial condition $X_0 \in \Omega$, *i.e.* that $N(0) - \frac{\Lambda}{\mu} \le 0$, then $N(t) \le \frac{\Lambda}{\mu}$. So, for all $t \in [0;t_e]$, we have

$$X_t = \left(S(t), E(t), I_1(t), I_2(t), R(t), V(t)\right) \in \left[0, \frac{\Lambda}{\mu}\right]^6 \quad \text{a.s.}$$

Now, let's show that $X_t \in \Omega$ for all $t \in [0;t_e]$ is global solution, *i.e.* that $t \in [0;t_e \to +\infty[$ a.s. Let's choose an integer $n_0 > 0$ sufficiently large for that $X_0 \in \left[\frac{1}{n_0}; n_0\right]^6$, and two others integers:

$$n_{\min} = \min\left\{S(t), E(t), I_1(t), I_2(t), R(t), V(t)\right\},$$

$$n_{\max} = \max\left\{S(t), E(t), I_1(t), I_2(t), R(t), V(t)\right\}$$

Let $\mho$ be the empty set such that set $\inf\{\mho\} = +\infty$. For each integer $n > n_0$, the stop-times is defined by

$$t_n = \left\{t \in [0;t_e]; n_{\min} \le \frac{1}{n} \text{ or } n_{\max} \ge n\right\}$$

Note that t_n is increasing as $n \to +\infty$, if $\lim_{n\to+\infty} t_n = t_\infty$, then we have $t_\infty \le t_e$ a.s.

It remains to show that $t_\infty = \infty$ a.s. Suppose by absurd that $t_\infty < +\infty$ a.s., there exists a constant $\tau > 0$ and for any $\epsilon \in [0;1]$ such that

$$\mathbb{P}(t_\infty \le \tau) > \epsilon. \tag{7}$$

Consequently, there exists an integer m such that

$$\mathbb{P}(t_\infty \le \tau) > \epsilon, \ \forall n \ge m.$$

Let's define a Lyapunov function $U : \mathbb{R}_+^6 \to \mathbb{R}_+$ by:

$$U(X_t) = -\sum_{i=1}^{6}(\ln X_i), \ \forall X = (X_i)_{i=1,\cdots,6} = (S, E, I_1, I_2, R, V_h)^{\mathrm{T}}.$$

Using the Itô's Formula, we compute

$$\begin{aligned}\mathrm{d}U(X(s)) &= 2(5\mu + \mu_d + \mu_v + 3\rho + \delta_1 + \delta_2 + \theta + u_4\nu + \lambda_1 + \lambda_2)\mathrm{d}s \\ &\quad - 2\left((\lambda_1+\lambda_2)\frac{S}{E} + \frac{1}{4}\sum_{i=1}^{6}\frac{\upsilon_i}{X_i}\right)\mathrm{d}s + \frac{1}{X_i}\left\langle g_j, \mathrm{d}W_s^j\right\rangle\end{aligned}$$

where

$$\begin{aligned}\frac{1}{X_i}\left\langle g_j, \mathrm{d}W_s^j\right\rangle = &-\frac{1}{S}\sum_{i\in\{3;16\}} g_i \mathrm{d}W_s^i - \frac{1}{E}\sum_{i=3}^{4} g_i \mathrm{d}W_s^i - \frac{1}{I_1}\sum_{i=5}^{6} g_i \mathrm{d}W_s^i \\ &-\frac{1}{I_2}\sum_{i=10}^{11} g_i \mathrm{d}W_s^i - \frac{1}{V}\sum_{i=17}^{18} g_i \mathrm{d}W_s^i - \frac{1}{R}\sum_{i\in\{8;9;13\}} g_i \mathrm{d}W_s^i \\ &+\frac{1}{S}\sum_{i=4}^{6} g_i \mathrm{d}W_s^i + \frac{1}{E}\sum_{i=6}^{8} g_i \mathrm{d}W_s^i + \frac{1}{I_1}\sum_{i=9}^{12} g_i \mathrm{d}W_s^i \\ &+\frac{1}{I_2}\sum_{i=13}^{14} g_i \mathrm{d}W_s^i + \frac{1}{R}\sum_{i=15}^{16} g_i \mathrm{d}W_s^i + \frac{1}{V}\sum_{i=17}^{18} g_i \mathrm{d}W_s^i.\end{aligned}$$

Using $\frac{X_i}{N-q\left(E+I_1+I_2\right)} \ll 1$, $\frac{\beta_v V}{K+V} \ll \beta_v$, $0 \le u_i \le 1$ for all i, and $\left(\lambda_1+\lambda_2\right)\frac{S}{E}+\frac{1}{4}\sum_{i=1}^{6}\frac{\upsilon_i}{X_i} \ge 0$ almost surely, then

$$\begin{aligned}\mathrm{d}U(X) \le\ & 2\left[5\mu+\mu_d+\mu_v+3\rho+\delta_1+\delta_2+\theta+\nu+\beta c\left(1+\eta_1=\eta_2\right)+\beta_v\right]\mathrm{d}s \\ &+\frac{1}{X_i}\left\langle g_j, \mathrm{d}W_s^j\right\rangle,\end{aligned}$$

from where, we find that for $s\in[0,t]$ almost surely

$$\mathrm{d}U(x) \le Cs + \frac{1}{X_i}\left\langle g_j, \mathrm{d}W_s^j\right\rangle, \tag{8}$$

where

$$C = 2\left[5\mu+\mu_d+\mu_v+3\rho+\delta_1+\delta_2+\theta+\nu+\beta c\left(1+\eta_1=\eta_2\right)+\beta_v\right].$$

Integrating inequality (8), then taking the mathematical expectation $\mathbb{E}[\cdot]$ in (8), we obtain that

$$\mathbb{E}\left[U\left(X(t)\right)\right] \le Ct + \mathbb{E}\left[\int_0^t \frac{1}{X_i}\left\langle g_j, \mathrm{d}W_s^j\right\rangle\right].$$

Because $\mathbb{E}\left[\int_0^t \frac{1}{X_i}\left\langle g_j, \mathrm{d}W_s^j\right\rangle\right] = 0$, we have:

$$\mathbb{E}\left(U\left(X(t)\right)\right) \le Ct.$$

Finally, we have all $t \ge 0$:

$$\mathbb{E}\left(U\left(X\left(\min\left(t_n,t\right)\right)\right)\right) \le C\min\left(t_n,t\right) \le Ct.$$

Let χ_{A_n} is the characteristic function of $A_n=\left(t_n\le\tau\right)$ with $A_n^c=\left(t_n>t\right)$ her complementary. From (7), we have $\mathbb{P}\left(A_n\right)\ge\epsilon$.

Denote $\wedge_{nt}=\min\left(t_n,t\right)$. Since $U\left(X\left(\wedge_{nt}\right)\right)>0$, then

$$\mathbb{E}\left(U\left(X\left(\wedge_{nt}\right)\right)\right) = \mathbb{E}\left(U\left(X\left(\wedge_{nt}\right)\right)\cdot\chi_{A_n}\right) + \mathbb{E}\left(U\left(X\left(\wedge_{nt}\right)\right)\cdot\chi_{A_n^c}\right).$$

Thus $\mathbb{E}\left(U\left(X\left(\wedge_{nt}\right)\right)\right) \ge \mathbb{E}\left(U\left(X\left(\wedge_{nt}\right)\right)\cdot\chi_{A_n}\right)$. It then follows that

$$Ct \ge \mathbb{E}\left(U\left(x\left(\min\left(t_n,t\right)\right)\right)\cdot\chi_{\left(t_n\le t\right)}\right).$$

Note that, for each $\varpi\in\left(t_n\le n\right)$ there is at least one of the components

$X_i(\varpi(t_n))$ of $X(\varpi(t_n)) = (S, E, I_1, I_2, R, V)(\varpi(t_n))$ equal either $\frac{1}{n}$ or n, therefore $U(x(\varpi(t_n)) \geq -\ln\frac{1}{n}$, hence

$$Ct \geq \mathbb{E}\left(U\left(X\left(\min(t_n, t)\right)\right) \cdot \chi_{A_n}\right) \geq \mathbb{P}(t_n \leq t)\ln n. \tag{9}$$

Passing to limit in (9) as $n \to +\infty$, we have that

$$\infty > Ct = \infty \text{ almost surely,}$$

is a contradiction. Thus $t_\infty = \infty$ almost surely. Therefore, this end the proof lemma. ☐

4.2.2. Global Behavior of Coronavirus Propagation Model

In this subsection, we analyse the global behavior, the stability of the endemic equilibrium for the stochastic model of COVID-19 (1). But before, we look at the behavior of a deterministic mean system of model (1)

Lemma 7. Let denoted $\eta_{X_i} = \mathbb{E}(X_i), i = 1, \cdots, 6$ such that $X_1 = S$, $X_2 = E$, $X_3 = I_1$, $X_4 = I_2$, $X_5 = R$ and $X_6 = V$. Then the determistic mean system (10) of model (1) gotten admits one disease-free equilibrium $\eta_0 = \varepsilon^0$ which is globally asymptotically attractive if $\mathcal{R}_0 < 1$ and $f_1 = f_2 = 0$.

Proof. Let denoted $m_{X_i} = \mathbb{E}(X_i), i = 1, \cdots, 6$ such that $X_1 = S$, $X_2 = E$, $X_3 = I_1$, $X_4 = I_2$, $X_5 = R$ and $X_6 = V$. Then the deterministic mean system of model (1) is given by:

$$\begin{cases} \dfrac{\mathrm{d}\mathbb{E}(S)}{\mathrm{d}t} = \dot{m}_S = \Lambda(1 - f_1 - f_2) + \theta m_R - \mu\eta_S - m_{\lambda_X}, \\ \dfrac{\mathrm{d}\mathbb{E}(E)}{\mathrm{d}t} = \dot{m}_E = \Lambda f_1 + m_{\lambda_X} - k_1 m_E, \\ \dfrac{\mathrm{d}\mathbb{E}(I_1)}{\mathrm{d}t} = \dot{m}_{I_1} = \Lambda f_2 + \delta_1(1 - g)m_E - k_2 m_{I_1}, \\ \dfrac{\mathrm{d}\mathbb{E}(I_2)}{\mathrm{d}t} = \dot{m}_{I_2} = \delta_2 m_{I_1} + g\delta_1 m_E - k_3 m_{I_2}, \\ \dfrac{\mathrm{d}\mathbb{E}(R)}{\mathrm{d}t} = \dot{m}_R = \rho\left(m_E + m_{I_1} + m_{I_2}\right) - k_4 m_R, \\ \dfrac{\mathrm{d}\mathbb{E}(V)}{\mathrm{d}t} = \dot{m}_V = \xi(1 - u_3)\left(m_E + m_{I_1} + \eta_3 m_{I_2}\right) - k_5 m_V, \end{cases} \tag{10}$$

wherein $m_{\lambda_X} = \mathbb{E}(\lambda_1 + \lambda_2)$ is the mean infectious for humans and coronaviris.

So, if $f_1 = f_2 = 0$, then from Equation (6), we compute mathematical expectation and we obtain for $s \in [0, t]$, $t \geq 0$ that:

$$\frac{\mathrm{d}\mathbb{E}(dN)}{\mathrm{d}s} = \dot{m}_{dN} = \Lambda - \mu m_N - \mu_d m_{I_2} - \xi(1 - u_3)\left(m_E + m_{I_1} + \eta_3 m_{I_2}\right) - u_4 \nu m_V$$

by Gronwall's lemma, one obtain

$$\frac{\Lambda}{\mu} + \left(m_N(0) - \frac{\Lambda}{\mu}\right)\mathrm{e}^{-\mu t} \geq m_N, \text{ a.s.}$$

Therefore, there exists a non negative real number $\sigma_\epsilon > 0$ and a time $t_\epsilon > 0$

such that

$$\frac{\Lambda}{\mu}+\left(m_N\left(0\right)-\frac{\Lambda}{\mu}\right)\mathrm{e}^{-\mu t}\geq m_N\geq\frac{\Lambda}{\mu}-\sigma_\epsilon \text{ a.s.}$$

Thus, for all $t\geq t_\epsilon$, the mean infectious force m_{λ_X} verifies the following inequality

$$m_{\lambda_X}\leq\frac{\beta c\left(1-u_1\right)\left(\eta_1 m_E+m_{I_1}+\eta_2 m_{I_2}\right)\mu}{\Lambda-\sigma_\epsilon\mu}+\frac{\beta_v\left(1-u_2\right)\Lambda}{\sigma_\epsilon\mu\left(u_4\nu+\mu_v\right)},\tag{11}$$

with $\sigma_\epsilon\to 0$ as $\epsilon\to 0^+$.

Let denoted here F_0^ϵ is a matrix satisfying the following linearzed system around ε^0:

$$\frac{\mathrm{d}X\left(t\right)}{\mathrm{d}t}=\mathcal{J}\left(\mathcal{D}_X\left(F\left(X\right)\right)_{X=\varepsilon^*}\right)X\left(t\right),\ X=\left(X_i\right)\in\mathbb{R}^6,$$

where F_0^ϵ is linear operator such that

$$\mathcal{J}\left(\mathcal{D}_X(F\left(X\right))_{X=\varepsilon^0}\right)=F_0^\epsilon\left(t\right),$$

where

$$F_0^\epsilon=\begin{pmatrix}\beta c\left(1-u_1\right)\eta_1-k_1 & \beta c\left(1-u_1\right) & \beta c\left(1-u_1\right)\eta_2 & 0 & \dfrac{\beta_v\left(1-u_2\right)}{K}\\ \left(1-g\right)\delta_1 & -k_2 & 0 & 0 & 0\\ g\delta_1 & \delta_2 & -k_3 & 0 & 0\\ \rho & \rho & \rho & -k_5 & 0\\ \xi\left(1-u_3\right) & \xi\left(1-u_3\right) & \xi\left(1-u_3\right)\eta_3 & 0 & -k_6\end{pmatrix}.$$

What remains is find the global stability of $\eta_0=\varepsilon^0$ and the basic reproduction number of mean system (10).

Let us choose $F_0^\epsilon=f_0^\epsilon-g_0$ such that for all variable state of the mean system denoted $m_X=\left(m_E,m_{I_1},m_{I_2},m_R,m_V\right)$, we have

$$\begin{cases}\dot{m}_X=\left(f_0^\epsilon-g_0\right)\left(t\right)m_X,\\ m_0\left(0\right)=\left(m_E\left(0\right),m_{I_1}\left(0\right),m_{I_2}\left(0\right),m_R\left(0\right),m_V\left(0\right)\right)\end{cases}$$

In the Stochastic model (1), since (11) is satisfied for or all $t\geq t_\epsilon$, we have

$$\dot{m}_X\leq\mathbb{E}\left(\mathcal{D}_X F_0\left(\upsilon\left(t\right)\right)\right)m_X.$$

In [1], it is shown that the basic reproduction number $\mathcal{R}_0=\varrho\left(\mathcal{L}_G\right)$ for the deterministic model (2), is given while using the Next-Generation matrix $G=\left(G_{nex-gen.}\right)$ in [13]. It's the spectral radius of a linear operator $\mathcal{L}_G=\mathcal{F}\mathcal{V}^{-1}$.

The comparison with the following auxiliary system

$$\begin{cases}\dot{m}_x\left(t\right)=\left(G_{nex-gen.}\right)m_x\left(t\right),\\ m_x\left(0\right)=\left(m_E\left(0\right),m_{I_1}\left(0\right),m_{I_2}\left(0\right),m_R\left(0\right),m_V\left(0\right)\right).\end{cases}\tag{12}$$

Then, us find that his system (12) has basic reproduction ratio $\mathcal{R}_0^m=\varrho\left(\mathcal{L}_\Phi\right)$. A globale solution of system (12) is given by

$$m_x(t) = \left(\Phi_{F_0^\epsilon}(s)\right)(t) \ \forall t \geq 0 \ \text{a.s.}$$

If $\mathcal{R}_0^m = \varrho(\mathcal{L}_\Phi) < 1$, then $\varrho\left(\Phi_{F_0^\epsilon}(s)\right) < 1 \ \Rightarrow 1 > \varrho\left(\Phi_{F_0^\epsilon}\right) = \lim_{\epsilon\to 0} \varrho\left(\Phi_{f_0^\epsilon - g_0}\right)$.

There exists $t_\epsilon > 0$ such that for all $t \in [0; t_\epsilon]$, $\varrho\left(\Phi_{f_0^\epsilon - g_0}\right) < 1$. Thus, there exists a positive solution bounded of system (12) and expressed by

$$m_x(t) = \mathcal{V}(t)\mathrm{e}^{\frac{1}{s}\ln\left(\varrho\left(\Phi_{f_0^\epsilon - g_0}\right)\right)}.$$

Consequently, because $\mathcal{R}_0 = \varrho\left(\Phi_{f_0^\epsilon - g_0}\right) < 1$, we have

$$m_x(0) = \left(m_E(t), m_{I_1}(t), m_{I_2}(t), m_R(t), m_V(t)\right) \to (0,0,0,0,0), \ \text{as } t \to +\infty$$

according the comparison theorem in [17]. Moreover we have

$$m_S - \frac{\Lambda}{\pi} = m_N - m_E - m_{I_1} - m_{I_2} - m_R - m_V - \frac{\Lambda}{\pi} \to 0, \ \text{as } t \to +\infty.$$

Hence $m_S \to \frac{\Lambda}{\pi}$, as $t \to +\infty$. Thus, $\eta_0 = \varepsilon^0$ is globally asymptotically attractive if $\mathcal{R}_0 < 1$ and $f_1 = f_2 = 0$. □

Theorem 8. The random disease-free equilibrium for the stochastic model (1) is globally asymptomatically stable and exponentially p-stable.

Proof. Let's define a Lyapunov $\mathcal{V}_X : \mathbb{R}_+^6 \to \mathbb{R}_+$ $\mathcal{V}_X = \frac{1}{p}\sum_{i=1}^{6} C_i X_i^p$, with $p \geq 2$ is an integer and $C_i > 0$ are non negative constants for all $i = 1,\cdots,6$.

That is for $X_1 = S$, $X_2 = E$, $X_3 = I_1$, $X_4 = I_2$, $X_5 = R$, $X_6 = V$,

$$\mathcal{V}_X = \frac{1}{p}\left[C_1 S^p + C_2 E^p + C_3 I_1^p + C_4 I_2^p + C_5 R^p + C_6 V^p\right]$$

Considering an infinitesimal generating operator $\mathcal{A}$ defined by

$$\begin{aligned}
\mathcal{A}\mathcal{V}_X = &-\left[C_1 \mu S^p + C_2 k_1 E^p + C_3 k_2 I_1^p + C_4 k_3 I_2^p + C_5 k_4 R^p + C_6 k_5 V^p\right] \\
&+ C_1 \lambda S^p + \left[C_1 \Lambda (1 - f_1 - f_2) + C_1 \theta R + C_2 \lambda\right] S^{p-1} + C_2 f_1 E^{p-1} \\
&+ C_3\left(f_2 \Lambda + (1-g)\delta_1 E\right) I_1^{p-1} + C_4\left(\delta_2 I_1 + g\delta_1 E\right) I_2^{p-1} + C_5 \rho (I_1 + I_2) R^{p-1} \\
&+ C_6 \xi (1 - u_3)(E + I_1 + \eta_3 I_2) V^{p-1} + \frac{1}{2}(p-1)\left[\sum_{i=1}^{6} C_i v_i X_i^{p-2}\right]
\end{aligned}$$

Using the inequalities of lemma 7 in [10] as follows:

Let an integer $p \geq 2$, $x, y \in \mathbb{R}_+$, and $\epsilon > 0$ chosen sufficiently small, we have

$$xy^{p-1} \leq \frac{\epsilon^{1-p}}{p} x^p + \frac{(p-1)\epsilon}{p} y^p,$$

$$x^2 y^{p-2} \leq \frac{2\epsilon^{\frac{2-p}{2}}}{p} x^p + \frac{(p-2)\epsilon}{p} y^p,$$

and Young's inequalities for $p, q > 0$

$$\frac{1}{p} + \frac{1}{q} = 1 \ \text{and} \ xy \leq \frac{x^p}{p} + \frac{y^p}{q}.$$

We get finally

$$\mathcal{A}\mathcal{V}_X \le -\left[C_1\mu S^p + C_2k_1E^p + C_3k_2I_1^p + C_4k_3I_2^p + C_5k_4R^p + C_6k_5V^p\right]$$

wherein all coefficients of $X_i, i=1,\cdots,6$ are non negative. Therefore, the random disease-free equilibrium for the stochastic model (1) is exponentially p-stable because there exists a function $\mathcal{V}_X \in \mathcal{C}^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^6\right)$ satisfying the following conditions for $K_1 > 0, K_2 > 0$ ans $p > 0$:

$$\|\mathcal{V}_X\| \le K_1\|x\|^p \text{ and } \mathcal{A}\mathcal{V}_X \le -K_2\|x\|^p.$$

When $p=2$, it's exponentially stable in mean square and is globally asymptotically stable. ❒

Theorem 9. The endemic equilibrium, for the stochastic model (1) $X^* = \varepsilon^*$, is locally asymptotically stable if $\mathcal{R}_0 > 1$.

Proof. Let suppose a stochastic perturbation of while noise type, directly proportional to distance $X - X^* = \left(X_i - X_i^*\right)_{i=1,\cdots,6}$. and influence model (1) expresses by the following stochastic equation:

$$\mathrm{d}X = f\left(t, X(t)\right)\mathrm{d}t + G\left(t, \left(X - X^*\right)(t)\right)\mathrm{d}W(t), \tag{13}$$

where $X = \left(S, E, I_1, I_2, R, V\right)$, and $X^* = \left(S^*, E^*, I_1^*, I_2^*, R^*, V^*\right) = \varepsilon^*$ Let be centred this dynamic at its endemic equilibria ε^* by change of random variables as: $Y = \left(Y_i\right) = \left(X_i - X_i^*\right)$, $i=1,\cdots,6$. Thus, we obtain an equivalent model because of the same law of probability of the random variables X and Y

$$\mathrm{d}Y = f_Y\left(t, Y(t)\right)\mathrm{d}t + G_Y\left(t, Y(t)\right)\mathrm{d}W_Y(t), \tag{14}$$

where $f_Y\left(t, Y(t)\right)$ and $G_Y\left(t, Y(t)\right)$ are determined by Itô formula as in [10] [11] [18].

It is easy to show that the equilibria $Y(0) \equiv 0$ of model (14) is locally asymptotically stable, so that X^* is locally asymptotically stable stable if $\mathcal{R}_0 > 1$. Let define a Lyapunov function $V_Y\left(Y(t)\right)$ given for p an integer by:

$$V_Y\left(Y(t)\right) = \sum_{i=1}^{6} Y_i^p$$

Using the differential operator $\mathcal{A}$ associated with model 1 for $V_Y\left(Y(t)\right)$ and by the Itô formula, we compute and obtain

$$\mathrm{d}\left(\mathrm{e}^t V_Y(Y)\right) = \mathrm{e}^t\left(V_Y(Y)\right) + \mathcal{A}V_Y(Y)\mathrm{d}t + p\mathrm{e}^t\sum_{i=1}^{18} fct\left(Y^{p-1}, g_i\mathrm{d}W_i\right)$$

where fct an know function and

$$\mathcal{A}(\cdot) = \frac{\partial}{\partial t}(\cdot) + f^{\mathrm{T}}(t, X)\frac{\partial}{\partial X}(\cdot) + \frac{1}{2}Tr\left(\left(G^{\mathrm{T}}G\right)(t, X)\frac{\partial^2}{\partial X^2}(\cdot)\right)$$

So that with M a non negative constant, we have

$$\mathrm{d}\left(\mathrm{e}^t V_Y\left(Y(t)\right)\right) = M\mathrm{e}^t\mathrm{d}t + p\mathrm{e}^t\sum_{i=1}^{18} fct\left(Y^{p-1}(t), g_i\mathrm{d}W_i(t)\right)$$

what implies clearly

$$\mathbb{E}\left[\mathrm{e}^t V_Y\left(Y(t)\right)\right] \le M \cdot V_Y\left(Y(0)\right) + M \cdot \mathbb{E}\left[\int_0^{\min(t, t_n)} \mathrm{e}^s\mathrm{d}s\right]$$

with t_n the stopping time defined for each integer $n \geq n_0$ by

$$t_n = \inf\left\{t \geq 0 : Y_i(t) \notin \left[\frac{1}{n_0}; n_0\right], i = 1, \cdots, 6\right\}$$

and $n_0 > 0$ be sufficiently large such that $Y(0)$ lying within the interval $\left[\frac{1}{n_0}; n_0\right]^6$. When $n \to +\infty$ then $\min(t, t_n) \to t$ yields:

$$\mathbb{E}\left[V_Y(Y(t))\right] \leq \mathrm{e}^{-t} V_Y(Y(0)) + M\left(1 - \mathrm{e}^{-t}\right) \leq \mathrm{e}^{-t} V_Y(Y(0)) + M.$$

According that $\mathbb{E}\left[V_Y\left(Y\left(\min(t, t_n)\right)\right)\right] \leq C^{te} \min(t, t_n) \leq C^{te} t$ and using the fact that equilibria $Y(0) = (0,0,0,0,0,0) \equiv 0$ is p-stable, then X^* is p-stable also. ☐

5. Stochastic Optimal Control of COVID-19 Spread

5.1. Optimal Control of Model without Random Noise

In this Subsection, the deterministic model is submitted into an optimal control problem by seeking to minimize an objective functional that measures total infected individuals and surface viral concentration, and the total cost associated with implementing the various controls.

To determine the best strategy adapted to the struggle against COVID-19 spread. The theory of the optimal control is used by formulating the optimization problem which minimize the objective functional as follows for $i = 1, 2, 3, 4$:

$$\begin{cases} \text{Find the admissible controls } u = (u_i) = (u_i^*) = u^* \text{ such that :} \\ \mathcal{J}\left(\left(u_i^*\right)_{i=1}^4\right) = \min\limits_{u_i \in [0;1]} \mathcal{J}\left(\left(u_i\right)_{i=1}^4\right), \\ \text{Subject to : } \dfrac{\mathrm{d}X}{\mathrm{d}t} = f\left(t, X(t), u(t)\right), \ \forall X \in \mathbf{R}_+^6, \end{cases} \tag{15}$$

wherein $X = (S, E, I_1, I_2, R, V)^{\mathrm{T}}$ is a variable of state and $\mathcal{J}$ is the functional cost:

$$\mathcal{J}\left(\left(u_i\right)_{i=1}^4\right) = \int_{t_0}^{T}\left(A_1 E + A_2 I_1 + A_3 I_2 + A_4 V + \sum_{i=1}^{4} B_i u_i^2\right)\mathrm{d}t. \tag{16}$$

the coefficients A_i (respectively B_i), $i = 1, 2, 3, 4$ are balancing factors accounting for the differences in the importance of the state variables (respectively the various controls) objective functional measures the total infections and total costs associated with the controls and without a random noise in model.

The Pontryagin's Maximum Principle produces, using the minimization problem of the Hamiltonian, the necessary optimality conditions for the quadruple optimal controls solutions of the optimization problem (15).

Theorem 10. Let $X^s = \left(X_i^s\right)_{i=1}^6 = \left(S^s, E^s, I_1^s, I_2^s, R^s, I_1^s\right)^{\mathrm{T}}$ be the solution associated with the quadruple optimal controls $\left(u_i^*\right)_{i=1}^4$, solution of (15). Then,

1) The Hamiltonian $\mathcal{H}^0$ is given by

$$\mathcal{H}^0 = \frac{\mathrm{d}\mathcal{J}\left(\left(u_i\right)_{i=1}^4\right)}{\mathrm{d}t} + \sum_{i=1}^6 p_{X_i} \frac{\mathrm{d}X_i}{\mathrm{d}t} \tag{17}$$

where, the $p_{X_i}, i = 1, \cdots, 6$ are the ad-joint variables associated with the state variables X_i such that

$$\frac{\mathrm{d}p_{X_i}}{\mathrm{d}t} = -\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}X_i} \tag{18}$$

2) The following transversality conditions hold

$$p_{X_i}\left(T\right) = 0,\ \forall \left(X_i\right)_{i=1}^6. \tag{19}$$

3) The admissible controls u_i^* is given by

$$u_i^* = \max\left\{0, \min\left\{1, \tilde{u}\right\}\right\}, \tag{20}$$

where $\tilde{u} = \tilde{u}\left(X^s\right)$ is found in [1]

These results gotten in [1] for the deterministic optimal control that we have just recalled are necessary for the analysis of the stochastic model (1).

5.2. Optimal Control of Model with Random Noise

The optimal strategies of the stochastic control model of COVID are gotten on the one hand here in this section, as minimizing under the stochastic dynamics controlled of the COVID-19 forces it of people infection in human population because this strength depends in majority by variables of people state exposed and tainted having some middle and stern symptoms, of the concentration of the coronavirus in surface and in minimising on the other hand, the total number of expositions and tainted as well as the concentration in surfaces of the coronavirus under the same stochastic dynamics.

5.2.1. Cost Function Resulting Some Development Limited

As minimizing subject to the stochastic dynamics controlled of the COVID-19, the mathematical expectation of infectious force $\lambda = \lambda_1 + \lambda_2$ of human population because this force depends in majority by the state variables vector $X_t = \left(S, E, I_1, I_2, R, V\right)$ and the control vector $u = \left(u_1, u_2, u_3, u_4\right)$.

To minimisze the total force of infection $\lambda = \lambda\left(S, E, I_1, I_2, R, V, u_1, u_2\right))$ over the finite time horizon $\left[0; T\right]$, dependent on random states, amounts to bringing it absolutely back to the controlled state disease free, *i.e.* by making $\lambda \to \lambda^s\left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, u_1, u_2\right) = \lambda^s\left(\varepsilon^0, u_1, u_2\right)$ when X_t tends towards ε^0 by u for all $t \in \left[0; T\right]$. As the minimization with respect to control u of the cost functional $\mathcal{J}\left(u\right)$ depends on this random force λ, then for the optimum to exist, the mathematical expectation of the cost to be minimized depends on a convex function. This is why, λ^s we add a quadratic combinaison of controls u_i, *i.e.* $\sum_{i=1}^4 B_i u^2$ and we obtain a cost functional qualified as a pseudo-quadratic given by (21).

Considering the pseudo-quadratic cost functional of the following form for each integer s:

$$\mathcal{J}^s(u) = \mathbb{E}\left[\int_0^T \left(\left|\lambda^s\left(X_t^u\right)\right| + C_0^s + \sum_{i=1}^4 B_i u_i^2\right)\mathrm{d}t\right], \tag{21}$$

wherein $\lambda^s\left(X_t^u\right)$ is a non constant functional term resulting some development limited of infectious force $\lambda = \lambda_1 + \lambda_2$ around ε^0 at order $s = 1$, $\left|\lambda^s\left(X_t^u\right)\right|$ is it absolute value and C_0^s it constant term representing the total force of human infection and viral spread at disease-free equilibrium ε^0, *i.e.* $C_0^s = \lambda^s\left(\varepsilon^0\right)$ without control. The choice of $s = 1$ follows from relation (16) giving the cost functional which measures the total number of exposed people, mildly and severely symptomatic people and surface concentration of virus; linear relation in which the constants A_i independent of the controls represent the important weights of the state variables of deterministic model. In the case of the stochastic model whose variables are random, these coefficients can vary depending on the control, this is the case here. Therefore, the choice $s = 1$ is explained by this linear relation similar to that of cost functional which measures total infected individuals and surface viral concentration, and the total cost associated with implementing the various controls.

Development limited of λ around ε^0 at order $s = 1$:

$$\lambda = \lambda\left(\varepsilon^0\right) + \nabla_X \lambda\left(\varepsilon^0\right)\left(X - \varepsilon^0\right) + 0\left(X - \varepsilon^0\right),$$

then, we find $C_0^{reds} = 0$ and

$$\left|\lambda^1\left(X_t^u\right)\right| = \beta c\left(1 - u_1\right)\frac{\mu}{\Lambda}\left(\eta_1 E + I_1 + \eta_2 I_2\right) + \beta_v\left(1 - u_2\right)V.$$

Let $T = t_f$ the terminal time is equal to final date of the epidemic by

$$t_f = \inf\left\{t \in \mathbb{R}_+ / E(t) + I_1(t) + I_2(t) + V(t) = I_{hv}(t) \ll 1\right\}$$

Let $\varphi_1 : \left[0; t_f\right] \times \mathbb{R}^6 \times \mathcal{U} \to \mathbb{R}$ and $\psi_1 : \mathbb{R}^6 \to \mathbb{R}$ are respectively the functions of instantaneous cost and terminal cost such that:

$$\varphi_1\left(t, X(t), u\right) = \left|\lambda^1\left(X_t^u\right)\right| + \sum_{i=1}^4 B_i u_i^2 \text{ and } \psi_1\left(X(t)\right) = 0\left(X - \varepsilon^0\right),$$

Thus, the cost functional is given by

$$\mathcal{J}^1(u) = \mathbb{E}\left[\int_0^{t_f} \varphi_1\left(t, X(t), u(t)\right)\mathrm{d}t + \psi_1\left(X\left(t_f\right)\right) / X(0) = x\right], \tag{22}$$

5.2.2. Controlled Evolution Process of COVID-19

- **Problem: Control of evolution by minimizing the cost functional $\mathcal{J}^1$**

Here, the controlled process $X = \left(S, E, I_1, I_2, R, V\right) \in \mathbb{R}^6$ of COVID-19 evolves like a diffusion non-controlled, but the "controller" can influence the behavior of X by modifying its evolution vector f or diffusion matrix G at each time t through choice of the value $u(t)$.

Thus, one of the main objectives in this part is to find the optimality controlling variables $u^* = \left(u_1^*, u_2^*, u_3^*, u_4^*\right)$ using cost functional (22) such that

$$\begin{cases} \mathcal{J}^{s=1}\left(u^*\right) = \min\limits_{u_i \in [0;1]} \mathcal{J}^{s=1}(u), \\ \text{Subject to} : \mathrm{d}X = f\left(t, X(t), u(t)\right)\mathrm{d}t + G^0\left(t, X(t)\right)\mathrm{d}W(t), \ \forall X \in \mathbf{R}_+^6, \end{cases} \tag{23}$$

where

$$f\left(t,X(t),u(t)\right)=\left(f_i\right)_{i=1}^{6}$$

$$G^0\left(t,X_t\right)=\begin{pmatrix} G^0_{3\times 8} & Q_{3\times 10} \\ O_{3\times 8} & G^0_{3\times 10}\end{pmatrix}=\left(G_{ij}\right)_{i,j=1}^{i=6,j=18}\left(t,X,u=0\right)\quad i.e\ u_1=u_2=u_3=u_4=0$$

wherein $G^0_{3\times 8}=\begin{pmatrix} g_1 & -g_2 & -g_3 & -g^0_4 & -g_5 & 0 & 0 & 0 \\ 0 & 0 & g_3 & g_4 & 0 & -g_6 & -g_7 & -g_8 \\ 0 & 0 & 0 & 0 & g_5 & g_6 & 0 & 0 \end{pmatrix}$

$$Q_{3\times 10}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -g_9 & -g_{10} & -g_{11} & -g_{12} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$O_{3\times 8}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G^0_{3\times 10}=\begin{pmatrix} 0 & g_{10} & g_{11} & 0 & -g_{13} & -g_{14} & 0 & 0 & 0 & 0 \\ g_9 & 0 & 0 & 0 & g_{13} & 0 & -g_{15} & -g_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g^0_{17} & -g^0_{18} \end{pmatrix}$$

such that $g_1=\sqrt{\Lambda}$, $g_2=\sqrt{\mu S}$, $g_3=\sqrt{f_1\Lambda}$, $g^0_4=\sqrt{\lambda^0 S}$, $g_5=\sqrt{f_2\Lambda}$, $g_6=\sqrt{\delta_1 E}$, $g_7=\sqrt{\mu E}$, $g_8=\sqrt{\rho E}$, $g_9=\sqrt{\rho I_1}$, $g_{10}=\sqrt{g\delta_1 E}$, $g_{11}=\sqrt{\delta_2 I_1}$, $g_{12}=\sqrt{\mu I_1}$, $g_{13}=\sqrt{\rho I_2}$, $g_{14}=\sqrt{\left(\mu+\mu_d\right)I_2}$, $g_{15}=\sqrt{\mu R}$, $g_{16}=\sqrt{\theta R}$, $g^0_{17}=\sqrt{\xi\left(E+I_1+\eta_3 I_2\right)}$, $g^0_{18}=\sqrt{\mu_v V}$.

$$\lambda^0=\lambda^0\left(X_t\right)=\frac{\beta c\left(\eta_1 E+I_1+\eta_2 I_2\right)}{S+(1-q)\left(E+I_1+I_2\right)+R+V}+\frac{\beta_v V}{K+V}$$

In applying the Pontryagin's Minimum Principle (PMP) for the stochastic model [7], we give definition of Hamiltonian $\mathcal{H}^s$ corresponding to optimal problem 23 as follows:

$$\mathcal{H}^{s=1}:\left[0;t_f\right]\times\mathbb{R}^6\times\mathcal{U}_{ad}\times L^2\left(\left[0;t_f\right],\mathbb{R}^6\right)\times L^2\left(\left[0;t_f\right],\mathbb{R}^{6\times 18}\right)\to\mathbb{R}$$

$$\begin{aligned}\mathcal{H}&=\mathcal{H}\left(t,X(t),u(t),p(t),q(t)\right)\\&=p^{\mathrm{T}}(t)\cdot f\left(t,X(t),u(t)\right)+Tr\left(G^{\mathrm{T}}G\left(t,X(t)\right)\cdot q(t)\right)-\psi\left(t,X(t),u(t)\right)\end{aligned}$$

$$\mathcal{H}=\sum_{i=1}^{6}p_i(t)f_i(t,X,u)+\sum_{i=1}^{6}q_i(t)v_i(t,X)-\varphi_1(t,X,u)$$

where $v_i=\left(G^{\mathrm{T}}G\right)_{ii}$, for $i=1,\cdots,6$ are the diagonal components of $G^{\mathrm{T}}G$ defined by:

$$GG^{\mathrm{T}}=\begin{pmatrix} v_1 & -g_3^2-\left(g^0\right)_4^2 & -g_5^2 & 0 & -g_{16}^2 & 0 \\ -g_3^2-\left(g^0\right)_4^2 & v_2 & -g_6^2 & 0 & -g_8^2 & 0 \\ -g_5^2 & -g_6^2 & v_3 & -g_{10}^2-g_{11}^2 & -g_9^2 & 0 \\ 0 & 0 & -g_{10}^2-g_{11}^2 6 & v_4 & -g_{13}^2 & 0 \\ -g_{16}^2 & -g_8^2 & -g_9^2 & -g_{13}^2 & v_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\left(g^0\right)_{17}^2-\left(g^0\right)_{18}^2 \end{pmatrix}$$

with

$$v_1 = g_1^2 + g_2^2 + g_3^2 + \left(g^0\right)_4^2 + g_5^2 + g_{16}^2 = \Lambda\left(1 + f_1 + f_2\right) + \left(\lambda + \mu\right)S + \theta R,$$

$$v_2 = g_3^2 + \left(g^0\right)_4^2 + g_6^2 + g_7^2 + g_8^2 = \Lambda f_1 + \lambda S + k_1 E,$$

$$v_3 = g_5^2 + g_6^2 + g_6^2 + g_9^2 + g_{10}^2 + g_{11}^2 + g_{12}^2 = \Lambda f_2 + \left(1 - g\right)\delta_1 E + k_2 I_1,$$

$$v_4 = g_{10}^2 + g_{11}^2 + g_{12}^2 + g_{14}^2 = g\delta_1 E + \delta_2 I_1 + k_3 I_2,$$

$$v_5 = g_8^2 + g_9^2 + g_{13}^2 + g_{15}^2 + g_{16}^2 = \rho\left(E + I_1 + I_2\right) + k_4 R,$$

$$v_6 = \left(g^0\right)_{17}^2 + \left(g^0\right)_{18}^2 = \xi\left(E + I_1 + \eta_3 I_2\right) + \mu_v V.$$

$p(t) = \left(p_i(t)\right)_{i=1}^{6}$ and $q(t) = \left(q_i(t)\right)_{i=1}^{6} = \left(q_{ij}(t)\right)_{i,j=1}^{i=6,j=18}$ are the adjoints process such that

$$\begin{cases} \mathrm{d}p(t) = -\nabla_X \mathcal{H}\left(t, X(t), u(t), p(t), q(t)\right)\mathrm{d}t + q(t)\mathrm{d}W(t) \\ p\left(t_f\right) = \psi_1\left(X\left(t_f\right)\right). \end{cases}$$

from where

$$\begin{cases} \mathrm{d}p(t) = \left(\dfrac{\partial \varphi_1}{\partial X}(t, X, u) - \sum_{i=1}^{6} p_i \dfrac{\partial f_i}{\partial X_i}(t, X, u) - \sum_{i=1}^{6} q_i \dfrac{\partial v_i}{\partial X_i}(t, X)\right)\mathrm{d}t + q(t)\mathrm{d}W(t) \\ p\left(t_f\right) = \psi_1\left(X\left(t_f\right)\right) = 0\left(X\left(t_f\right) - \varepsilon^0\right). \end{cases} \tag{24}$$

Theorem 11. Let u^* the optimal control, $X^* = \left(S^*, E^*, I_1^*, I_2^*, R^*, V^*\right)$ an optimal way of X solution of EDS in (23) and $\left(p^*, q^*\right)$ solution of Equation (24). Then under the above hypotheses, for $t \in \left[0; t_f\right]$ the optimal control u^* verify the following optimality controlling:

$$\begin{aligned} u^* &= \arg\max_{u \in \mathcal{U}_{ad}} \mathcal{H}\left(t, X^*(t), u(t), p^*(t), q^*(t)\right) \\ &= \arg \mathcal{H}\left(t, X^*(t), u^*(t), p^*(t), q^*(t)\right) \end{aligned} \tag{25}$$

Proof. See in Book [6]-[8]. ☐

Corollary 2. Let's suppose that hypotheses of Theorem 11 are true. Then, for $t \in \left[0; t_f\right]$ the optimal control u^* ts characterised by

$$\mathbb{E}\left(\int_0^{t_f} \frac{\partial \mathcal{H}}{\partial u}\left(t, X^*(t), u^*(t), p^*(t), q^*(t)\right) \cdot \left(\overline{u} - u(t)\right)\mathrm{d}t\right) \geq 0$$

what implies

$$\frac{\partial \mathcal{H}}{\partial u}\left(t, X^*(t), u^*(t), p^*(t), q^*(t)\right) \cdot \left(\overline{u} - u(t)\right) \geq 0 \quad \forall \overline{u} \in \mathcal{U}_{ad} \quad a.s.$$

By application of Theorem 11 and Corollary 2 to the stochastic model of COVID-19, *i.e.* that on derivative of Hamiltonian formula [8] [9] with respect to $u = \left(u_1, u_2, u_3, u_4\right)$ and we obtain $u^* = \left(u_1^*, u_2^*, u_3^*, u_4^*\right)$ such that:

$$u_i^* = \max\left\{0; \min\left\{1; \overline{\overline{u}}_i\right\}\right\}, \quad i = 1, 2, 3, 4.$$

where

$$\begin{aligned}
\bar{\bar{u}}_1 &= \frac{\beta c\left(\eta_1 E^* + I_1^* + \eta_2 I_2^*\right)}{2B_1}\left(\frac{\mu}{\Lambda} + \frac{\left(p_1 - p_2\right)S^*}{S^* + (1-q)\left(E^* + I_1^* + I_2^*\right) + R^* + V^*}\right), \\
\bar{\bar{u}}_2 &= \frac{\beta_v V^* S^*\left(1 + p_1 - p_2\right)}{2B_2\left(K + V^*\right)}, \\
\bar{\bar{u}}_3 &= \frac{p_6 \xi\left(E^* + I_1^* + \eta_3 I_2^*\right)}{2B_3}, \\
\bar{\bar{u}}_4 &= \frac{p_6 \nu V^*}{2B_4}
\end{aligned} \tag{26}$$

We note this result depends $X^* = \left(S^*, E^*, I_1^*, I_2^*, R^*, V^*\right)$ and $p(t)$ only but doesn't depend q because $G^0\left(t, X(t)\right) = \left(G_{ij}\right)_{i,j=1}^{i=6,j=18}\left(t, X, u=0\right)$ *i.e.* $u_1 = u_2 = u_3 = u_4 = 0$.

Although the objective aimed very well to control the evolution and the diffusion of the propagation of the COVID-19 at a time, the control of the evolution has just been done while waiting for the control of the evolution and the diffusion of the same in next publications.

6. Numerical Implementation of Data and Stochastic Model Controlled

6.1. Curve Fitting for the Estimation Values of Parameters

The new model 1 that goes simulated is similar to the stochastic in [10] and [11]. In Table 3, we have two values assumed of parameters of our models, Λ and g. These assumed values determine from the mean model 10 depending on the mathematical expectation. For g e.g., from 4th equation of the mean model, we used the mathematical expectaion of state variables E, I_1, I_2 known in two instants t_0 (initial time) and t (intermediate time) and the parameters $g, \delta_1, \delta_2, \rho, \mu, \mu_d$. The other parameters being already estimated and chosen of data in the literature as $\delta_1, \delta_2, \rho, \mu, \mu_d$, then parameter unknown g is calculated therefore.

From real data of the COVID-19 pandemic in 4 localities in Chad, data 1, 2, 3 and 4 given from 2020-01-03 to 2021-01-10 (see [19]), we extracted that of data 2 from 2020-03-15 to 2020-10-02 (see Figure 2) for to validate our model.

With a particular case of sensitivity and parameters value evaluated, we constructed a curve fitting in Figure 2. Extension of this case permitted us to identify (assumed) and to estimate the values of certain parameters on the Table 3.

6.2. Numerical Simulation of Stochastic Model of COVID-19 Control

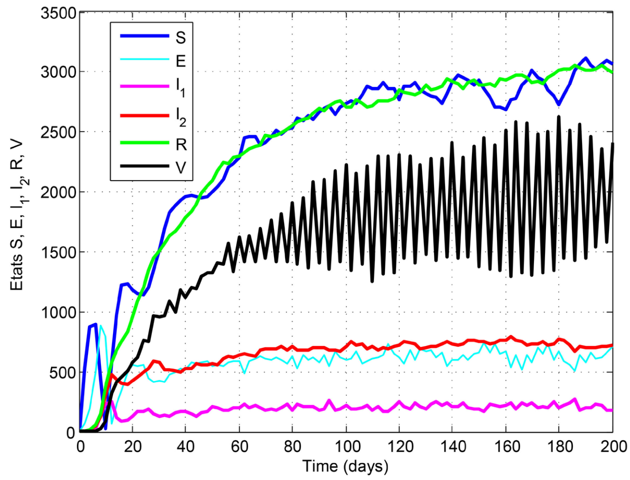
To observe what would happen with the numerical simulations of states S, E, I_1, I_2, R and V of our model, a certain number of scenarios in relation with the government's decision is predefined in order to eradicate COVID-19.

Scenario No. 1: ***Simulation of model 1 with implementation the various controls.***

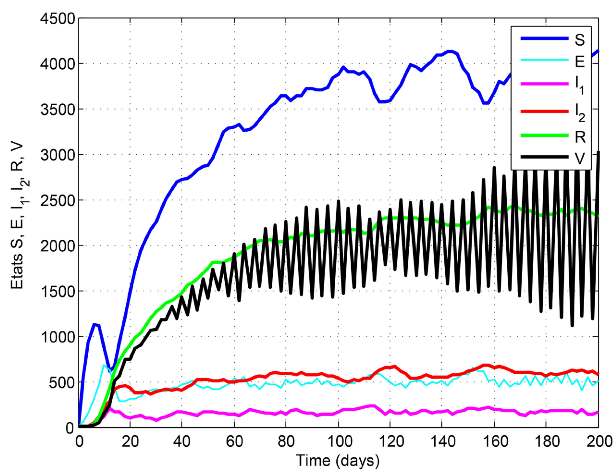
With 0%, 25%, 50%, 95% and 100% implementation of the controls constant $u_1 = u_2 = u_3 = u_4$ the simulation of stochastic model 1 is performed in Figure 3 revealing that human infection force decreases when values of four controls are constantly increasing. Consequently E, I_1, I_2, R, and V have the tendency to disappear (become hopeless) when theses controls constantly offer toward 100%. Therefore an effective implementation of all controls at 100% by the government, COVID-19 can be eradicated.

Scenario No. 2: ***Effective implementation of controls*** $(u_i)_{i=1}^4$***, but with infective immigrants.***

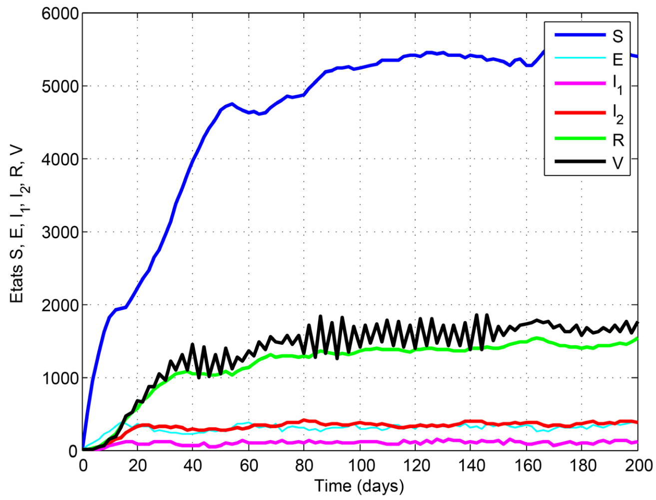
With infective immigrants, *i.e.* that the proportions of exposed immigrants and mildly symptomatic are not controlled are such that $f_1 \neq 0$ and $f_2 \neq 0$. Figure 4 compares the simulation at 100% implementation of controls with and without infective immigrants: $f_1 \neq 0$ and $f_2 \neq 0$ on the hand and $f_1 = 0$ and $f_2 = 0$ on the other hand, then the simulations of states S, E, I_1, I_2, R and V are performed in Figure 4. It is shown that disease persists in the population during the period of simulation when infective immigrants are allowed into the country, even when $\mathcal{R}_0 < 1$ and $u_1 = u_2 = u_3 = u_4 = 100\%$ implementation of controls. This persistence is reduced distinctly to the neighborhood of zero, even stable when the infective immigrants are not allowed in the country.



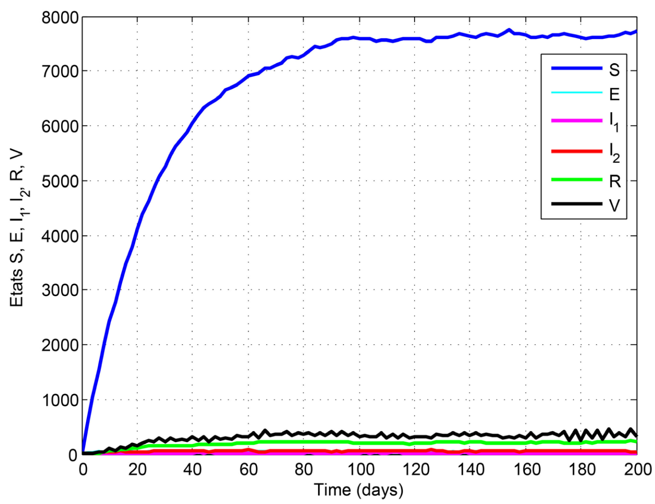
(a) 0% controls



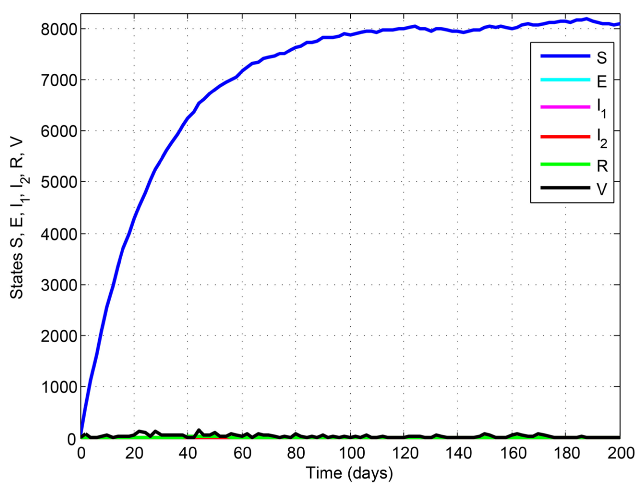
(b) 25% controls



(c) 50% controls

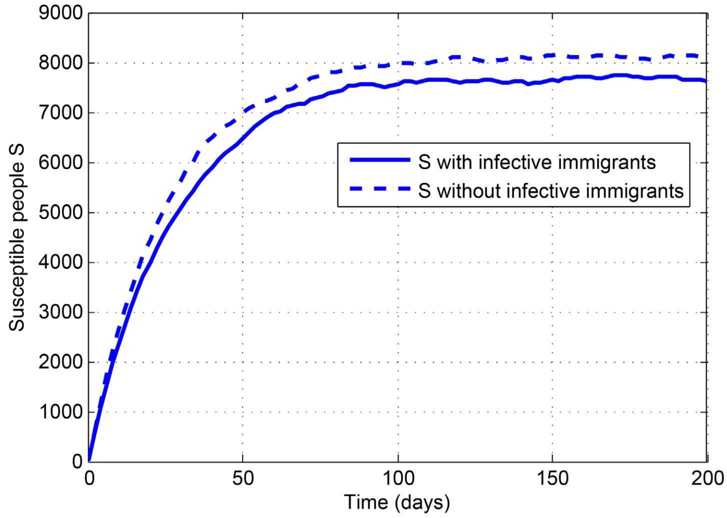


(d) 95% controls

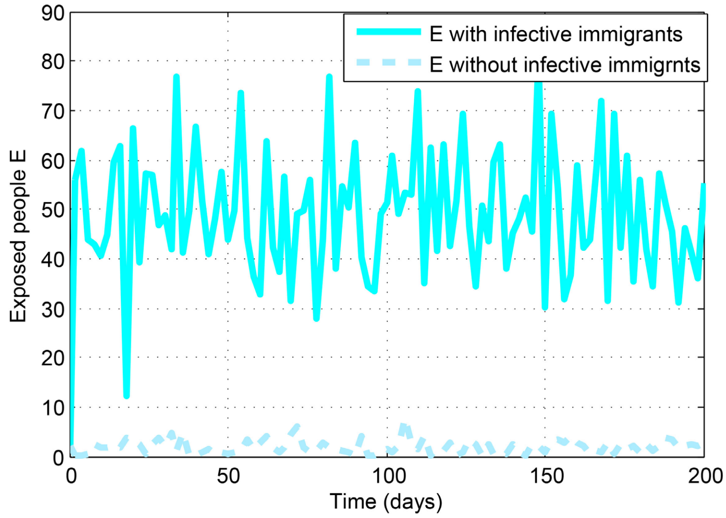


(e) 100% controls

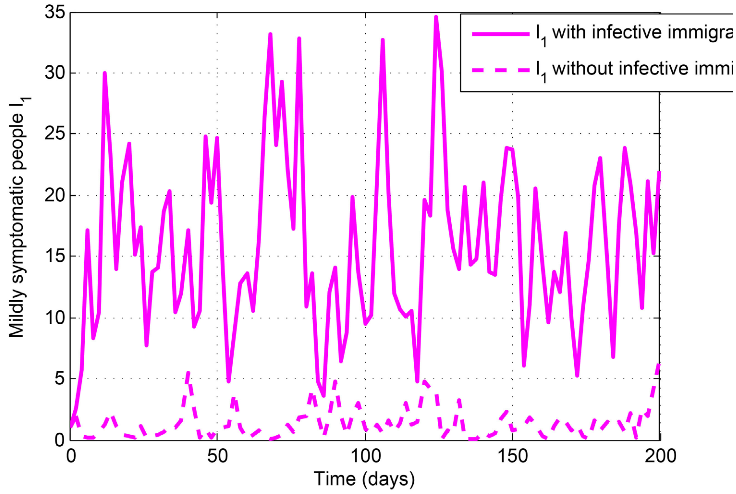
Figure 3. Numerical simulation of model 1 for $u_1 = u_2 = u_3 = u_4 = 0\%, 25\%, 50\%, 95\%$ and 100%.



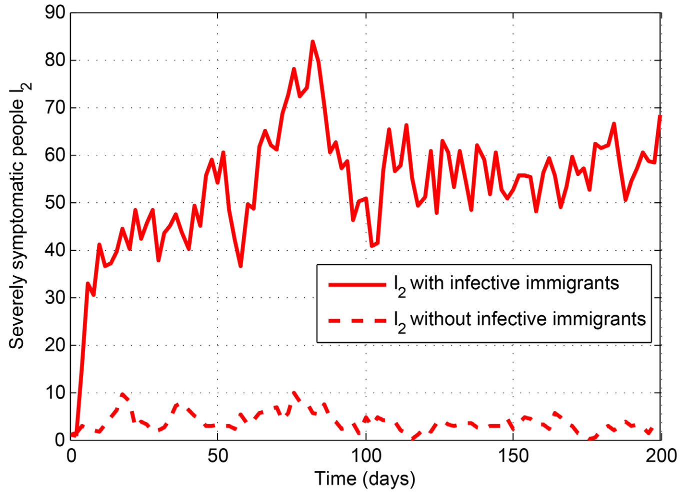
(a) $u_1 = u_2 = 100\%$ controls



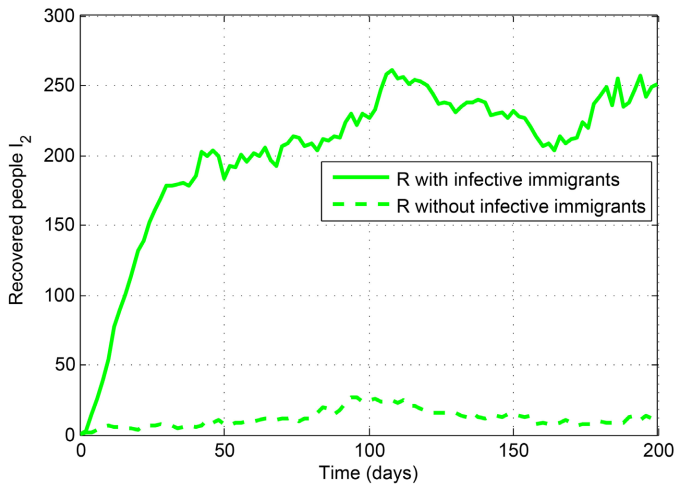
(b) $u_1 = u_2 = 100\%$ controls



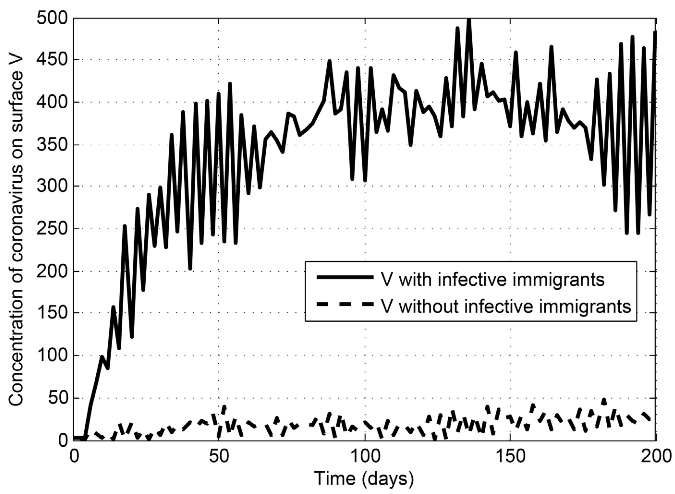
(c) 100% controls



(d) 100% controls



(e) 100% controls



(f) $u_3 = u_4 = 100\%$ controls

Figure 4. Numerical simulation of model 1 with and without infective immigrants for 100% controls.

Scenario No. 3: ***No-control with immigration temporary or extended border restriction.***

This scenario is illustrated by two scripts as follows:

• 0% implementation of all control with immigration permanent border restriction during the period of simulation for all $t \in [0;t_f]$;

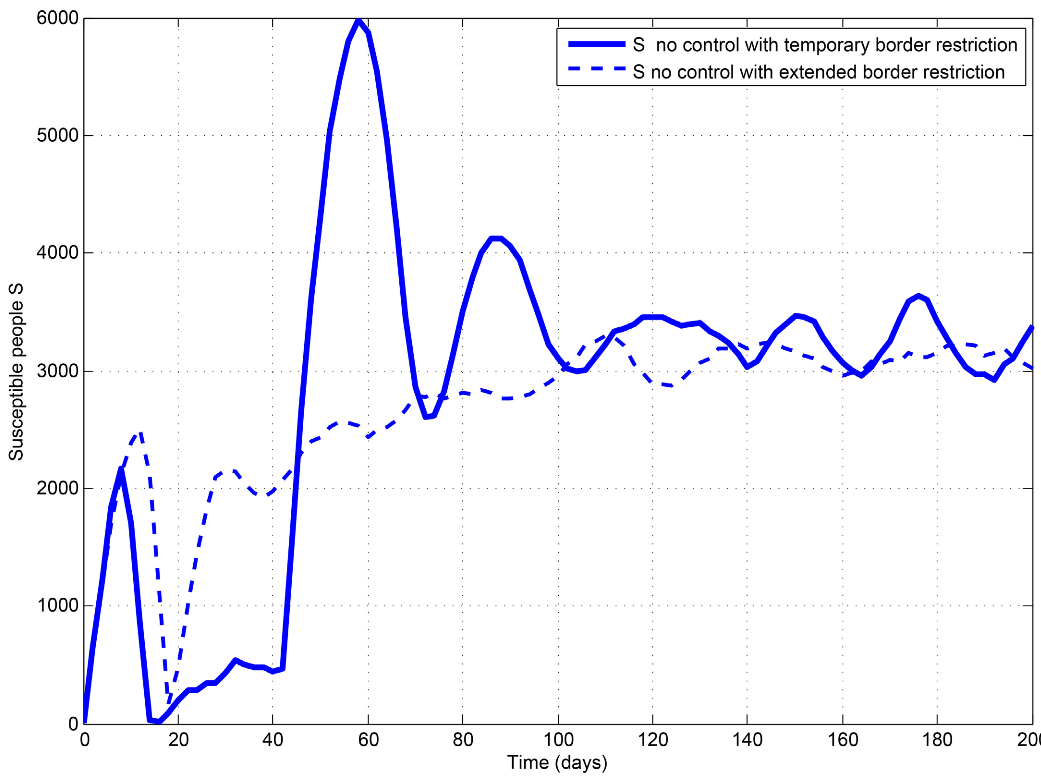
• Before a short stopping time $t_s \in [0;t_f]$, immigration temporary border restiction was allowed. *i.e.* that a proportion of exposed immigrants f_1 and another proportion of those mildly symptomatic f_2 verifying

$$\begin{cases} f_1 = f_0^1 \cdot \mathbf{I}_{[0;t_s]} \\ f_2 = f_0^2 \cdot \mathbf{I}_{[0;t_s]} \end{cases} \tag{27}$$

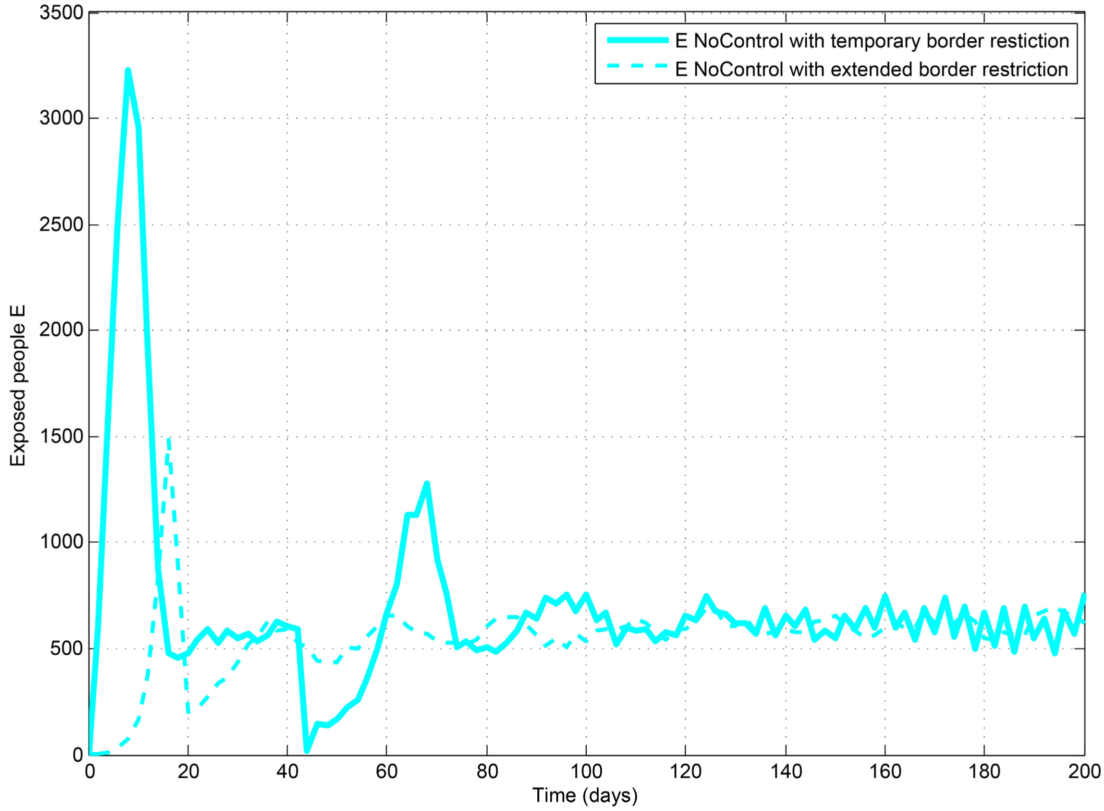
where $f_0^1 > 0$ and $f_0^2 > 0$ are non negatives values; $\mathbf{I}_{[0;t_s]}$ is the characteristic function in $[0;t_s]$.

With $f_0^1 = f_1 = 0.01$ (resp $f_1 = 0.5$) and $f_0^2 = f_2 = 0.01$ (resp $f_1 = 0.05$) the simulations of states S, E, I_1, I_2, R and V are performed in Figure 5 to compare these two scripts avec $\mathcal{R}_0 > 1$. It is observed from Figure 5 that after a short stopping time t_s, each state of model 1 oscillate around the uncertain equilibria that is locally steady asymptotically stable.

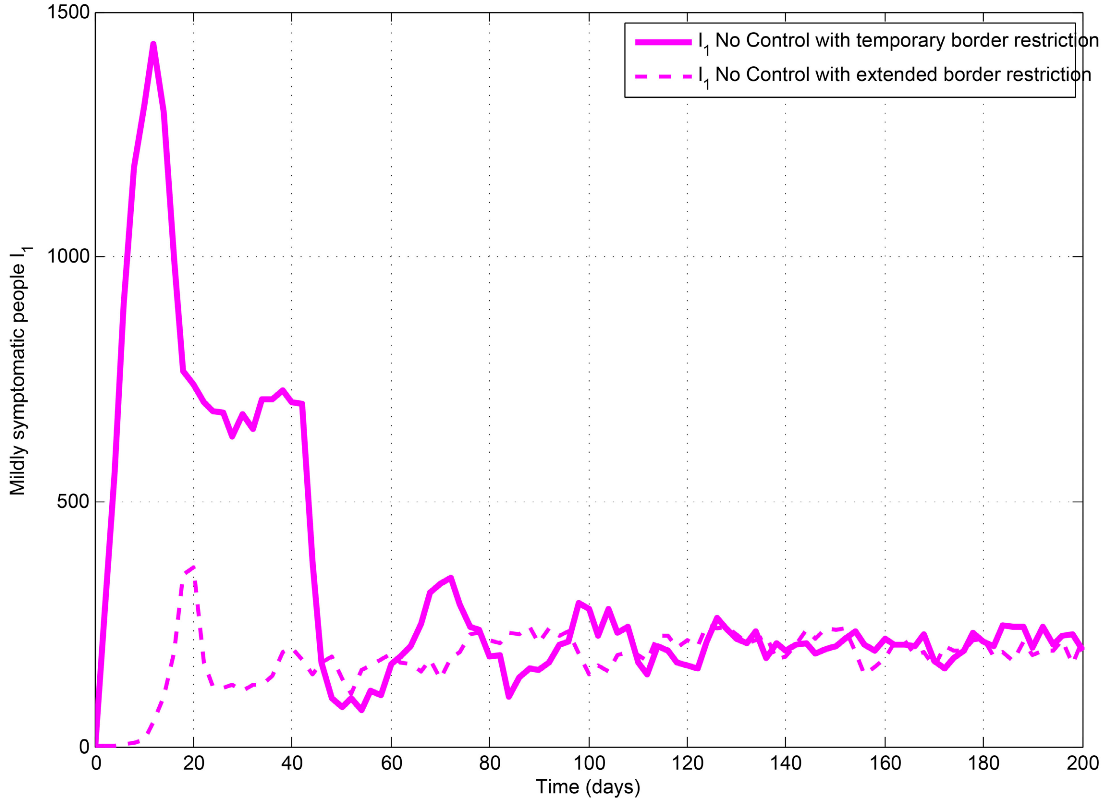
When one changes 0% in the first script appropriate by 100% implementation of controls, then the simulations of states S, E, I_1, I_2, R and V are performed in Figure 6 showing that after a short stopping time t_s, COVID-19 is eradicated with $\mathcal{R}_0 = 0 < 1$.



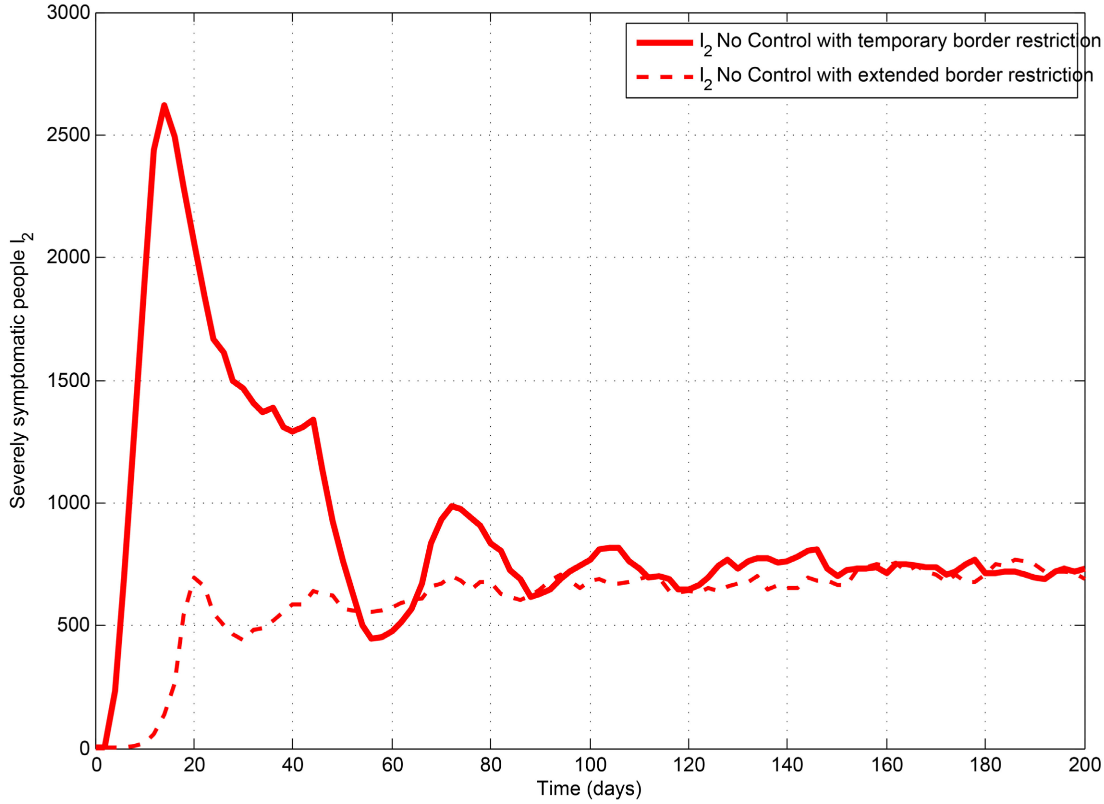
(a) 50% controls



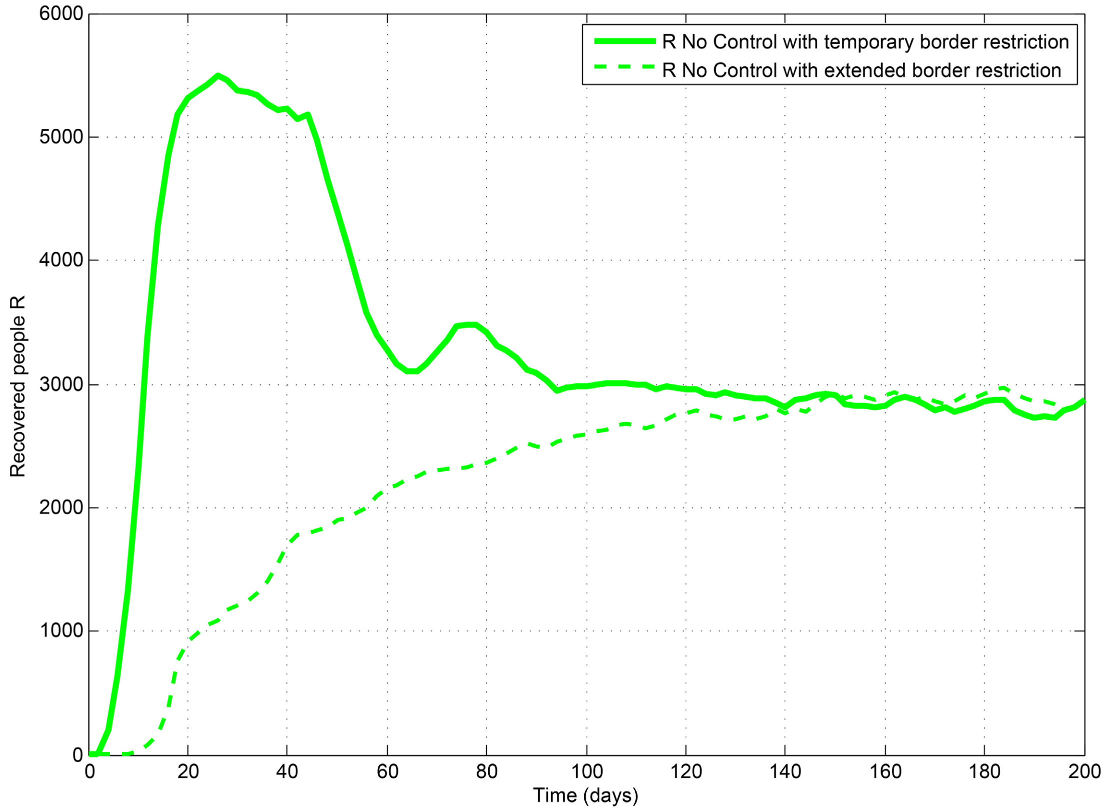
(b) 50% controls



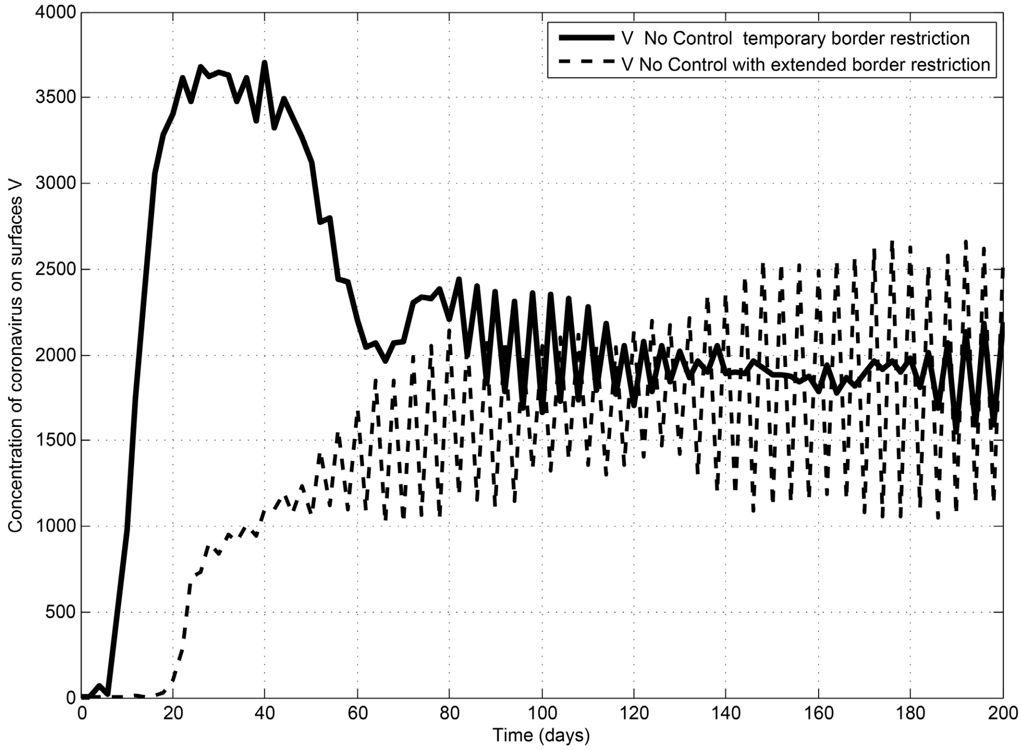
(c) 50% controls



(d) 50% controls

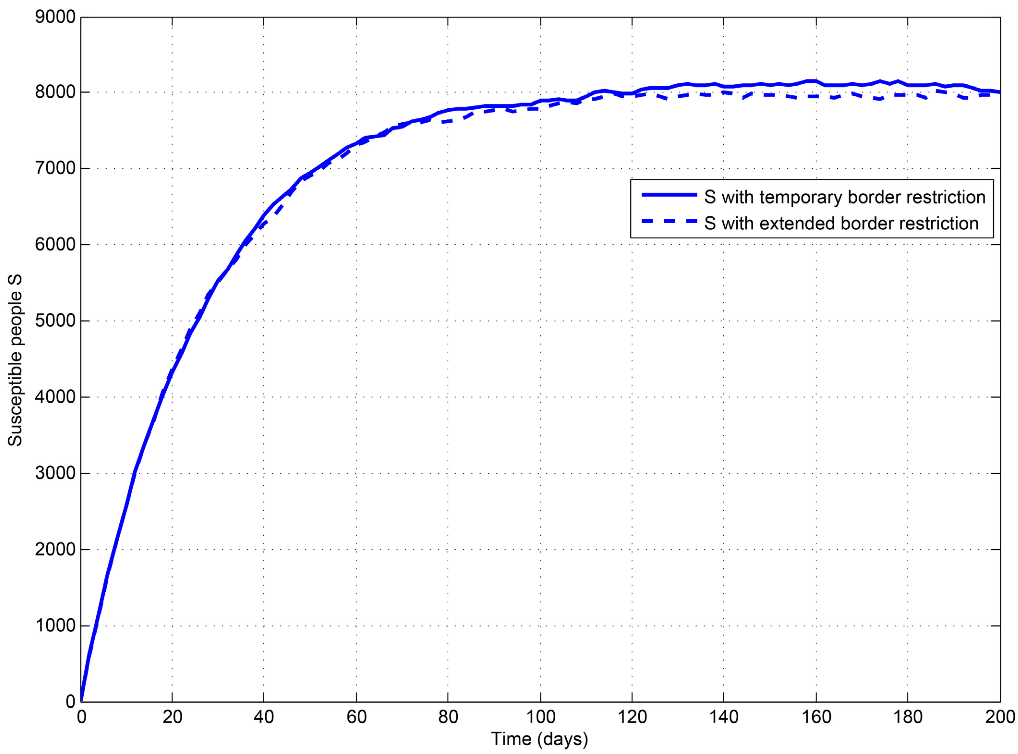


(e) 50% controls

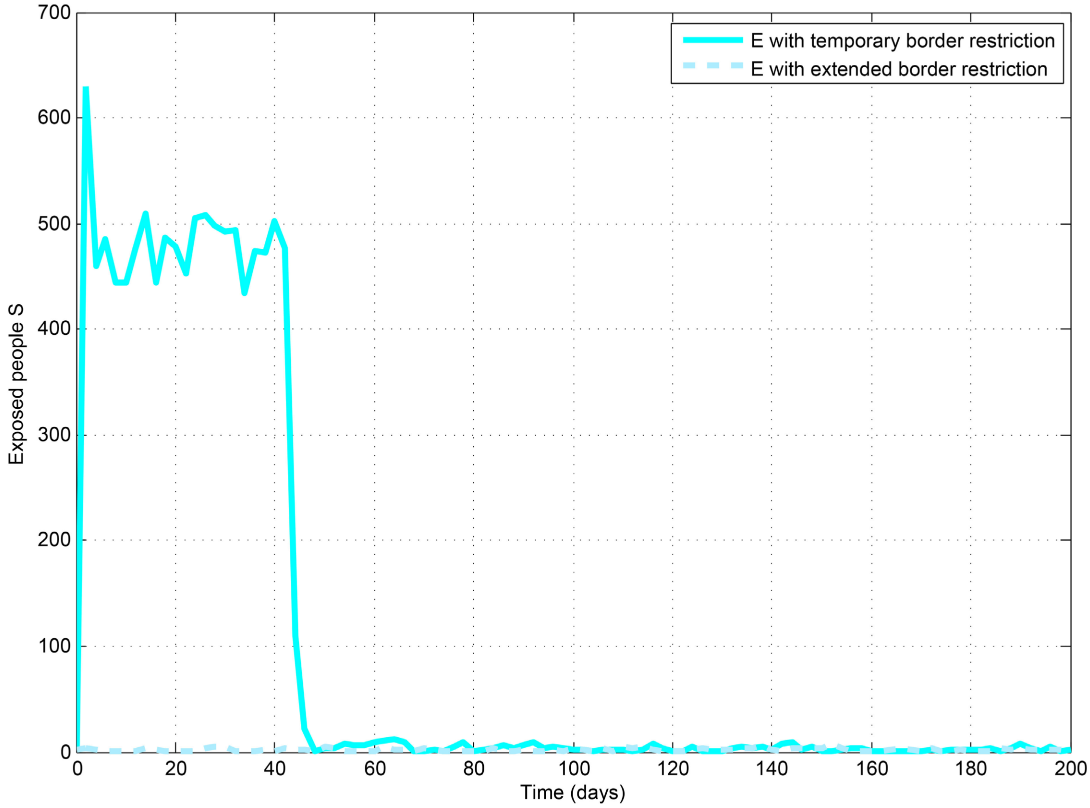


(f) 50% controls

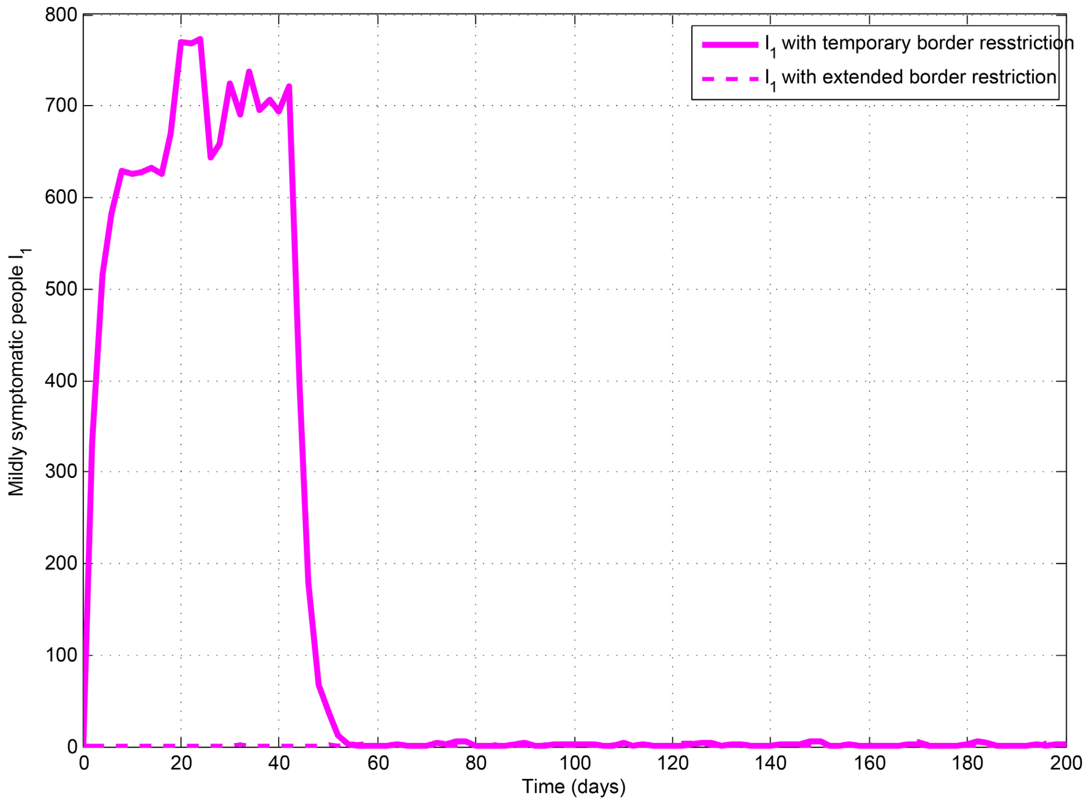
Figure 5. States S, E, I_1, I_2, R and V simulated at $u_1 = u_2 = u_3 = u_4 = 50\%$. implementation of controls with extended and temporary border closure for viral shedding rate of infected persons $\xi = 100$ cells/day.



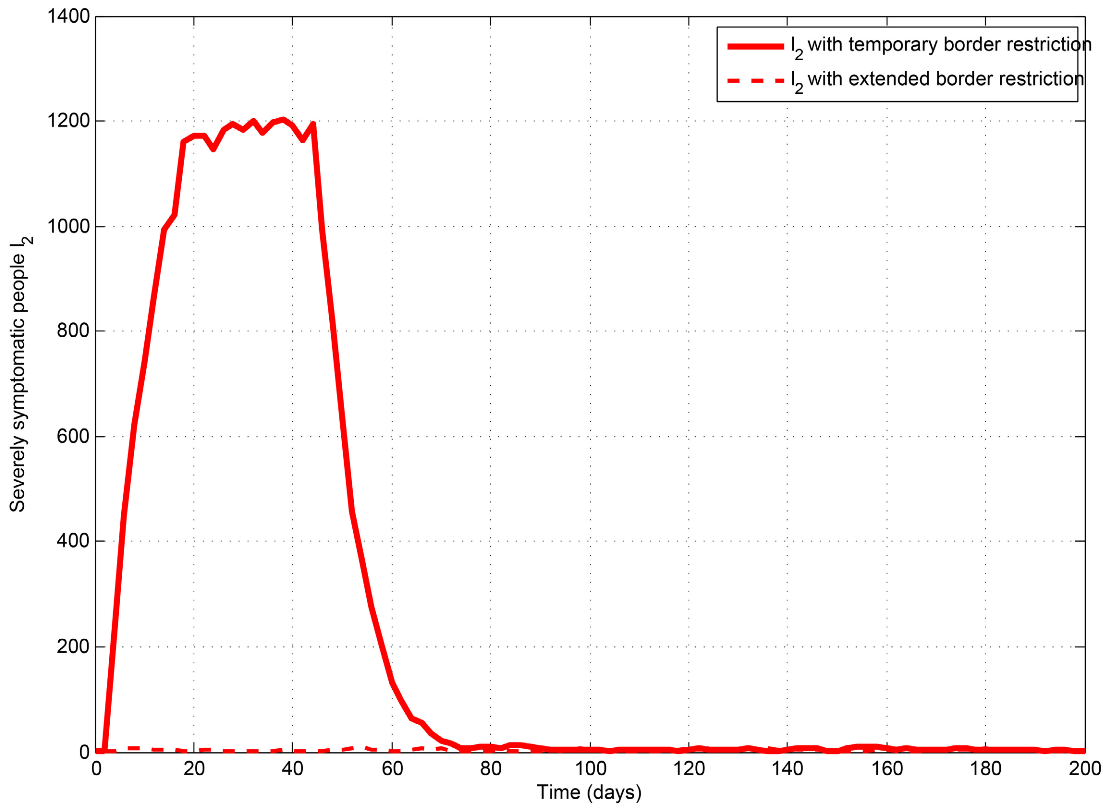
(a) 100% controls



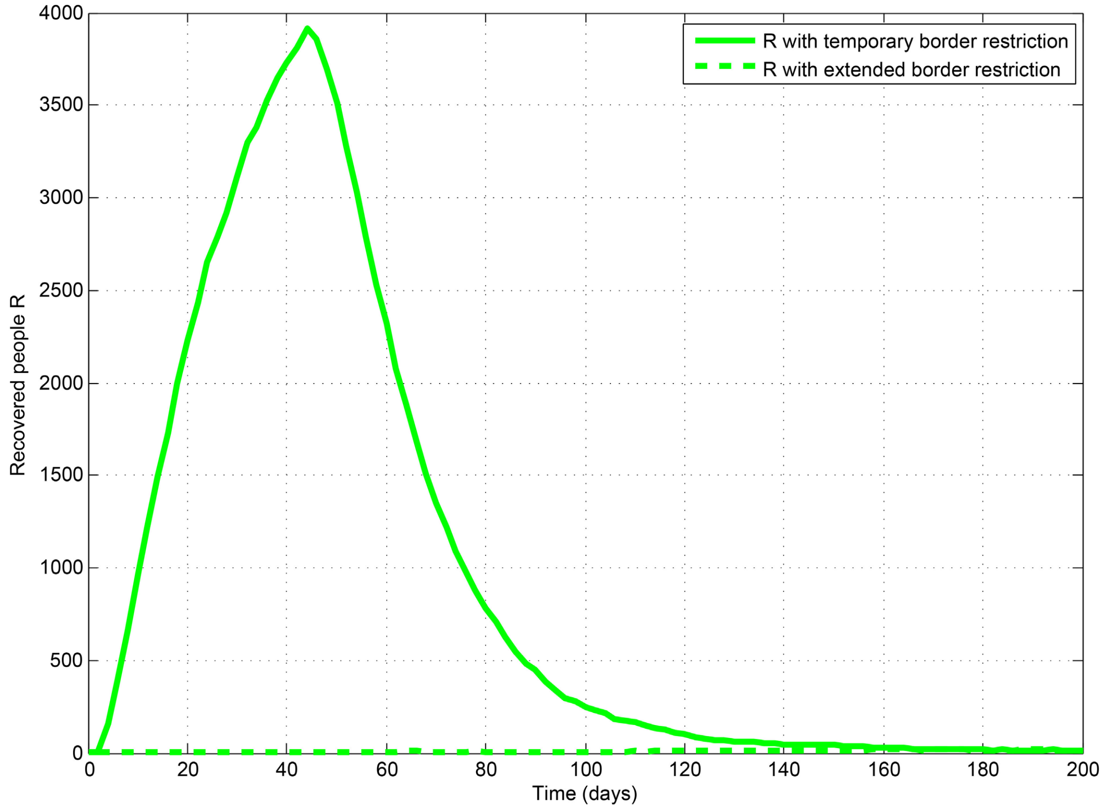
(b) 100% controls



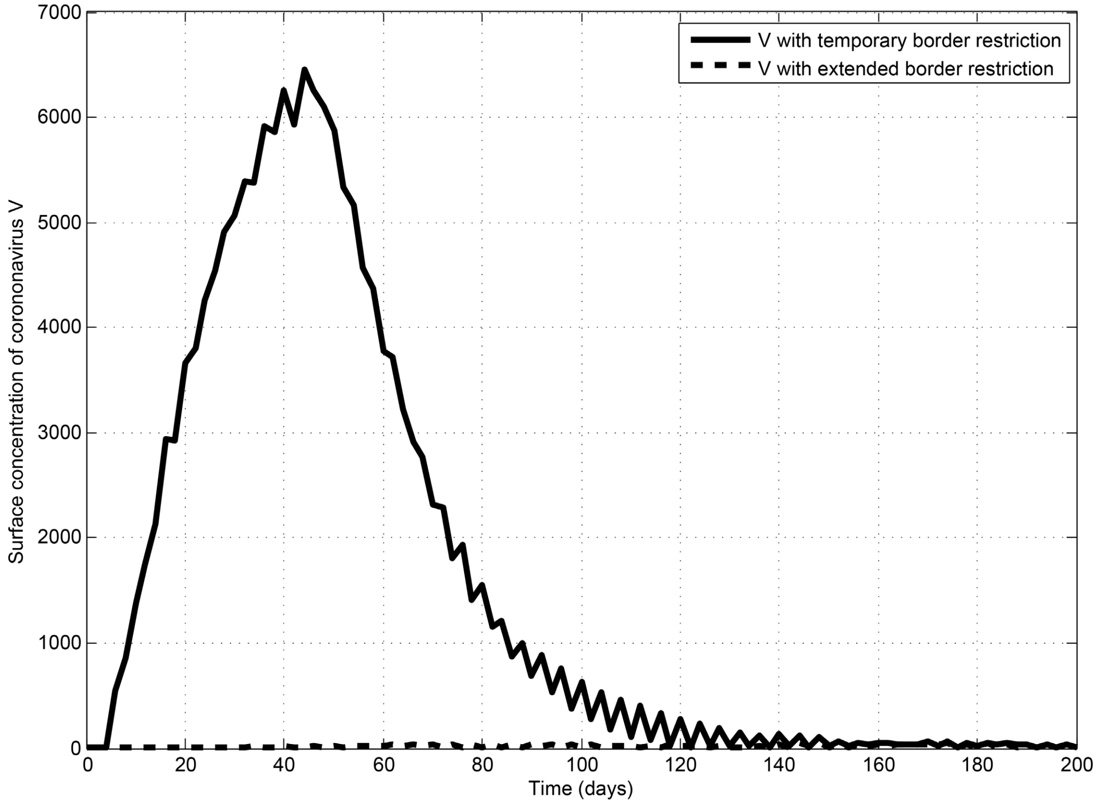
(c) 100% controls



(d) 100% controls



(e) 100% controls



(f) 100% controls

Figure 6. States S, E, I_1, I_2, R and V simulated at $u_1 = u_2 = u_3 = u_4 = 100\%$ implementation of controls with extended and temporary border closure for viral shedding rate of infected persons $\xi = 100$ cells/day.

Scenario No. 4: 100% ***-control with immigration temporary or extended border restriction.***

- 100% implementation of all control with immigration permanent border restriction during the period of simulation for all $t \in \left[0; t_f\right]$;
- After a short stopping time $t_s \in \left[0; t_f\right]$, unrestricted immigration was allowed.

After a short stopping time t_s, immigration no-restraining was allowed. *i.e.* that a proportion of exposed immigrants f_1 and another proportion of those mildly symptomatic f_2 verifying

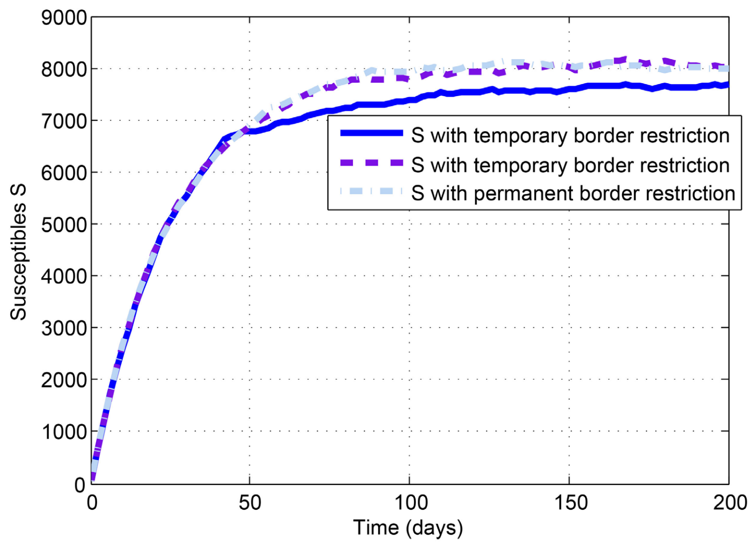
$$\begin{cases} f_1 = f_0^1 \cdot \mathbf{I}_{\left]t_s; t_f\right]} \\ f_2 = f_0^2 \cdot \mathbf{I}_{\left]t_s; t_f\right]} \end{cases} \tag{28}$$

where $f_0^1 > 0$ and $f_0^2 > 0$ are non negatives values; $\mathbf{I}_{\left]t_s; t_f\right]}$ is the characteristic function in interval $\left]t_s; t_f\right]$.

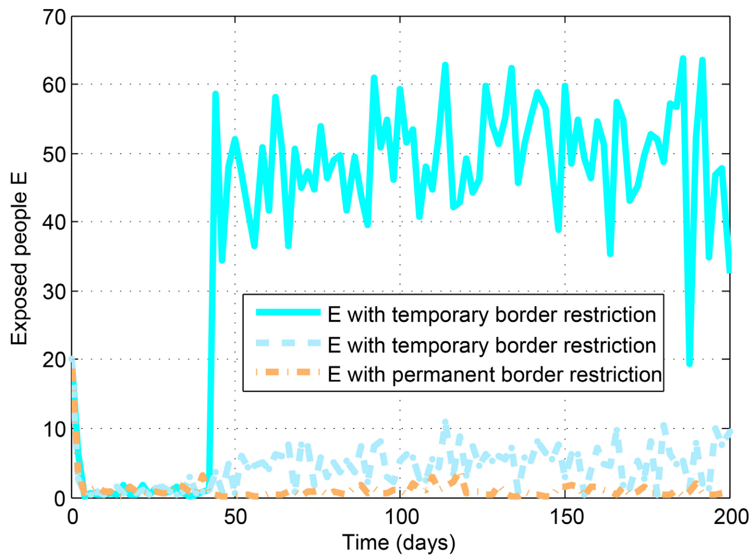
With 100% implementation of controls and with the decreasing values of the proportions $f_i, i = 1; 2$ ($f_1 = f_0^1 = 0.100; 0.010; 0.000$ and $f_2 = f_0^2 = 0.010; 0.001; 0.000$ per e.g.), we obtain **Figure 7**. It shows that when f_1 and f_2 converge toward zero, then the different states of the model 1 oscillate around random disease-free equilibria. Therefore with 100% implementation of controls plus a strict restriction for exposed immigrants and those mildly symptomatic infected, then COVID-19 can be eradicate (e.g. see Figure 7 in a period $[0;t_s]$).

Scenario No. 5: ***Simulation of controlled model with combination of the four controls***

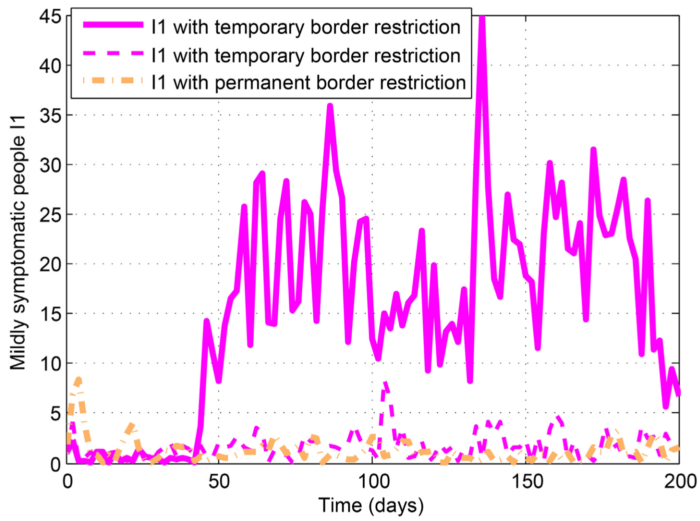
Simulations of S, E, I_1, I_2, R and V are performed with controls combination u_1, u_2, u_3, and u_4 at two values $u_i \in \{0;1\}$ in Table 4 and Table 5 for each i in Figure 8. It is observed that with these Strategies of control combination in $u\{0;1\}$ are unprofitable in certain cases because of the environment that remained contaminated strongly by coronavirus with a growth important of the viral concentration V (see the Strategies 3, 4, 7, 8, 11, 12 and 15 in Table 5 represented by Figure 8).



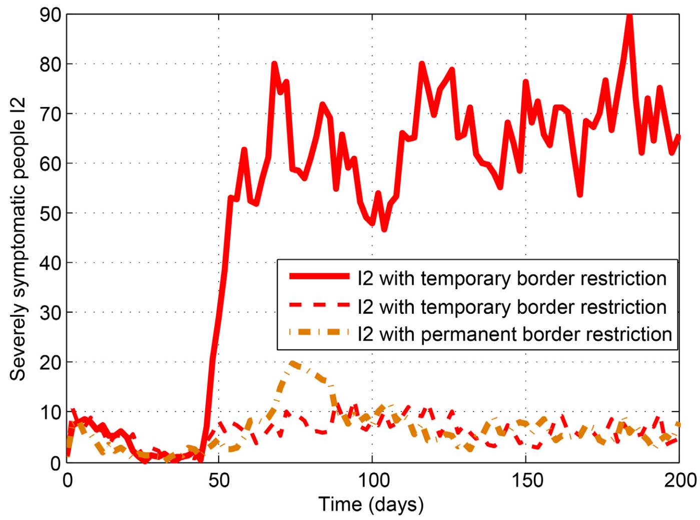
(a) 100% controls with temporary and permanent immigration



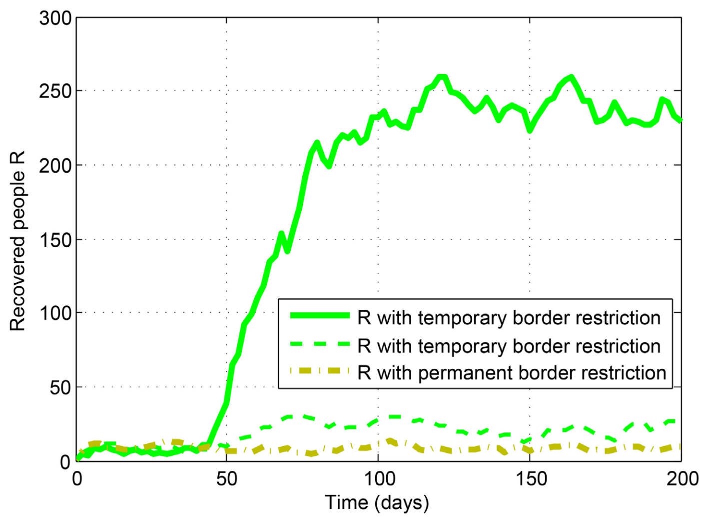
(b) 100% controls controls with temporary and permanent immigration



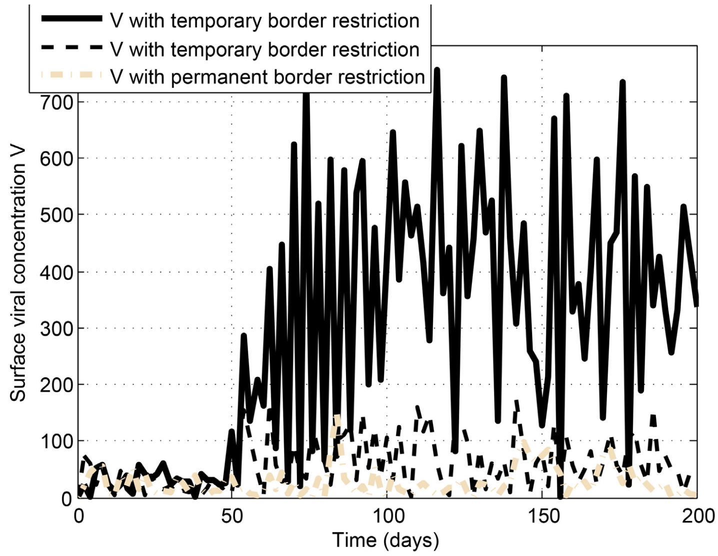
(c) 100% controls with temporary and permanent immigration



(d) 100% controls controls with temporary and permanent immigration



(e) 100% controls with temporary and permanent immigration



(f) 100% controls controls with temporary and permanent immigration

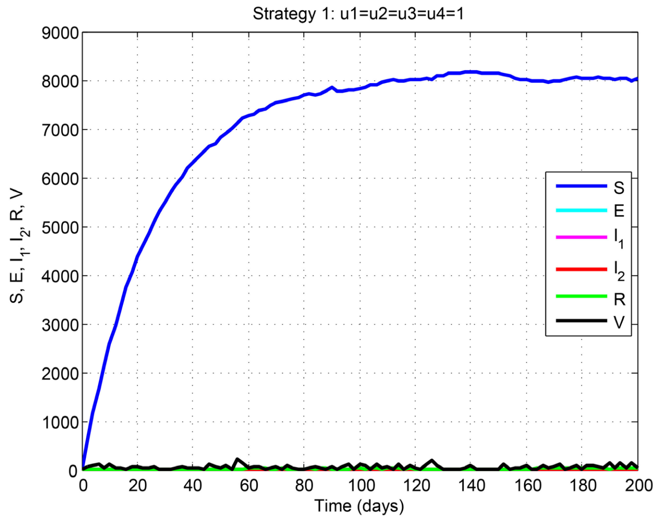
Figure 7. States S, E, I_1, I_2, R and V are simulated at $u_1 = u_2 = u_3 = u_4 = 100\%$ implementation of controls with permanent border restriction on $\left[0; t_f\right]$ / solid curve (–·–·): $f_1 = f_2 = 0$ and temporary border restriction before a time $t_s = 21$ days/dash-dash curve (–––): $f_1 = 0.01$; $f_2 = 0.01$ and dash-dot (~): $f_1 = 0.10$; $f_2 = 0.01$.

Table 4. Numerical results (**Figure 8**) of optimal strategies: $\forall u_i \in \{0;1\}$, $u_3 = 1$.

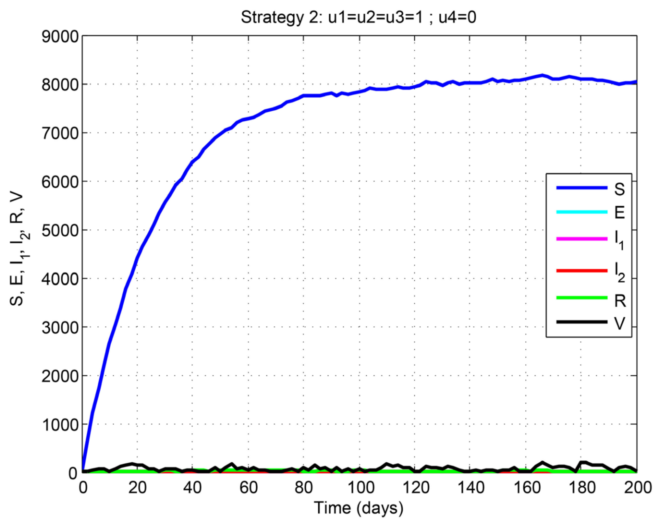
Strategy	Control combination	$\mathbb{E}[E]$	$\mathbb{E}[I_1]$	$\mathbb{E}[I_2]$	$\mathbb{E}[V]$	$J^1(u)$
1	$(1\ \ 1\ \ 1\ \ 1)$	3.0038	1.8453	3.0604	46.3145	103,100,000
2	$(1\ \ 1\ \ 1\ \ 0)$	2.8367	1.9724	2.5193	50.8881	13,100,000
5	$(1\ \ 0\ \ 1\ \ 1)$	3.4390	2.3352	3.1835	72.6912	102,800,000
6	$(1\ \ 0\ \ 1\ \ 0)$	3.3159	1.9459	3.4645	53.4757	12,800,000
9	$(0\ \ 1\ \ 1\ \ 1)$	3.3188	2.2539	2.7147	51.9319	93,100,000
10	$(0\ \ 1\ \ 1\ \ 0)$	2.8744	2.2348	3.4293	64.3991	3,100,000
13	$(0\ \ 0\ \ 1\ \ 1)$	3.7935	3.1679	3.7962	60.9383	92,800,000
14	$(0\ \ 0\ \ 1\ \ 0)$	3.4447	2.7469	3.6237	57.0152	2,800,000

Table 5. Numerical results (**Figure 8**) of unprofitable strategies: $\forall u_i \in \{0;1\}$, $u_3 = 0$.

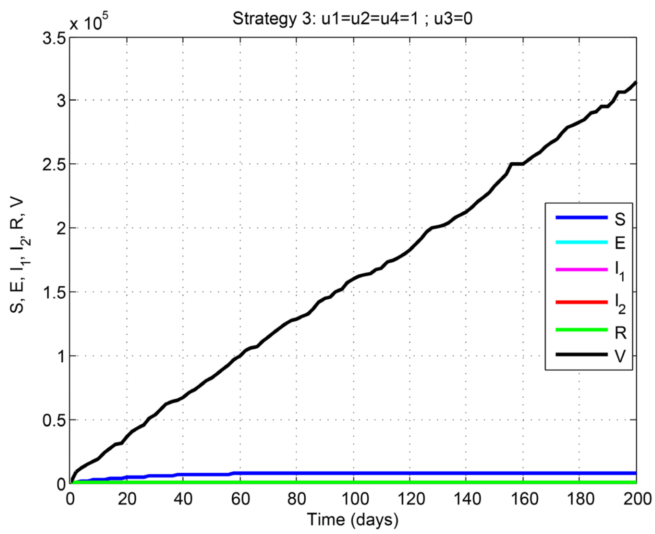
Strategy	Control combination	$\mathbb{E}[E]$	$\mathbb{E}[I_1]$	$\mathbb{E}[I_2]$	$\mathbb{E}[V]$	$J^1(u)$
3	$(1\ \ 1\ \ 0\ \ 1)$	3.7212	1.9645	3.0746	89,416	100,300,000
4	$(1\ \ 1\ \ 0\ \ 0)$	3.5647	2.0380	3.2427	91,561	10,300,000
7	$(1\ \ 0\ \ 0\ \ 1)$	8.6163	4.7340	9.5011	427,330	10,040,000
8	$(1\ \ 0\ \ 0\ \ 0)$	7.7826	5.2573	9.7446	431,480	10,043,000
11	$(0\ \ 1\ \ 0\ \ 1)$	3.9965	2.0731	2.4844	98,057	90,300,000
12	$(0\ \ 1\ \ 0\ \ 0)$	3.8869	2.3990	2.9101	90,493	300,000
15	$(0\ \ 0\ \ 0\ \ 1)$	8.2471	6.0573	9.7234	4.25710	90,043,000



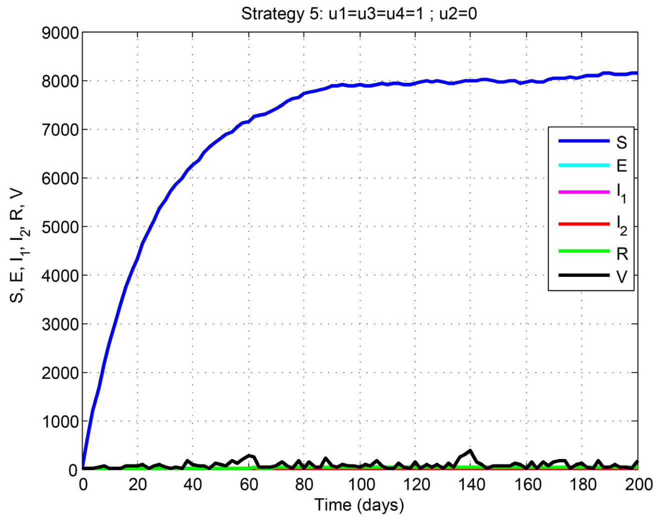
(a) Strategy 1



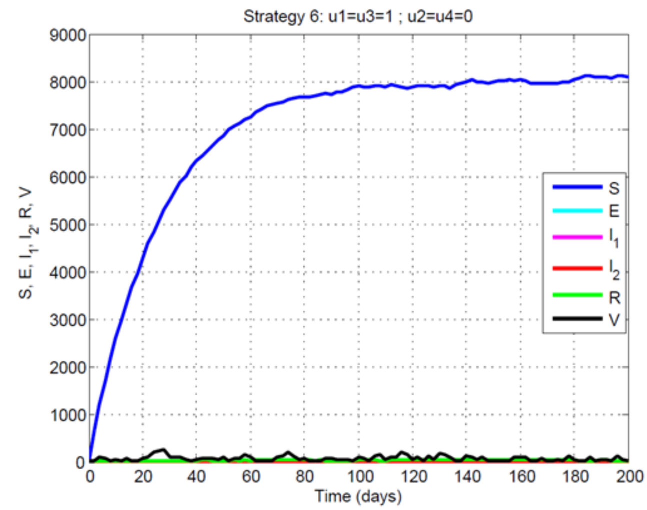
(b) Strategy 2



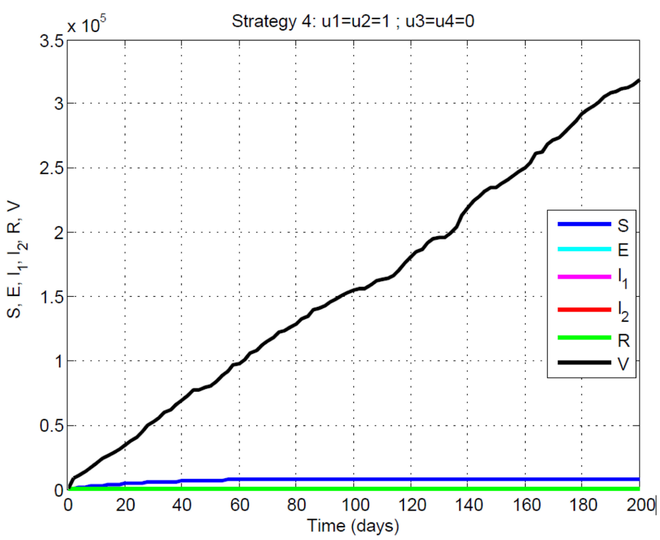
(c) Strategy 3



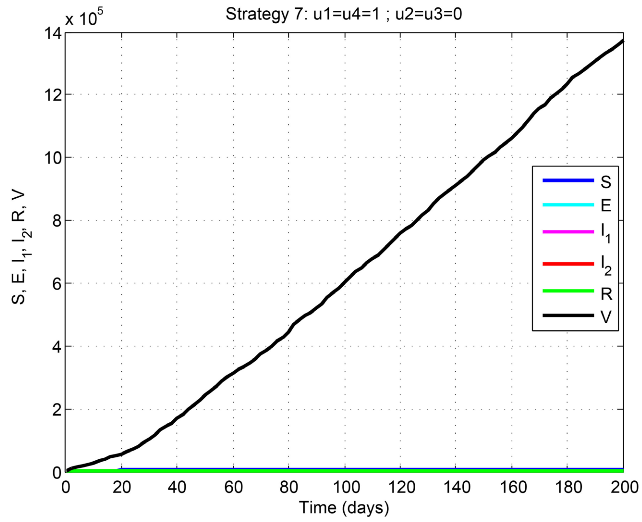
(d) Strategy 5



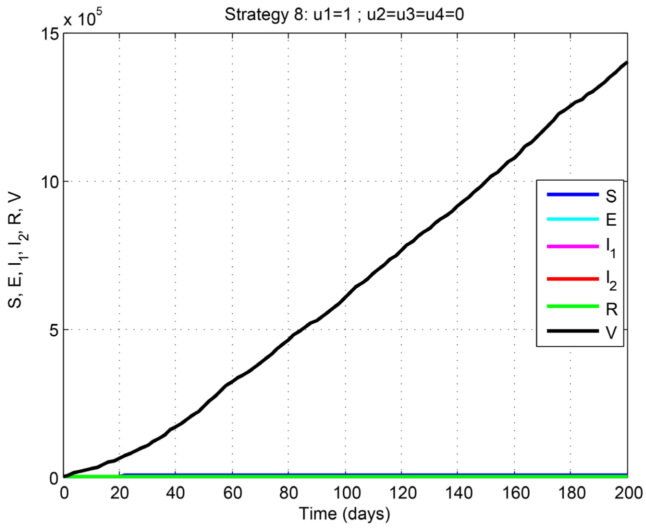
(e) Strategy 6



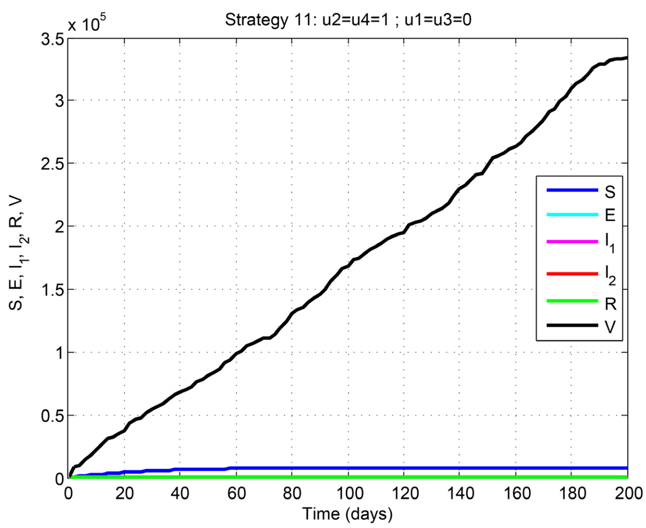
(f) Strategy 4



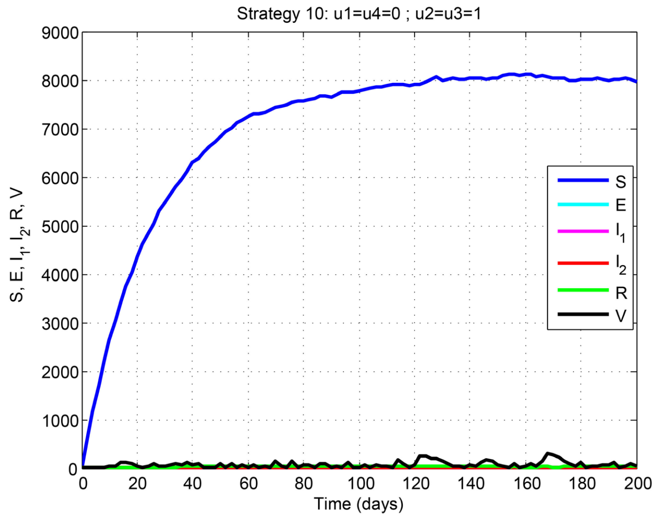
(g) Strategy 7



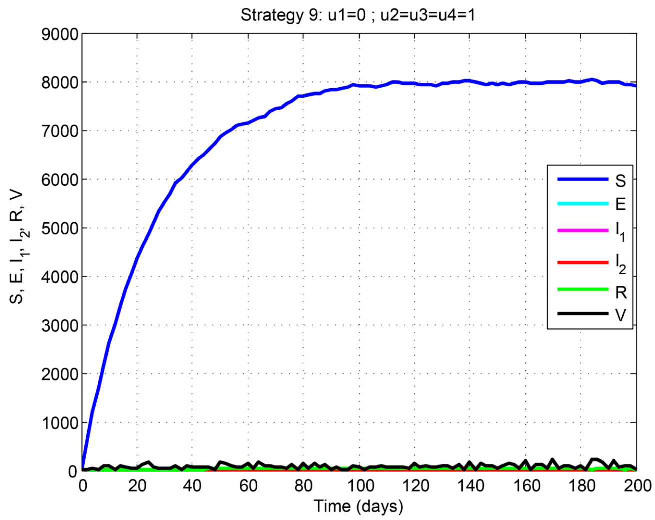
(h) Strategy 8



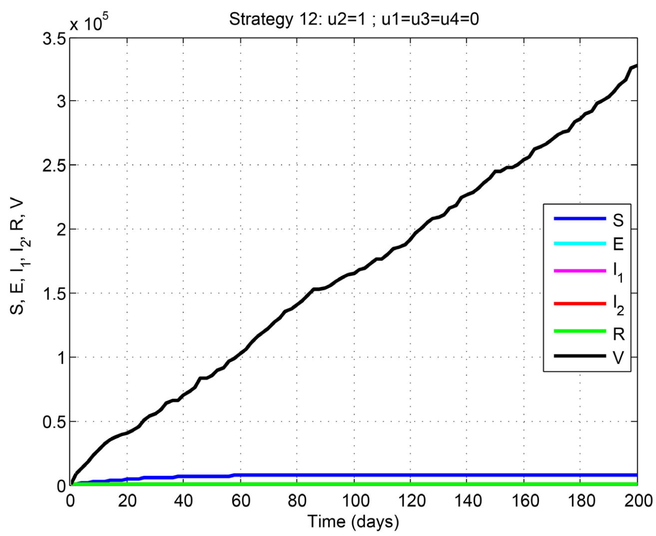
(i) Strategy 11



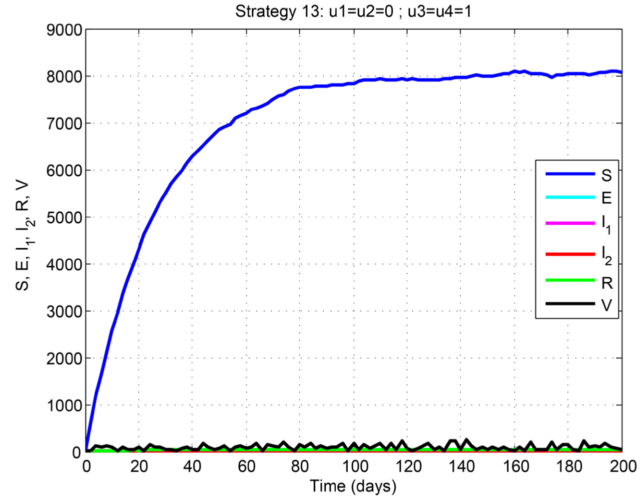
(j) Strategy 10



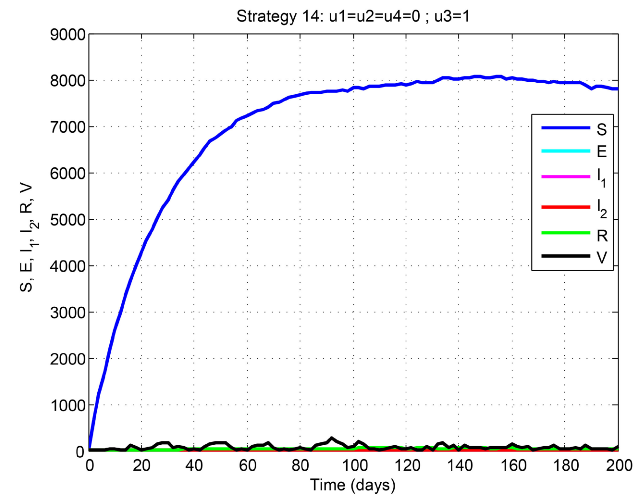
(k) Strategy 9



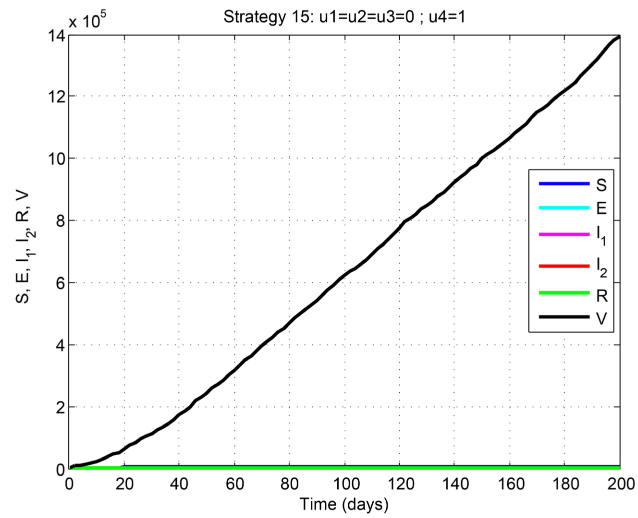
(l) Strategy 12



(m) Strategy 13



(n) Strategy 14

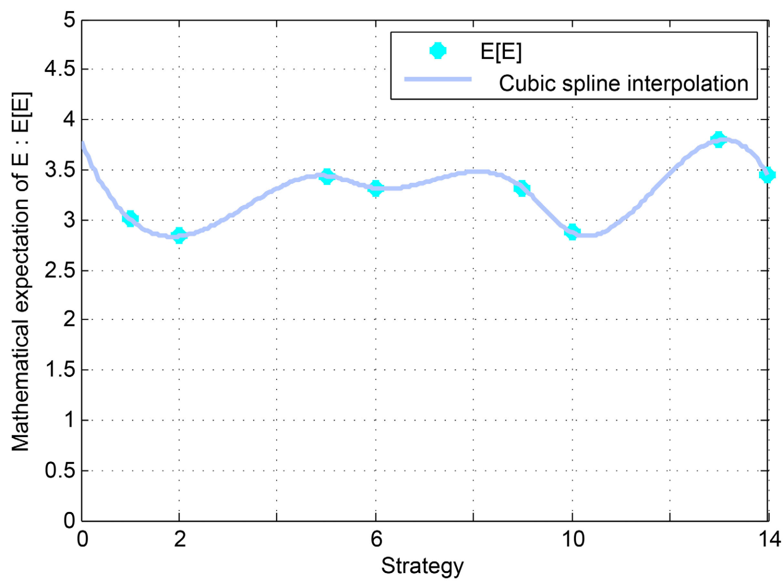


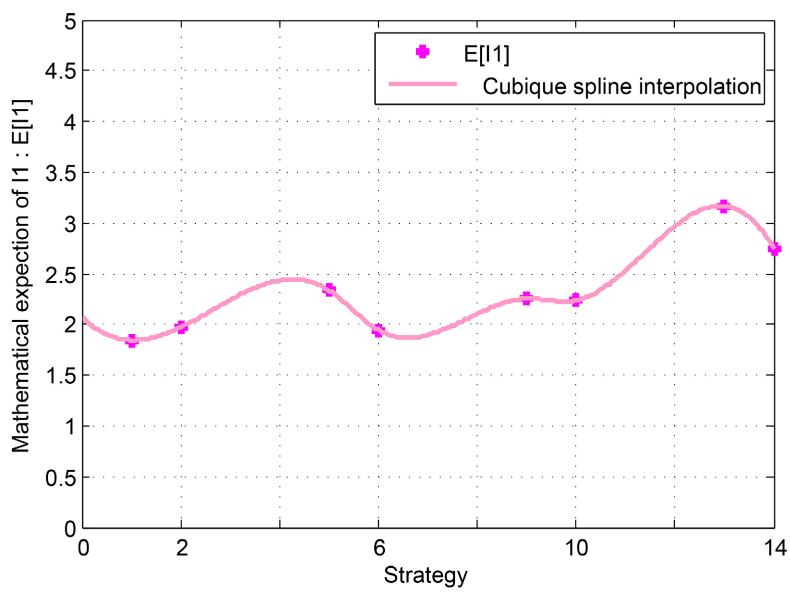
(o) Strategy 15

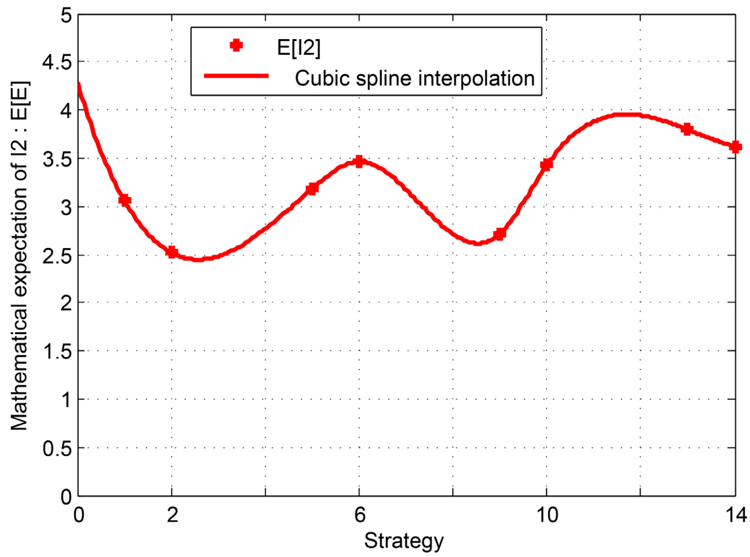
Figure 8. States S, E, I_1, I_2, R and V are simulated with controls combination u_1, u_2, u_3, and u_4 at two values $u_i \in \{0;1\}$ in Table 4 and Table 5 for each i.

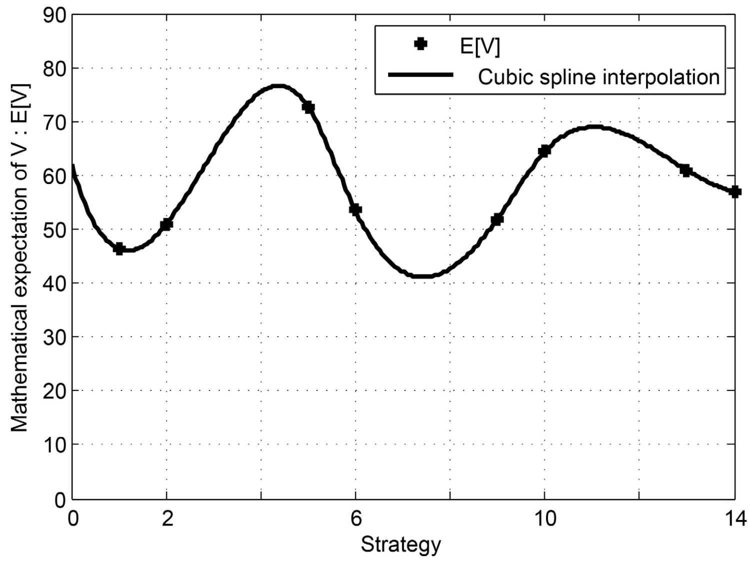
As for the eight strategies summarized in Table 4, it is about the Optimal Strategies 1, 2, 5, 6, 9, 10, 13 and 14 of which simulations in Figure 8 are profitable while minimizing the cost bound to the human infection force and the viral spread force at a time. Among these profitable strategies for COVID-19 control with controls combination taking values extreme 0 or 1, only one strategy is optimal for the minimal mathematical expectation of state $\mathbb{E}[E]$, $\mathbb{E}[I_1]$, $\mathbb{E}[I_2]$, and $\mathbb{E}[V]$ (see Figure 9. This optimal strategy is represented at 14th strategy, with the optimal cost $J^*(u) = 48400000$, consistent of 10th then 6th strategy).

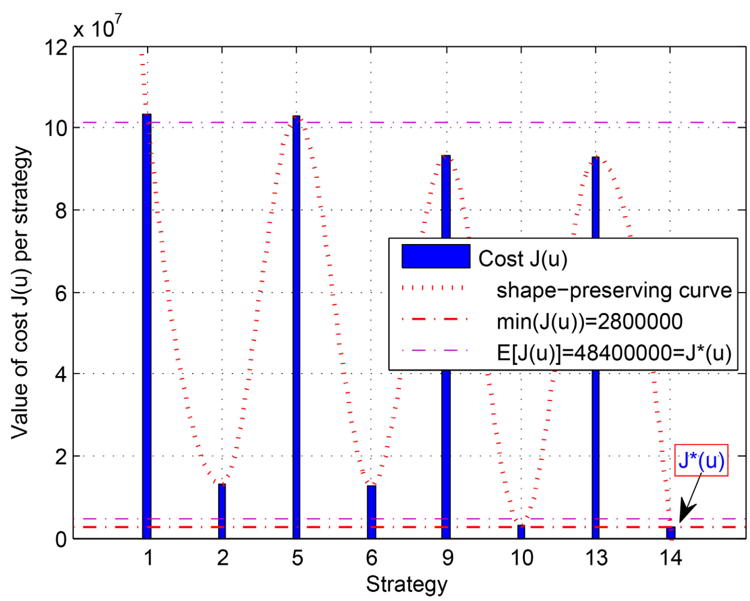
The last point of this scenario that remains to be explored is the case of the control combination with controls taking intermediate values of the optimal control $u^* = (u_i^*) \in]0;1[$ linked to the values $\bar{\bar{u}}_i$ given by Equation (26) such that $0 < \bar{\bar{u}}_i < 1$, *i.e.* $\min\{1; \bar{\bar{u}}_i\} = \bar{\bar{u}}_i = u^* = u_i \in]0,1[$.






E[I2]
Cubic spline interpolation
Mathematical expectation of I2 : E[E]
Strategy


E[V]
Cubic spline interpolation
Mathematical expectation of V : E[V]
Strategy


x 10^7
Cost J(u)
shape−preserving curve
min(J(u))=2800000
E[J(u)]=48400000=J*(u)
J*(u)
Value of cost J(u) per strategy
Strategy

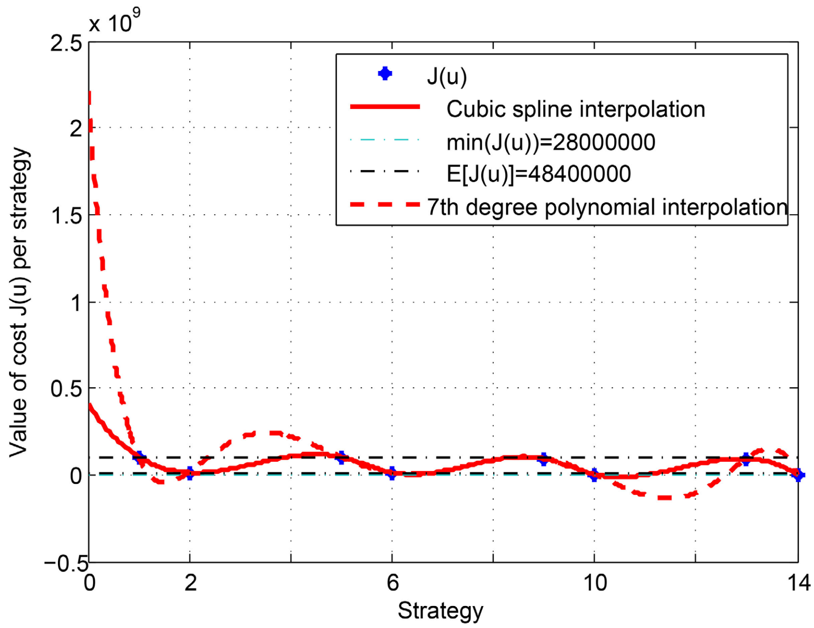
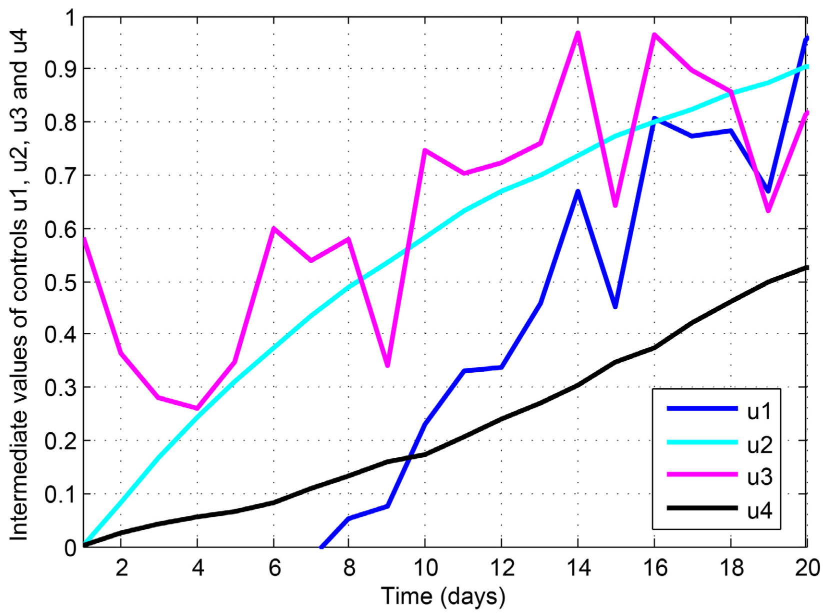


Figure 9. Simulation (Interpolation and fitting) of $\mathbb{E}[E]$, $\mathbb{E}[I_1]$, $\mathbb{E}[I_2]$, $\mathbb{E}[V]$—Histogram, Interpolation and fitting of cost per strategies in Table 4.

Considering the intermediate value multiplicity combined that can take controls $u_i = \overline{\overline{u}}_i \in]0;1[$ such that $\overline{\overline{u}}_i < 1$, a numeric script in MATLAB is written to simulate the behavior of these intermediate values so the functional associated cost. For each intermediate value taking by the controls, Figure 10 simulates these optimal controls so the cost. The optimal values are given in Figure 10 by Software MATLAB. Then, the optimal value of cost is $J^{1*}(u) = 94630000$ situated between 10th and 12th day of beginning of disease for this scenario with controls combination taking intermediate values between 0 and 1, *i.e.* $u_i \in]0;1[$.



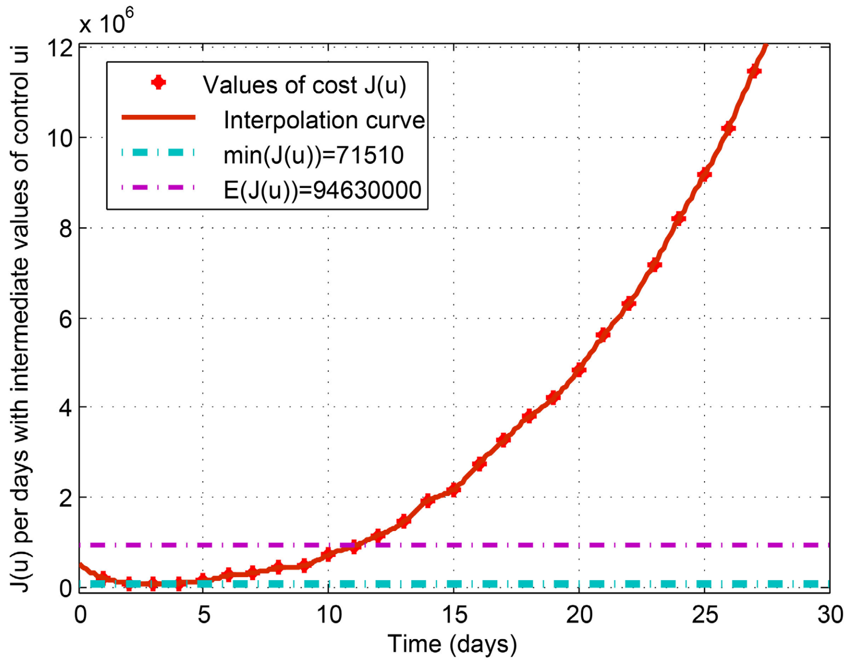
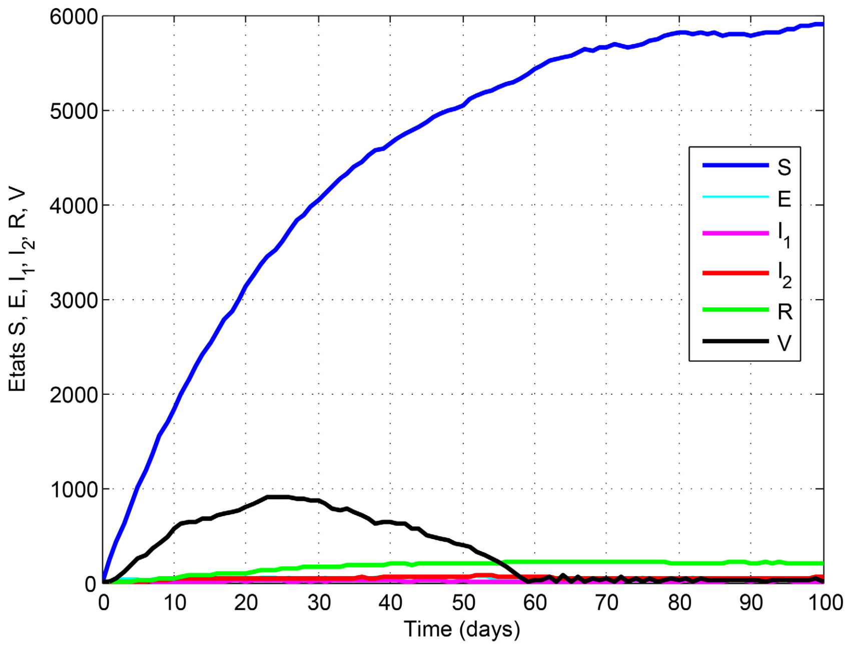


Figure 10. Simulation of cost $J(u)$ with the intermediate values of controls u_i per day.

The optimal strategy first recommended for theses scenarios consists in applying control u_3 and u_2, then to make efficient controls u_4 and u_1 with a permanent restriction of border closing preventing the entry of exposed immigrants and those mildly symptomatic. Results given in **Figure 11** are favourable to the minimization of the cost and compliant those given by **Figure 10**.

The optimal values are given in **Figure 10** by Software MATLAB e.g.
$J^*(u) = 48400000$ by 14th strategy.



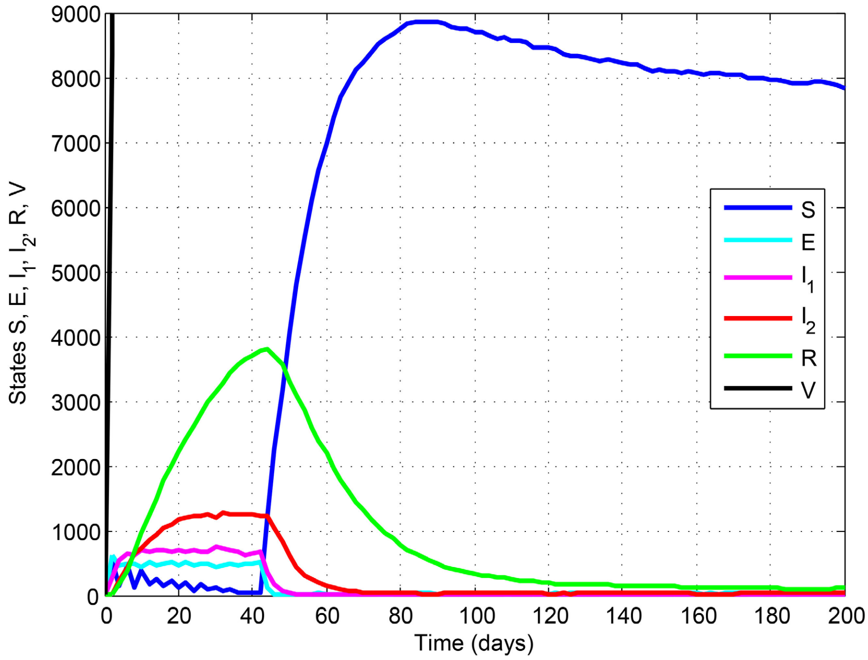


Figure 11. Simulation of S, E, I_1, I_2, R and V with 100% implementation of controls and with a permanent restriction of border closing after 21 th days.

7. Conclusions and Perspectives

A new stochastic model of COVID-19 with two terms is formulated and presented. The mathematical analysis of this model permitted to get some important results focused on the positivity and boundedness of the solutions, the global behavior of the intermediate system, under the sole condition of stability of the disease-free equilibria. These results reveal that disease-free equilibria and the endemic equilibria of model are exponentially p-stable and globally asymptotically stable; finally, that the endemic equilibria are locally asymptotically stable.

In this dynamics of coronavirus propagation described by this new model, the first term determines its evolution and the second, which is a matrix, expresses its diffusion within a susceptible population and exposed vulnerable people. The force of human infection, accentuated by that of viral propagation, dangerously affects the population. Minimizing the cost of these forces due to the coronavirus becomes a necessity, even an obligation, in government action. Thus, in order to eradicate this pandemic, an optimal control problem is formulated by minimizing the cost evaluated in the form of a mathematical expectation of the integral over a finite time horizon of the approximate value of the total forces due to COVID-19. Pontryagin's Minimum Principle is used to solve this minimization problem and characterize the optimal control as well as the mathematical expectation of the states of exposed, moderately and severely symptomatic people. However, control without optimal strategies seems ineffective, which is why a class of 5 scenarios is proposed and explored in order to determine the optimal control strategies. This results in eight favorable strategies, including an optimal and least costly one, strategy 14, which requires compliance with and application of hygiene, health and treatment measures as priority, then the other strategies 10, 2 and 6 which, in turn, also require compliance with and application of barrier measures and physical and social distancing such as strict compliance with border closure restrictions with quarantine, not shaking hands; disinfection measures and preventive protocols such as hand washing, wearing helmets. In addition to these optimal strategies, strategy 1, which is similar to the first scenario, is certainly advantageous because it requires 100% compliance and application of all controls, but it is very costly. In the absence of significant means of combating the coronavirus, we encourage the government to pursue the policy of implementing these optimal strategies, which are effective ways to eradicate the coronavirus at a lower cost.

Finally, it was proved that this class of 5 scenarios, provided optimal strategies for the control of COVID-19 in a process that is not only deterministic (not regarding diffusion), but also exists in a stochastic process focused on both evolution and diffusion.

The aspect of diffusion control in this process is part of the future prospects with another control article on this same model.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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