Applied Mathematics, 2025, 16(11), 805-830
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Constitutive Theories for Linear Micromorphic Thermoviscoelastic Solids

Karan S. Surana[1]*, Sri Sai Charan Mathi[2]

[1]Department of Mechanical Engineering, University of Kansas, Lawrence, KS, USA
[2]Trane Technologies, La Crosse, WI, USA
Email: *kssurana@ku.edu

How to cite this paper: Surana, K.S. and Mathi, S.S.C. (2025) Constitutive Theories for Linear Micromorphic Thermoviscoelastic Solids. *Applied Mathematics*, **16**, 805-830.
https://doi.org/10.4236/am.2025.1611042

Received: August 28, 2025
Accepted: November 17, 2025
Published: November 20, 2025

Copyright © 2025 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

Abstract

This paper presents constitutive theories for linear micromorphic microcontinuum thermoviscoelastic solids in which elasticity and dissipation are considered for the microconstituents, the solid medium and the interaction between the microconstituents and the solid medium. The conservation and the balance laws derived by Surana *et al.* in a recent paper in which the derivations is initiated for micro deformation using the conservation and balance laws of classical continuum mechanics followed by "integral-average" definitions valid at macro level permitting derivation of conservation and balance laws at macro level are utilized in the present work. Significant aspects of this theory are: 1) Microconstituent rigid rotation physics is treated identically in all 3M theories; 2) The balance of moment of moments balance law, essential in all 3M theories, is used in the present work; 3) Only the symmetric part of nonsymmetric macro Cauchy tensor can be a constitutive tensor; 4) The smoothing weighting function $\phi^{(\alpha)}$ used by Eringen is neither needed nor used in present work; 5) Constitutive tensors of rank two are always symmetric; 6) All constitutive theories are derived using theory of isotropic tensors in conjunction with entropy inequality, and are therefore always thermodynamically and mathematically consistent 7) Conservation of microinertia, as advocated by Eringen, is neither needed nor used in the present work. All three dissipation mechanisms are based on higher order rates up to a desired order of the strain tensors, and hence represent a comprehensive ordered mechanism yielding three ordered spectra of dissipation coefficients. Constitutive theories are first derived using integrity, the complete basis of the constitutive tensor space, and the representation theorem. These are then followed by simplified yet general forms of the constitutive theories, in which physical meaning of the material coefficients can be clearly established. The linear micromorphic theory presented here for thermoviscoelastic solids is compared with Eringen's theory to identify differences, evaluate their validity based on thermodynamic and mathematical principles, and ultimately determine the thermodynamic and mathematical consistency of the published micromorphic theories.

Keywords

Micromorphic, Micro, Macro, Deformation/Strain Measures, Conservation and Balance Laws, Balance of Moment of Moments, Integral-Average, Representation Theorem, Constitutive Theories, Dissipation

1. Introduction

In a recent paper, Surana *et al.* [1] presented a review of published works on microcontinuum theories, including micromorphic theories. The published works discussed in this reference are included here in the list of references [2]-[32] for the convenience of the reader, but the details of these works are not repeated here for the sake of brevity. The majority of the published works on 3M microcontinuum theories are due to Eringen and Eringen *et al.* [7]-[24]. The other published works generally follow the concepts and the theories published in references [7]-[24]. Surana *et al.* [1] presented detailed derivations of micro and macro conservation and balance laws from the first principles for linear micromorphic microcontinuum solid medium.

In the following, we summarize the basic concepts and steps used in reference [1] in deriving the conservation and balance laws for linear micromorphic continua for micro and macro deformation physics. 1) the deformation/strain measures derived by Surana *et al.* [33] serve as the basic measures of deformation for the micromorphic theory. The microconstituents are deformable, hence there is a micro deformation gradient tensor associated with them. 2) All conservation and balance laws are initiated for the micro deformation of the microconstituent, using laws of thermodynamics in classical continuum mechanics, yielding micro conservation and balances. From the micro conservation and balance laws, "integral-average" definitions are introduced that permit derivation of macro conservation and balance laws and constitutive theories using principles of thermodynamics and well-established concepts in applied mathematics. 3) In deriving conservation and the balance laws and constitutive theories for micromorphic continua, we maintain and adhere to the concepts of classical rotations, the Cauchy moment tensor, and the theory of isotropic tensors, introduced and used by Surana *et al.* [33]-[55] in conjunction with linear and nonlinear micropolar theories for solid and fluid continua. This is necessary because the physics of rigid rotations of microconstituents exists in all 3M theories, arising from the skew-symmetric part of the micro deformation gradient tensor. Thus, we must have exactly the same mathematical treatment of rigid rotation physics in 3M theories, requiring that we maintain the micropolar theory as a subset of micromorphic theory.

Surana *et al.* presented conservation and balance laws for micro as well as macro deformation physics with clarity of valid "integral-average" definitions that are essential for deriving the conservation and balance laws at the macro level. This was followed by constitutive theories for the macro Cauchy stress tensor, the microconstituent Cauchy stress tensor, the Cauchy moment tensor, and the heat vector for thermoelastic medium. Constitutive theories were initiated using conjugate pairs in the entropy inequality establishing constitutive tensors and their argument tensors. In all three constitutive theories (microconstituent Cauchy stress tensor, macro Cauchy stress tensor and macro Cauchy moment tensor), mechanisms of elasticity for the micro Cauchy stress tensor ($\boldsymbol{S}$) and the macro stress tensor $_s\boldsymbol{\sigma}$ (symmetric part) were considered. All constitutive theories were derived using the representation theorem [56]-[67], and are therefore always thermodynamically (not in violation of entropy inequality) and mathematically consistent. Constitutive theories and material coefficients were first derived using integrity (complete basis of the spaces of constitutive tensors), after which their simplified forms were presented, linear in the components of the argument tensors.

The micromorphic theory derived by Surana *et al.* was shown to be thermodynamically and mathematically consistent, and is therefore a valid and physical linear micromorphic theory. This linear micromorphic theory was also compared with Eringen's micromorphic theory. The differences in the two theories were identified, discussed and evaluated for their validity based on thermodynamic principles and well accepted mathematical concepts to establish their validity or lack thereof, and hence the validity of the resulting micromorphic theory. In the context of Eringen's microcontinuum theories, authors showed certain omissions, use of incorrect approaches, inconsistencies, and the thermodynamic and mathematical inconsistency of the resulting micromorphic theories, casting serious doubts on their validity. All of these shortcomings and issues were addressed and corrected in reference [1] in the linear micromorphic microcontinuum theory for solid continua presented by the authors.

In this paper, the conservation and balance laws for linear micromorphic microcontinuum solids derived by Surana *et al.* and the strain/deformation measures derived in reference [33] are utilized to derive the constitutive theories for the microconstituent Cauchy stress tensor, the macro Cauchy stress tensor, the macro Cauchy moment tensor and the heat vector for thermoviscoelastic physics without rheology. Important aspects of the scope of work in this paper are summarized below:

1) Microconstituents have elasticity and dissipation physics. The dissipation physics is a function of micro strain rates up to order $n_s^{(\alpha)}$ yielding an ordered rate dissipation theory with a micro dissipation spectrum consisting of $n_s^{(\alpha)}$ pairs of dissipation coefficients corresponding to $n_s^{(\alpha)}$ micro strain rates.

2) The solid medium also has elasticity and dissipation (macro dissipation). The dissipation mechanism is a function of macro strain rates up to order n, yielding an ordered rate dissipation theory with a macro dissipation spectrum consisting of n pairs of dissipation coefficients corresponding to n_σ macro strain rates.

3) The resistance offered by the elastic and viscous medium to the rigid rota-

tions of the micro constituents is the third mechanism of elasticity and dissipation. The dependence of the Cauchy moment tensor on the symmetric part of the classical rotation gradient tensor provides additional elasticity, *i.e.*, stiffness, and its dependence on rates of up to order n_m of the symmetric part of the rotation gradient tensor provides an additional ordered rate dissipation mechanism with a dissipation spectrum consisting of n_m pairs of dissipation coefficients corresponding to n_m rotation gradient rates.

4) All constitutive theories are derived using the representation theorem, integrity (the complete basis of the space of the constitutive tensors), and the conjugate pairs in the entropy inequality, hence are always thermodynamically and mathematically consistent.

5) Material coefficients are derived for the constitutive theories based on integrity, using the principle of smooth neighborhood and a Taylor series expansion of the coefficients used in the linear combination of the invariants and temperature about a known configuration.

6) Simplified (yet general) forms of the constitutive theories are presented, in which the physical meaning of the material coefficients can be established.

7) The constitutive theories derived here are compared with those of Eringen to highlight differences and identify serious shortcomings of the constitutive theories of Eringen *et al.*

2. Micro and Macro Deformations, Some Basic Considerations

In microcontinuum theories microdeformation of microconstituents influences the macro response of a volume of matter. Consideration of each microconstituent deformation at different locations within a volume of matter is a formidable task. Instead, we consider the entire volume of matter divided into material points, each material point containing microconstituents, with each microconstituent having its own volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ (**Figure 1**). Still, the problem is intractable, as each microconstituent location and deformation within the material point would require individual microconstituent considerations. To address this complex problem we assume that the center of mass $\bar{P}$ of the material point only sees the statistically average response of all microconstituents in its volume $\bar{V} + \partial\bar{V}$. We further assume that there is a surrogate configuration of microconstituents in which the response of each microconstituent at the material point $\bar{P}$ is the same, and this response is also the same as the statistically average response of the original configuration of microconstituents. With this assumption, we only need to consider the microdeformation of one microconstituent in a material point. If the matter is isotropic and homogeneous then this treatment for a material point holds for all material points within the volume, and therefore for the entire volume of matter.

Details of this approach are discussed in refs [1] [33] and have been used in ref [33] for deriving nonlinear deformation measures for 3M physics. In **Figure 1**, P

is the center of mass of the material point, $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and $\boldsymbol{x}^{(\alpha)}$ are the locations of microconstituent α from the center of mass P of the material point and with respect to the fixed x-frame; $\boldsymbol{x}$ is the location of the center of mass P of the material point in the reference or undeformed configuration. Similarly, $\bar{P}, \underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}, \bar{\boldsymbol{x}}^{(\alpha)}$ and $\bar{\boldsymbol{x}}$ are the corresponding quantities in the current configuration. $V + \partial V$ and $\bar{V} + \partial \bar{V}$ are the undeformed and deformed volumes of the material point. Likewise $V^{(\alpha)} + \partial V^{(\alpha)}$ and $\bar{V}^{(\alpha)} + \partial \bar{V}^{(\alpha)}$ are the undeformed and deformed volumes associated with microconstituent α (see **Figure 1**).

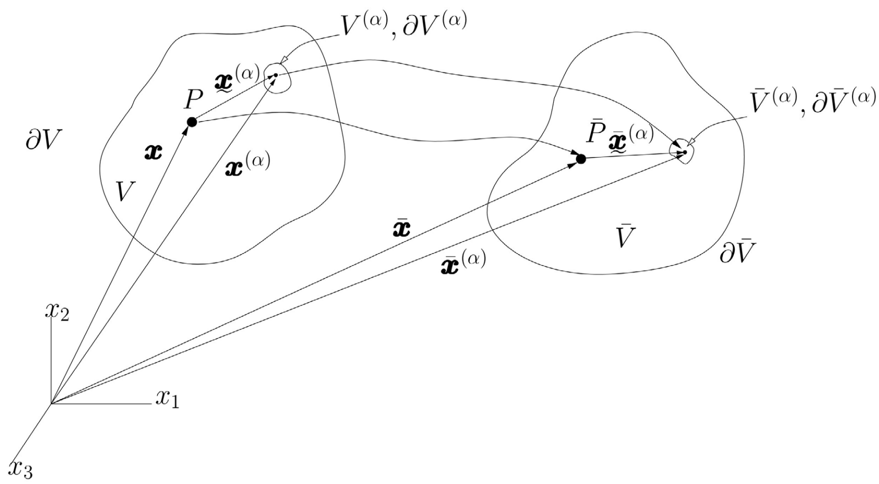

Figure 1. Undeformed and deformed configurations of a material point volume.

$\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ are called directors in the reference and current configurations. The deformation of $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ due to $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ characterizes the micro-mechanics of the microconstituent α. Deformation measures using this concept have been presented by Surana *et al.* in reference [33].

We note that rigid rotation physics of the microconstituents is present in all 3M theories and appears in exactly the same form. Thus, this physics requires a consistent treatment in the development of 3M theories, independent of type of microcontinuum theory. Eringen uses rigid rotation ${}_{\alpha}\boldsymbol{\Theta}$ of the microconstituents as an unknown degrees of freedom to account for rigid rotation physics. Our view and approach to incorporating this physics in 3M theories [1] [33]-[55] [68]-[70] differs from than that of Eringen.

In every deforming isotropic, homogeneous solid matter classical rotations ${}_{c}\boldsymbol{\Theta}$ due to $\nabla \times \boldsymbol{u} = {}_{c}\Theta_i \boldsymbol{e}_i$ constitute a free field in classical continuum mechanics. This field is always present in every deforming solid but does not influence classical continuum physics as it is a free field [1] [33]. Due to the presence of microconstituents, this free field is no longer a free field, since the microconstituent offer resistance to this free field. Surana *et al.* [1] [33]-[55] [68]-[70] have shown that in all microcontinuum theories the classical rotations ${}_{c}\boldsymbol{\Theta}$ (known) describe rigid rotations of the microconstituent, thus eliminating the need for ${}_{\alpha}\boldsymbol{\Theta}$ as unknown degrees of freedom for the microconstituents. Many other measures, definitions and notations used in this paper are described in references [1] [40] and are not repeated here for the sake of brevity.

3. Degrees of Freedom in Micro Deformation Physics

These have also been discussed by Surana *et al.*, but are essential to describe here as they play a crucial role in the derivation of the constitutive theories. Our views and approach here also differ completely from those of Eringen and are based more on the physics of deformation and available means in the theory to describe it. Conceptually, if we knew the microconstituent displacements, hence the microconstituent strain measures and what follows would be straight forward, but this is not the case, hence an alternate approach is necessary. ${}_s^d\boldsymbol{J}^{(\alpha)}$, the symmetric part of the microconstituent displacement gradient tensor ${}^d\boldsymbol{J}^{(\alpha)}$, is completely defined if $\boldsymbol{u}^{(\alpha)}$ are known, but in the absence of $\boldsymbol{u}^{(\alpha)}$ we have no choice but to consider six independent components of ${}_s^d\boldsymbol{J}^{(\alpha)}$ as unknown deformational degrees of freedom for the microconstituents. Thus, in our micromorphic theory, a microconstituent has nine degrees of freedom: three rigid rotations ${}_c\boldsymbol{\Theta}$ (known) and six independent components of ${}_s^d\boldsymbol{J}^{(\alpha)}$ that are unknown. This choice of degrees of freedom is valid for deriving physically and mathematically consistent constitutive theories.

In Eringen's work all nine components of $J^{(\alpha)}$ are considered as unknown deformational degrees of freedom in addition to three unknown rigid rotations ${}_\alpha\boldsymbol{\Theta}$. We remark that $J^{(\alpha)}$ contains rigid rotation field of the microconstituents, and therefore should not be used in deriving constitutive theories for the microconstituent stress tensor, as was is in Eringen's work [16].

4. Conservation and Balance Laws for Linear Micromorphic Continua

The derivation of the conservation and the balance laws for a linear micromorphic microcontinuum solid medium begins by applying the conservation and balance laws of classical continuum mechanics to the microconstituents. From these microconstituent conservation and balance laws, "integral-average" definitions are introduced to derive the corresponding macro conservation and balance laws. There are major differences between the approach used by Surana *et al.* [1] compared to Eringen [7]-[24]. Major weaknesses in the micromorphic theory presented by Eringen [7]-[24] are summarized in section 9. The conservation and balance laws for a linear micromorphic solid in the Lagrangian descriptions are given in the following.

Conservation of mass, balance of linear momenta, balance of angular momenta, first and second law of thermodynamics and balance of moment of moments are given in the following.

$$\rho_0(\boldsymbol{x}) = |\boldsymbol{J}|\rho(\boldsymbol{x},t) \tag{1}$$

$$\rho_0 a_k - \rho_0 {}^b F_k - \sigma_{lk,l} = 0 \tag{2}$$

$$\epsilon_{nmk}\left(\sigma_{mk} \pm S_{mk}\right) + m_{ln,l} = 0 \tag{3}$$

$$\rho_0 \dot{e} - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{J}}^{(\alpha)} - \nabla \cdot \boldsymbol{q} - \left({}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}\right) + \boldsymbol{m} : {}^{c\Theta}\dot{\boldsymbol{J}} \right) = 0 \tag{4}$$

$$\rho_0 \left(\dot{\phi} + \eta \dot{\theta}\right) - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{J}}^{(\alpha)} + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \left({}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}\right) + \boldsymbol{m} : {}^{c\Theta}\dot{\boldsymbol{J}} \right) \leq 0 \tag{5}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{6}$$

This mathematical model consists of seven partial differential equations: balance of linear momenta (3), balance of angular momenta (3) and the energy Equation (1) in thirty four dependent variables: $\boldsymbol{u}(3), \boldsymbol{\sigma}(9), \boldsymbol{S}(6), \boldsymbol{m}(6), \boldsymbol{q}(3), \theta(1), {}_{s}\boldsymbol{J}^{(\alpha)}(6)$. Thus, additional twenty seven equations are needed for closure. Constitutive theories provide twenty one equations: $\boldsymbol{\sigma}(6), \boldsymbol{S}(6), \boldsymbol{m}(6), \boldsymbol{q}(3)$. Thus, additional six equations are needed for closure. These are discussed in the following section.

5. Additional Six Equations in the Mathematical Model

From the conservation and the balance laws, we note that the microconstituent stress $\boldsymbol{S}$ only appears in the energy equation and the entropy inequality. This, of course, implies that if we were to solve a boundary value problem for isothermal physics, in which case the energy equation is not part of the mathematical model, then the microconstituent stress $\boldsymbol{S}$ is completely absent from the mathematical model. This certainly is not physical, as the microconstituent deformation contributes to macro physics for stationary processes as well as evolutionary processes. Thus, we must have another relationship that considers symmetric stress tensor $\boldsymbol{S}$ and the symmetric part of $\boldsymbol{\sigma}$.

There are many differences between our work and Eringen's work on nonclassical theories.

1) Moment tensor (nonclassical mechanics) is defined using $\boldsymbol{\sigma}^{(\alpha)}$, due to classical mechanics, thus this definition is in error.

2) Due to not using balance of moment of moments balance law, the moment tensor is nonsymmetric.

3) Moment tensor in balance of angular momenta contains permutation tensor. This is obviously in error as the permutation tensor only appears in force terms due to their cross product with distance. This is obviously not needed in case of moment tensor as it is already a moment.

4) In Eringen's work in the derivation of balance of angular momenta, the skew symmetric components of $\boldsymbol{\sigma}$ are balanced by the gradients of the skew symmetric part of the moment tensor (as the moment tensor has permutation tensor in Eringen's derivations resulting in three equations). Eringen [7]-[24] and those following his work also suggest that in the derivation of the balance of angular momenta, the permutation tensor must be dropped to obtained another balance law, moments of $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ (only symmetric part) that must balance with gradients of the symmetric part of the moment tensor to obtain additional equations.

5) In references [7]-[24] it is stated that the nine equations in 4) are suitable for determining nine components of $\boldsymbol{J}^{(\alpha)}$.

6) It has been pointed, discussed and demonstrated that in 3M theories, balance of moment of moments balance law is essential [55] [71]. Due to this balance law, the Cauchy moment tensor is symmetric. Thus, in the balance of angular momenta ${}_a\boldsymbol{\sigma}$ are balanced by the gradients of symmetric moment tensor. This is the correct balance of angular momenta.

7) We must recognize that the permutation tensor in balance of angular moment only appears with force terms due to their cross product with distance vector, we just cannot discard it (as suggested by Eringen) as it is due to the physics of moment of forces. It is obvious that what is suggested in (4) has no basis, hence will not lead to any meaningful relations.

8) Thus, in Eringen's work on balance of angular momenta as well as the six additional equations, both are in error. Our view, approach and outcome to obtain the six additional equations is completely different than Eringen.

Derivation of Additional Six Equations

From the derivation of balance of angular momenta leading to (3) (in Lagrangian description), we note that $\boldsymbol{\sigma}$ has nine independent components, three in ${}_a\boldsymbol{\sigma}$ and six in ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$ has six independent components. However, presence of permutation tensor on the left side of Equation (3) forces us to discard six symmetric components of $\boldsymbol{\sigma}$ as well as $\boldsymbol{S}$. This is an important observation that suggests that some how ϵ_{mkn} from the left side of (3) must be eliminated. This of course can be done by premultiplying Equation (3) with $(\epsilon_{mkn})^{-1}$, the inverse of ϵ_{mkn}. Symbolically we can write

$$(\epsilon_{mkn})^{-1}\epsilon_{mkn}(\sigma_{mk} \pm S_{mk}) + (\epsilon_{mkn})^{-1} m_{ln,l} = 0 \tag{7}$$

$$\text{or} \quad \sigma_{mk} \pm S_{mk} + (\epsilon_{mkn})^{-1} m_{ln,l} = 0 \tag{8}$$

But inverse of ϵ_{mkn} (for values of 1, 2, 3 for m,k,n) is ϵ_{mkn}, thus we can write (8) as

$$\sigma_{mk} \pm S_{mk} + \epsilon_{mkn} m_{ln,l} = 0 \tag{9}$$

$$\text{or} \quad {}_a\sigma_{mk} + {}_s\sigma_{mk} \pm S_{mk} + \epsilon_{mkn} m_{ln,l} = 0 \tag{10}$$

Since

$${}_a\sigma_{mk} + \epsilon_{mkn} m_{ln,l} = 0 \tag{11}$$

is balance of angular momenta, (10) reduces to the following.

$${}_s\sigma_{mk} \pm S_{mk} = 0 \tag{12}$$

At this point choice of negative sign is physical as it would suggest that symmetric part of $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ balance each other, this obviously has to be true at an interface between the microconstituent and the medium, recalling that ${}_a\boldsymbol{\sigma}$ are balanced by the gradients of $\boldsymbol{m}$. Thus, we rewrite (12) with only negative sign.

$${}_s\sigma_{mk} - S_{mk} = 0 \tag{13}$$

Equation (13) are additional six equations that provide closure of the mathematical model.

Remarks

1) First, we note that (3) (balance of angular momenta) only contains nonclassical physics, both ${}_a\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are due to nonclassical physics, whereas (13) contains stresses due to classical mechanics. This is necessary for maintaining consistency of physics in the derivations.

2) When the microconstituents and the medium are of the same material, then naturally (13) must hold. When the microconstituents and the medium are of different material, (13) must also hold at the interface, continuity of stress due to classical physics, while ${}_a\boldsymbol{\sigma}$ is taken care by the gradients of moment tensor, both ${}_a\boldsymbol{\sigma}$ and moment tensor are due to nonclassical physics.

6. Microconstituent Stress Tensor $\boldsymbol{S}$ Due to Micro Cauchy Stress Tensor $\bar{\sigma}^{(\alpha)}$ (or $\sigma^{(\alpha)}$)

The integral average definition

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} \mathrm{d}\bar{V}^{(\alpha)} = \bar{S}_{mk} \mathrm{d}\bar{V} \tag{14}$$

considers total stresses $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ and $\bar{\boldsymbol{S}}$ without any additive decomposition into equilibrium and deviatoric components. Secondly, if we consider a decomposition of the stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ and $\bar{\boldsymbol{S}}$ to define volumetric and distortional deformation of the microconstituent, then the microconstituent density $\bar{\rho}^{(\alpha)}$ would be needed to describe volumetric deformation, but $\bar{\rho}^{(\alpha)}$ is removed by the integral-average definition. Both of these aspects suggest that we must consider $\bar{\boldsymbol{S}}$ or $\boldsymbol{S}$ purely as arising from mechanical loading, and therefore as a function of the work conjugate strain tensor and the elastic properties of the microconstituents. Thus, in the following, we consider $\bar{\boldsymbol{S}}$ or $\boldsymbol{S}$ as a constitutive tensor with the work conjugate strain tensor and temperature θ as its arguments.

7. Constitutive Theories for Linear Micromorphic Solid Continua

In deriving the constitutive theories we consider comprehensive thermoelastic behavior with dissipation (without rheology). We assume that the medium is elastic and has dissipation mechanism. The micro constituents naturally have elasticity but we assume that each microconstituent has its own dissipation mechanism. Additionally, rigid rotations of the microconstituents in a viscous elastic medium result in elasticity as well as a dissipation mechanism. Thus, in the derivation of the constitutive theory for $\boldsymbol{\sigma}, \boldsymbol{S}$ and $\boldsymbol{m}$, elasticity and dissipation physics are considered for the microconstituents, the solid medium, and the interaction of rigid rotations of the microconstituents with the viscoelastic solid medium. Furthermore, all three dissipation mechanisms use higher order rates of strains, and therefore result in ordered rate constitutive theories with dissipation spectra for each of the three constitutive theories, corresponding to the strain rates considered.

7.1. Initial Determination of Constitutive Tensors and Their Argument Tensors

In deriving constitutive theories, we always begin with rate of work or the corresponding conjugate pairs in entropy inequality for determination of constitutive tensors based on causality axiom of constitutive theory and their possible argument tensors. The choice of constitutive tensors can be altered if the physics requires so and the argument tensors of the constitutive tensors can be augmented with additional tensors such physics was not considered when deriving the entropy inequality. We follow the details and guidelines presented in references [72] [73]. Once the constitutive tensors and their argument tensors are established, we follow the theory of isotropic tensors or the representation theorem in deriving the constitutive theories, and the standard procedure of a Taylor series expansion of the coefficients used in the linear combination of the basis of the constitutive tensor space [72] [73].

Consider entropy inequality (5)

$$\rho_0\left(\dot{\phi}+\eta\dot{\theta}\right)-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{J}}^{(\alpha)}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\left({}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}\right)\leq 0 \tag{15}$$

The macro stress tensor $\boldsymbol{\sigma}$ is nonsymmetric, and therefore cannot serve as a constitutive tensor due to the representation theorem. Thus, we need an additive decomposition of $\boldsymbol{\sigma}$ into the symmetric tensor ${}_{s}\boldsymbol{\sigma}$ and the skew-symmetric tensor ${}_{a}\boldsymbol{\sigma}$. There cannot be a constitutive theory for ${}_{a}\boldsymbol{\sigma}$, as it is already defined by gradients of Cauchy moment tensor due to the balance of angular momenta. Thus, ${}_{s}\boldsymbol{\sigma}$ is the constitutive tensor and not $\boldsymbol{\sigma}$ or ${}_{a}\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma}={}_{s}\boldsymbol{\sigma}+{}_{a}\boldsymbol{\sigma} \tag{16}$$

Secondly

$$\dot{\boldsymbol{J}}={}^{d}\dot{\boldsymbol{J}}={}^{d}_{s}\dot{\boldsymbol{J}}+{}^{d}_{a}\dot{\boldsymbol{J}}=\dot{\boldsymbol{\varepsilon}}+{}^{d}_{a}\dot{\boldsymbol{J}} \tag{17}$$

in which ${}^{d}\boldsymbol{J}$ is the displacement gradient tensor and ${}^{d}_{s}\boldsymbol{J}$ and ${}^{d}_{a}\boldsymbol{J}$ are the symmetric and skew symmetric tensors obtained by the additive decomposition of ${}^{d}\boldsymbol{J}$.

$${}^{d}\boldsymbol{J}={}^{d}_{s}\boldsymbol{J}+{}^{d}_{a}\boldsymbol{J} \tag{18}$$

Likewise, additive decomposition of ${}^{c\Theta}\boldsymbol{J}$ and ${}^{d}\boldsymbol{J}^{(\alpha)}$ into symmetric and skew symmetric tensors gives:

$${}^{c\Theta}\boldsymbol{J}={}^{c\Theta}_{s}\boldsymbol{J}+{}^{c\Theta}_{a}\boldsymbol{J} \tag{19}$$

Also

$$\dot{\boldsymbol{J}}^{(\alpha)}={}^{d}\dot{\boldsymbol{J}}^{(\alpha)} \tag{20}$$

$$\text{and}\quad {}^{d}\boldsymbol{J}^{(\alpha)}={}^{d}_{s}\boldsymbol{J}^{(\alpha)}+{}^{d}_{a}\boldsymbol{J}^{(\alpha)} \tag{21}$$

in which ${}^{d}\boldsymbol{J}^{(\alpha)}$ is micro displacement gradient tensor and ${}^{d}_{s}\boldsymbol{J}^{(\alpha)}$ and ${}^{d}_{a}\boldsymbol{J}^{(\alpha)}$ are symmetric and skew symmetric tensors due to additive decomposition of ${}^{d}\boldsymbol{J}^{(\alpha)}$. Furthermore,

$$ {}^{d}\boldsymbol{J}^{(\alpha)} = {}^{d}_{s}\boldsymbol{J}^{(\alpha)} + {}^{d}_{a}\boldsymbol{J}^{(\alpha)} = \boldsymbol{\varepsilon}^{(\alpha)} + {}^{d}_{a}\boldsymbol{J}^{(\alpha)} \tag{22} $$

also

$$ {}^{c\Theta}\dot{\boldsymbol{J}} = {}^{c\Theta}_{s}\dot{\boldsymbol{J}} + {}^{c\Theta}_{a}\dot{\boldsymbol{J}} \tag{23} $$

Substituting (16)-(23) in the entropy inequality (5) and noting that

$$ {}_{s}\boldsymbol{\sigma} : {}^{d}_{a}\dot{\boldsymbol{J}} = 0;\ \ {}_{a}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = 0;\ \ \boldsymbol{S} : {}^{d}_{a}\dot{\boldsymbol{J}}^{(\alpha)} = 0;\ \ \boldsymbol{m} : {}^{c\Theta}_{a}\dot{\boldsymbol{J}} = 0 \tag{24} $$

We can write (15) as follows:

$$ \rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{a}\boldsymbol{\sigma} : {}^{d}_{a}\dot{\boldsymbol{J}} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} + {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}\right) \le 0 \tag{25} $$

From balance of angular momenta

$$ \nabla \cdot \boldsymbol{m} = -\epsilon : \boldsymbol{\sigma} \tag{26} $$

Substituting (26) in (25)

$$ \rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{a}\boldsymbol{\sigma} : \left({}^{d}_{a}\dot{\boldsymbol{J}}\right) - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} - {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\epsilon : \boldsymbol{\sigma}\right) - \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} \le 0 \tag{27} $$

A simple calculation shows that

$$ {}_{a}\boldsymbol{\sigma} : {}^{c\Theta}_{a}\dot{\boldsymbol{J}} = {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\epsilon : \boldsymbol{\sigma}\right) \tag{28} $$

Using (28) in (27), (28) reduces to

$$ \rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} - \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} \le 0 \tag{29} $$

Further additive decomposition of ${}_{s}\boldsymbol{\sigma}$ into equilibrium and deviatoric stress ${}^{e}_{s}\boldsymbol{\sigma}, {}^{d}_{s}\boldsymbol{\sigma}$ is needed to derive constitutive theory for volumetric and distortional deformation physics that are mutually exclusive.

$$ {}_{s}\boldsymbol{\sigma} = {}^{e}_{s}\boldsymbol{\sigma} + {}^{d}_{s}\boldsymbol{\sigma} \tag{30} $$

substituting (30) in (29)

$$ \rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}^{e}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} - \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} \le 0 \tag{31} $$

rate of work conjugate pairs and the last term in (31) in conjunction with the axiom of causality [72] [73] imply that ${}^{e}_{s}\boldsymbol{\sigma}, {}^{d}_{s}\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ and $\boldsymbol{q}$ are valid choices of constitutive tensors provided volumetric change for microconstituents *i.e.*, ${}_{e}\boldsymbol{S}$ (equilibrium micro constituent stress) is not considered in which case $\boldsymbol{S} = {}_{d}\boldsymbol{S}$. The initial choice of constitutive tensors and their argument tensors is (with θ included as an argument tensor in all constitutive tensors because of non-isothermal physics):

$$ {}^{e}_{s}\boldsymbol{\sigma} = {}^{e}_{s}\boldsymbol{\sigma}\left(\rho, \theta\right) \tag{32} $$

$$ {}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}, \theta\right) \tag{33} $$

$$ \boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)}, \theta\right) \tag{34} $$

$$ \boldsymbol{m} = \boldsymbol{m}\left({}^{c\Theta}_{s}\boldsymbol{J}, \theta\right) \tag{35} $$

$$ \boldsymbol{q} = \boldsymbol{q}\left(\boldsymbol{g}, \theta\right) \tag{36} $$

Even though we do not need a constitutive theory for Φ, its argument tensors are essential to establish, since it is used to simplify the entropy inequality (31) as well as to derive the constitutive theory for ${}_s^e\boldsymbol{\sigma}$. The presence of η in (31) must be addressed as well. The Helmholtz free energy density Φ must show dependence on ρ and θ. In the Lagrangian description ρ is not permissible as an argument tensor as it is not a dependent variable, but we use it in (32) in a symbolic sense. Other argument tensors of Φ and η are chosen based on the principle of equipresence as we do not have any other basis for their choice. However, the principle of equipresence is not used in (32) - (36) as the conjugate pairs in the entropy inequality (31) clearly dictate the choices of argument tensors:

$$\Phi = \Phi\left(\rho, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}, {}_s^{c\Theta}\boldsymbol{J}, \boldsymbol{q}, \theta\right) \tag{37}$$

$$\eta = \eta\left(\rho, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}, {}_s^{c\Theta}\boldsymbol{J}, \boldsymbol{q}, \theta\right) \tag{38}$$

7.2. Constitutive Theory for Equilibrium Cauchy Stress Tensor ${}_s^e\boldsymbol{\sigma}$

In Lagrangian description, density $\rho(\boldsymbol{x},t)$ is deterministic from the conservation of mass $\rho(\boldsymbol{x},t) = \dfrac{\rho_0}{|\boldsymbol{J}|}$ once the deformation gradient tensor $\boldsymbol{J}$ is known, hence $\rho(\boldsymbol{x},t)$ cannot be an argument tensor of the constitutive tensors [73]. However, compressibility and incompressibility physics is related to density and temperature. Thus, the constitutive theory for ${}_s^e\boldsymbol{\sigma}$ cannot be derived using entropy inequality (31) in Lagrangian description, instead we must consider entropy inequality similar to (31) in Eulerian description.

$$\bar{\rho}\left(\dot{\bar{\Phi}} + \eta\dot{\bar{\theta}}\right) - {}_s^e\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_s^d\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}} : \left({}_s^{r}{}_c^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{39}$$

In this case $\bar{\rho}$ is unknown, hence is a dependent variable in the mathematical model. Following same procedure as for Lagrangian description, the constitutive tensors and their argument tensors (including $\bar{\Phi}$ and $\bar{\eta}$) are given by:

$${}_s^e\bar{\boldsymbol{\sigma}}^{(0)} = {}_s^e\bar{\boldsymbol{\sigma}}^{(0)}\left(\bar{\rho}, \bar{\theta}\right) \tag{40}$$

$${}_s^d\bar{\boldsymbol{\sigma}}^{(0)} = {}_s^d\bar{\boldsymbol{\sigma}}^{(0)}\left(\bar{\rho}, \bar{\boldsymbol{D}},, \bar{\theta}\right) \tag{41}$$

$$\bar{\boldsymbol{S}} = \bar{\boldsymbol{S}}\left(\bar{\boldsymbol{D}}^{(\alpha)}, \bar{\theta}\right) \tag{42}$$

$$\bar{\boldsymbol{m}} = \bar{\boldsymbol{m}}\left(\bar{\rho}, {}_s^{r}{}_c^{\bar{\Theta}}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{43}$$

$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{44}$$

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{D}}^{(\alpha)}, {}_s^{r}{}_c^{\bar{\Theta}}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{45}$$

$$\bar{\eta} = \bar{\eta}\left(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{D}}^{(\alpha)}, {}_s^{r}{}_c^{\bar{\Theta}}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{46}$$

Using (45) we can write

$$\dot{\bar{\Phi}} = \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \dot{\bar{\Phi}} + \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{D}}} : \dot{\bar{\boldsymbol{D}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{D}}^{(\alpha)}} : \dot{\bar{\boldsymbol{D}}}^{(\alpha)} + \frac{\partial \bar{\Phi}}{\partial {}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}}} : {}_{s}^{c}{}^{r}\!\Theta \dot{\bar{\boldsymbol{J}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} \tag{47}$$

From conservation of mass in Eulerian description

$$\dot{\bar{\rho}} = -\bar{\rho}\left(\bar{\nabla} \cdot \bar{\boldsymbol{v}}\right) = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\bar{D}_{kk}\delta_{lk} = \bar{\rho}\bar{\boldsymbol{D}} : \boldsymbol{\delta} \tag{48}$$

substituting from (48) for $\dot{\bar{\rho}}$ in (47) and then substituting (47) in (39), we obtain the following after regrouping the terms

$$\begin{aligned} &\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)}\right) : \bar{\boldsymbol{D}} + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{D}}} : \dot{\bar{\boldsymbol{D}}} + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{D}}^{(\alpha)}} : \dot{\bar{\boldsymbol{D}}}^{(\alpha)} + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}}\right)} : {}_{s}^{c}{}^{r}\!\Theta \dot{\bar{\boldsymbol{J}}} \\ &+ \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \bar{\rho}\left(\bar{\eta} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\right)\dot{\bar{\theta}} - {}_{s}^{d}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}} : {}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \end{aligned} \tag{49}$$

The entropy inequality (49) holds for arbitrary but admissible choices of $\dot{\bar{\boldsymbol{D}}}, \dot{\bar{\boldsymbol{D}}}^{(\alpha)}, {}_{s}^{c}{}^{r}\!\Theta \dot{\bar{\boldsymbol{J}}}, \dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$ if the following conditions hold:

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{D}}} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi}\left(\bar{\boldsymbol{D}}\right) \tag{50}$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{D}}^{(\alpha)}} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi}\left(\bar{\boldsymbol{D}}^{(\alpha)}\right) \tag{51}$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial {}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}}} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi}\left({}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}}\right) \tag{52}$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} = 0 \Rightarrow \bar{\Phi} \ne \bar{\Phi}\left(\boldsymbol{g}\right) \tag{53}$$

$$\bar{\rho}\left(\bar{\eta} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\right) = 0 \Rightarrow \bar{\eta} = -\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \tag{54}$$

Equations (50)-(54) implies that $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}, \bar{\boldsymbol{D}}^{(\alpha)}, {}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}}$ and $\bar{\boldsymbol{g}}$. Equation (54) implies that $\bar{\eta}$ is deterministic from $\bar{\Phi}$, hence $\bar{\eta}$ is not a constitutive or dependent variable. Using (50)-(54), the constitutive tensor and their argument tensors in (40)-(44) remain the same, but the argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ can be modified:

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\theta}\right) \tag{55}$$

$$\bar{\eta} = \bar{\eta}\left(\bar{\rho}, \bar{\Phi}\right) \tag{56}$$

and the entropy inequality (49) reduces to

$$\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{e}\boldsymbol{\sigma}\right) : \bar{\boldsymbol{D}} - {}_{s}^{d}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}}^{(0)} : {}_{s}^{c}{}^{r}\!\Theta \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{57}$$

Constitutive theory for ${}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)}$ for compressible matter can be obtained by setting coefficient of $\bar{\boldsymbol{D}}$ in the first term of (57) to zero.

$${}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)}\left(\bar{\rho}, \bar{\theta}\right) = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} = \bar{p}\left(\bar{\rho}, \bar{\theta}\right)\boldsymbol{\delta} \tag{58}$$

$$\bar{p}\left(\bar{\rho}, \bar{\theta}\right) = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \tag{59}$$

in which $\bar{p}(\bar{\rho},\bar{\theta})$ is thermodynamic pressure or equation of state for the compressible matter. When the deforming matter is incompressible, there is no change in volume. Thus, for a fixed mass, the density is constant *i.e.*,
$\bar{\rho}(\boldsymbol{x},t)=\rho(\boldsymbol{x},t)=\bar{\rho}=\rho_0$. For this case, from conservation of mass, we have:

$$\dot{\bar{\rho}}=-\bar{\rho}(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{v}})=0 \tag{60}$$

and

$$\frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\bar{\rho}}=\frac{\partial\bar{\Phi}(\rho_0,\theta)}{\partial\bar{\rho}}=0 \tag{61}$$

Hence, for incompressible solid, the constitutive theory for ${}_s^e\boldsymbol{\sigma}$ cannot be derived using (58) and (59). First, using (61), the entropy inequality (57) reduces to

$$-{}_s^e\bar{\boldsymbol{\sigma}}^{(0)}:\bar{\boldsymbol{D}}-{}_s^d\bar{\boldsymbol{\sigma}}^{(0)}:\bar{\boldsymbol{D}}-\bar{\boldsymbol{S}}:\bar{\boldsymbol{D}}^{(\alpha)}-\bar{\boldsymbol{m}}^{(0)}:{}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}}+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}\leq 0 \tag{62}$$

In order to derive constitutive theory for ${}_s^e\bar{\boldsymbol{\sigma}}^{(0)}$ for incompressible solid matter, we must introduce incompressibility condition in (62). From continuity equation, the velocity field for incompressible matter is divergence free *i.e.*,

$$\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}}=\bar{D}_{kk}=\bar{D}_{kk}\delta_{lk}=\boldsymbol{\delta}:\bar{\boldsymbol{D}}=0. \tag{63}$$

If (63) holds, then the following holds too:

$$\bar{p}(\bar{\theta})\boldsymbol{\delta}:\bar{\boldsymbol{D}}=0 \tag{64}$$

in which, $\bar{p}(\bar{\theta})$ is a Lagrange multiplier. Adding (64) to (62) and regrouping terms

$$\left(\bar{p}(\bar{\theta})\boldsymbol{\delta}-{}_s^e\bar{\boldsymbol{\sigma}}^{(0)}\right):\bar{\boldsymbol{D}}-{}_s^d\bar{\boldsymbol{\sigma}}^{(0)}:\bar{\boldsymbol{D}}-\bar{\boldsymbol{S}}:\bar{\boldsymbol{D}}^{(\alpha)}+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-\bar{\boldsymbol{m}}^{(0)}:{}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}}\leq 0 \tag{65}$$

Entropy inequality (65) holds for arbitrary but admissible $\bar{\boldsymbol{D}}$, if the coefficient of $\bar{\boldsymbol{D}}$ in the first term in (65) is set to zero, giving:

$${}_s^e\bar{\boldsymbol{\sigma}}^{(0)}=\bar{p}(\bar{\theta})\boldsymbol{\delta} \tag{66}$$

In (66), $\bar{p}$ is mechanical pressure. The reduced form of entropy inequality is given by:

$$-{}_s^d\bar{\boldsymbol{\sigma}}^{(0)}:\bar{\boldsymbol{D}}-\bar{\boldsymbol{S}}:\bar{\boldsymbol{D}}^{(\alpha)}-\bar{\boldsymbol{m}}^{(0)}:{}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}}+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}\leq 0 \tag{67}$$

In Lagrangian description, the constitutive theory for ${}_s^e\boldsymbol{\sigma}$ can be obtained directly from (58), (59) and (66).

$${}_s^e\boldsymbol{\sigma}^{(0)}=p(\rho,\theta)\boldsymbol{\delta};\ p(\rho,\theta)=-\rho^2\frac{\partial\Phi}{\partial\rho}\ (\text{compressible}) \tag{68}$$

$${}_s^e\boldsymbol{\sigma}^{(0)}=p(\theta)\boldsymbol{\delta}\ (\text{incompressible}) \tag{69}$$

The reduced form of entropy inequality in Lagrangian description follows directly from (67).

$$-{}_s^d\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-\boldsymbol{S}:\boldsymbol{\varepsilon}^{(\alpha)}-\boldsymbol{m}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}\leq 0 \tag{70}$$

In the following, we present derivation of constitutive theories for ${}^{d}_{s}\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ and $\boldsymbol{q}$ using representation theorem [56]-[67]. ${}^{d}_{s}\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ are symmetric tensors of rank two and their conjugate $\dot{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}^{(\alpha)}$ and ${}^{c\Theta}_{s}\dot{\boldsymbol{J}}$ are also symmetric tensors of rank two. $\boldsymbol{q}$ and $\boldsymbol{g}$ are tensors of rank one. Thus, there is no difficulty in deriving constitutive theories for all four constitutive tensors using representation theorem. Furthermore, in the constitutive theories for ${}^{d}_{s}\boldsymbol{\sigma}, \boldsymbol{S}$ and $\boldsymbol{m}$ we consider elasticity and dissipation mechanisms. Dissipation mechanisms are ordered rate mechanism, hence yield dissipation spectrum in each constitutive theory.

7.3. Constitutive Theory for ${}^{d}_{s}\boldsymbol{\sigma}$ Cauchy Stress Tensor

We consider the medium to be linear elastic. We begin with conjugate pair ${}^{e}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$ in the reduced form of the entropy inequality (70). This conjugate pair in conjunction with axiom of causality suggest that ${}^{d}_{s}\boldsymbol{\sigma}$ is the constitutive tensor and $\boldsymbol{\varepsilon}$ as its argument tensor. Thus, we can write (θ is included as the argument tensors due to non isothermal physics)

$$ {}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{71} $$

We know from physics of viscous fluids that dissipation requires strain rate, same as rate of strain in Lagrangian description, thus $\dot{\boldsymbol{\varepsilon}}$ or $\boldsymbol{\varepsilon}_{(1)}$ should be argument tensor of ${}^{d}_{s}\boldsymbol{\sigma}$. We generalize the dissipation mechanism by considering strain rates up to orders n *i.e.*, by considering $\boldsymbol{\varepsilon}_{(i)}; i = 1, 2, \cdots, n$ as argument tensors of ${}^{d}_{s}\boldsymbol{\sigma}$. Thus, we have

$$ {}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{(i)}, \theta\right); i = 1, 2, \cdots, n \tag{72} $$

${}^{d}_{s}\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{(i)}; i = 1, 2, \cdots, n$ are symmetric tensors of rank two and θ is a tensor of rank zero. Thus, we can use representation theorem to derive constitutive theory for ${}^{d}_{s}\boldsymbol{\sigma}$.

Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}; i = 1, 2, \cdots, N^{\sigma}$ be the combined generators of the argument tensors of ${}^{d}_{s}\boldsymbol{\sigma}$ in (72) that are symmetric tensors of rank two and let ${}^{\sigma}\underset{\sim}{I}^{j}; j = 1, 2, \cdots, M^{\sigma}$ be the combined invariants of the same argument tensors of ${}^{d}_{s}\boldsymbol{\sigma}$ in (72). Then, $\boldsymbol{I}, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}; i = 1, 2, \cdots, N^{\sigma}$ constitute the basis of the space of tensor ${}^{d}_{s}\boldsymbol{\sigma}$, also referred to as integrity. Now, ${}^{d}_{s}\boldsymbol{\sigma}$ can be expressed as a linear combination of the basis in the current configuration.

$$ {}^{d}_{s}\boldsymbol{\sigma} = {}^{\sigma}\alpha^{0}\boldsymbol{I} + \sum_{i=1}^{N^{\sigma}} {}^{\sigma}\alpha^{i}\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right); \;\; {}^{\sigma}\underset{\sim}{\alpha}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}\left({}^{\sigma}\underset{\sim}{I}^{j}, \theta\right); \;\; i = 0, 1, \cdots, N^{\sigma}; j = 1, 2, \cdots, M^{\sigma} \tag{73} $$

in which ${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\left({}^{\sigma}\underset{\sim}{I}^{j}, \theta\right)$ are coefficients in the linear combination (73). The material coefficients in (73) are determined by expanding ${}^{\sigma}\underset{\sim}{\alpha}^{i}; i = 0, 1, \cdots, {}^{\sigma}N$ in the invariants ${}^{\sigma}\underset{\sim}{I}^{j}; j = 1, 2, \cdots, {}^{\sigma}M$ and the temperature θ about a known configuration $\underset{\sim}{\Omega}$ and only retaining up to linear terms in ${}^{\sigma}\underset{\sim}{I}^{j}; j = 1, 2, \cdots, {}^{\sigma}M$ and temperature θ (to simplify the resulting constitutive theory).

$$ {}^{\sigma}\underset{\sim}{\alpha}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial^{\sigma}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right) + \frac{\partial^{\sigma}\alpha^{i}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right); \;\; i = 0, 1, \cdots, {}^{\sigma}N \tag{74} $$

Substituting ${}^{\sigma}\underset{\sim}{\alpha}^{0}$ and ${}^{\sigma}\underset{\sim}{\alpha}^{i}; i=1,\cdots,{}^{\sigma}N$ from (74) into (73)

$$
\begin{aligned}
{}_{s}^{d}\boldsymbol{\sigma} = & \left({}^{\sigma}\underset{\sim}{\alpha}^{0}\Big|_{\Omega} + \sum_{j=1}^{M^{\sigma}} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{0}}{\partial^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\Omega} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\Omega} \right) + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial\theta}\bigg|_{\Omega} \left(\theta - \theta|_{\Omega} \right) \right) \boldsymbol{I} \\
& + \sum_{i=1}^{N^{\sigma}} \left({}^{\sigma}\underset{\sim}{\alpha}^{i}\Big|_{\Omega} + \sum_{j=1}^{M^{\sigma}} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\Omega} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\Omega} \right) + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial\theta}\bigg|_{\Omega} \left(\theta - \theta|_{\Omega} \right) \right) {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}
\end{aligned}
\tag{75}
$$

Collecting coefficients of $\boldsymbol{I}, {}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{I}, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}, {}^{\sigma}\underset{\sim}{I}^{j}\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}, \left(\theta-\theta|_{\Omega}\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$ and $\left(\theta-\theta|_{\Omega}\right)\boldsymbol{I}$, we can write (75) as follows:

$$
\begin{aligned}
{}_{s}^{d}\boldsymbol{\sigma} = & \sigma_0 \boldsymbol{I} + \sum_{i=1}^{N^{\sigma}} {}^{\sigma}\underset{\sim}{a}_{j} \left({}^{\sigma}\underset{\sim}{I}^{j} \right) \boldsymbol{I} + \sum_{j=1}^{{}^{\sigma}M} {}^{\sigma}\underset{\sim}{b}_{j} \left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) + \sum_{j=1}^{{}^{\sigma}M} \sum_{i=1}^{N^{\sigma}} {}^{\sigma}\underset{\sim}{c}_{ij} \left({}^{\sigma}\underset{\sim}{I}^{j} \right) \left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) \\
& - \sum_{i=1}^{N^{\sigma}} {}^{\sigma}\underset{\sim}{d}^{i} \left(\theta - \theta|_{\Omega} \right) \left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) - \left({}^{\sigma}\alpha_{tm} \right)_{\Omega} \left(\theta - \theta|_{\Omega} \right) \boldsymbol{I}
\end{aligned}
\tag{76}
$$

The material coefficients ${}^{\sigma}\underset{\sim}{a}_{j}, {}^{\sigma}\underset{\sim}{b}_{i}, {}^{\sigma}\underset{\sim}{c}_{ij}, {}^{\sigma}\underset{\sim}{d}_{i}$ and ${}^{\sigma}\alpha_{tm}; i=1,2,\cdots,N; j=1,2,\cdots,M$ are defined in the following:

$$
\begin{aligned}
& \sigma_0 = \left({}^{\sigma}\underset{\sim}{\alpha}^{0}\Big|_{\Omega} - \sum_{j=1}^{{}^{\sigma}M} \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\Omega} \right) \left(- {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\Omega} \right); \quad \underset{\sim}{a}_{j} = \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\Omega} \\
& \underset{\sim}{b}_{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}\Big|_{\Omega} + \sum_{j=1}^{{}^{\sigma}M} \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\Omega} \left(- {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\Omega} \right); \quad \underset{\sim}{c}_{ij} = \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\Omega} \\
& {}^{\sigma}\underset{\sim}{d}_{i} = - \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\theta}\Bigg|_{\Omega}; \quad {}^{\sigma}\alpha_{tm} = - \frac{\partial^{\sigma}\alpha^{0}}{\partial\theta}\bigg|_{\Omega}
\end{aligned}
\tag{77}
$$

The constitutive theory (76) with material coefficients (77) is based on integrity, complete basis of the space of constitutive tensor ${}_{s}^{d}\boldsymbol{\sigma}$. Desired simplified forms can be obtained from (76) by retaining specific generators and invariants. This constitutive theory is ordered rate constitutive theory of orders n of strain tensors. Material coefficients can be functions of ${}^{\sigma}\underset{\sim}{I}^{j}; j=1,2,\cdots,M^{\sigma}$ and θ in a known configuration.

Simplified form of (76) can be obtained by retaining desired generators and the invariants. Perhaps a simplified yet most general constitutive theory for ${}_{s}^{d}\boldsymbol{\sigma}$ is one in which ${}_{s}^{d}\boldsymbol{\sigma}$ is a linear function of the components of its argument tensors. Redefining material coefficients and rearranging terms in (76) we can write the following:

$$
{}_{s}^{d}\boldsymbol{\sigma} = \sigma_0 \boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}\boldsymbol{\varepsilon} I + \sum_{i=1}^{n} \left(2\eta_i^{\sigma} \boldsymbol{\varepsilon}_{(i)} + \kappa_i^{\sigma} \left(\operatorname{tr}\left(\boldsymbol{\varepsilon}_{(i)} \right) \right) \boldsymbol{I} \right) - {}^{\sigma}\alpha_{tm} \left(\theta - \theta|_{\Omega} \right) \boldsymbol{I} \tag{78}
$$

in which σ_0 is initial stress field, μ^{σ} and λ^{σ} are Lames constants, η_i^{σ} and $\kappa_i^{\sigma}; i=1,2,\cdots,n$ is the spectrum of damping coefficients corresponding to strain rates $\boldsymbol{\varepsilon}_{(i)}; i=1,2,\cdots,n$, ${}^{\sigma}\alpha_{tm}$ is thermal modulus.

A further simplified model that is commonly used is obtained for $n=1$ *i.e.*, strain rate of order one only. In this case (78) reduces to

$$
{}_{s}^{d}\boldsymbol{\sigma} = \sigma_0 \boldsymbol{I} + 2\mu^{\sigma}(\boldsymbol{\varepsilon}) + \lambda^{\sigma}\operatorname{tr}(\boldsymbol{\varepsilon}) + 2\eta_1^{\sigma}\boldsymbol{\varepsilon}_{(1)} + \kappa_1^{\sigma}\operatorname{tr}\left(\boldsymbol{\varepsilon}_{(1)}\right)\boldsymbol{I} - {}^{\sigma}\alpha_{tm}\left(\theta - \theta|_{\Omega}\right)\boldsymbol{I} \tag{79}
$$

7.4. Constitutive Theory for Micro Stress Tensor $\boldsymbol{S}$

We consider microconstituent to have elasticity and dissipation mechanisms. Thus, following Section 7.3, we can choose the following for the constitutive tensor and its argument tensors related to microconstituent stress.

$$\boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)}, \boldsymbol{\varepsilon}_{(i)}^{(\alpha)}, \theta\right); i = 1,2,\cdots,n^s \tag{80}$$

where n^s is the highest order of rate of strain $\boldsymbol{\varepsilon}^{(\alpha)}$. Let ${}^s\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^s$ be the combined generators of the argument tensors of $\boldsymbol{S}$ in (80) and let ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ be the combined invariants of the same argument tensors of $\boldsymbol{S}$ in (80), then $\boldsymbol{I}, {}^s\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^s$ constitutes the basis of the space of constitutive tensor $\boldsymbol{S}$ and we can write the following for $\boldsymbol{S}$.

$$\boldsymbol{S} = {}^s\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N^s} {}^s\alpha_i \left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) \tag{81}$$

in which

$${}^s\alpha_i = {}^s\alpha_i\left({}^s\underset{\sim}{I}^j, \theta\right); j = 1,2,\cdots,M^s \tag{82}$$

Following the procedure described in Section 7.3 (Taylor series expansion) we can derive the following constitutive theory for $\boldsymbol{S}$

$$\begin{aligned}\boldsymbol{S} = S_0\boldsymbol{I} &+ \sum_{j=1}^{M^s} {}^s\underset{\sim}{a}_j\left({}^s\underset{\sim}{I}^j\right) + \sum_{i=1}^{N^s} {}^s\underset{\sim}{b}_i\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N^s}\sum_{j=1}^{M^s} {}^s\underset{\sim}{c}_{ij}\left({}^s I^j\right)\left({}^s\underset{\sim}{G}^i\right) \\ &- \sum_{i=1}^{N^s} {}^s\underset{\sim}{d}_i\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right){}^s\underset{\sim}{\boldsymbol{G}}^i - {}^s\alpha_{tm}\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{83}$$

in which material coefficients are given by (77) after replacing ${}^\sigma\alpha^i; i = 0,1,\cdots,N^\sigma$ with ${}^s\alpha^i; i = 0,1,\cdots,N^s$ and replacing ${}^\sigma\underset{\sim}{a}_j, {}^\sigma\underset{\sim}{b}_i, {}^\sigma\underset{\sim}{c}_{ij}, {}^\sigma\underset{\sim}{d}_i$ and ${}^\sigma\alpha_{tm}; i = 1,2,\cdots,N^\sigma; j = 1,2,\cdots,M^\sigma$ by ${}^s\underset{\sim}{a}_j, {}^s\underset{\sim}{b}_i, {}^s\underset{\sim}{c}_{ij}, {}^s\underset{\sim}{d}_i$ and ${}^s\alpha_{tm}; i = 1,2,\cdots,N^s; j = 1,2,\cdots,M^s$ and σ_0 by S_0. The material coefficients can be functions of ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and θ in a known configuration $\underset{\sim}{\Omega}$. This constitutive theory is based on integrity. A constitutive theory for $\boldsymbol{S}$ that is linear in the components of the argument tensors and is of order n^s is given by (after redefining material coefficients)

$$\boldsymbol{S} = S_0\boldsymbol{I} + 2\mu^s\left(\boldsymbol{\varepsilon}^{(\alpha)}\right) + \lambda^s\left(\text{tr}\left(\boldsymbol{\varepsilon}^{(\alpha)}\right)\right)\boldsymbol{I} + \sum_{i=1}^{n^s}\left(2\eta_i^s\boldsymbol{\varepsilon}_{(i)}^{(\alpha)} + \kappa_i^s\left(\text{tr}\boldsymbol{\varepsilon}_{(i)}^{(\alpha)}\right)\boldsymbol{I}\right) - {}^s\alpha_{tm}\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \tag{84}$$

This constitutive theory is of orders n^s in rates of microconstituent strain $\boldsymbol{\varepsilon}^{(\alpha)}$. When $n^s = 1$, we obtain the most simplified constitutive theory for $\boldsymbol{S}$.

$$\boldsymbol{S} = S_0\boldsymbol{I} + 2\mu^s\left(\boldsymbol{\varepsilon}^{(\alpha)}\right) + \lambda^s\left(\text{tr}\left(\boldsymbol{\varepsilon}^{(\alpha)}\right)\right)\boldsymbol{I} + 2\eta_1^s\boldsymbol{\varepsilon}_{(1)}^{(\alpha)} + \kappa_1^s\,\text{tr}\left(\boldsymbol{\varepsilon}_{(1)}^{(\alpha)}\right)\boldsymbol{I} - {}^s\alpha_{tm}\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \tag{85}$$

7.5. Constitutive Theory for Moment Tensor $\boldsymbol{m}$

Rigid rotations and rotation rates of the microconstituents in the elastic and viscous medium result in: 1) elasticity due to rotation gradient tensor, 2) dissipation due to viscous drag experienced by the microconstituents due to rates of the rotation gradients. If ${}_s^{c\Theta}\boldsymbol{J}_{(i)}; i = 0,1,\cdots,n^m$ are the rate of symmetric part of rotation gradient tensor, then we can write:

$$\boldsymbol{m} = \boldsymbol{m}\left({}_{s}^{c\Theta}\boldsymbol{J}, {}_{s}^{c\Theta}\boldsymbol{J}_{(i)}, \theta \right); i = 1,2,\cdots,n^m \tag{86}$$

Let ${}^{m}\underset{\sim}{\boldsymbol{G}}^{i}; i = 1,2,\cdots,N^m$ be the combined generators and let ${}^{m}\underset{\sim}{I}^{j}; j = 1,2,\cdots,M^m$ be the combined invariants of the same argument tensors, then $\boldsymbol{I}, {}^{m}\underset{\sim}{\boldsymbol{G}}^{i}; i = 1,2,\cdots,N^m$ forms the basis of the space (integrity) of constitutive tensor $\boldsymbol{m}$ and we can write the following for $\boldsymbol{m}$:

$$\boldsymbol{m}^{[0]} = {}^{m}\underset{\sim}{\alpha}^{0}\boldsymbol{I} + \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{\alpha}^{i}\left({}^{m}\underset{\sim}{\boldsymbol{G}}^{i} \right) \tag{87}$$

in which coefficients

$${}^{m}\alpha^{i} = {}^{m}\alpha^{i}\left({}^{m}\underset{\sim}{I}^{j}, \theta \right); i = 0,1,\cdots,N^m; j = 1,2,\cdots,M^m \tag{88}$$

Following the procedure described in section 7.3 (Taylor series expansion), we can derive the following constitutive theory for $\boldsymbol{m}$:

$$\begin{aligned}\boldsymbol{m} = {} & m_0\boldsymbol{I} + \sum_{j=1}^{M^m} {}^{m}\underset{\sim}{a}_{i}\, {}^{m}\underset{\sim}{I}^{j}\boldsymbol{I} + \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{b}_{i}\, {}^{m}\underset{\sim}{\boldsymbol{G}}^{i} + \sum_{i=1}^{N^m}\sum_{j=1}^{M^m} {}^{m}\underset{\sim}{c}_{ij}\, {}^{m}\underset{\sim}{I}^{j}\, {}^{m}\underset{\sim}{\boldsymbol{G}}^{i} \\ & - \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{d}_{i}\left(\theta - \theta|_{\Omega} \right) {}^{m}\underset{\sim}{\boldsymbol{G}}^{i} - {}^{m}\alpha_{tm}\left(\theta - \theta|_{\Omega} \right)\boldsymbol{I}\end{aligned} \tag{89}$$

in which material coefficients are given by after replacing ${}^{\sigma}\alpha^{i}; i = 0,1,\cdots,N^m$ and ${}^{\sigma}a_0, {}^{\sigma}\underset{\sim}{b}_{i}, {}^{\sigma}\underset{\sim}{c}_{ij}, {}^{\sigma}\underset{\sim}{d}_{i}; i = 1,2,\cdots,N^m; j = 1,2,\cdots,M^m$ with ${}^{m}\alpha^{i}; i = 0,1,\cdots,N^m$ and ${}^{m}\underset{\sim}{a}_{i}, {}^{m}\underset{\sim}{b}_{i}, {}^{m}\underset{\sim}{c}_{ij}, {}^{m}\underset{\sim}{d}_{i}, {}^{m}\alpha_{tm}; i = 1,2,\cdots,N^m; j = 1,2,\cdots,M^m$ and σ_0 by m_0. The material coefficients can be functions of ${}^{m}\underset{\sim}{I}^{\theta}; j = 1,2,\cdots,M^m$ and θ in a known configuration $\underset{\sim}{\Omega}$.

A constitutive theory that is linear in the components of the argument tensor is given by:

$$\begin{aligned}\boldsymbol{m}^{[0]} = {} & m_0\boldsymbol{I} + 2\left(\underset{\sim}{\mu}^{m} \right)\left({}_{s}^{c\Theta}\boldsymbol{J} \right) + \left({}^{m}\underset{\sim}{\lambda} \right)\left(\mathrm{tr}\left({}_{s}^{c\Theta}\boldsymbol{J} \right) \right)\boldsymbol{I} \\ & + \sum_{i=1}^{n^m}\left(\eta_i^m\left({}_{s}^{c\Theta}\boldsymbol{J}_{(i)} \right) + \kappa_i^m \mathrm{tr}\left({}_{s}^{c\Theta}\boldsymbol{J}_{(i)} \right) \right) - \left({}^{m}\alpha_{tm} \right)\left(\theta - \theta|_{\Omega} \right)\boldsymbol{I}\end{aligned} \tag{90}$$

When $n^m = 1$, we have the simplest possible constitutive theory for $\boldsymbol{m}$:

$$\begin{aligned}\boldsymbol{m} = {} & m_0\boldsymbol{I} + 2\mu^m\left({}_{s}^{c\Theta}\boldsymbol{J} \right) + \lambda^m\left(\mathrm{tr}\left({}_{s}^{c\Theta}\boldsymbol{J} \right) \right)\boldsymbol{I} + 2\eta_1^m\left({}_{s}^{c\Theta}\boldsymbol{J}_{(1)} \right) \\ & + \kappa_1^m\left(\mathrm{tr}\left({}_{s}^{c\Theta}\boldsymbol{J}_{(1)} \right) \right)\boldsymbol{I} - {}^{m}\alpha_{tm}\left(\theta - \theta|_{\Omega} \right)\boldsymbol{I}\end{aligned} \tag{91}$$

7.6. Constitutive Theory for q

In this derivation, we consider (based on conjugate pairs in the reduced entropy inequality)

Considering

$$\boldsymbol{q} = \boldsymbol{q}\left(\boldsymbol{g}, \theta \right) \tag{92}$$

Tensors $\boldsymbol{q}$ and $\boldsymbol{g}$ are tensors of rank one and θ is a tensor of rank zero. The only combined generator of rank one of the argument tensor $\boldsymbol{g}$ and θ is

$\boldsymbol{g}$, hence based on representation theorem, we can write:

$$\boldsymbol{q} = -{}^{q}\alpha \boldsymbol{g} \tag{93}$$

The coefficient ${}^{q}\alpha$ is a function of the combined invariants of $\boldsymbol{g}, \theta$ *i.e.*, $\boldsymbol{g}^{\mathrm{T}}\boldsymbol{g}$ and temperature θ. Let us define ${}^{q}\underset{\sim}{I} = \boldsymbol{g}^{\mathrm{T}}\boldsymbol{g}$ to simplify the details of further derivation. We note that (93) holds in the current configuration in which the deformation is not known. Hence, in (93), ${}^{q}\alpha = {}^{q}\alpha\left({}^{q}\underset{\sim}{I}, \theta\right)$ is not yet deterministic and it is not a material coefficient. To determine material coefficients in (93), we expand ${}^{q}\alpha\left({}^{q}\underset{\sim}{I}, \theta\right)$ in Taylor series about a known configuration $\underline{\Omega}$ in ${}^{q}\underset{\sim}{I}$ and θ and retain only up to linear terms in ${}^{q}\underset{\sim}{I}$ and θ (for simplicity)

$${}^{q}\alpha = {}^{q}\alpha\Big|_{\underline{\Omega}} + \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left({}^{q}\underset{\sim}{I} - \left({}^{q}\underset{\sim}{I}\right)_{\underline{\Omega}}\right) + \left.\frac{\partial {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right) \boldsymbol{g} \tag{94}$$

Substituting (94) into (93)

$$\boldsymbol{q} = -\left({}^{q}\alpha\Big|_{\underline{\Omega}} + \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left({}^{q}\underset{\sim}{I} - \left({}^{q}\underset{\sim}{I}\right)_{\underline{\Omega}}\right) + \left.\frac{\partial {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right)\right) \boldsymbol{g} \tag{95}$$

We note that ${}^{q}\alpha\Big|_{\underline{\Omega}}, \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}}$ and $\left.\frac{\partial {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}}$ are functions of $\left({}^{q}\underset{\sim}{I}\right)_{\underline{\Omega}}$ and $\theta\Big|_{\underline{\Omega}}$, whereas ${}^{q}\alpha$ in (93) is a function of ${}^{q}\underset{\sim}{I}$ and θ in the current configuration. From (95), we can write the following, noting that ${}^{q}\underset{\sim}{I} = \boldsymbol{g}^{\mathrm{T}}\boldsymbol{g}$

$$\boldsymbol{q} = -{}^{q}\alpha\Big|_{\underline{\Omega}} \boldsymbol{g} - \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left(\{g\}^{\mathrm{T}}\{g\}\right)\boldsymbol{g} - \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left(\{\boldsymbol{g}\}^{\mathrm{T}}\{\boldsymbol{g}\}\right)_{\underline{\Omega}}\boldsymbol{g} - \left.\frac{\partial {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right)\boldsymbol{g} \tag{96}$$

or

$$\boldsymbol{q} = -\left({}^{q}\alpha\Big|_{\underline{\Omega}} \boldsymbol{g} + \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left(\{g\}^{\mathrm{T}}\{g\}\right)\right)\boldsymbol{g} - \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left(\{\boldsymbol{g}\}^{\mathrm{T}}\{\boldsymbol{g}\}\right)_{\underline{\Omega}}\boldsymbol{g} - \left.\frac{\partial {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right)\boldsymbol{g} \tag{97}$$

Let

$$\kappa\left(\theta\Big|_{\underline{\Omega}}, \left({}^{q}\underset{\sim}{I}\right)_{\underline{\Omega}}\right) = {}^{q}\alpha\Big|_{\underline{\Omega}} - \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \left(\{g\}^{\mathrm{T}}\{g\}\right)_{\underline{\Omega}} \tag{98}$$

$$\kappa_1\left(\theta\Big|_{\underline{\Omega}}, \left({}^{q}\underset{\sim}{I}\right)_{\underline{\Omega}}\right) = \left.\frac{\partial {}^{q}\alpha}{\partial {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} \tag{99}$$

$$\kappa_2\left(\theta\Big|_{\underline{\Omega}}, \left({}^{q}\underset{\sim}{I}\right)_{\underline{\Omega}}\right) = \left.\frac{\partial {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}} \tag{100}$$

Then,

$$\boldsymbol{q} = -\kappa \boldsymbol{g} - \kappa_1\left(\{g\}^{\mathrm{T}}\{g\}\right)\boldsymbol{g} - \kappa_2\left(\theta - \theta_{\underline{\Omega}}\right)\boldsymbol{g} \tag{101}$$

This is the simplest possible constitutive theory based on conjugate pairs in the entropy inequality, representation theorem and (92). This constitutive theory uses integrity, the complete basis of the space of $\boldsymbol{q}$. The only assumption in this theory beyond (92) is the truncation of the Taylor series in (94) beyond linear terms in ${}^{q}\underset{\sim}{I}$ and θ. The constitutive theory for $\boldsymbol{q}$ in (101) is cubic in $\boldsymbol{q}$. It contains linear and cubic terms in $\boldsymbol{g}$, but does not contain a quadratic term in $\boldsymbol{g}$. Simplified linear theory is given by (101) by retaining only the first term on the right hand side of (101) (Fourier heat conduction law).

8. Thermodynamic and Mathematical Consistency of the Linear Micromorphic Theory Presented in the Paper

Surana *et al.* have shown in their earlier paper that the linear micromorphic microcontinuum theory presented for thermoelastic solids is thermodynamically and mathematically consistent. This is also the case for thermoviscoelastic solids. Significant aspects of the linear micromorphic theory presented in ref [1] are: strict use of classical continuum mechanics for microconstituents, valid integral-average definitions and their use in the derivation of macro conservation and balance laws, additively separating microconstituent deformation and its rigid rotations, use of classical rotations ${}_{c}\boldsymbol{\Theta}$ to define microconstituent rigid rotations, use of conjugate pairs in the entropy inequality to determine constitutive tensors and the initial choice of their argument tensors in conjunction with the axiom of causality, ensuring that constitutive tensors of rank two are symmetric tensors and their argument tensors of rank two are also symmetric tensors of rank two, strict adherence to the representation theorem in deriving constitutive theories, use of valid deformation measures derived in ref [33], appropriate additive decomposition of stress tensors to accommodate correct physics and to obtain valid constitutive tensors. Additionally, derivations of the constitutive theories presented in this paper for thermoviscoelastic solids with dissipation are carried out strictly using the theory of isotropic tensors and the entropy inequality, and are therefore thermodynamically and mathematically consistent. Thus, the complete mathematical model including the constitutive theories is thermodynamically and mathematically consistent.

9. Linear Micromorphic Theory of Eringen

Surana *et al.* in a recent paper have also presented an extensive discussion of Eringen's micromorphic theory including various issues, inconsistencies, incorrect definitions, questionable derivations of some balance laws and constitutive theories, which suggest that the micromorphic theory presented by Eringen may have serious concerns. Some of these, reported in ref [1], are summarized here: the microconstituents have nine deformational degrees of freedom, six due to $\boldsymbol{J}^{(\alpha)}$ and three unknown rigid rotations ${}_{\alpha}\boldsymbol{\Theta}$; use of a weighted integral of balance of micro linear momenta has no physical or mathematical basis and yields balance of angular momenta that cannot be derived using well known conventional approach; introduction of a third rank moment tensor for nonclassical physics using $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ which is due to classical continuum mechanics, is invalid; use of the permutation tensor with the moment tensor in the balance of linear momenta is in error, use of nonsymmetric constitutive tensors (of rank two) and their nonsymmetric argument tensors is not supported by the theory of isotropic tensors, necessity of an additional balance due to a new kinematically conjugate pair of rotations and moments is completely ignored, leading to spurious constitutive theories; use of phenomenologically constructed potentials containing nonsymmetric tensors of rank two to derive constitutive theories for nonsymmetric tensors of rank two has no basis based on the representation theorem; additive decomposition of $\boldsymbol{\sigma}$ is not employed to separate ${}_a\boldsymbol{\sigma}$, which cannot be part of a constitutive tensor as it is defined by the balance of angular momenta; the principle of equipresence introduces non physical coupling between classical and nonclassical physics; conservation of micro inertia as a conservation law introduced to obtain additional equations for closure of the mathematical model is neither needed nor used in the theory presented in ref [1]. It is thus conclusive, based on the points discussed in ref [1] and summarized above, that Eringen's micromorphic theory has many serious concerns and lacks thermodynamic and mathematical consistency.

10. Summary and Conclusions

The thermodynamically and mathematically consistent linear micromorphic microcontinuum theory derived by Surana $\mathcal{S}$ting of conservation and balance laws is utilized in the present work to derive constitutive theories for a linear micromorphic elastic continuum with dissipation. The significant aspects of the constitutive theories are:

1) Classical rotations ${}_c\boldsymbol{\Theta}$ describing rigid rotations of the microconstituents are separated from the argument tensor of the constitutive stress tensors.

2) Appropriate additive decomposition of the stress tensor $\boldsymbol{\sigma}$ is considered to separate volumetric and distortional physics and to ensure that ${}_a\boldsymbol{\sigma}$ is not part of the constitutive tensor as it is defined by the balance of angular momenta.

3) All constitutive tensors of rank two are symmetric with symmetric argument tensors of rank two, as necessitated by the representation theorem.

4) The constitutive theories naturally provide a mechanism of elasticity for the microconstituent, for the medium of the volume of matter as well as for the interaction of the microconstituent with the medium (due to rigid rotations ${}_c\boldsymbol{\Theta}$). The dissipation mechanisms for microconstituents (due to $\boldsymbol{\varepsilon}_{(i)}^{(\alpha)}$), for the volume of the medium (due to $\boldsymbol{\varepsilon}_{(i)}$) and due to interaction of the microconstituents with the medium (function of rates of the symmetric part of the rotation gradient tensor) are all ordered rate mechanisms. That is each dissipation physics can be considered to be dependent on up to the desired orders of the conjugate rates. Thus, in these constitutive theories there is a spectrum of dissipation coefficients for each of the three dissipation mechanisms.

5) All constitutive theories are initiated using conjugate pairs in the entropy inequality and are derived using the representation theorem, and are therefore thermodynamically and mathematically consistent.

Acknowledgements

The first author is grateful for his endowed professorships and the department of mechanical engineering of the University of Kansas for providing financial support to the second author. The computational facilities provided by the Computational Mechanics Laboratory of the mechanical engineering departments are also acknowledged.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Surana, K.S. and Mathi, S.S.C. (2025) Thermodynamically and Mathematically Consistent Linear Micromorphic Microcontinuum Theory for Solid Continua. *Journal of Applied Mathematics and Physics*, **13**, 3616-3661. https://doi.org/10.4236/jamp.2025.1310202

[2] Voigt, W. (1887) Theoretische studien uber die elastirifikaterhaltnisse der krystalle. *Abhandlungen der Wissenschaften Gesellschaft*, **34**, 3-51.

[3] Grad, H. (1952) Statistical Mechanics, Thermodynamics, and Fluid Dynamics of Systems with an Arbitrary Number of Integrals. *Communications on Pure and Applied Mathematics*, **5**, 455-494. https://doi.org/10.1002/cpa.3160050405

[4] Gunther, W. (1958) Zur statik und kinematik des cosseratschen kontinuums. *Abh. Braunschweig. Wiss. Ges.*, **10**, 195-213.

[5] Cosserat, E. and Cosserat, P. (1909) Théorie des Corps Déformables. Hermann.

[6] Schaefer, H. (1967) Das Cosserat Kontinuum. *Zeitschrift für Angewandte Mathematik und Mechanik*, **47**, 485-498. https://doi.org/10.1002/zamm.19670470802

[7] Eringen, A.C. (1967) Linear Theory of Micropolar Viscoelasticity. *International Journal of Engineering Science*, **5**, 191-204. https://doi.org/10.1016/0020-7225(67)90004-3

[8] Eringen, A. (1969) Compatibility Conditions of the Theory of Micromorphic Elastic Solids. *Indiana University Mathematics Journal*, **19**, 473-481. https://doi.org/10.1512/iumj.1970.19.19044

[9] Eringen, A.C. (1967) Theory of Micropolar Plates. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, **18**, 12-30. https://doi.org/10.1007/bf01593891

[10] Eringen, A.C. and Suhubi, E.S. (1964) Nonlinear Theory of Simple Micro-Elastic Solids—I. *International Journal of Engineering Science*, **2**, 189-203. https://doi.org/10.1016/0020-7225(64)90004-7

[11] Eringen, A.C. (1964) Simple Microfluids. *International Journal of Engineering Science*, **2**, 205-217. https://doi.org/10.1016/0020-7225(64)90005-9

[12] Eringen, A.C. (1964) Mechanics of Micromorphic Materials. In: Görtler, H., Ed., *Applied Mechanics*, Springer-Verlag, 131-138.

[13] Eringen, A.C. (1965) Theory of Micropolar Continua. *Proceedings of the 9th Mid-Western Mechanics Congress*, Madison, 16-18 August 1965, 23-40.

[14] Eringen, A.C. (1966) Linear Theory of Micropolar Elasticity. *Journal of Mathematics and Mechanics*, **15**, 909-923.

[15] Eringen, A. (1966) Theory of Micropolar Fluids. *Indiana University Mathematics Journal*, **16**, 1-18. https://doi.org/10.1512/iumj.1967.16.16001

[16] Eringen, A.C. (1968) Theory of Micropolar Elasticity. In: Liebowitz, H., Ed., *Fracture*, Academic Press, 621-729.

[17] Eringen, A.C. (1968) Mechanics of Micromorphic Continua. In: Kröner, E., Ed., *Mechanics of Generalized Continua*, Springer, 18-35. https://doi.org/10.1007/978-3-662-30257-6_2

[18] Eringen, A.C. (1970) Foundations of Micropolar Thermoelasticity. International Centre for Mechanical Studies, Course and Lectures No. 23. Springer-Verlag.

[19] Bringen, A.C. (1970) Balance Laws of Micromorphic Mechanics. *International Journal of Engineering Science*, **8**, 819-828. https://doi.org/10.1016/0020-7225(70)90084-4

[20] Eringen, A.C. (1972) Theory of Micromorphic Materials with Memory. *International Journal of Engineering Science*, **10**, 623-641. https://doi.org/10.1016/0020-7225(72)90089-4

[21] Eringen, A.C. (1992) Balance Laws of Micromorphic Continua Revisited. *International Journal of Engineering Science*, **30**, 805-810. https://doi.org/10.1016/0020-7225(92)90109-t

[22] Eringen, A.C. and Claus, W.D. (1970) A Micromorphic Approach to Dislocation Theory and Its Relation to Several Existing Theories. In: Shimron, J.A., DeWit, R. and Bullough, R., Eds., *Fundamental Aspects of Dislocation Theory*, Volume 2, National Bureau of Standards Publications, 1023-1040.

[23] Eringen, A.C. (1999) Microcontinuum Field Theories I. Foundations and Solids. Springer.

[24] Eringen, A.C. (2001) Microcontinuum Field Theories II. Fluent Media. Springer.

[25] Vernerey, F., Liu, W.K. and Moran, B. (2007) Multi-Scale Micromorphic Theory for Hierarchical Materials. *Journal of the Mechanics and Physics of Solids*, **55**, 2603-2651. https://doi.org/10.1016/j.jmps.2007.04.008

[26] Chen, Y., Lee, J. and Xiong, L. (2009) A Generalized Continuum Theory and Its Relation to Micromorphic Theory. *Journal of Engineering Mechanics*, **135**, 149-155. https://doi.org/10.1061/(asce)0733-9399(2009)135:3(149)

[27] Regueiro, R.A. (2010) On Finite Strain Micromorphic Elastoplasticity. *International Journal of Solids and Structures*, **47**, 786-800. https://doi.org/10.1016/j.ijsolstr.2009.11.006

[28] Wang, X. and Lee, J.D. (2010) Micromorphic Theory: A Gateway to Nano World. *International Journal of Smart and Nano Materials*, **1**, 115-135. https://doi.org/10.1080/19475411.2010.484207

[29] Lee, J.D. and Wang, X. (2011) Generalized Micromorphic Solids and Fluids. *International Journal of Engineering Science*, **49**, 1378-1387. https://doi.org/10.1016/j.ijengsci.2011.04.001

[30] Isbuga, V. and Regueiro, R.A. (2011) Three-Dimensional Finite Element Analysis of Finite Deformation Micromorphic Linear Isotropic Elasticity. *International Journal of Engineering Science*, **49**, 1326-1336. https://doi.org/10.1016/j.ijengsci.2011.04.006

[31] Reges, P.D.N., Pitangueira, R.L.S. and Silva, L.L. (2024) Modeling of Micromorphic Continuum Based on a Heterogeneous Microscale. *International Journal of Non-Linear Mechanics*, **167**, Article ID: 104881. https://doi.org/10.1016/j.ijnonlinmec.2024.104881

[32] McAvoy, R.C. (2024) Consistent Linearization of Micromorphic Continuum Theories. *Mathematics and Mechanics of Solids*, **30**, 1366-1392. https://doi.org/10.1177/10812865241280280

[33] Surana, K. and Mathi, S.S.C. (2025) Nonlinear Deformation/Strains for 3M Continua and Consistency of Linear Micropolar Theories. *Journal of Applied Mathematics and*

Physics, **13**, 933-989. https://doi.org/10.4236/jamp.2025.133049

[34] Surana, K.S. and Carranza, C.H. (2022) Nonclassical Continuum Theories for Fluent Media Incorporating Rotation Rates and Their Thermodynamic Consistency. *Journal of Applied Mathematics and Mechanics*, **103**, e202200079. https://doi.org/10.1002/zamm.202200079

[35] Surana, K.S. and Kendall, J.K. (2020) Existence of Rotational Waves in Non-Classical Thermoelastic Solid Continua Incorporating Internal Rotations. *Continuum Mechanics and Thermodynamics*, **32**, 1659-1683. https://doi.org/10.1007/s00161-020-00872-6

[36] Surana, K.S. and Carranza, C.H. (2020) Dynamic Behavior of Thermoelastic Solid Continua Using Mathematical Model Derived Based on Non-Classical Continuum Mechanics with Internal Rotations. *Meccanica*, **56**, 1345-1375. https://doi.org/10.1007/s11012-020-01221-2

[37] Surana, K.S. and Kendall, J.K. (2022) Rotational Inertial Physics in Non-Classical Thermoviscous Fluent Continua Incorporating Internal Rotation Rates. *Applied Mathematics*, **13**, 453-487. https://doi.org/10.4236/am.2022.136030

[38] Surana, K.S. and Kendall, J.K. (2023) NCCT for Micropolar Solid and Fluid Media Based on Internal Rotations and Rotation Rates with Rotational Inertial Physics: Model Problem Studies. *Applied Mathematics*, **14**, 612-651. https://doi.org/10.4236/am.2023.149037

[39] Surana, K.S. and Long, S.W. (2020) Ordered Rate Constitutive Theories for Non-Classical Thermofluids Based on Convected Time Derivatives of the Strain and Higher Order Rotation Rate Tensors Using Entropy Inequality. *Entropy*, **22**, Article No. 443. https://doi.org/10.3390/e22040443

[40] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) A Finite Deformation, Finite Strain Nonclassical Internal Polar Continuum Theory for Solids. *Mechanics of Advanced Materials and Structures*, **26**, 1-13.

[41] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) Non-classical Continuum Theory for Solids Incorporating Internal Rotations and Rotations of Cosserat Theories. *Continuum Mechanics and Thermodynamics*, **29**, 665-698. https://doi.org/10.1007/s00161-017-0554-1

[42] Surana, K.S., Joy, A.D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Thermoviscoelastic Solids without Memory Incorporating Internal and Cosserat Rotations. *Acta Mechanica*, **229**, 3189-3213. https://doi.org/10.1007/s00707-018-2163-x

[43] Surana, K.S., Long, S.W. and Reddy, J.N. (2016) Rate Constitutive Theories of Orders N and 1n for Internal Polar Non-Classical Thermofluids without Memory. *Applied Mathematics*, **7**, 2033-2077. https://doi.org/10.4236/am.2016.716165

[44] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Fluids with Internal Rotation Rates. *Applied Mathematics*, **9**, 907-939. https://doi.org/10.4236/am.2018.98063

[45] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Necessity of Law of Balance/Equilibrium of Moment of Moments in Non-Classical Continuum Theories for Fluent Continua. *Acta Mechanica*, **229**, 2801-2833. https://doi.org/10.1007/s00707-018-2143-1

[46] Surana, K.S., Mohammadi, F., Reddy, J.N. and Dalkilic, A.S. (2016) Ordered Rate Constitutive Theories for Non-Classical Internal Polar Thermoviscoelastic Solids without Memory. *International Journal of Mathematics, Science, and Engineering Applications*, **20**, 99-121.

[47] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories

for Non-Classical Thermoviscoelastic Solids with Dissipation and Memory Incorporating Internal Rotations. *Polytechnica*, **1**, 19-35. https://doi.org/10.1007/s41050-018-0004-2

[48] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Non-Classical Continuum Theories for Solid and Fluent Continua and Some Applications. *International Journal of Smart and Nano Materials*, **10**, 28-89. https://doi.org/10.1080/19475411.2018.1530700

[49] Surana, K., Powell, M.J. and Reddy, J.N. (2015) A More Complete Thermodynamic Framework for Fluent Continua. *Journal of Thermal Engineering*, **1**, 460-475. https://doi.org/10.18186/jte.00314

[50] Surana, K., Powell, M.J. and Reddy, J.N. (2015) A More Complete Thermodynamic Framework for Solid Continua. *Journal of Thermal Engineering*, **1**, 446-459. https://doi.org/10.18186/jte.17430

[51] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A Polar Continuum Theory for Fluent Continua. *International Journal of Engineering Research and Interdisciplinary Applications*, **6**, 107-146.

[52] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Constitutive Theories for Internal Polar Thermoelastic Solid Continua. *Journal of Pure and Applied Mathematics Advances and Applications*, **14**, 89-150.

[53] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Ordered Rate Constitutive Theories for Internal Polar Thermofluids. *International Journal of Mathematics*, *Science*, *and Engineering Applications*, **9**, 51-116.

[54] Surana, K.S., Reddy, J.N., Nunes, D. and Powell, M.J. (2015) A Polar Continuum Theory for Solid Continua. *International Journal of Engineering Research and Industrial Applications*, **8**, 77-106.

[55] Surana, K.S., Shanbhag, R. and Reddy, J.N. (2018) Necessity of Law of Balance of Moment of Moments in Non-Classical Continuum Theories for Solid Continua. *Meccanica*, **53**, 2939-2972. https://doi.org/10.1007/s11012-018-0851-1

[56] Smith, G.F. (1965) On Isotropic Integrity Bases. *Archive for Rational Mechanics and Analysis*, **18**, 282-292. https://doi.org/10.1007/bf00251667

[57] Smith, G.F. (1970) On a Fundamental Error in Two Papers of C.-C. Wang "on Representations for Isotropic Functions, Parts I and II". *Archive for Rational Mechanics and Analysis*, **36**, 161-165. https://doi.org/10.1007/bf00272240

[58] Smith, G.F. (1971) On Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. *International Journal of Engineering Science*, **9**, 899-916. https://doi.org/10.1016/0020-7225(71)90023-1

[59] Spencer, A.J.M. (1971) Theory of Invariants. In: Eringen, A.C., Ed., *Treatise on Continuum Physics*, *I*, Elsevier, 239-353. https://doi.org/10.1016/b978-0-12-240801-4.50008-x

[60] Spencer, A.J.M. and Rivlin, R.S. (1958) The Theory of Matrix Polynomials and Its Application to the Mechanics of Isotropic Continua. *Archive for Rational Mechanics and Analysis*, **2**, 309-336. https://doi.org/10.1007/bf00277933

[61] Spencer, A.J.M. and Rivlin, R.S. (1959) Further Results in the Theory of Matrix Polynomials. *Archive for Rational Mechanics and Analysis*, **4**, 214-230. https://doi.org/10.1007/bf00281388

[62] Wang, C.C. (1969) On Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **33**, 249-267. https://doi.org/10.1007/bf00281278

[63] Wang, C.C. (1969) On Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **33**, 268-287. https://doi.org/10.1007/bf00281279

[64] Wang, C.C. (1970) A New Representation Theorem for Isotropic Functions, Part I and II. *Archive for Rational Mechanics and Analysis*, **36**, 166-223.

[65] Wang, C.C. (1971) Corrigendum to My Recent Papers on "Representations for Isotropic Functions". *Archive for Rational Mechanics and Analysis*, **43**, 392-395. https://doi.org/10.1007/bf00252004

[66] Zheng, Q.S. (1993) On the Representations for Isotropic Vector-Valued, Symmetric Tensor-Valued and Skew-Symmetric Tensor-Valued Functions. *International Journal of Engineering Science*, **31**, 1013-1024. https://doi.org/10.1016/0020-7225(93)90109-8

[67] Zheng, Q.S. (1993) On Transversely Isotropic, Orthotropic and Relative Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. *International Journal of Engineering Science*, **31**, 1399-1453.

[68] Surana, K.S. and Mathi, S.S.C. (2025) A Nonlinear Micropolar Continuum Theory for Thermoviscoelastic Solid Medium Based on Classical Rotations. *Applied Mathematics*, **16**, 235-261. https://doi.org/10.4236/am.2025.163012

[69] Surana, K.S. and Mathi, S.S.C. (2025) Finite Deformation, Finite Strain Nonlinear Micropolar NCCT for Thermoviscoelastic Solids with Rheology. *Applied Mathematics*, **16**, 143-168. https://doi.org/10.4236/am.2025.161006

[70] Surana, K.S. and Mathi, S.S.C. (2022) Thermodynamic Consistency of Nonclassical Continuum Theories for Solid Continua Incorporating Rotations. *Continuum Mechanics and Thermodynamics*, **35**, 17-59. https://doi.org/10.1007/s00161-022-01163-y

[71] Yang, F., Chong, A.C.M., Lam, D.C.C. and Tong, P. (2002) Couple Stress Based Strain Gradient Theory for Elasticity. *International Journal of Solids and Structures*, **39**, 2731-2743. https://doi.org/10.1016/s0020-7683(02)00152-x

[72] Surana, K.S. (2015) Advanced Mechanics of Continua. CRC/Taylor and Francis.

[73] Surana, K.S. (2022) Classical Continuum Mechanics. 2nd Edition, CRC/Taylor and Francis.